Integral HOSM Semiglobal Controller for Finite-Time Exact Compensation of Unmatched Perturbations

Antonio Estrada and Leonid M. Fridman

Abstract—The present technical note studies a class of nonlinear systems with unmatched perturbations. The combination of integral high-order sliding modes with the hierarchical quasi-continuous controller is proposed allowing finite-time exact compensation of unmatched perturbations with semiglobal convergence features for both regulation and tracking.

Index Terms—High-order sliding modes (HOSM), nonlinear block controllable form (NBC-form), sliding mode (SM) control.

I. INTRODUCTION

It is a known issue that classical sliding mode (SM) control [22] is not able to compensate unmatched perturbations [25].

Combinations of different robust techniques and SM have been applied to deal with systems with unmatched uncertainties [11]–[14]. In order to reduce the effects of the unmatched uncertainties [9] proposes a method that combines $\mathcal{H}_\infty$ and integral sliding mode control. The main idea is to choose a projection matrix, which ensures that unmatched perturbations are not amplified and moreover minimized. For uncertain nonlinear systems in strict-feedback form [19], [20] develop the technique known as backstepping where a virtual control based on Lyapunov methods is constructed step by step. In a similar manner to backstepping, Multiple surface sliding control is proposed in [21] to simplify the controller design of systems where model differentiation is difficult.

The combination of the backstepping design and sliding mode control is studied in [6] for systems in strict-feedback form with parameter uncertainties and extended to the multi-input case in [7]. The procedure proposed in [6], [7] reduces the computational load, as compared with the standard backstepping strategy, because it only retains $n-2$ steps of the original backstepping technique, coupling them with with an auxiliary second order subsystem to which a second order sliding mode control is applied. In [8] the combination of dynamical adaptive backstepping and first and second order sliding mode control is applied to both triangular and nontriangular uncertain observable minimum phase nonlinear systems.

Another approach to the problem of unmatched uncertainty compensation is based on the nonlinear block controllable form (NBC-form) [1]. In [1] the sliding mode technique is applied to compensate the matched perturbations. A high gain approach is used to achieve compensation of unmatched uncertainties and stabilization of the sliding mode dynamics. In [15] a sliding mode controller is designed using the combination of block control [16], a sigmoid aproximation to the integral sliding mode control [17] and nested sliding mode control [18]. A

coordinate transformation is applied to design a nonlinear sliding manifold. This transformation requires smoothness of each virtual control, that is why sigmoid, instead of signum functions are used. With the use of the high gain approach in [1] and sigmoid functions in [15], [18] they prove that asymptotic tracking is achieved.

In [2] a new design algorithm for systems in strict-feedback form, a special case of the BC-form, is proposed. This algorithm achieves finite-time **exact** tracking of the desired output in the presence of smooth unmatched perturbations. These features are accomplished via the usage of quasi-continuous high-order sliding modes (HOSM) and a hierarchical design approach. In the first step the desired dynamic for the first state is defined by the desired tracking signal. After the first step the desired dynamic for each state is defined by the previous one. Each virtual control is divided into two parts, the first one is intended to compensate the nominal nonlinear part of the system and the second one is aimed to achieve the desired dynamics in spite of perturbations.

Difficulties arise when initial conditions lead to big initial errors because then the smoothness needed to achieve and keep the HOSM of each virtual control could be broken in some of them leading to loss of control. One possible solution is to increase the gains of the HOSM term included in each virtual control, nevertheless it goes against a key motivation of the algorithm which is to reduce discontinuous control gain via the use of information on the known nominal part of the system [24]. The solution proposed in the present technical note is to apply the integral HOSM approach reported in [3] in which the desired reference is reached by means of a previously designed auxiliary smooth trajectory that depends on the initial conditions of the error and its derivatives up to the order of the HOSM control used. Thus each state starts in the proper auxiliary sliding motion and the entire internal dynamics remain controlled from the beginning.

The present technical note proceeds as follows. In Section II the class of nonlinear systems to be treated and the problem formulation are described. Section III introduces the Quasi-continuous control [4], the hierarchical design algorithm proposed in [2] and finally introduces the integral HOSM [3]. Section IV presents the modification to the hierarchical design algorithm and its convergence proof. In Section V the algorithm is applied and simulation results are presented. The note then concludes with a brief comment on the proposed algorithm.

