Chapter 19

Ordinary Differential Equations

19.1 Classes of ODE

In this chapter we will deal with the class of functions satisfying the following *ordinary differential equation*

$$\boxed{\begin{array}{c} \dot{x}(t) = f(t, x(t)) \text{ for almost all } t \in [t_0, t_0 + \theta] \\ x(t_0) = x_0 \\ f : \mathbb{R} \times \mathcal{X} \to \mathcal{X} \end{array}} \tag{19.1}$$

where f is a nonlinear function and $\mathcal{X}$ is a Banach space (any concrete space of functions). The *Cauchy's problem* for (19.1) consists in resolving of (19.1), or, in other words, in finding a function $x(t)$ which satisfies (19.1).

For the simplicity we also will use the following abbreviation:

- ODE meaning an *ordinary differential equation*,

- DRHS meaning the *discontinuous right-hand side*.

Usually the following three classes of ODE (19.1) are considered:

1) *Regular ODE*:

$f(t,x)$ is continuous in both variables. In this case $x(t)$, satisfying (19.1), should be continuous differentiable, i.e.,

$$\boxed{x(t) \in C^1[t_0, t_0+\theta]} \tag{19.2}$$

2) *ODE of the Caratheodory's type*:

   $f(t,x)$ in (19.1) is measurable in t and continuous in x.

3) *ODE with discontinuous right-hand side*:

   $f(t,x)$ in (19.1) is continuous in t and discontinuous in x. In fact, this type of ODE equation is related to the *differential inclusion*:

$$\boxed{\dot{x}(t) \in F(t, x(t))} \tag{19.3}$$

   where $F(t,x)$ is a set in $\mathbb{R} \times \mathcal{X}$. If this set for some pair (t,x) consists of one point, then $F(t,x) = f(t,x)$.

19.2 Regular ODE

19.2.1 Theorems on existence

Theorem based on the contraction principle

Theorem 19.1 (on local existence and uniqueness) *Let $f(t,x)$ be a continuous in t on $[t_0, t_0+\theta]$, $(\theta \geq 0)$ and for any $t \in [t_0, t_0+\theta]$ it satisfies the, so-called,* ***local Lipschitz condition*** *in x, that is, there exists constant $c, L_f > 0$ such that*

$$\boxed{\begin{array}{c} \|f(t,x)\| \leq c \\ \\ \|f(t,x_1) - f(t,x_2)\| \leq L_f \|x_1 - x_2\| \end{array}} \tag{19.4}$$

for all $t \in [t_0, t_0+\theta]$ and all $x, x_1, x_2 \in B_r(x_0)$ where

$$B_r(x_0) := \{x \in \mathcal{X} \mid \|x - x_0\| \leq r\}$$

Then the Cauchy's problem (19.1) has a unique solution on the time-interval $[t_0, t_0+\theta_1]$, where

$$\boxed{\theta_1 < \min\left\{r/c, L_f^{-1}, \theta\right\}} \tag{19.5}$$

Proof.

1) First, show that the Cauchy's problem (19.1) is equivalent to finding the continuos solution to the following integral equation

$$x(t) = x_0 + \int\limits_{s=t_0}^{t} f(s, x(s))\, ds \tag{19.6}$$

Indeed, if $x(t)$ is a solution of (19.1), then, obviously, it is a differentiable function on $[t_0, t_0 + \theta_1]$. By integration of (19.1) on $[t_0, t_0 + \theta_1]$ we obtain (19.6). Inverse, suppose $x(t)$ is continuos function satisfying (19.6). Then, by the assumption (19.4) of the theorem, it follows

$$\begin{gathered}
\|f(s, x(s)) - f(s_0, x(s_0))\| = \\
\|[f(s, x(s)) - f(s, x(s_0))] + [f(s, x(s_0)) - f(s_0, x(s_0))]\| \leq \\
\|f(s, x(s)) - f(s, x(s_0))\| + \|f(s, x(s_0)) - f(s_0, x(s_0))\| \leq \\
L_f \|x(s) - x(s_0)\| + \|f(s, x(s_0)) - f(s_0, x(s_0))\|
\end{gathered}$$

This implies that if $s, s_0 \in [t_0, t_0 + \theta_1]$ and $s \to s_0$, then the right hand-side of the last inequality tends to zero, and, hence, $f(s, x(s))$ is continuous at each point of the interval $[t_0, t_0 + \theta_1]$. And, moreover, we also obtain that $x(t)$ is differentiable on this interval, satisfies (19.1) and $x(t_0) = x_0$.

2) Using this equivalence, let us introduce the Banach space $C[t_0, t_0 + \theta_1]$ of abstract continuous functions $x(t)$ with values in $\mathcal{X}$ and with the norm

$$\|x(t)\|_C := \max_{t \in [t_0, t_0 + \theta_1]} \|x(t)\|_{\mathcal{X}} \tag{19.7}$$

Consider in $C[t_0, t_0 + \theta_1]$ the ball $B_r(x_0)$ and notice that the nonlinear operator $\Phi : C[t_0, t_0 + \theta_1] \to C[t_0, t_0 + \theta_1]$ defined by

$$\Phi(x) = x_0 + \int\limits_{s=t_0}^{t} f(s, x(s))\, ds \tag{19.8}$$

transforms $B_r(x_0)$ in to $B_r(x_0)$ since

$$\|\Phi(x) - x_0\| = \max_{t\in[t_0,t_0+\theta_1]} \left\| \int_{s=t_0}^{t} f(s, x(s))\, ds \right\| \leq$$

$$\max_{t\in[t_0,t_0+\theta_1]} \int_{s=t_0}^{t} \|f(s, x(s))\|\, ds \leq \theta_1 c < r$$

Moreover, the operator Φ is a contraction (see Definition 14.20) on $B_r(x_0)$. Indeed, by the local Lipschitz condition (19.4), it follows

$$\|\Phi(x_1) - \Phi(x_2)\| = \max_{t\in[t_0,t_0+\theta_1]} \left\| \int_{s=t_0}^{t} [f(s, x_1(s)) - f(s, x_2(s))]\, ds \right\| \leq$$

$$\max_{t\in[t_0,t_0+\theta_1]} \int_{s=t_0}^{t} \|f(s, x_1(s)) - f(s, x_2(s))\|\, ds \leq$$

$$\theta_1 L_f \|x_1 - x_2\|_C = q\|x_1 - x_2\|_C$$

where $q := \theta_1 L_f < 1$ for small enough r. Then, by Theorem (the contraction principle) 14.17, we conclude that (19.6) has a unique solution $x(t) \in C[t_0, t_0 + \theta_1]$. Theorem is proven. ■

Corollary 19.1 *If in the conditions of Theorem 19.1 the Lipschitz condition (19.4) is fulfilled not locally, but* ***globally****, that is for all* $x_1, x_2 \in \mathcal{X}$ *(that corresponds with the case* $r = \infty$*), then the Cauchy's problem (19.1) has a unique solution for* $[t_0, t_0 + \theta]$ *for any* θ *big enough.*

Proof. It directly follows from Theorem 19.1 if take $r \to \infty$. But here we prefer to present also another proof based on another type of norm different from (19.7). Again, let us use the integral equivalent form (19.6). Introduce in the Banach space $C[t_0, t_0 + \theta_1]$ the following norm equivalent to (19.7):

$$\|x(t)\|_{\max} := \max_{t\in[t_0,t_0+\theta]} \left\|e^{-L_f t} x(t)\right\|_{\mathcal{X}} \tag{19.9}$$

