Proofs for the paper

"A HYBRID ROBUST NON-HOMOGENEOUS FINITE-TIME DIFFERENTIATOR"

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All references and formulas enumeration are introduced in the maternal work.

Proof of theorem 1. Consider for this system the following Lyapunov function candidate:

$$V(\mathbf{e})=\gamma_1\beta|e_1|+\alpha|e_1|^{1.5}+0.5\psi(\mathbf{e})^2+0.5\gamma_2 e_2^2,\ \psi(\mathbf{e})=e_1-\alpha\sqrt{|e_1|}sign(e_1)+e_2,$$

$$\gamma_1\beta|e_1|+\alpha|e_1|^{1.5}+0.5\gamma_2 e_2^2\le V(\mathbf{e})\le(\gamma_1\beta+\alpha^2)|e_1|+\alpha|e_1|^{1.5}+2e_1^2+(2+0.5\gamma_2)e_2^2,$$

where $\gamma_1=\gamma_2+1$, $\gamma_2=1+\chi/\delta$. This function is continuously differentiable outside of the origin. Note that

$$(\gamma_1\beta+\alpha^2)|e_1|+\alpha|e_1|^{1.5}+2e_1^2\le(\gamma_1\beta+\alpha^2+\alpha)|e_1|+(2+\alpha)e_1^2,$$

then

$$\sqrt{V}\le\sqrt{\gamma_1\beta+\alpha^2+\alpha}\sqrt{|e_1|}+\sqrt{2+\alpha}|e_1|+\sqrt{2+0.5\gamma_2}|e_2|.$$

The full time derivative of the function V for the system (3) can be written as follows:

$$\begin{aligned}\dot V&=\gamma_1\beta\, sign(e_1)\dot e_1+1.5\alpha\sqrt{|e_1|}sign(e_1)\dot e_1+\psi\dot\psi+\gamma_2 e_2\dot e_2=\\&=-\alpha\gamma_1\beta\sqrt{|e_1|}+[\gamma_1\beta-\gamma(t)\gamma_2]sign(e_1)e_2-1.5\alpha^2|e_1|+1.5\alpha\sqrt{|e_1|}sign(e_1)e_2-\chi\gamma_2|e_2|-\gamma_2 e_2^2+\psi\dot\psi.\end{aligned}$$

Since

$$\begin{aligned}\dot\psi&=\dot e_1-0.5\alpha\dot e_1\, sign(e_1)^2/\sqrt{|e_1|}+\dot e_2=\\&=-\alpha\sqrt{|e_1|}sign(e_1)+[0.5\alpha^2-\gamma(t)]sign(e_1)-0.5\alpha e_2\, sign(e_1)^2/\sqrt{|e_1|}-\chi\, sign(e_2)\end{aligned}$$

and

$$\begin{aligned}\psi\dot\psi=&-1.5\alpha\sqrt{|e_1|}sign(e_1)e_2-\alpha|e_1|^{1.5}+[1.5\alpha^2-\gamma(t)]|e_1|-\chi\, sign(e_2)e_1-\\&-0.5\alpha e_2^2\, sign(e_1)^2/\sqrt{|e_1|}+[\alpha^2-\gamma(t)]sign(e_1)e_2-\chi|e_2|+\\&+\alpha[\gamma(t)-0.5\alpha^2]\sqrt{|e_1|}+\chi\alpha\sqrt{|e_1|}sign(e_1)sign(e_2)\end{aligned}$$

we have

$$\begin{aligned}\dot V=&-\alpha[\gamma_1\beta+0.5\alpha^2-\gamma(t)]\sqrt{|e_1|}+[\gamma_1\beta+\alpha^2-(1+\gamma_2)\gamma(t)]sign(e_1)e_2-\gamma(t)|e_1|-\alpha|e_1|^{1.5}-\\&-\chi(1+\gamma_2)|e_2|-\gamma_2 e_2^2-\chi\, sign(e_2)e_1-0.5\alpha e_2^2\, sign(e_1)^2/\sqrt{|e_1|}+\chi\alpha\sqrt{|e_1|}sign(e_1)sign(e_2).\end{aligned}$$

Substituting $-\chi\, sign(e_2)e_1\le\chi|e_1|$ and $\sqrt{|e_1|}sign(e_1)sign(e_2)\le\sqrt{|e_1|}$ we obtain:

$$\begin{aligned}\dot V\le&-\alpha[\gamma_1\beta+0.5\alpha^2-\gamma(t)-\chi\alpha]\sqrt{|e_1|}+[\gamma_1\beta+\alpha^2-(1+\gamma_2)\gamma(t)]sign(e_1)e_2-\\&-[\gamma(t)-\chi]|e_1|-\alpha|e_1|^{1.5}-\chi(1+\gamma_2)|e_2|-\gamma_2 e_2^2-0.5\alpha e_2^2\, sign(e_1)^2/\sqrt{|e_1|}.\end{aligned}$$