II. PROBLEM STATEMENT

Consider a class of nonlinear systems presented in the special NBC-form [1]

$$\left.\begin{aligned}
\dot{x}_1 &= f_1(x_1, t) + B_1(x_1, t)x_2 + \omega_1(x_1, t) \\
\dot{x}_i &= f_i(\overline{x}_i, t) + B_i(\overline{x}_i, t)x_{i+1} + \omega_i(\overline{x}_i, t) \\
\dot{x}_n &= f_n(x, t) + B_n(x, t)u + \omega_n(x, t) \\
i &= 2, \dots, n-1
\end{aligned}\right\} \quad (1)$$

where $x \in R^n$ is the state vector, $x_i \in R$, $\overline{x}_i = [x_1 \dots x_i]^T$; $u \in R$ is the control vector. Moreover $f_i(\overline{x}_i, t)$ and $B_i(\overline{x}_i, t)$ are smooth functions, ω_i is a bounded unknown perturbation term due to parameter variations and external disturbances with at least $n-i$ bounded derivatives w.r.t. system (1). $B_i \neq 0$ for $x \in X \subset R^n$, $t \in [0, \infty)$.

The relative degree, ϱ_d, of the output is assumed to be constant and known. To simplify the exposition, only the case of a $\rho_d = n$ is considered i.e. $y = x_1$; however, if zero dynamics stability is assumed, the approach can be used in the case of $\rho_d < n$. The whole state is assumed to be known. The goal is to achieve exact tracking of a smooth desired signal, y_d by the output y.

In [2], HOSM and feedback compensation is introduced through several virtual controls in order to ensure the desired tracking. The results

achieved in [2] are local and, for the sake of completeness, described in next section.

III. HIERARCHICAL QUASI-CONTINUOUS CONTROLLER DESIGN [2]

In the first part of this section the Quasi-continuous controller is briefly introduced, after that the design algorithm reported in [2] is re-stated for reader convenience and in the third part the integral HOSM approach is introduced.

A. Quasi-Continuous Controller [4]

Consider a single-input-single-output system of the form

$$\dot{\xi} = a(t,\xi) + b(t,\xi)u, \quad \xi \in R^n, \quad u \in R$$
$$\sigma : (t,\xi) \mapsto \sigma(t,\xi) \in R \tag{2}$$

where σ is the measured output of the system, u is the control. Smooth functions a, b, σ are assumed to be unknown, the dimension n can also be uncertain. The task is to achieve $\sigma \equiv 0$.

It is assumed that system (2) has a constant and known relative degree r. From [23] it follows that $\sigma^{(r)} = h(t,\xi) + g(t,\xi)u, g(t,\xi) \neq 0$ holds, where $h(t,\xi) = \sigma^{(r)}|_{u=0}, g(t,\xi) = (\partial/\partial u)\sigma^{(r)}$. If the inequalities $0 < K_m \le (\partial/\partial u)\sigma^{(r)} \le K_M, |\sigma^{(r)}|_{u=0}| \le C$ are fulfilled for some $K_m, K_M, C > 0$. Trajectories of (2) are assumed infinitely extendible in time for any Lebesgue-measurable bounded control u(t,x). The next differential inclusion is implied

$$\sigma^{(r)} \in [-C,C] + [K_m, K_M]u. \tag{3}$$

The above problem can be solved by the Quasi-continuous controller [4], that is constructed such that $\sigma = \dot{\sigma} = \ldots = \sigma^{(r-1)} = 0$ is established in finite time

$$u = -\alpha \Psi_{r-1,r}\left(\sigma, \dot{\sigma}, \ldots, \sigma^{(r-1)}\right) \tag{4}$$

$$\varphi_{0,r} = \sigma, \quad N_{0,r} = |\sigma|, \quad \Psi_{0,r} = \varphi_{0,r}/N_{0,r} = sign\sigma$$
$$\varphi_{i,r} = \sigma^{(i)} + \beta_i N_{i-1,r}^{(r-i)/(r-i+1)} \Psi_{i-1,r}$$
$$N_{i,r} = \left|\sigma^{(i)}\right| + \beta_i N_{i-1,r}^{(r-i)/(r-i+1)}$$
$$\Psi_{i,r}(\cdot) = \varphi_{i,r}/N_{i,r}; \quad i = 0, \ldots, r-1 \tag{5}$$

where $\beta_1, \ldots, \beta_{r-1}$, are positive numbers.