Then

$$\|\Phi(x_1) - \Phi(x_2)\| \le L_f \int_{s=t_0}^{t} e^{-L_f s} e^{L_f s} \|x_1(s) - x_2(s)\|_C =$$

$$L_f \int_{s=t_0}^{t} e^{L_f s} \left(e^{-L_f s} \|x_1(s) - x_2(s)\|_C \right) ds \le$$

$$L_f \int_{s=t_0}^{t} e^{L_f s} \|x_1(t) - x_2(t)\|_{\max} ds =$$

$$L_f \int_{s=t_0}^{t} e^{L_f s} ds \|x_1(t) - x_2(t)\|_{\max} = \left(e^{L_f t} - 1 \right) \|x_1(t) - x_2(t)\|_{\max}$$

Multiplying this inequality by $e^{-L_f t}$ and taking $\max\limits_{t \in [t_0, t_0 + \theta]}$ we get

$$\|\Phi(x_1) - \Phi(x_2)\|_{\max} \le \left(1 - e^{-L_f \theta} \right) \|x_1(t) - x_2(t)\|_{\max}$$

Since $q := 1 - e^{-L_f \theta} < 1$ we conclude that Φ is a contraction. Taking then θ big enough we obtain the result. Corollary is proven. ■

Remark 19.1 *Sure, the global Lipschitz condition (19.4) with* $r = \infty$ *holds for very narrow class of functions which is known as the class of "**quasi-linear**" functions, that's why Corollary 19.1 is too conservative. On the other hand, the conditions of Theorem 19.1 for finite (small enough)* $r < \infty$ *is not so restrictive valid for any function satisfying somewhat mild smoothness conditions.*

Remark 19.2 *The main disadvantage of Theorem 19.1 is that the solution of the Cauchy's problem (19.1) exists only on the interval* $[t_0, t_0 + \theta_1]$ *(where* θ_1 *satisfies (19.5)), but not at the complete interval* $[t_0, t_0 + \theta]$*, that is very restrictive. For example, the Cauchy problem*

$$\dot{x}(t) = x^2(t), \ x(0) = 1$$

has the exact solution $x(t) = \dfrac{1}{1-t}$ *that exists only on* $[0, 1)$ *but not for all* $[0, \infty)$.

The theorem presented below gives a constructive (direct) method of finding a unique solution of the problem (19.1). It has several forms. Here we present the version of this result which does not use any Lipschitz conditions: neither local, no global.

Theorem 19.2 (Picard-Lindelöf, 1890) *Consider the Cauchy's problem (19.1) where the function $f(t,x)$ is continuous on*

$$\boxed{S := \left\{(t,x) \in \mathbb{R}^{1+n} \mid |t-t_0| \le \theta \le t_0,\ \|x-x_0\|_C \le r\right\}} \tag{19.10}$$

and the partial derivative $\frac{\partial}{\partial x} f : S \to \mathbb{R}^n$ is also continuous on S. Define the sets

$$\boxed{\mathcal{M} := \max_{(t,x)\in S} \|f(t,x)\|,\ L := \max_{(t,x)\in S} \left\| \frac{\partial}{\partial x} f(t,x) \right\|} \tag{19.11}$$

and choose the real number θ such that

$$\boxed{0 < \theta \le r,\ \theta\mathcal{M} \le r,\ q := \theta L < 1} \tag{19.12}$$

Then

1) *the Cauchy's problem (19.1) has unique solution on S;*

2) *the sequence $\{x_n(t)\}$ of functions generated iteratively by*

$$\boxed{\begin{array}{c} x_{n+1}(t) = x_0 + \displaystyle\int_{s=t_0}^{t} f(s, x_n(s))\, ds \\ x_0(t) = x_0,\ n = 0, 1, ...;\ t_0 - \theta \le t \le \theta + t_0 \end{array}} \tag{19.13}$$

to $x(t)$ in the Banach space $\mathcal{X}$ with the norm (19.7) converges geometrically as

$$\boxed{\|x_{n+1}(t) - x(t)\|_C \le q^{n+1} \|x_0 - x(t)\|_C} \tag{19.14}$$

Proof. Consider the integral equation (19.6) and the integral operator Φ (19.8) given on S. So, (19.6) can be represented as

$$\Phi(x(t)) = x(t),\ x(t) \in B_r(x_0)$$

where $\Phi : M \to \mathcal{X}$. For all $t \in [t_0, t_0 + \theta]$ we have

$$\|\Phi(x(t)) - x_0\| = \max_{t \in [t_0, t_0+\theta]} \left\| \int_{s=t_0}^{t} f(s, x_n(s))\, ds \right\| \leq$$
$$\max_{t \in [t_0, t_0+\theta]} (t - t_0) \max_{(t,x) \in S} \|f(t, x)\| \leq \theta \mathcal{M} \leq r$$

i.e., $\Phi(M) \subseteq M$. By the classical mean value theorem 16.5

$$\|f(t, x) - f(t, y)\| = \left\| \frac{\partial}{\partial x} f(t, z) \mid_{z \in [x,y]} (x - y) \right\| \leq L \|x - y\|$$

and, hence,

$$\|\Phi(x(t)) - \Phi(y(t))\| = \max_{t \in [t_0, t_0+\theta]} \left\| \int_{s=t_0}^{t} [f(s, x(s)) - f(s, y(s))]\, ds \right\|$$
$$\leq \theta L \max_{t \in [t_0, t_0+\theta]} \|x(t) - y(t)\| = q \|x(t) - y(t)\|_C$$

Applying now the contraction principle we obtain that (19.6) has a unique solution $x \in B_r(x_0)$. We also have

$$\|x_{n+1}(t) - x(t)\|_C = \max_{t \in [t_0, t_0+\theta]} \int_{s=t_0}^{t} [f(s, x_n(s)) - f(s, x(s))]\, ds$$
$$\leq q \|x_n(t) - x(t)\|_C \leq q^{n+1} \|x_0 - x(t)\|_C$$

Theorem is proven. ■

Theorem based on the Schauder fixed-point theorem

The next theorem to be proved drops the assumption of Lipschitz continuity but, also, the assertion of uniqueness.