Completing squares we get

$$\begin{aligned}\dot V\le&-\alpha[\gamma_1\beta+0.5\alpha^2-\gamma(t)-\chi-0.5\rho(t)]\sqrt{|e_1|}-[\gamma(t)-\chi]|e_1|-\\&-\alpha|e_1|^{1.5}-\chi(1+\gamma_2)|e_2|-\gamma_2 e_2^2-\boldsymbol{\varepsilon}(t)\mathbf{R}(t)\boldsymbol{\varepsilon}(t)^T.\end{aligned}$$

for $\boldsymbol{\varepsilon}(t)=[|e_1|^{0.25}\ \ |e_1|^{-0.25}sign(e_1)e_2]$,

$$\mathbf{R}(t)=\frac{1}{2}\begin{bmatrix}\alpha\rho(t) & -\nu(t)\\ -\nu(t) & \alpha\end{bmatrix},\ \nu(t)=\gamma_1\beta+\alpha^2-(1+\gamma_2)\gamma(t),\ \rho(t)=[\nu(t)/\alpha]^2.$$

Such choice of the function ρ ensures that the matrix function $\mathbf{R}$ is not negative definite for all $t\ge 0$. Let

$$\gamma_1\beta+0.5\alpha^2-\gamma(t)-\chi-0.5\rho(t)\ge\eta>0, \tag{13}$$

then for $\mu=\min\{\alpha\eta/\sqrt{\gamma_1\beta+\alpha^2+\alpha},[\delta-\chi]/\sqrt{2+\alpha},\chi(1+\gamma_2)/\sqrt{2+0.5\gamma_2}\}$

$$\dot{V} \le -\alpha\eta\sqrt{|e_1|} - [\delta - \chi]|e_1| - \chi(1+\gamma_2)|e_2| \le -\mu\sqrt{V},$$

that for any initial conditions $\mathbf{e}(0) \in R^2$ implies stability and finite-time convergence to the origin of the system (3) trajectories with the desired upper estimate on T_0. To verify (13) consider the function

$$\lambda(\gamma_1, \gamma_2, \gamma) = \gamma_1\beta + 0.5\alpha^2 - \gamma - \chi - 0.5[(\gamma_1\beta + \alpha^2 - (1+\gamma_2)\gamma)/\alpha]^2,$$

it is necessary to find the values $\gamma_1 > 0$ and $\gamma_2 > 0$ such that $\lambda(\gamma_1, \gamma_2, \gamma) \ge \eta > 0$ for all $\delta \le \gamma \le \kappa$. This function is quadratic with respect to all its arguments, the roots with respect to the variable γ_1 has form:

$$\gamma_1(\gamma_2, \gamma) = \beta^{-1}[\gamma(1+\gamma_2) \pm \alpha\sqrt{2(\gamma\gamma_2 - \chi)}].$$

For the chosen value of $\gamma_2 = 1 + \chi/\delta$ the roots are real since $\gamma\gamma_2 - \chi > 0$. Analysis of the function λ shows that it is negative for large enough negative and positive values of γ_1, thus, the function is positive for

$$\beta^{-1}[\gamma(1+\gamma_2) - \alpha\sqrt{2(\gamma\gamma_2 - \chi)}] \le \gamma_1 \le \beta^{-1}[\gamma(1+\gamma_2) + \alpha\sqrt{2(\gamma\gamma_2 - \chi)}] \qquad (14)$$

and its maximum positive value is reached for $\gamma_1^{opt} = \beta^{-1}\gamma(1+\gamma_2)$. The chosen value $\gamma_1 = \gamma_2 + 1 = 0.5\beta^{-1}(\kappa+\delta)(\gamma_2+1)$ is the average of γ_1^{opt} for $\delta \le \gamma \le \kappa$. Next, we have to verify that the value $\gamma_1 = \gamma_2 + 1$ for any $\delta \le \gamma \le \kappa$ belongs to the interval (14), i.e.

$$\beta^{-1}[\gamma(1+\gamma_2) - \alpha\sqrt{2(\gamma\gamma_2 - \chi)}] \le 0.5\beta^{-1}(\kappa+\delta)(\gamma_2+1) \le \beta^{-1}[\gamma(1+\gamma_2) + \alpha\sqrt{2(\gamma\gamma_2 - \chi)}].$$