Theorem 1: [4] Provided that $\beta_1, \ldots, \beta_{r-1}, \alpha > 0$ are chosen sufficiently large in the listed order, the above design results in the r-sliding homogeneous controller

$$u = -\alpha \Psi_{r-1,r}\left(\sigma, \dot{\sigma}, \ldots, \sigma^{(r-1)}\right) \tag{6}$$

providing for the finite-time stability of (3), (6). The finite-time stable r-sliding mode $\sigma \equiv 0$ is established in system (2), (6).

B. Hierarchical Quasi-Continuous Controller Design Algorithm [2]

Each state is seen as a function of the previous states, $x_{i+1}(t) = \phi_i(\overline{x}_i(t))$, i.e., as a virtual control. For example consider the state x_2 in (1) as a virtual control, then uncertainty $\omega_1(x_1, t)$ is seen as matched. The next algorithm is proposed to achieve tracking of y_d.

Step 1: Defining $x_2 = \phi_1$, the n-sliding homogeneous quasi-continuous controller is included in ϕ_1, where ϕ_1 is an $n-1$ times differentiable function defined as

$$\phi_1(x_1, t, u_{1,1}) = B_1(x_1, t)^{-1}\{-f_1(x_1, t) + u_{1,1}\}$$
$$\dot{u}_{1,1} = u_{1,2}$$
$$\vdots$$
$$\dot{u}_{1,n-1} = -\alpha_1 \Psi_{n-1,n}\left(\sigma_1, \dot{\sigma}_1, \ldots, \sigma_1^{(n-1)}\right) \tag{7}$$

where $\sigma_1 = x_1 - y_d$. The first part of control $\phi_1(x_1)$ is aimed at compensating the nominal part of the system. The second part, $u_{1,1}$, is aimed at compensating perturbations; it is introduced through $n-1$ integrators in order to maintain the relative degree of x_1. $\Psi_{n-1,n}(\cdot)$ is defined as in (5) with the substitutions $r = n, \sigma = \sigma_1$.

The derivatives $\sigma_1, \dot{\sigma}_1, \ldots, \sigma_1^{(n-1)}$ are calculated by means of robust differentiators with finite-time convergence [5].

The convergence proof is based on the assumption of a constant and known relative degree, equal to n, of the output which leads to the existence of a differential inclusion $\sigma_1^n \in [-C_1, C_1] + [K_{m1}, K_{M1}]\dot{u}_{1,n-1}$ for some constants C_1, K_{m1}, K_{M1} and thus the establishment of the finite-time stable n-sliding mode for the constrain σ_1. An analog procedure is applied for each state. As it was mentioned before only local convergence can be assured.

Step i: The desired dynamics for x_i is $\phi_{i-1}(x_i)$. The control proposed is analogous to (7), but with some changes in the order

$$\phi_i(\overline{x}_i, t, u_{i,1}) = B_i(\overline{x}_i, t)^{-1}\{-f_i(\overline{x}_i, t) + u_{i,1}\}$$
$$\dot{u}_{i,1} = u_{i,2}$$
$$\vdots$$
$$\dot{u}_{i,n-i} = -\alpha_i \Psi_{n-i,n-i+1}\left(\sigma_i, \dot{\sigma}_i, \ldots, \sigma_i^{(n-i)}\right) \tag{8}$$

where $\sigma_i = x_i - \phi_{i-1}$ and $\Psi_{n-i,n-i+1}(\cdot)$ is defined as in (5), with the proper substitution $\sigma = \sigma_i$ in those equations.

Step n: In this step the real control is calculated, $\sigma_n = x_n - \phi_{n-1}$ then

$$u = B_n(x,t)^{-1}\{-f_n(x,t) + u_{n,1}\}$$
$$\text{where} \quad u_{n,1} = -\alpha_n sign(\sigma_n). \tag{9}$$

Remark 1: If the existence of perturbations and desired signal derivatives of the proper order can be assured, the order of the HOSM in each virtual control could be increased and thus the real control can be softened.

Theorem 2: [2] Provided that ω_i in system (1) and y_d are smooth functions with $n-i$ and n bounded derivatives respectively the above hierarchic design results in an ultimate controller $u = \phi_n(x_n)$ providing for the finite time stability of $\sigma_1 = x_1 - y_d = \dot{\sigma}_1 = \ldots = \sigma_1^{(n-1)} = 0$ in system (1).