Theorem 19.3 (Peano, 1890) *Consider the Cauchy´s problem (19.1) where the function $f(t, x)$ is continuous on S (19.10) where the real parameter θ is selected in such a way that*

$$\boxed{0 < \theta \leq r,\ \theta \mathcal{M} \leq r} \tag{19.15}$$

Then the Cauchy's problem (19.1) has ***at least one solution*** *on S.*

Proof.

a) The Schauder fixed-point theorem use (Zeidler 1995). By the same arguments as in the proof of Theorem 19.2 it follows that the operator $\Phi : B_r(x_0) \to B_r(x_0)$ is compact (see Definition). So, by the *Schauder fixed-point theorem* 18.20 we conclude that the operator equation $\Phi(x(t)) = x(t)$, $x(t) \in B_r(x_0)$ has at least one solution. This completes the proof.

b) Direct proof (Hartman 2002). Let $\delta > 0$ and $x_0(t) \in C^1[t_0 - \delta, t_0]$ satisfy on $[t_0 - \delta, t_0]$ the following conditions: $x_0(t) = x_0$, $\|x_0(t) - x_0\| \leq r$ and $\|x_0'(t)\| \leq d$. For $0 < \varepsilon \leq \delta$ define a function $x_\varepsilon(t)$ on $[t_0 - \delta, t_0 + \varepsilon]$ by putting $x_\varepsilon(t_0) = x_0$ on $[t_0 - \delta, t_0]$ and

$$x_\varepsilon(t) = x_0 + \int_{s=t_0}^{t} f(s, x_\varepsilon(s - \varepsilon))\, ds \qquad (19.16)$$

on $[t_0, t_0 + \varepsilon]$. Note that $x_\varepsilon(t)$ is C^0-function on $[t_0 - \delta, t_0 + \varepsilon]$ satisfying

$$\|x_\varepsilon(t) - x_0\| \leq r \text{ and } \|x_\varepsilon(t) - x_\varepsilon(s)\| \leq d\,|t - s|$$

Thus, for the family of functions $\{x_{\varepsilon_n}(t)\}$, $\varepsilon_n \to 0$ whereas $n \to \infty$ it follows that the limit $x(t) = \lim_{n\to\infty} x_{\varepsilon_n}(t)$ exists uniformly on $[t_0 - \delta, t_0 + \theta]$, that implies that

$$\|f(t, x_{\varepsilon_n}(t - \varepsilon_n)) - f(t, x(t))\| \to 0$$

uniformly as $n \to \infty$. So, term-by-term integration of (19.16) with $\varepsilon = \varepsilon_n$, gives (19.6) and, hence, $x(t)$ is a solution of (19.1). ■

The following corollary of the Peano's existence theorem is often used.

Corollary 19.2 ((Hartman 2002)) *Let $f(t, x)$ be a continuous on an open (t, x) - set of $\mathbb{E} \subseteq \mathbb{R}^{1+n}$ satisfying $\|f(t, x)\| \leq \mathcal{M}$. Let also $\mathbb{E}_0$ be a compact subset of $\mathbb{E}$. Then there exists a $\theta > 0$, depending on $\mathbb{E}$, $\mathbb{E}_0$ and $\mathcal{M}$, such that if $(t_0, x_0) \in \mathbb{E}_0$, then (19.6) has a solution on $|t - t_0| \leq \theta$.*

19.2.2 Differential inequalities, extension and uniqueness

The most important technique in ODE-theory involves the "integration" of the, so-called, differential inequalities. In this subsection we present results dealing with this integration process and extensively used throughout, and, then there will be presented its immediate application to the extension and uniqueness problems.

Bihari and Gronwall-Bellman inequalities

Lemma 19.1 ((Bihari 1956)) *Let*

1) $v(t)$ *and* $\xi(t)$ *be nonnegative continuous functions on* $[t_0, \infty)$*, that is,*

$$\boxed{v(t) \geq 0,\ \xi(t) \geq 0\ \forall t \in [t_0, \infty)\,,\ v(t)\,, \xi(t) \in C[t_0, \infty)} \tag{19.17}$$

2) *for any* $t \in [t_0, \infty)$ *the following inequality holds*

$$\boxed{v(t) \leq c + \int_{\tau=t_0}^{t} \xi(\tau)\,\Phi(v(\tau))\,d\tau} \tag{19.18}$$

where c *is a positive constant* $(c > 0)$ *and* $\Phi(v)$ *is a positive non-decreasing continuous function, that is,*

$$\boxed{0 < \Phi(v) \in C[t_0, \infty)\ \forall v \in (0, \bar{v})\,,\ \bar{v} \leq \infty} \tag{19.19}$$

Denote

$$\boxed{\Psi(v) := \int_{s=c}^{v} \frac{ds}{\Phi(s)}\ (0 < v < \bar{v})} \tag{19.20}$$

If in addition

$$\boxed{\int_{\tau=t_0}^{t} \xi(\tau)\,d\tau < \Psi(\bar{v} - 0)\,, t \in [t_0, \infty)} \tag{19.21}$$

then for any $t \in [t_0, \infty)$

$$\boxed{v(t) \le \Psi^{-1}\left(\int\limits_{\tau=t_0}^{t} \xi(\tau)\, d\tau\right)} \tag{19.22}$$

where $\Psi^{-1}(y)$ *is the function inverse to* $\Psi(v)$*, that is,*

$$y = \Psi(v), v = \Psi^{-1}(y) \tag{19.23}$$

In particular, if $\bar{v} = \infty$ *and* $\Psi(\infty) = \infty$*, then the inequality (19.22) is fulfilled without any constraints.*

Proof. Since $\Phi(v)$ is a positive non decreasing continuous function the inequality (19.18) implies that

$$\Phi(v(t)) \le \Phi\left(c + \int\limits_{\tau=t_0}^{t} \xi(\tau)\, \Phi(v(\tau))\, d\tau\right)$$

and

$$\frac{\xi(t)\, \Phi(v(t))}{\Phi\left(c + \int\limits_{\tau=t_0}^{t} \xi(\tau)\, \Phi(v(\tau))\, d\tau\right)} \le \xi(t)$$

Integrating the last inequality, we obtain

$$\int\limits_{s=t_0}^{t} \frac{\xi(s)\, \Phi(v(s))}{\Phi\left(c + \int\limits_{\tau=t_0}^{s} \xi(\tau)\, \Phi(v(\tau))\, d\tau\right)} ds \le \int\limits_{s=t_0}^{t} \xi(s)\, ds \tag{19.24}$$

Denote

$$w(t) := c + \int\limits_{\tau=t_0}^{t} \xi(\tau)\, \Phi(v(\tau))\, d\tau$$

Then evidently

$$\dot{w}(t) = \xi(t)\, \Phi(v(t))$$

Hence, in view of (19.20), the inequality (19.24) may be represented as

$$\int_{s=t_0}^{t} \frac{\dot{w}(s)}{\Phi(w(s))} ds = \int_{w=w(t_0)}^{w(t)} \frac{dw}{\Phi(w)}$$

$$= \Psi(w(t)) - \Psi(w(t_0)) \leq \int_{s=t_0}^{t} \xi(s)\, ds$$

Taking into account that $w(t_0) = c$ and $\Psi(w(t_0)) = 0$, from the last inequality it follows

$$\Psi(w(t)) \leq \int_{s=t_0}^{t} \xi(s)\, ds \tag{19.25}$$

Since

$$\Psi'(v) = \frac{1}{\Phi(v)} \quad (0 < v < \bar{v})$$

the function $\Psi(v)$ has the uniquely defined continuous monotonically increasing inverse function $\Psi^{-1}(y)$ defined within the open interval $(\Psi(+0), \Psi(\bar{v} - 0))$. Hence, (19.25) directly implies

$$w(t) = c + \int_{\tau=t_0}^{t} \xi(\tau)\Phi(v(\tau))\, d\tau \leq \Psi^{-1}\left(\int_{s=t_0}^{t} \xi(s)\, ds\right)$$

that, in view of (19.18), leads to (19.22). Indeed,

$$v(t) \leq c + \int_{\tau=t_0}^{t} \xi(\tau)\Phi(v(\tau))\, d\tau \leq \Psi^{-1}\left(\int_{s=t_0}^{t} \xi(s)\, ds\right)$$

The case $\bar{v} = \infty$ and $\Psi(\infty) = \infty$ is evident. Lemma is proven. ■