The latter is true if $\beta^{-1}[\kappa(1+\gamma_2) - \alpha\sqrt{2(\delta\gamma_2 - \chi)}] \le 0.5\beta^{-1}(\kappa+\delta)(\gamma_2+1) \le \beta^{-1}[\delta(1+\gamma_2) + \alpha\sqrt{2(\delta\gamma_2 - \chi)}]$, or, equivalently, $\kappa - \alpha\sqrt{2(\delta\gamma_2 - \chi)}/(1+\gamma_2) \le 0.5(\kappa+\delta) \le \delta + \alpha\sqrt{2(\delta\gamma_2 - \chi)}/(1+\gamma_2)$, that gives the conditions

$$\alpha > 0.5(\kappa - \delta)(1+\gamma_2)/\sqrt{2(\delta\gamma_2 - \chi)} = (2 + \chi/\delta)(L_1 + L_2 + 2\chi)/\sqrt{2\delta}.$$

The last thing to do is to evaluate $\eta = \inf_{\delta \le \gamma \le \kappa} \mathcal{F}(\gamma)$, $\mathcal{F}(\gamma) = \lambda(2 + \chi/\delta, 1 + \chi/\delta, \gamma)$. The function $\mathcal{F}$ is quadratic with respect to γ and has the negative coefficient for γ^2, thus, the function $\mathcal{F}$ reaches for its minimum on the ends of the interval $\delta \le \gamma \le \kappa$ and $\eta = \min\{\mathcal{F}(\delta), \mathcal{F}(\kappa)\}$.

P r o o f o f t h e o r e m 2. Consider for the system (3) the following Lyapunov function candidate:

$$W(\mathbf{e}) = \Gamma(\mathbf{e})|e_1| + 0.5e_2^2,$$

$$\Gamma(\mathbf{e}) = \begin{cases} 0.5[(\theta\sqrt{|e_1|})^{-1} sign(e_1)(\delta - \kappa)e_2 + (\delta + \kappa)] & \text{if } |e_2| < \theta\sqrt{|e_1|}; \\ \Lambda(\mathbf{e}) & \text{if } |e_2| \ge \theta\sqrt{|e_1|}, \end{cases} \quad \Lambda(\mathbf{e}) = \begin{cases} \delta & \text{if } e_1e_2 \ge 0; \\ \kappa & \text{if } e_1e_2 < 0, \end{cases}$$

where $\theta > 0$ is the design parameter. From the function Γ definition $\delta \le \Gamma(\mathbf{e}) \le \kappa$ for all $\mathbf{e} \in R^2$, hence,

$$\delta|e_1| + 0.5e_2^2 \le W(\mathbf{e}) \le \kappa|e_1| + 0.5e_2^2, \quad \sqrt{W} \le \sqrt{\kappa}\sqrt{|e_1|} + \sqrt{0.5}\,|e_2|.$$

The function W is continuous, but not continuously differentiable. For the case $|e_2| \ge \theta\sqrt{|e_1|}$ we have:

$$W(\mathbf{e}) = \Lambda(\mathbf{e})|e_1| + 0.5e_2^2, \; \dot{W} = \alpha\Lambda(\mathbf{e})\sqrt{|e_1|} + [\Lambda(\mathbf{e}) - \gamma(t)]e_2 sign(e_1) - \chi|e_2| - e_2^2 \le -\alpha\delta\sqrt{|e_1|} - \chi|e_2| - e_2^2.$$

The case $|e_2| < \theta\sqrt{|e_1|}$ is more complicated:

$$W(\mathbf{e}) = 0.5[\theta^{-1}(\delta - \kappa)\sqrt{|e_1|} sign(e_1)e_2 + (\delta + \kappa)|e_1|] + 0.5e_2^2,$$

$$\dot W = 0.5\theta^{-1}(\kappa-\delta)\sqrt{|e_1|}\,sign(e_1)e_2 - 0.25\theta^{-1}(\kappa-\delta)|e_1|^{-0.5}\,sign(e_1)^2 e_2^2 -$$
$$-0.5[\alpha(\kappa+\delta)-(\gamma(t)+\chi\, sign(e_1 e_2))\theta^{-1}(\kappa-\delta)]\sqrt{|e_1|} - \chi|e_2| - e_2^2 +$$
$$+[0.5(\kappa+\delta)-\gamma(t)-0.25\alpha\theta^{-1}(\kappa-\delta)]sign(e_1)e_2.$$

Since $\sqrt{|e_1|}\,sign(e_1)e_2 \le 0.5|e_1|^{-0.5}\,sign(e_1)^2 e_2^2 + 0.5|e_1|^{1.5}$ we obtain:

$$\dot W \le 0.25\theta^{-1}(\kappa-\delta)|e_1|^{1.5} - 0.5[\alpha(\kappa+\delta)-(\kappa+\chi)\theta^{-1}(\kappa-\delta)]\sqrt{|e_1|} - \chi|e_2| - e_2^2 +$$
$$+0.5(1+0.5\alpha\theta^{-1})(\kappa-\delta)|e_2| \le 0.25\theta^{-1}(\kappa-\delta)|e_1|^{1.5} - \chi|e_2| - e_2^2 -$$
$$-0.5[\alpha(\kappa+\delta)-[(\kappa+\chi)\theta^{-1}+(\theta+0.5\alpha)](\kappa-\delta)]\sqrt{|e_1|}.$$