Remark 2: The above theorem is valid only locally and stability is strongly dependent on the initial conditions of the system.

C. Integral HOSM [3]

In order to illustrate the convenience of the integral HOSM approach in combination with the above described design algorithm, consider the state x_n and recall the first step of the convergence proof for the control (9) as stated in [2]:

- For the state n

$$\dot{x}_n = f_n + B_n u + \omega_n(x,t)$$
$$\text{with} \quad u = B_n^{-1}\{-f_n + \alpha_n sign(\sigma_n)\}$$
$$\sigma_n = x_n - \phi_{n-1}; \quad \phi_{n-1} \quad \text{sufficiently smooth.}$$

Then $\dot{\sigma}_n = -\alpha_n sign(\sigma_n) + \omega_n(x,t) + \dot{\phi}_{n-1}$ and taking $\alpha_n \ge |\omega_n| + |\dot{\phi}_{n-1}|$ provides for the appearance of a 1-sliding mode for the constraint σ_n after a finite time Tn. Thus the subsystem

$$\dot{x}_{n-1} = f_{n-1} + B_{n-1}x_{n-1} + \omega_{n-1}(\overline{x}_{n-1}, t)$$

could be unstable in the transient, when $x_n \neq \phi_{n-1}$. The same can be said, unless the system is bounded-input-bounded-state (BIBS), for each of the remaining states used as virtual controls before they reach the desired dynamics.

In order to overcome the problem of transient dynamics, use of integral High-order sliding modes is proposed. The main idea is that each virtual control starts in the sliding mode of the proper order from the very beginning. The procedure is as follows.

Consider the r-sliding controller (6) and suppose a transient trajectory $\sigma(t, x(t)) = \rho(t)$, $t_0 \leq t \leq t_f$ such that

$$\left. \begin{array}{l} \rho(t_0) = \sigma(t_0), \ldots, \rho^{(j)}(t_0) = \sigma^{(j)}(t_0) \\ j = 1, \ldots, r-1; \quad \rho(t) \equiv 0 \ \forall \, t \geq t_f \end{array} \right\}. \quad (10)$$

Integral r-Sliding Mode: Let $\rho^{(r-1)}(t)$ be a Lipschitz function, then it has a globally bounded derivative $\rho^{(r)}(t)$ almost everywhere, and the new output $\Sigma(t, x) = \sigma(t, x) - \rho(t)$ satisfies $0 < K_m \leq (\partial/\partial u)\Sigma^{(r)} \leq K_M$, $|\Sigma^{(r)}|_{u=0}| \leq C$ with some changed constants $K_m, K_M, C > 0$.

Let the $(r-1)-$smooth function $\rho(t)$ satisfying (10) have the form

$$\rho = (t - t_f)^r \left(c_0 + c_1(t - t_0) + \ldots + c_{r-1}(t - t_0)^{r-1} \right). \quad (11)$$

Parameters c_i are now to be found from conditions (10) after t_f is assigned. In order to avoid the necessity of very large control values to reach the r-sliding mode $\overrightarrow{\sigma}(t_0) = 0$ (i.e. $\sigma = \dot{\sigma} = \ldots = \sigma^{(r-1)} = 0$) due to far distanced initial values, or a very low convergence rate if $\overrightarrow{\sigma}(t_0)$ is close to zero, instead of a constant let $t_f - t_0$ be a continuous positive-definite r-sliding homogeneous function of the initial conditions $\overrightarrow{\sigma}(t_0)$ of homogeneity degree 1, i.e.

$$t_f - t_0 = T\left(\overrightarrow{\sigma}(t_0) \right), \quad T(d_\kappa \overrightarrow{\sigma}) = \kappa T(\overrightarrow{\sigma}) \quad \forall \kappa > 0. \quad (12)$$

Theorem 3: [3] The function $\rho(t - t_0, \overrightarrow{\sigma}(t_0))$ is uniquely determined by (10), (11), (12). Then with any sufficiently large α, independent of the initial conditions $\overrightarrow{\sigma}(t_0)$, the controller (6):

$$u = \alpha \Psi_{r-1, r} \left(\Sigma, \dot{\Sigma}, \ldots, \Sigma^{(r-1)} \right)$$

$$\Sigma(t, x) = \begin{cases} \sigma(t, x) \\ \quad -\rho\left(t - t_0, \overrightarrow{\sigma}(t_0) \right), & t_0 \leq t \leq t_0 + T\left(\overrightarrow{\sigma}(t_0) \right) \\ \sigma(t, x), & t \geq t_0 + T\left(\overrightarrow{\sigma}(t_0) \right) \end{cases}$$

$$(13)$$

establishes the finite-time-stable r-sliding mode $\sigma \equiv 0$ with the transient time (12). The equality $\sigma(t, x(t)) = \rho(t - t_0, \overrightarrow{\sigma}(t_0)))$ is kept during the transient process.