Corollary 19.3 *Taking in (19.22)*

$$\boxed{\Phi(v) = v^m \ (m > 0, m \neq 1)}$$

it follows

$$v(t) \leq \left[c^{1-m} + (1-m)\int_{\tau=t_0}^{t} \xi(\tau)\,d\tau\right]^{\dfrac{1}{m-1}} \quad \text{for } 0 < m < 1 \tag{19.26}$$

and

$$v(t) \leq c\left[1-(1-m)\,c^{m-1}\int_{\tau=t_0}^{t} \xi(\tau)\,d\tau\right]^{-\dfrac{1}{m-1}}$$

for

$$m > 1 \text{ and } \int_{\tau=t_0}^{t} \xi(\tau)\,d\tau < \frac{1}{(m-1)\,c^{m-1}}$$

Corollary 19.4 ((Gronwall 1919)) *If $v(t)$ and $\xi(t)$ are nonnegative continuous functions on $[t_0, \infty)$ verifying*

$$v(t) \leq c + \int_{\tau=t_0}^{t} \xi(\tau)\,v(\tau)\,d\tau \tag{19.27}$$

then for any $t \in [t_0, \infty)$ the following inequality holds:

$$v(t) \leq c\exp\left(\int_{s=t_0}^{t} \xi(s)\,ds\right) \tag{19.28}$$

This results remains true if $c = 0$.

Proof. Taking in (19.18) and (19.20)

$$\Phi(v) = v$$

we obtain (19.193) and, hence, for the case $c > 0$

$$\Psi(v) := \int_{s=c}^{v} \frac{ds}{s} = \ln\left(\frac{v}{c}\right)$$

and

$$\Psi^{-1}(y) = c \cdot \exp(y)$$

that implies (19.28). The case $c = 0$ follows from (19.28) applying $c \to 0$. ■

Differential inequalities

Here we completely follow (Hartman 2002).

Definition 19.1 *Let $f(t,x)$ be a continuous function on a plane (t,x)-set $\mathbb{E}$. By a* ***maximal solution*** *$x^0(t)$ of the Cauchy's problem*

$$\boxed{\dot{x}(t) = f(t,x),\ x(t_0) = x_0 \in \mathbb{R}} \qquad (19.29)$$

is meant a solution of (19.29) on a maximal interval of existence such that if $x(t)$ is any solution of (19.29) then

$$\boxed{x(t) \le x^0(t)} \qquad (19.30)$$

holds (by component-wise) on the common interval of existence of $x(t)$ and $x^0(t)$. The ***minimal solution*** *is similarly defined.*

Lemma 19.2 *Let $f(t,x)$ be a continuous function on a rectangle*

$$S^+ := \left\{(t,x) \in \mathbb{R}^2 \mid t_0 \le t \le t_0 + \theta \le t_0,\ \|x - x_0\|_C \le r\right\} \qquad (19.31)$$

and on S^+

$$\|f(t,x)\| \le \mathcal{M} \text{ and } \alpha := \min\{\theta; r/\mathcal{M}\}$$

Then the Cauchy's problem (19.29) has a solution on $[t_0, t_0 + \alpha)$ such that every solution $x = x(t)$ of $\dot{x}(t) = f(t,x)$, $x(t_0) \le x_0$ satisfies (19.30) on $[t_0, t_0 + \alpha)$.

Proof. Let $0 < \alpha' < \alpha$. Then, by the Peano's existence theorem 19.3, the Cauchy problem

$$\dot{x}(t) = f(t,x) + \frac{1}{n},\ x(t_0) = x_0 \qquad (19.32)$$

has a solution $x = x_n(t)$ on $[t_0, t_0 + \alpha']$ if n is sufficiently large. Then there exists a subsequence $\{n_k\}_{k=1,2,..}$ such that the limit $x^0(t) =$

$\lim_{k\to\infty} x_{n_k}(t)$ exists uniformly on $[t_0, t_0 + \alpha']$ and $x^0(t)$ is a solution of (19.29). To prove that (19.30) holds on $[t_0, t_0 + \alpha']$ it is sufficient to verify

$$x(t) \le x_n(t) \text{ on } [t_0, t_0 + \alpha'] \tag{19.33}$$

for large enough n. If this is not true, then there exists a $t = t_1 \in (t_0, t_0 + \alpha')$ such that $x(t_1) > x_n(t_1)$. Hence there exists a largest t_2 on $[t_0, t_1)$ such that $x(t_2) = x_n(t_2)$ and $x(t) > x_n(t)$. But by (19.32) $x_n'(t_2) = x'(t_2) + \frac{1}{n}e$, so that $x_n(t) > x(t)$ for $t > t_2$ near t_2. This contradiction proves (19.33). Since $\alpha' < \alpha$ is arbitrary, the lemma follows. ■

Corollary 19.5 *Let $f(t, x)$ be a continuous function on an open set $\mathbb{E}$ and $(t_0, x_0) \in \mathbb{E} \subseteq \mathbb{R}^2$. Then the Cauchy's problem (19.29) a maximal and minimal solution near (t_0, x_0).*

Right derivatives

Lemma 19.3

1. *If $n = 1$ and $x \in C^1[a, b]$ then $|x(t)|$ has a right derivative*

$$\boxed{D_R|x(t)| := \lim_{0<h\to 0}\frac{1}{h}\left[|x(t+h)| - |x(t)|\right]} \tag{19.34}$$

such that

$$\boxed{D_R|x(t)| = \begin{cases} x'(t)\operatorname{sign}(x(t)) & \text{if } x(t) \ne 0 \\ |x'(t)| & \text{if } x(t) = 0 \end{cases}} \tag{19.35}$$

and

$$\boxed{|D_R|x(t)|| = |x'(t)|} \tag{19.36}$$

2. *If $n > 1$ and $x \in C^1[a, b]$ then $\|x(t)\|$ has a right derivative*

$$\boxed{D_R\|x(t)\| := \lim_{0<h\to 0}\frac{1}{h}\left[\|x(t+h)\| - \|x(t)\|\right]} \tag{19.37}$$

such that on $t \in [a, b)$

$$\boxed{\begin{array}{c}\|D_R\|x(t)\|\| = \max\limits_{k=1,\dots,n} D_R|x_k(t)| \le \\ \|x'(t)\| := \max\{|x_1'(t)|, \dots, |x_n'(t)|\}\end{array}} \tag{19.38}$$

Proof. The assertion 1) is clear when $x(t) \neq 0$ and the case $x(t) = 0$ follows from the identity

$$x(t+h) = x(t) + hx'(t) + o(h) = h[x'(t) + o(1)]$$

so that, in general, when $h \to 0$

$$|x(t+h)| = |x(t)| + h[|x'(t)| + o(1)]$$

The multidimensional case 2) follows from 1) if take into account that

$$|x_k(t+h)| = |x_k(t)| + h[|x'_k(t)| + o(1)]$$

Taking the $\max\limits_{k=1,\ldots,n}$ of these identities, we obtain

$$\|x(t+h)\| = \|x(t)\| + h\left[\max_{k=1,\ldots,n} |x'_k(t)| + o(1)\right]$$

whereas $h \to 0$. This proves (19.38). ■

Example 19.1 *Let $x(t) := (t - t_0)^2$. Then $x'(t) := 2(t - t_0)$ is continuous and, hence, $x(t) \in C^1$. By Lemma 19.3 it follows $D_R|x(t)| = 2|t - t_0|$.*

Differential inequalities

The next theorem deals with the integration of differential inequalities and is most used in the ODE-theory.