For some $\rho > 0$, to be specified later, choose

$$\alpha = \{\rho + [\theta + (\kappa+\chi)\theta^{-1}](\kappa-\delta)\}/(1.5\delta+0.5\kappa),$$

then $\rho = \alpha(\kappa+\delta) - [(\kappa+\chi)\theta^{-1} + (\theta+0.5\alpha)](\kappa-\delta)$ and

$$\dot W \le 0.25\theta^{-1}(\kappa-\delta)|e_1|^{1.5} - \chi|e_2| - e_2^2 - 0.5\rho\sqrt{|e_1|}.$$

Let the constraint $|e_1| \le \rho\theta(\kappa-\delta)^{-1}$ hold, then

$$\dot W \le -0.25\rho\sqrt{|e_1|} - \chi|e_2| - e_2^2$$

and combining it with the estimate computed for the case $|e_2| \ge \theta\sqrt{|e_1|}$ we finally obtain:

$$\dot W \le -\mu\sqrt{W},\ \mu = \min\{\alpha\delta/\sqrt{\kappa}, 0.25\rho/\sqrt{\kappa}, \sqrt{2}\chi\},$$

that gives the required upper estimate for the time of convergence to zero T_0. Since

$$\delta|e_1(t)| \le \delta|e_1(t)| + 0.5e_2^2(t) \le W(\mathbf{e}(t)) \le W(\mathbf{e}(0)) \le \kappa|e_1(0)| + 0.5e_2^2(0)$$

for the initial conditions $\mathbf{e}(0) \in \Omega_0$, $\Omega_0 = \{\mathbf{e} \in R^2 : \kappa|e_1(0)| + 0.5e_2^2(0) \le \rho\theta\delta(\kappa-\delta)^{-1}\}$ the constraint

$$|e_1(t)| \le \rho\theta(\kappa-\delta)^{-1}$$

holds for all $t \ge 0$ and the derived estimates are valid.

The last thing to do is to optimize values of the parameters θ and ρ. Again, the value of μ is not changing if $0.25\rho/\sqrt{\kappa} = \sqrt{2}\chi$, therefore, $\rho = 4\sqrt{2\kappa}\chi$ and $\alpha = \{4\sqrt{2\kappa}\chi + [\theta + (\kappa+\chi)\theta^{-1}](\kappa-\delta)\}/(1.5\delta+0.5\kappa)$. The function α reaches for its minimum $\alpha = 2\{\sqrt{8\kappa}\chi + \sqrt{\chi+\kappa}(\kappa-\delta)\}/(1.5\delta+0.5\kappa)$ for $\theta = \sqrt{\chi+\kappa}$, that leads to the estimates given in the theorem formulation.

P r o o f o f c o r o l l a r y 1. The value $f(t)$, $t \in R$ is assumed to be accessible for a designer, thus $e_1(0) = 0$ is an admissible choice. The value $e_2(0) \in [-L_1, L_1]$ if $x_2(0) = 0$, that gives the estimate on the set of initial conditions:

$$0.5L_1^2 \le 4\sqrt{2\kappa(\chi+\kappa)}\chi\delta(\kappa-\delta)^{-1}.$$

Since $4\sqrt{2}\chi^2 \le 2\sqrt{2}(\beta+L_1+L_2+2\chi)\chi(\beta-L_1-L_2-2\chi)(L_1+L_2+2\chi)^{-1} \le 4\sqrt{2\kappa(\chi+\kappa)}\chi\delta(\kappa-\delta)^{-1}$ this estimate is satisfied if $0.5L_1^2 \le 4\sqrt{2}\chi^2$, that gives the proposed choice of χ admissible for any nonnegative υ.