The function used in this technical note for $T(\overrightarrow{\sigma}(t_0))$ is the one reported in [3], which expression is

$$T\left(\overrightarrow{\sigma}(t_0) \right) = \lambda \left(|\sigma(t_0)|^{p/r} + |\dot{\sigma}(t_0)|^{p/(r-1)} + \right.$$

$$\left. \ldots + \left| \sigma^{(r-1)}(t_0) \right|^p \right)^{1/p}; \quad p, \lambda > 0. \quad (14)$$

As previously mentioned the initial conditions independence feature achieved by the integral HOSM will be used in the Hierarchical Quasi-continuous design; the details are explained in the next section.

IV. INTEGRAL HIERARCHICAL QUASI-CONTINUOUS CONTROLLER

A. Hierachical Quasi-Continuous Controller Modification

The only modification to the above described hierarchical quasi-continuous controller is the substitution of each σ_i for $\Sigma_i = \sigma_i - \rho_i$, as described next.

Step i: The i-th sliding surface is chosen as $\Sigma_i = \sigma_i - \rho_i$ where $\sigma_i = x_i - \phi_{i-1}$ (with the exception $\sigma_1 = x_1 - y_d$)

$$\phi_i(\overline{x}_i, t, u_{i,1}) = B_i(\overline{x}_i, t)^{-1} \left\{ -f_i(\overline{x}_i, t) + u_{i,1} \right\}$$

$$\dot{u}_{i,1} = u_{i,2}$$

$$\vdots$$

$$\dot{u}_{i,n-i} = -\alpha_i \Psi_{n-i, n-i+1}$$

$$\times \left(\Sigma_i, \dot{\Sigma}_i, \ldots, \Sigma_i^{(n-i)} \right). \quad (15)$$

$\Psi_{n-i, n-i+1}$ is defined as in (5), obviously using $\sigma = \Sigma_i$ in those equations and where ρ_i fulfills condition (10):

$$\left. \begin{array}{l} \rho_i(t_0) = \sigma(t_0), \ldots, \rho_i^{(n-i)}(t_0) = \sigma_i^{(n-i)}(t_0) \\ \rho_i(t) \equiv 0 \quad \forall \, t \geq t_{fi} \end{array} \right\} \quad (16)$$

and is constructed according to (11) setting $r = n - i + 1$

$$T\left(\overrightarrow{\sigma_i}(t_0) \right) = \lambda_i \left(|\sigma_i(t_0)|^{p/n-i+1} + |\dot{\sigma}_i(t_0)|^{p/(n-i)} + \right.$$

$$\left. \ldots + \left| \sigma_i^{(n-i)}(t_0) \right|^p \right)^{1/p}. \quad (17)$$

Step n: $\Sigma_n = \sigma_n - \rho_n$ where $\sigma_n = x_n - \phi_{n-1}$

$$u = B_n(x, t)^{-1} \left\{ -f_n(x, t) + u_{n,1} \right\}$$

$$\text{where} \quad u_{n,1} = -\alpha_n \, sign(\Sigma_n). \quad (18)$$

Theorem 4: If system (1) is BIBS then provided that ω_i and y_d are smooth functions with $n - i$ and n bounded derivatives respectively the above hierarchic design results in the controller (18) that assures the finite time stability of $\sigma_1 = x_1 - y_d = \dot{\sigma}_1 = \ldots = \sigma_1^{(n-1)} = 0$ in system (1) independently of their initial conditions $x(t_0)$.

Remark 3: Observe that the BIBS condition is only a sufficient but not necessary condition as it can be seen in the convergence proof.