Theorem 19.4 ((Hartman 2002)) *Let $f(t,x)$ be continuous on an open (t,x)-set $\mathbb{E} \subseteq \mathbb{R}$ and $x^0(t)$ be the maximal solution of (19.6). Let $v(t)$ be a continuous on $[t_0, t_0 + \alpha]$ function such that*

$$\boxed{\left.\begin{array}{c} v(t_0) \leq x_0, \ \ (t,v) \in \mathbb{E} \\ D_R v(t) \leq f(t, v(t)) \end{array}\right\}} \tag{19.39}$$

Then, on the common interval of existence of $x^0(t)$ and $v(t)$

$$\boxed{v(t) \leq x^0(t)} \tag{19.40}$$

Remark 19.3 *If the inequalities (19.39) are reversed with the* ***left-derivative*** $D_L v(t)$ *instead of* $D_R v(t)$*, then the conclusion (19.40) must be replaced by* $v(t) \geq x_0(t)$ *where* $x_0(t)$ *is the* ***minimal solution*** *of (19.6).*

Proof. It is sufficient to show that there exists a $\delta > 0$ such that (19.40) holds for $[t_0, t_0 + \delta]$. Indeed, if this is the case and if $v(t)$ and $x^0(t)$ are defined on $[t_0, t_0 + \beta]$, then it follows that the set of t-values, where (19.40) holds, can not have an upper bound different from β. Let in Lemma 19.2 $n > 0$ be large enough and δ be chosen independent of n such that (19.32) has a solution $x = x_n(t)$ on $[t_0, t_0 + \delta]$. In view of Lemma 19.2 it is sufficient to verify that $v(t) \leq x_n(t)$ on $[t_0, t_0 + \delta]$. But the proof of this fact is absolutely identical to the proof of (19.33). Theorem is proven. ■

In fact, the following several consequences of this theorem are widely used in the ODE-theory.

Corollary 19.6 *If* $v(t)$ *is continuous on* $[t_0, t_f]$ *and* $D_R v(t) \leq 0$ *when* $t \in [t_0, t_f]$*, then*

$$\boxed{v(t) \leq v(t_0) \text{ for any } t \in [t_0, t_f]} \tag{19.41}$$

Corollary 19.7 (Lemma on differential inequalities) *Let* $f(t, x)$*,* $x^0(t)$ *be as in Theorem 19.4 and* $g(t, x)$ *be continuous on an open* (t, x)*-set* $\mathbb{E} \subseteq \mathbb{R}^2$ *satisfying*

$$\boxed{g(t, x) \leq f(t, x)} \tag{19.42}$$

Let also $v(t)$ *be a solution of the following ODE:*

$$\boxed{\dot{v}(t) = g(t, v(t)),\ v(t_0) := v_0 \leq x_0} \tag{19.43}$$

on $[t_0, t_0 + \alpha]$*. Then*

$$\boxed{v(t) \leq x^0(t)} \tag{19.44}$$

holds on any common interval of existence of $v(t)$ *and* $x^0(t)$ *to the right of* t_0*.*

Corollary 19.8 *Let $x^0(t)$ be the maximal solution of*

$$\dot{x}(t) = f(t, x(t)), \;\; x(t_0) := x^0 \in \mathbb{R}$$

and $x_0(t)$ be the minimal solution of

$$\dot{x}(t) = -f(t, x(t)), \;\; x(t_0) := x_0 \geq 0$$

Let also $y = y(t)$ be a C^1 vector valued function on $[t_0, t_0 + \alpha]$ such that

$$\boxed{\begin{array}{c} x_0 \leq \|y(t_0)\| \leq x^0 \\ (t, y) \in \mathbb{E} \subseteq \mathbb{R}^2 \\ \frac{d}{dt}(\|y(t)\|) \leq f(t, \|y(t)\|) \end{array}} \tag{19.45}$$

Then the first (second) of two inequalities

$$\boxed{x_0(t) \leq \|y(t)\| \leq x^0(t)} \tag{19.46}$$

holds on any common interval of existence of $x_0(t)$ and $y(t)$ (or $x^0(t)$ and $y(t)$).

Corollary 19.9 *Let $f(t, x)$ be continuous and non-decreasing on x when $t \in [t_0, t_0 + \alpha]$. Let $x^0(t)$ be a maximal solution of (19.6) which exists on $[t_0, t_0 + \alpha]$. Let another continuous function $v(t)$ satisfies on $[t_0, t_0 + \alpha]$ the integral inequality*

$$\boxed{v(t) \leq v_0 + \int_{s=t_0}^{t} f(s, v(s))\, ds} \tag{19.47}$$

where $v_0 \leq x_0$. Then

$$\boxed{v(t) \leq x^0(t)} \tag{19.48}$$

holds on $[t_0, t_0 + \alpha]$. This results is false if we omit: $f(t, x)$ is non-decreasing on x.

Proof. Denote by $V(t)$ the right-hand side of (19.47), so that $v(t) \leq V(t)$, and, by the monotonicity property with respect the second argument, we have

$$\dot{V}(t) = f(t, v(t)) \leq f(t, V(t))$$

By Theorem 19.4 we have $V(t) \leq x^0(t)$ on $[t_0, t_0 + \alpha]$. Thus $v(t) \leq x^0(t)$ that completes the proof. ■

Existence of solutions on the complete axis $[t_0, \infty)$

Here we show that the condition $\|f(t,x)\| \leq k\|x\|$ guaranties the existence of the solutions of ODE $\dot{x}(t) = f(t, x(t))$, $x(t_0) = x_0 \in R^n$ for any $t \geq t_0$. In fact, the following more general result holds.

Theorem 19.5 ((Wintner 1945)) *Let for any* $t \geq t_0$ *and* $x \in \mathbb{R}^n$

$$\boxed{(x, f(t,x)) \leq \Psi\left(\|x\|^2\right)} \tag{19.49}$$

where the function Ψ *satisfies the condition*

$$\boxed{\int\limits_{s=s_0}^{\infty} \frac{ds}{\Psi(s)} = \infty,\ \Psi(s) > 0 \ \text{as} \ s \geq s_0 \geq 0} \tag{19.50}$$

Then the Cauchy's problem

$$\dot{x}(t) = f(t, x(t)),\ x(t_0) = x_0 \in R^n$$

has a solution on the complete semi-axis $[t_0, \infty)$ *for any* $x_0 \in \mathbb{R}^n$.