In accordance with theorem 2 result the function

$$\mu = \min\{\alpha(\beta-L_1-L_2-2\chi)/\sqrt{\beta+L_1+L_2+2\chi}, \sqrt{2}\chi\}.$$

It is easy to see, that the first term under the minimum sign is an increasing function of β, therefore, if we are able to show that for the minimum value $L_1+L_2+3\chi$ of β the first term is always bigger than $\sqrt{2}\chi$, then the expression for μ can be simplified:

$$\alpha(\beta-L_1-L_2-2\chi)/\sqrt{\beta+L_1+L_2+2\chi}\Big|_{\beta=L_1+L_2+3\chi}\geq$$
$$\geq 4\{\sqrt{2(2L_1+2L_2+5\chi)}\,\chi+\sqrt{2L_1+2L_2+6\chi}\,(L_1+L_2+2\chi)\}\chi/\sqrt{2L_1+2L_2+5\chi}/(L_1+L_2+4\chi)\geq$$
$$\geq 4\chi(L_1+L_2+(2+\sqrt{2})\chi)/(L_1+L_2+4\chi).$$

The function $(L_1+L_2+(2+\sqrt{2})\chi)/(L_1+L_2+4\chi)$ is strictly decreasing in $\chi>0$ and its minimum is $0.25(2+\sqrt{2})$, therefore $\alpha(\beta-L_1-L_2-2\chi)/\sqrt{\beta+L_1+L_2+2\chi}\Big|_{\beta=L_1+L_2+3\chi}\geq(2+\sqrt{2})\chi>\sqrt{2}\chi$, that gives $\mu=\sqrt{2}\chi=\sqrt{2}(0.58^{-0.25}L_1+\upsilon)$ and the required upper estimate for T_0 follows by theorem 2 result.

P r o o f o f l e m m a 1 . Let us start with the system (5b), considering the Lyapunov function $U(x_2)=0.5x_2^2$, which time derivative takes form

$$\dot{U}\leq-2U+(\beta-\chi)|x_2|\leq-U+0.5(\beta-\chi)^2.$$

That implies the following time estimate

$$|x_2(t)|\leq|x_2(t_0)|e^{-0.5t}+|\beta-\chi|.$$

Next consider the same Lyapunov function for the system (5b), $U(x_1)=0.5(x_1-\tilde{f}(t))^2$, with the time derivative (since $\tilde{f}$ is Lipschitz continuous by Rademacher's theorem its derivative $\dot{\tilde{f}}$ exists almost everywhere in the sense of Lebesgue measure [4], [15], $|\dot{\tilde{f}}(t)|\leq C$ for almost all $t\in R$):

$$\dot{U}\leq-\alpha\sqrt{|x_1-\tilde{f}(t)|}\,|x_1-\tilde{f}(t)|+|x_1-\tilde{f}(t)|\,|x_2(t)+\dot{\tilde{f}}(t)|.$$

Since $|x_2(t)|\leq|x_2(t_0)|+|\beta-\chi|$ for $|x_2(t_0)|+|\beta-\chi|+C\leq0.5\alpha\sqrt{|x_1-\tilde{f}(t)|}$ we have:

$$\dot{U}\leq-0.5\alpha\sqrt{|x_1-\tilde{f}(t)|}\,|x_1-\tilde{f}(t)|<0,$$

that implies the result.

P r o o f o f t h e o r e m 3 . The proof of this theorem follows from theorem 1 and corollary 1 under observation that in coordinates $e_1(t)=x_1(t)-\tilde{f}(t)$, $e_2(t)=x_2(t)-\tilde{f}'(t)$ the system (5) is reduced to (3).

P r o o f o f c o r o l l a r y 2 . The result is a direct consequence of theorem 3 taking in mind that $\lambda_i=0$, $i=\overline{1,2}$.

P r o o f o f f a c t 1 . Consider the function $p(z,q)=\sqrt{|z|}sign(z)-\sqrt{|z-q|}sign(z-q)$ for any $z\in R$, $q\in R$. The quantity $p(z,q)=w(z,q)\sqrt{|q|}sign(q)$ holds for $w(z,q)=p(z,q)/[\sqrt{|q|}sign(q)]$. To show that the function w is defined for all $z\in R$, $q\in R$, note that

$$\lim_{q\to0}w(z,q)=\lim_{q\to0}\sqrt{|q|}/\sqrt{|z-q|}=0,\ \lim_{z\to0}w(z,q)=1,\ \lim_{z\to q}w(z,q)=1.$$

The function w is discontinuous, but well defined for the case $q=0$. To compute maximal values of the function w fix a $q\in R$ assuming that $q\neq0$, $z\neq0$, $z\neq q$ (the values of w at these points are already computed), then

$$\partial w/\partial z=0\ \Rightarrow\ z=0.5q$$

and $w_{\max}=\sup_{z\in R,q\in R}w(z,q)=w(0.5q,q)=\sqrt{2}$, that implies the claim.