B. Convergence Proof

* Consider the state x_n

$$\dot{x}_n = f_n + B_n u + \omega_n(x, t)$$

$$\text{with} \quad u = B_n^{-1} \left\{ -f_n + \alpha_n sign(\Sigma_n) \right\}$$

$$\Sigma_n = \sigma_n - \rho_n, \quad \sigma_n = x_n - \phi_{n-1}, \quad \phi_{n-1}$$

$$\text{sufficiently smooth.}$$

Then $\dot{\Sigma}_n = -\alpha_n sign(\sigma_n) + \omega_n(x, t) + \dot{\phi}_{n-1} - \dot{\rho}_n$ and taking $\alpha_n \geq |\omega_n| + |\dot{\phi}_{n-1}| + |\dot{\rho}_n|$ provides for the appearance of a 1-sliding mode for the constraint Σ_n after $t = t_0$, i.e. since the beginning, and for σ_n after $T_n = t_{fn} - t_0 = \lambda_n(|\sigma_n(t_0)|)$ choosing $p = 1$ for (17).

* Now for the state x_{n-1}, with ϕ_{n-1} defined according to (15) and $\Sigma_{n-1} = \sigma_{n-1} - \rho_{n-1}$, $\sigma_{n-1} = x_{n-1} - \phi_{n-2}$ then

$$\dot{\Sigma}_{n-1} = \dot{x}_{n-1} - \dot{\phi}_{n-2} - \dot{\rho}_{n-1}$$

$$= f_{n-1} + B_{n-1}x_n + \omega_{n-1} - \dot{\rho}_{n-1} - \dot{\phi}_{n-2}$$

$$= f_{n-1} + B_{n-1}(\phi_{n-1} + \rho_{n-1}) + \omega_{n-1} - \dot{\rho}_{n-1} - \dot{\phi}_{n-2}.$$

The function f_{n-1}, may not be compensated right from $t = t_0$ because of the arbitrary initial condition of x_n. However due to the

BIBS condition x_{n-1} remains bounded and after $t = t_0 + T_n$, when $x_n = \phi_{n-1}$

$$\dot{\Sigma}_{n-1} = u_{n-1,1} + \omega_{n-1}(\overline{x}_{n-1}, t) - \dot{\rho}_n - \dot{\phi}_{n-2}$$
$$\ddot{\Sigma}_{n-1} = \dot{u}_{n-1,1} + \dot{\omega}_{n-1}(\overline{x}_{n-1}, t) - \ddot{\rho}_n - \ddot{\phi}_{n-2} \qquad (19)$$

that is (19) takes the form

$$\ddot{\Sigma}_{n-1} = h_{n-1}(t, x) + g_{n-1}(t, x)u_{n-1} \qquad (20)$$

$$\text{with} \quad h_{n-1}(t, x) = \ddot{\Sigma}_{n-1}|_{u_{n-1}=0};$$
$$g_{n-1}(t, x) = \frac{\partial}{\partial u_{n-1}} \ddot{\Sigma}_{n-1}$$
$$u_{n-1} = \dot{u}_{n-1,1}$$
$$= -\alpha_{n-1} \Psi_{1,2}(\Sigma_{n-1}, \dot{\Sigma}_{n-1}). \qquad (21)$$

If for some $K_{m_{n-1}}, K_{M_{n-1}}, C_{n-1} > 0$ the inequalities $0 < K_{m_{n-1}} \le (\partial/\partial u_{n-1})\ddot{\Sigma}_{n-1} \le K_{M_{n-1}}, |\ddot{\Sigma}_{n-1}|_{u_{n-1}=0}| \le C_{n-1}$ holds the next differential inclusion is implied

$$\ddot{\Sigma}_{n-1} \in [-C_{n-1}, C_{n-1}] + [K_{m_{n-1}}, K_{M_{n-1}}] u_{n-1} \qquad (22)$$

and controller (21) keeps (since it was established from t_0) stability of (22), (21). The finite-time stable 2-sliding mode is maintained for the constraint Σ_{n-1} from t_0 and for σ_{n-1} after $t_{fn-1} = t_0 + T_n$.

The same procedure can be applied to each one of the states of (1).

Remark 4: As it was previously mentioned it becomes clear that the BIBS condition is not a necessary one, it will suffice that in each subsystem of (1)

$$\dot{x}_i = f_i(\overline{x}_i, t) + B_i(\overline{x}_i, t)x_{i+1} + \omega_i(\overline{x}_i, t)$$

x_i remains bounded with the input x_{i+1} bounded, at least during the time interval $t < t_{fi}$; because after that time f_i is compensated. This BIBS condition may be relaxed with the application of the new advances in HOSM adjustment [10].