Proof. Notice that for the function $w(t) := \|x(t)\|^2$ in view of (19.49) we have

$$\frac{d}{dt}w(t) = 2(x(t), \dot{x}(t)) = 2(x(t), f(t, x(t))) \leq 2\Psi\left(\|x(t)\|^2\right) = 2\Psi(w(t))$$

Then by Theorem 19.4 (see (19.44)) it follows that $w(t_0) \leq s_0$ implies $w(t) \leq s(t)$, where $s(t)$ satisfies

$$\dot{s}(t) = 2\Psi(s(t)),\ s(t_0) = s_0 := \|x_0\|^2$$

But the solution of the last ODE is always bounded for any finite $t \geq t_0$. Indeed,

$$\int\limits_{s=s_0}^{s(t)} \frac{ds}{\Psi(s)} = 2(t - t_0) \tag{19.51}$$

and $\Psi(s) > 0$ as $s \geq s_0$ implies that $\dot{s}(t) > 0$, and, hence, $s(t) > 0$ for all $t > t_0$. But the solution $s(t)$ can fail to exist on a bounded interval

$[t_0, t_0 + a]$ only if it exists on $[t_0, t_0 + \alpha]$ with $\alpha < a$ and $s(t) \to \infty$ if $t \to t_0 + a$. But this gives the contradiction to (19.50) since the left-hand side of (19.51) tend to infinity and the right-hand side of (19.51) remains finite and equal to $2a$. ■

Remark 19.4 *The admissible choices of $\Psi(s)$ may be, for example, C, Cs, $Cs \ln s$,... for large enough s and C as a positive constant.*

Remark 19.5 *Some generalizations of this theorem can be found in (Hartman 2002).*

Example 19.2 *If $A(t)$ is a continuous $n \times n$ matrix and $g(t)$ is continuous on $[t_0, t_0 + a]$ vector function, then the Cauchy's problem*

$$\dot{x}(t) = A(t) x(t) + g(t), x(t_0) = x_0 \in R^n \tag{19.52}$$

has a unique solution $x(t)$ on $[t_0, t_0 + a]$. It follows from the Wintner theorem 19.5 if take $\Psi(s) := C(1 + s)$ with $C > 0$.

The continuous dependence of the solution on a parameter and on the initial conditions

Theorem 19.6 *If the right-hand side of ODE*

$$\boxed{\dot{x}(t) = f(t, x(t), \mu),\ x(t_0) = x_0 \in R^n} \tag{19.53}$$

*is **continuous** with respect to μ on $[\mu^-, \mu^+]$ and satisfies the condition of Theorem 19.1 with the Lipschitz constant L_f which is independent of μ, then the solution $x(t, \mu)$ of (19.53) depends **continuously** on $\mu \in [\mu^-, \mu^+] \in \mathbb{R}^m$ as well as on x_0 in some neighborhood.*

Proof. The proof of this assertion repeats word by word the proof of Theorem 19.1. Indeed, by the same reasons as in Theorem 19.1, the solution $x(t, \mu)$ is a continuous function of both t and μ if L_f is independent of μ. As for the proof of the continuous dependence of the solution on the initial conditions, it can be transform to the proof of the continuous dependence of the solution on the parameter. Indeed, putting

$$\tau := t - t_0,\ z := x(t, \mu) - x_0$$

we obtain that (19.53) is converted to

$$\frac{d}{d\tau}z = f(\tau + t_0, z + x_0, \mu), \ z(0) = 0$$

where x_0 may be considered as a new parameter so that f is continuous on x_0 by the assumption. This proves the theorem. ■

19.2.3 Linear ODE

Linear vector ODE

Lemma 19.4 *The solution $x(t)$ of the linear ODE (or, the corresponding Cauchy's problem)*

$$\boxed{\dot{x}(t) = A(t)\,x(t), \ x(t_0) = x_0 \in \mathbb{R}^{n\times n}, \ t \geq t_0} \tag{19.54}$$

where $A(t)$ is a continuous $n \times n$-matrix function, may be presented as

$$\boxed{x(t) = \Phi(t, t_0)\,x_0} \tag{19.55}$$

*where the matrix $\Phi(t, t_0)$ is, so-called, the **fundamental matrix** of the system (19.54) and satisfies the following matrix ODE*

$$\boxed{\frac{d}{dt}\Phi(t, t_0) = A(t)\,\Phi(t, t_0), \ \Phi(t_0, t_0) = I} \tag{19.56}$$

*and fulfills the **group property***

$$\boxed{\Phi(t, t_0) = \Phi(t, s)\,\Phi(s, t_0) \ \forall s \in (t_0, t)} \tag{19.57}$$

Proof. Assuming (19.55), the direct differentiation of (19.55) implies

$$\dot{x}(t) = \frac{d}{dt}\Phi(t, t_0)\,x_0 = A(t)\,\Phi(t, t_0)\,x_0 = A(t)\,x(t)$$

So, (19.55) verifies (19.54). Uniqueness of such presentation follows from Example 19.2. The property (19.57) results from the fact that

$$x(t) = \Phi(t, s)\,x_s = \Phi(t, s)\,\Phi(s, t_0)\,x(t_0) = \Phi(t, t_0)\,x(t_0)$$

Lemma is proven. ■

Liouville's theorem

The next results serves for the demonstration that the transformation $\Phi(t, t_0)$ is non-singular (or, has its inverse) on any finite time interval.

Theorem 19.7 (Liouville, 1836) *If* $\Phi(t, t_0)$ *is the solution to (19.56), then*

$$\boxed{\det \Phi(t, t_0) = \exp\left\{ \int_{s=t_0}^{t} \mathrm{tr} A(s)\, ds \right\}} \tag{19.58}$$

Proof. The usual expansion for the determinant $\det \Phi(t, t_0)$ and the rule for differentiating the product of scalar functions show that

$$\frac{d}{dt} \det \Phi(t, t_0) = \sum_{j=1}^{n} \det \tilde{\Phi}_j(t, t_0)$$

where $\tilde{\Phi}_j(t, t_0)$ is the matrix obtained by replacing the j-th row $\Phi_{j,1}(t, t_0)$, ..., $\Phi_{j,n}(t, t_0)$ of $\Phi(t, t_0)$ by its derivatives $\dot{\Phi}_{j,1}(t, t_0)$, ..., $\dot{\Phi}_{j,n}(t, t_0)$. But since

$$\dot{\Phi}_{j,k}(t, t_0) = \sum_{i=1}^{n} a_{j,i}(t)\, \Phi_{i,k}(t, t_0), \;\; A(t) = \|a_{j,i}(t)\|_{j.i=1,\ldots,n}$$

it follows

$$\det \tilde{\Phi}_j(t, t_0) = a_{j,j}(t) \det \Phi(t, t_0)$$

that gives

$$\frac{d}{dt} \det \Phi(t, t_0) = \sum_{j=1}^{n} \frac{d}{dt} \det \tilde{\Phi}_j(t, t_0)$$
$$= \sum_{j=1}^{n} a_{j,j}(t) \det \Phi(t, t_0) = tr\{A(t)\} \det \Phi(t, t_0)$$

and, as a result, we obtain (19.58) that completes the proof. ■

Corollary 19.10 *If for the system (19.54)*

$$\boxed{\int_{s=t_0}^{T} \mathrm{tr} A(s)\, ds > -\infty} \tag{19.59}$$

then for any $t \in [t_0, T]$

$$\boxed{\det \Phi(t, t_0) > 0} \tag{19.60}$$

Proof. It is the direct consequence of (19.58). ■

Lemma 19.5 *If (19.59) is fulfilled, namely,* $\int_{s=t_0}^{T} \mathrm{tr} A(s)\, ds > -\infty$, *then the solution* $x(t)$ *on* $[0, T]$ *of the linear non autonomous ODE*

$$\boxed{\dot{x}(t) = A(t)\, x(t) + g(t),\ x(t_0) = x_0 \in \mathbb{R}^{n \times n},\ t \geq t_0} \tag{19.61}$$

where $A(t)$ *and* $f(t)$ *are assumed to be continuous matrix and vector functions, may be presented by the* ***Cauchy formula***

$$\boxed{x(t) = \Phi(t, t_0) \left[x_0 + \int_{s=t_0}^{t} \Phi^{-1}(s, t_0)\, g(s)\, ds \right]} \tag{19.62}$$

where $\Phi^{-1}(t, t_0)$ *exists for all* $t \in [t_0, T]$ *and satisfies*

$$\boxed{\frac{d}{dt} \Phi^{-1}(t, t_0) = -\Phi^{-1}(t, t_0)\, A(t),\ \Phi^{-1}(t_0, t_0) = I} \tag{19.63}$$