P r o o f o f t h e o r e m 4 . Consider for the system (6) the same Lyapunov function as previously:

$$W(\mathbf{e})=\Gamma(\mathbf{e})|e_1|+0.5e_2^2,$$

where the function Γ is defined in the proof of theorem 2. Let us start with the variant $|e_2|<\theta\sqrt{|e_1|}$, then

$$W(\mathbf{e}) = 0.5[\theta^{-1}(\delta-\kappa)\sqrt{|e_1|}sign(e_1)e_2 + (\delta+\kappa)|e_1|] + 0.5e_2^2,$$

$$\begin{aligned}\dot W = {} & 0.5\theta^{-1}(\kappa-\delta)\sqrt{|e_1|}sign(e_1)e_2 - 0.25\theta^{-1}(\kappa-\delta)|e_1|^{-0.5}sign(e_1)^2e_2^2 - \\ & -0.5[\alpha(\kappa+\delta)-(\gamma(t)+\chi\, sign(e_1e_2))\theta^{-1}(\kappa-\delta)]\sqrt{|e_1|} - \chi|e_2| - e_2^2 + \\ & +[0.5(\kappa+\delta)-\gamma(t)-0.25\alpha\theta^{-1}(\kappa-\delta)]sign(e_1)e_2 + \\ & +[0.5(\kappa+\delta)sign(e_1)+0.25(\delta-\kappa)\theta^{-1}|e_1|^{-0.5}sign(e_1)^2e_2]\delta_1 + \\ & +[0.5\theta^{-1}(\delta-\kappa)\sqrt{|e_1|}sign(e_1)+e_2]\delta_2 .\end{aligned}$$

Taking in mind inequalities

$$\sqrt{|e_1|}sign(e_1)e_2 \le 0.5|e_1|^{-0.5}sign(e_1)^2e_2^2 + 0.5|e_1|^{1.5},$$

$$\begin{aligned}&[0.5(\kappa+\delta)sign(e_1)+0.25(\delta-\kappa)\theta^{-1}|e_1|^{-0.5}sign(e_1)^2e_2]\delta_1 \le \\ &\le [0.5(\kappa+\delta)+0.25(\kappa-\delta)]|\delta_1| = (0.25\delta+0.75\kappa)|\delta_1|,\end{aligned}$$

$$[0.5\theta^{-1}(\delta-\kappa)\sqrt{|e_1|}sign(e_1)+e_2]\delta_2 \le e_2\delta_2,\ \alpha = \{\rho+[\theta+(\kappa+\chi)\theta^{-1}](\kappa-\delta)\}/(1.5\delta+0.5\kappa)$$

for some $\rho > 0$ (to be specified later) we obtain:

$$\dot W \le 0.25\theta^{-1}(\kappa-\delta)|e_1|^{1.5} - \chi|e_2| - e_2^2 - 0.5\rho\sqrt{|e_1|} + (0.25\delta+0.75\kappa)|\delta_1| + e_2\delta_2 .$$

Let the constraint $|e_1| \le \rho\theta(\kappa-\delta)^{-1}$ hold, then

$$\begin{aligned}\dot W &\le -0.25\rho\sqrt{|e_1|} - \chi|e_2| - e_2^2 + (0.25\delta+0.75\kappa)|\delta_1| + e_2\delta_2 \le \\ &\le -0.25\rho\sqrt{|e_1|} - \chi|e_2| - e_2^2 + [(0.25\delta+0.75\kappa)\alpha + \sqrt{2}\beta\theta]\sqrt{2\lambda_0},\end{aligned}$$

where we used the series of relations $e_2\delta_2 \le 2\beta|e_2| < 2\beta\theta\sqrt{|e_1|} \le 2\beta\theta\sqrt{\lambda_0}$ (the last step follows by observation that $\delta_2 = 0$ for $|e_1| > \lambda_0$).

For the case $|e_2| \ge \theta\sqrt{|e_1|}$ we have:

$$W(\mathbf{e}) = \Lambda(\mathbf{e})|e_1| + 0.5e_2^2,$$

$$\begin{aligned}\dot W &= \alpha\Lambda(\mathbf{e})\sqrt{|e_1|} + [\Lambda(\mathbf{e})-\gamma(t)]e_2 sign(e_1) - \chi|e_2| - e_2^2 + \Lambda(\mathbf{e})sign(e_1)\delta_1 + e_2\delta_2 \le \\ &\le -\alpha\delta\sqrt{|e_1|} - \chi|e_2| - e_2^2 + \kappa|\delta_1| + e_2\delta_2 .\end{aligned}$$