Notice that, with the use of integral HOSM in each virtual control, it is possible to introduce suitable dynamics on each of them. If direct application is used, in which only the input and the output is considered, this is not possible. The algorithm proposed in this technical note can be applied in physical systems such as synchronous machines and some underactuated manipulators that can be brought to the form (1).

V. EXAMPLE

Consider the perturbed third order system

$$\dot{x}_1 = 2\sin(x_1) + 1.5x_2 + \omega_1(x_1, t)$$
$$\dot{x}_2 = 0.8x_1x_2 + x_3 + \omega_2(\overline{x}_2, t)$$
$$\dot{x}_3 = -1.5x_3^2 + 2u + \omega_3(x, t) \qquad (23)$$

where functions ω_1, ω_2 are the unmatched bounded perturbations and function ω_3 is the matched perturbation. These functions were defined as follows:

$$\omega_1(x_1, t) = 0.2\sin(t) + 0.1x_1 + 0.12$$
$$\omega_2(\overline{x}_2, t) = 0.3\sin(2t) + 0.2x_1 + 0.2x_2 - 0.4$$
$$\omega_3(x, t) = 0.2\sin(2t) + 0.2x_1 + 0.3x_2 + 0.2x_3 + 0.3$$

a controller that achieves tracking of $y_d = 2\sin(0.15t) + 4\cos(0.1t) - 4$ by x_1 is desired. In addition to the previous perturbations the nominal
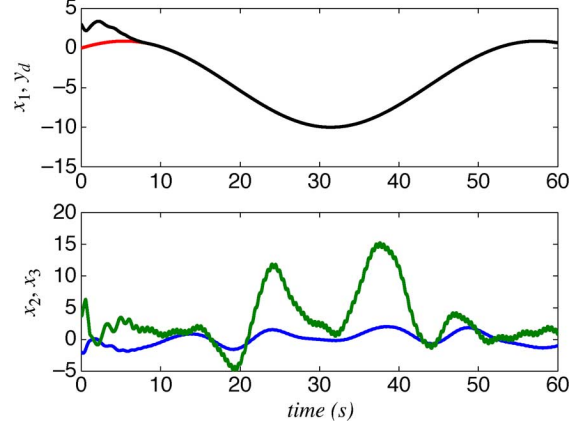

Fig. 1.　States evolution $(x(0) = [3\ -2\ 4]^T)$.

compensation term of the first two virtual controls is not exact. The first sliding surface is $\Sigma_1 = \sigma_1 - \rho_1$, $\sigma_1 = x_1 - y_d$ and the virtual control for x_1

$$\phi_1(x_1, t, u_{11}) = \frac{1}{1.5}\{-1.8\sin(x_1) + u_{11}\}$$
$$\dot{u}_{11} = u_{12}$$
$$\dot{u}_{12} = -\alpha_1 \Psi_{2,3}(\Sigma_1, \dot{\Sigma}_1, \ddot{\Sigma}_1);$$
$$\Psi_{2,3}(\Sigma_1, \dot{\Sigma}_1, \ddot{\Sigma}_1)$$
$$= \frac{\ddot{\Sigma}_1 + 2\left(|\dot{\Sigma}_1| + |\Sigma_1|^{2/3}\right)^{-1/2}\left(\dot{\Sigma}_1 + |\Sigma_1|^{2/3}sign(\Sigma_1)\right)}{|\ddot{\Sigma}_1| + 2\left(|\dot{\Sigma}_1| + |\Sigma_1|^{2/3}\right)^{1/2}}$$
$$\rho_1 = (t - t_{f1})^3 \left(c_{10} + c_{11}(t - t_0) + c_{12}(t - t_0)^2\right)$$
$$T_1 = \lambda_1 \left(|\sigma_1(t_0)|^2 + |\dot{\sigma}_1(t_0)|^3 + |\ddot{\sigma}_1(t_0)|^6\right)^{1/6}.$$

For the next state $\Sigma_2 = \sigma_2 - \rho_2$, $\sigma_2 = x_2 - \phi_1$ then

$$\phi_2(x_2, t, u_{21}) = -0.7x_1x_2 + u_{21}$$
$$\dot{u}_{21} = -\alpha_2 \Psi_{1,2}(\Sigma_2, \dot{\Sigma}_2)$$
$$\Psi_{1,2}(\Sigma_2, \dot{\Sigma}_2) = \frac{\dot{\Sigma}_2 + |\Sigma_2|^{1/2}sign(\Sigma_2)}{|\dot{\Sigma}_2| + |\Sigma_2|^{1/2}}$$
$$\rho_2 = (t - t_{f2})^2 \left(c_{20} + c_{21}(t - t_0)\right)$$
$$T_2 = \lambda_2 \left(|\sigma_2(t_0)| + |\dot{\sigma}_2(t_0)|^2\right)^{1/2}.$$