Proof. By the previous corollary, $\Phi^{-1}(t, t_0)$ exists within the interval $[t_0, T]$. The direct derivation of (19.62) implies

$$\begin{gathered}
\dot{x}(t) = \dot{\Phi}(t, t_0) \left[x_0 + \int_{s=t_0}^{t} \Phi^{-1}(s, t_0)\, g(s)\, ds \right] \\
+ \Phi(t, t_0)\, \Phi^{-1}(t, t_0)\, g(t) = \\
A(t)\Phi(t, t_0) \left[x_0 + \int_{s=t_0}^{t} \Phi^{-1}(s, t_0)\, g(s)\, ds \right] + g(t) \\
= A(t) x(t) + g(t)
\end{gathered}$$

that coincides with (19.61). Notice that the integral in (19.62) is well defined in view of the continuity property of the participating functions to be integrated. By identities

$$\begin{aligned} \Phi(t,t_0)\,\Phi^{-1}(t,t_0) &= I \\ \frac{d}{dt}\left[\Phi(t,t_0)\,\Phi^{-1}(t,t_0)\right] &= \dot{\Phi}(t,t_0)\,\Phi^{-1}(t,t_0) \\ &+ \Phi(t,t_0)\,\frac{d}{dt}\Phi^{-1}(t,t_0) = 0 \end{aligned}$$

it follows

$$\frac{d}{dt}\Phi^{-1}(t,t_0) = -\Phi^{-1}(t,t_0)\left[\dot{\Phi}(t,t_0)\right]\Phi^{-1}(t,t_0) =$$

$$-\Phi^{-1}(t,t_0)\left[A(t)\,\Phi(t,t_0)\right]\Phi^{-1}(t,t_0) = -\Phi^{-1}(t,t_0)\,A(t)$$

Lemma is proven. ■

Remark 19.6 *The solution (19.62) can be rewritten as*

$$\boxed{x(t) = \Phi(t,t_0)\,x_0 + \int\limits_{s=t_0}^{t} \Phi(t,s)\,g(s)\,ds} \qquad (19.64)$$

since by (19.57)

$$\boxed{\Phi(t,s) = \Phi(t,t_0)\,\Phi^{-1}(s,t_0)} \qquad (19.65)$$

Bounds for norm of ODE solutions

Let $\|A\| := \sup\limits_{\|x\|=1} \|Ax\|$ where $\|x\|$ is Euclidean or Chebishev's type.

Lemma 19.6 *Let $x(t)$ be a solution of (19.61). Then*

$$\boxed{\|x(t)\| \leq (\|x(t_0)\| + \int\limits_{s=t_0}^{t} \|g(s)\|\,ds)\exp(\int\limits_{s=t_0}^{t} \|A(s)\|\,ds)} \qquad (19.66)$$

Proof. By (19.61) it follows

$$\|\dot{x}(t)\| \leq \|A(t)\| \|x(t)\| + \|g(t)\|$$

Let $v(t)$ be the unique solution of the following ODE:

$$\dot{v}(t) = \|A(t)\| v(t) + \|g(t)\|, \; v(t_0) = \|x(t_0)\|$$

which solution is

$$v(t) = [v(t_0) + \int_{s=t_0}^{t} \|f(s)\| \exp(- \int_{r=t_0}^{s} \|A(r)\| dr)] \exp(\int_{s=t_0}^{t} \|A(s)\| ds)$$

Then, by Lemma 19.7, it follows that $\|x(t)\| \leq v(t)$ for any $t \geq t_0$ that gives (19.66). ■

Corollary 19.11 *Similarly, if $w(t)$ is the solution of*

$$\dot{v}(t) = -\|A(t)\| v(t) - \|g(t)\|, \; w(t_0) = \|x(t_0)\|$$

then $\|x(t)\| \geq v(t)$ for any $t \geq t_0$ that gives

$$\boxed{\|x(t)\| \geq (\|x(t_0)\| - \int_{s=t_0}^{t} \|g(s)\| ds) \exp(- \int_{s=t_0}^{t} \|A(s)\| ds)} \tag{19.67}$$

Stationary linear ODE

If in (19.1) $A(t) = A$ is a constant matrix, then it is easily to check that

$$\boxed{\Phi(t, t_0) := e^{A(t-t_0)} \text{ where } e^{At} = \sum_{k=0}^{\infty} \frac{1}{k!} A^k t^k} \tag{19.68}$$

and (19.62), (19.64) become

$$\boxed{\begin{aligned} x(t) &= e^{A(t-t_0)} \left[x_0 + \int_{s=t_0}^{t} e^{-A(s-t_0)} g(s) \, ds \right] \\ &= e^{A(t-t_0)} x_0 + \int_{s=t_0}^{t} e^{A(t-s)} g(s) \, ds \end{aligned}} \tag{19.69}$$

Linear ODE with periodic matrices

In this subsection we show that the case of variable, but periodic, coefficients can be reduced to the case of constant coefficients.

Theorem 19.8 (Floquet 1883) *Let in ODE*

$$\boxed{\dot{x}(t) = A(t)\,x(t)} \tag{19.70}$$

the matrix $A(t) \in \mathbb{R}^{n\times n}$ $(-\infty < t < \infty)$ *be a continuous and periodic of period* T*, that is, for any* t

$$\boxed{A(t+T) = A(t)} \tag{19.71}$$

Then the fundamental $\Phi(t, t_0)$ *of (19.70) has a representation of the form*

$$\boxed{\begin{array}{c} \Phi(t,t_0) = \tilde{\Phi}(t-t_0) = Z(t-t_0)\,e^{R(t-t_0)} \\ Z(\tau) = Z(\tau+T) \end{array}} \tag{19.72}$$

and R *is a constant* $n \times n$ *matrix.*

Proof. Since $\tilde{\Phi}(\tau)$ is fundamental matrix of (19.70), then $\tilde{\Phi}(\tau+T)$ is fundamental too. By the group property (19.57) it follows $\tilde{\Phi}(\tau+T) = \tilde{\Phi}(\tau)\,\tilde{\Phi}(T)$. Since $\det\tilde{\Phi}(T) \neq 0$ one can represent $\tilde{\Phi}(T)$ as $\tilde{\Phi}(T) = e^{RT}$ and hence

$$\tilde{\Phi}(\tau+T) = \tilde{\Phi}(\tau)\,e^{RT} \tag{19.73}$$

So, defining $Z(\tau) := \tilde{\Phi}(\tau)\,e^{-R\tau}$, we get

$$\begin{gathered} Z(\tau+T) = \tilde{\Phi}(\tau+T)\,e^{-R(\tau+T)} = \\ \left[\tilde{\Phi}(\tau+T)\,e^{-RT}\right]e^{-R\tau} = \tilde{\Phi}(\tau)\,e^{-R\tau} = Z(\tau) \end{gathered}$$

that completes the proof. ■

First integrals and related adjoint linear ODE

Definition 19.2 *A function* $F = F(t,x) : \mathbb{R}\times\mathbb{C}^n \to \mathbb{C}$*, belonging to* $C^1[\mathbb{R}\times\mathbb{C}^n]$*, is called the* ***first integral*** *to ODE (19.1) if it is constant* *over trajectories of* $x(t)$ *generated by (19.1), that is, if for any* $t \geq t_0$ *and any* $x_0 \in \mathbb{C}^n$

$$\boxed{\begin{array}{l}\frac{d}{dt}F(t,x(t)) = \frac{\partial}{\partial t}F(t,x(t)) + \left(\frac{\partial}{\partial x}F(t,x(t)),\dot{x}(t)\right) \\ \quad = \frac{\partial}{\partial t}F(t,x) + \left(\frac{\partial}{\partial x}F(t,x(t)), f(t,x(t))\right) = 0\end{array}} \tag{19.74}$$