According to fact 1, $|\delta_1| \le \alpha\sqrt{2\lambda_0}$, while in general case $|\delta_2| \le 2\beta$. The main issue of the last estimate is how to treat the disturbance δ_2 computing the required bounds on the trajectories convergence dependent on λ_0 only. Fortunately, the disturbance δ_2 affects negatively on the system dynamics in two compact sets only. In Fig. 4 the partition of the planar state space of the system (6) is shown, where $\delta_2 = 0$ for $|e_1| \ge \lambda_0$ (more precisely for $|e_1| \ge |\varphi|$). Since always $\delta_2 e_1 \ge 0$ by construction of δ_2, then $e_2\delta_2 \le 0$ in two quadrants with $e_1e_2 \le 0$. Finally, for $|e_2| \ge 2\beta$ the inequality $e_2\delta_2 - e_2^2 \le 0$ is satisfied provided that always $|\delta_2| \le 2\beta$, and $|e_2\delta_2| \le 6\alpha\beta\sqrt{2\lambda_0}$ for $|e_2| \le 3\alpha\sqrt{2\lambda_0}$. Thus, appearance of the destructive amplitude 2β of the disturbance δ_2 is possible into the compact set $\Upsilon = \{|e_1| < \lambda_0 \wedge 3\alpha\sqrt{2\lambda_0} < |e_2| < 2\beta \wedge e_1e_2 > 0\}$ only (see Fig. 4). Thus, for the cases $|e_1| \ge \lambda_0$, $|\delta_2| \ge 2\beta$ or $e_1e_2 \le 0$ we have

$$\dot W \le -\alpha\delta\sqrt{|e_1|} - \chi|e_2| + \kappa\alpha\sqrt{2\lambda_0} .$$

For the case $|e_2| \le 3\alpha\sqrt{2\lambda_0}$ we obtain

$$\dot W \le -\alpha\delta\sqrt{|e_1|} - \chi|e_2| - e_2^2 + \alpha(\kappa+6\beta)\sqrt{2\lambda_0} .$$

Combining these estimates with the one computed for the case $|e_2|<\theta\sqrt{|e_1|}$ we finally obtain:

$$\dot{W}\le-\mu\sqrt{W}+[(0.25\delta+\kappa)\alpha+\max\{\sqrt{2}\theta,6\}\beta]\sqrt{2\lambda_0}\ ,$$

where as in theorem 2 proof $\mu=\min\{\alpha\delta/\sqrt{\kappa},0.25\rho/\sqrt{\kappa},\sqrt{2}\chi\}$, this estimate is valid for all $\mathbf{e}\in\{\mathbf{e}\in R^2\setminus\Upsilon:|e_1|\le\rho\theta(\kappa-\delta)^{-1}\}$, that gives the following time estimate for any $t\ge t_0\ge 0$:

$$W(t)\le\max\{[\max\{\sqrt{W(t_0)}-0.25\mu(t-t_0)]^2,0\},8\mu^{-2}[(0.25\delta+\kappa)\alpha+\max\{\sqrt{2}\theta,6\}\beta]^2\lambda_0\}\ .$$

Now let us compute the estimate on the system (6) solutions into the set Υ. This set is composed by two disjoin subsets, where for the first one the constraints $0<e_1<\lambda_0$, $3\alpha\sqrt{2\lambda_0}<e_2<2\beta$, $e_1e_2>0$ hold. Then

$$\dot{e}_1=-\alpha\sqrt{|e_1|}sign(e_1)+e_2+\delta_1\ge-\alpha\sqrt{\lambda_0}+3\alpha\sqrt{2\lambda_0}-\alpha\sqrt{2\lambda_0}=\alpha\sqrt{2\lambda_0}$$

and the time T_1 of the set Υ crossing by the system (6) trajectories is upper bounded as follows (the time of passing from 0 to λ_0 with the minimal rate $\alpha\sqrt{2\lambda_0}$):

$$T_1\le\sqrt{\lambda_0}/(\alpha\sqrt{2})\,.$$

For the second subset where $0>e_1>-\lambda_0$, $-3\alpha\sqrt{2\lambda_0}>e_2>-2\beta$, $e_1e_2>0$ the same estimate can be computed similarly. The set Υ also can be left in the direction of the variable e_2, but we are looking for the maximal time of the trajectories stay into the set Υ, and the estimate on T_1 is sufficient (if the system exits from the set Υ faster in the direction e_2 than in e_1, then T_1 is still the maximal time of stay into the set; if the time of the set Υ crossing for the variable e_2 is bigger than T_1, then it is not important since the trajectories exit the set in time T_1 at the maximum). The following estimate for the Lyapunov function holds in Υ (actually being the worst case estimate it is satisfied for all $|e_2|\ge\theta\sqrt{|e_1|}$):

$$\dot{W}\le-\alpha\delta\sqrt{|e_1|}-\chi|e_2|+\kappa\alpha\sqrt{2\lambda_0}+\beta^2\le-\mu\sqrt{W}+\kappa\alpha\sqrt{2\lambda_0}+\beta^2,$$

where we used the fact that for $|\delta_2|\le 2\beta$ the inequality $e_2\delta_2-e_2^2\le\beta^2$ holds for any $e_2\in R$. Let us introduce into consideration for any $t\ge t_0\ge 0$ the variable