Finally for state x_3, $\Sigma_3 = \sigma_3 - \rho_3$, $\sigma_3 = x_3 - \phi_2$

$$u = \frac{1}{2}\{1.5x_3^2 + u_{31}\}$$
$$u_{31} = -\alpha_3 sign(\Sigma_3)$$
$$\rho_3 = (t - t_{f3})(c_{30}); \quad T_3 = \lambda_3 (|\sigma_3(t_0)|).$$

Results obtained in simulations are shown in Figs. 1 and 2, using $\alpha_1 = 4$, $\alpha_2 = 10$, $\alpha_3 = 8$, $\lambda_1 = 6$, $\lambda_2 = 0.5$, $\lambda_3 = 1$. In Fig. 1, $x(0) = [3\ -2\ 4]^T$ is used. In Fig. 2, $x(0) = [-3\ 1.5\ 2]^T$ and a phase lead of 30 s is introduced in y_d (seeFigs. 3 and 4).

VI. CONCLUSION

The problem of control design for nonlinear systems with unmatched perturbations is treated in this technical note. A design algorithm providing finite-time exact tracking of a smooth desired signal is given.
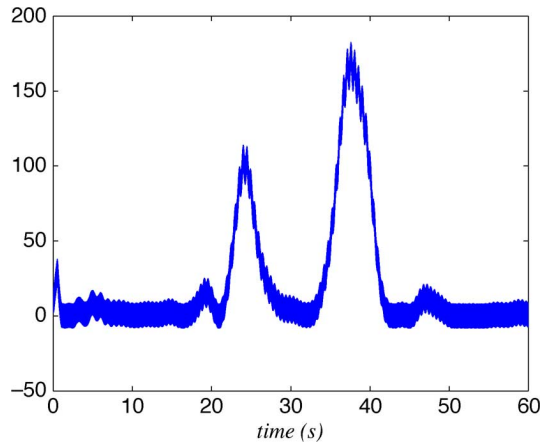
Fig. 2. Control signal u.
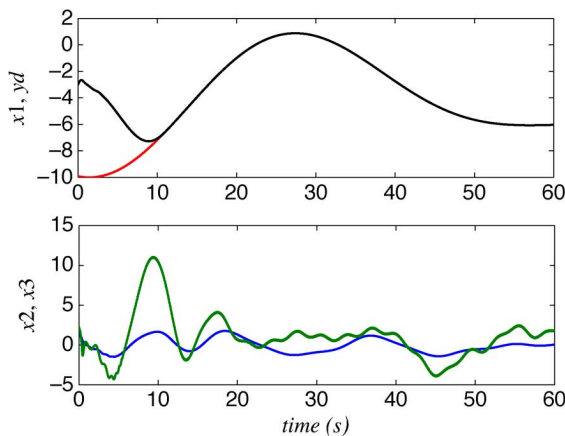
Fig. 3. States evolution ($x(0) = [-3 \ 1.5 \ 2]^T$).
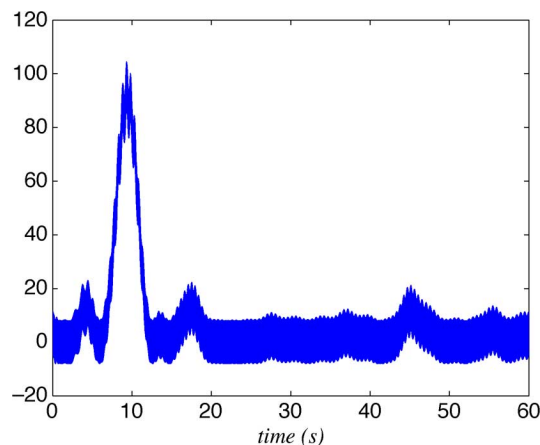
Fig. 4. Control signal u.

The proposed algorithm uses nested quasi-continuous HOSM in a hierarchical manner. The combination with integral HOSM in each virtual control allows introducing suitable dynamics on each of them thus providing semiglobal convergence.

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