$$S(t)=W(t)-s(t)\,,\ s(t)=\int_{t_0}^{t}\kappa\alpha\sqrt{2\lambda_0}+\beta^2d\tau\,,\ s\in R_+,$$

then $\dot{S}=\dot{W}-\kappa\alpha\sqrt{2\lambda_0}+\beta^2\le-\mu\sqrt{W}\le-\mu\sqrt{\max\{S+s,0\}}\le-\mu\sqrt{S}$. Therefore, for any $t\ge t_0\ge 0$ we have $S(t)\le\max\{[\sqrt{S(t_0)}-0.5\mu(t-t_0)]^2,0\}$ and

$$W(t)=S(t)+s(t)\le\max\{[\sqrt{W(t_0)}-0.5\mu(t-t_0)]^2,0\}+\int_{t_0}^{t}\kappa\alpha\sqrt{2\lambda_0}+\beta^2d\tau\,.$$

For the trajectories into the set Υ we know that $t\le t_0+\sqrt{\lambda_0}/(\alpha\sqrt{2})$, consequently

$$W(t)\le\max\{[\sqrt{W(t_0)}-0.5\mu(t-t_0)]^2,0\}+(\kappa\sqrt{\lambda_0}+\beta^2/(\alpha\sqrt{2}))\sqrt{\lambda_0}\ .$$

Taking maximum over all estimates obtained for the Lyapunov function W we obtain for all $t\ge 0$:

$$W(t)\le\max\{[\sqrt{W(0)}-0.25\mu t]^2,0\}+c_1\lambda_0+c_2\sqrt{\lambda_0}\ ,$$

$$c_1=\max\{8\mu^{-2}[(0.25\delta+\kappa)\alpha+\max\{\sqrt{2}\theta,6\}\beta]^2,\kappa\}\,,\ c_2=\beta^2/(\alpha\sqrt{2})\,.$$

Since

$$\delta|e_1(t)|\le\delta|e_1(t)|+0.5e_2^2(t)\le W(\mathbf{e}(t))\le\kappa|e_1(0)|+0.5e_2^2(0)+c_1\lambda_0+c_2\sqrt{\lambda_0}$$

for the initial conditions $\mathbf{e}(0)\in\Omega_0$, $\Omega_0=\{\mathbf{e}\in R^2:\kappa|e_1(0)|+0.5e_2^2(0)\leq 0.5\rho\theta\delta(\kappa-\delta)^{-1}\}$ and the measurement noise $c_1\lambda_0+c_2\sqrt{\lambda_0}\leq 0.5\rho\theta\delta(\kappa-\delta)^{-1}$ the constraint $|e_1(t)|\leq\rho\theta(\kappa-\delta)^{-1}$ holds for all $t\geq 0$ and the derived estimates are valid.

The last thing to do is to optimize values of the design parameters θ and ρ as in theorem 2. The value of μ is not changing if $0.25\rho/\sqrt{\kappa}=\sqrt{2}\chi$, then $\rho=4\sqrt{2\kappa}\chi$ and $\alpha=\{4\sqrt{2\kappa}\chi+[\theta+(\kappa+\chi)\theta^{-1}](\kappa-\delta)\}/(1.5\delta+0.5\kappa)$. The function α reaches its minimum $\alpha=2\{\sqrt{8\kappa}\chi+\sqrt{\chi+\kappa}(\kappa-\delta)\}/(1.5\delta+0.5\kappa)$ for $\theta=\sqrt{\chi+\kappa}$, that leads to the estimates given in the theorem formulation.

P r o o f o f t h e o r e m 5. The dynamics of the system (5) can be reduced to (3) and if $e_1(t)=0$, $t\geq 0$ then necessarily $e_2(t)=0$, $t\geq 0$. From lemma 1 even for wrongly chosen parameters of the system (5) the solutions are always bounded. According to theorem 3, if the conditions $\tilde{L}_i^j\geq\tilde{L}_i$, $i=\overline{1,2}$ are satisfied, then for initial conditions (10) in the system (5) it holds that $x_1(t)=\tilde{f}(t)$ ($e_1(t)=0$) for all $t\geq t_j+T_0^j$ and the desired estimates (12) hold. If a finite instant t_{j+1} appears in (11), then the values $\tilde{L}_i^{j+1}$, $i=\overline{1,2}$ increase in accordance with (7). Since $\tilde{L}_i\in R_+$, $i=\overline{1,2}$, then either there exists $N>0$ such that $\tilde{L}_i^N\geq\tilde{L}_i$, $i=\overline{1,2}$ and the result of theorem 3 holds, either the conditions (12) are satisfied after some step $j\geq 0$ for all $t\geq t_j+T_0^j$. In both cases there exists a finite time $T_0\geq 0$.