In the case of linear ODE (19.54) the condition (19.74) is converted into the following:

$$\boxed{\frac{\partial}{\partial t}F(t,x) + \left(\frac{\partial}{\partial x}F(t,x(t)), A(t)\right) = 0} \tag{19.75}$$

Let us try to find a first integral for (19.54) as a linear form of $x(t)$, i.e., let us try to satisfy (19.75) selecting F as

$$\boxed{F(t,x) = (z(t), x(t)) := z^*(t)\,x(t)} \tag{19.76}$$

where $z^*(t) \in \mathbb{C}^n$ is from $C^1[\mathbb{C}^n]$.

The existence of the first integral for ODE (19.1) permits to decrease the order of the system to be integrated since if the equation $F(t,x(t)) = c$ can be resolved with respect one of components, say,

$$x_\alpha(t) = \varphi(t, x_1(t), ..., x_{\alpha-1}(t), x_{\alpha+1}(t), ..., x_n(t))$$

then the order of ODE (19.1) becomes to be equal to $(n-1)$. If one can find all n first integrals $F_\alpha(t,x(t)) = c_\alpha$ $(\alpha = 1, ..., n)$ which are linearly independent, then the ODE system (19.1) can be considered to be solved.

Lemma 19.7 *A first integral* $F(t,x)$ *for (19.54) is linear on* $x(t)$ *as in (19.76) if and only if*

$$\boxed{\dot{z}(t) = -A^*(t)\,z(t),\ z(t_0) = z_0 \in \mathbb{R}^{n\times n},\ t \geq t_0} \tag{19.77}$$

Proof. *a) Necessity.* If a linear $F(t,x) = (z(t), x(t))$ is a first integral, then

$$\begin{array}{c}\frac{d}{dt}F(t,x(t)) = (\dot{z}(t), x(t)) + (z(t), \dot{x}(t)) = \\ (\dot{z}(t), x(t)) + (z(t), A(t)\,x(t)) = (\dot{z}(t), x(t)) + \\ (A^*(t)\,z(t), x(t)) = (\dot{z}(t) + A^*(t)\,z(t), x(t))\end{array} \tag{19.78}$$

Suppose that $\dot{z}(t') + A^*(t') z(t') \neq 0$ for some $t' \geq t_0$. Put

$$x(t') := \dot{z}(t') + A^*(t') z(t')$$

Since $x(t') = \Phi(t', t_0) x_0$ and $\Phi^{-1}(t', t_0)$ always exists, then for $x_0 = \Phi^{-1}(t', t_0) x(t')$ we obtain

$$\begin{aligned} \frac{d}{dt} F(t', x(t')) &= (\dot{z}(t') + A^*(t') z(t'), x(t')) \\ &= \|\dot{z}(t') + A^*(t') z(t')\|^2 \neq 0 \end{aligned}$$

that is in the contradiction with the assumption that $F(t, x(t))$ is a first integral.

b) Sufficiency. It directly results from (19.78) Lemma is proven. ■

Definition 19.3 *The system (19.77) is called the ODE system* ***adjoint*** *to (19.54). For the corresponding inhomogeneous system (19.61) the adjoint system is*

$$\boxed{\dot{z}(t) = -A^*(t) z(t) - \tilde{g}(t), \ z(t_0) = z_0 \in \mathbb{R}^{n \times n}, \ t \geq t_0} \quad (19.79)$$

There are several results concerning the joint behavior of (19.54) and (19.77).

Lemma 19.8 *A matrix $\Phi(t, t_0)$ is a fundamental matrix for the linear ODE (19.54) if and only if $(\Phi^*(t, t_0))^{-1} = (\Phi^{-1}(t, t_0))^*$ is a fundamental matrix for the adjoint system (19.77).*

Proof. Since $\Phi(t, t_0) \Phi^{-1}(t, t_0) = I$ by differentiation it follows that

$$\frac{d}{dt} \Phi^{-1}(t, t_0) = -\Phi^{-1}(t, t_0) \frac{d}{dt} \Phi(t, t_0) \Phi^{-1}(t, t_0) = -\Phi^{-1}(t, t_0) A(t)$$

and, taking the complex conjugate transpose of the last identity gives

$$\frac{d}{dt} \left(\Phi^{-1}(t, t_0)\right)^* = -A^*(t) \left(\Phi^{-1}(t, t_0)\right)^*$$

The converse is proved similarly. ■

Lemma 19.9 *The direct (19.61) and the corresponding adjoint (19.79) linear systems can be presented in the Hamiltonian form, i.e.,*

$$\boxed{\dot{x}(t) = \frac{\partial}{\partial z} H(z,x), \ \dot{z}(t) = -\frac{\partial}{\partial x} H(z,x)} \tag{19.80}$$

where

$$\boxed{H(t,z,x) := (z, f(t,x)) = (z, A(t)x + g(t))} \tag{19.81}$$

is called the ***Hamiltonian function*** *for the system (19.61). In the stationary homogeneous case when*

$$\boxed{\dot{x}(t) = Ax(t), \ x(t_0) = x_0 \in \mathbb{R}^{n\times n}, \ t \geq t_0} \tag{19.82}$$

the Hamiltonian function is a first integral for (19.82).

Proof. The representation (19.80) follows directly from (19.81). In the stationary, when $\frac{\partial}{\partial t} H(t,z,x) = 0$, we have

$$\begin{aligned} \frac{d}{dt} H(t,z,x) &= \frac{\partial}{\partial t} H(t,z,x) + \left(\frac{\partial}{\partial z} H(z,x), \dot{z}\right) + \left(\frac{\partial}{\partial x} H(z,x), \dot{x}\right) \\ &= \left(\frac{\partial}{\partial z} H(z,x), -\frac{\partial}{\partial x} H(z,x)\right) + \left(\frac{\partial}{\partial x} H(z,x), \frac{\partial}{\partial z} H(z,x)\right) = 0 \end{aligned}$$

So, $H(t,z,x)$ is a constant.

Lemma 19.10 *If $A(t) = -A^*(t)$ is skew Hermitian, then*

$$\boxed{\|x(t)\| = \underset{t}{\mathrm{const}}} \tag{19.83}$$

Proof. One has directly

$$\begin{gathered} \frac{d}{dt} \|x(t)\|^2 = (\dot{x}(t), x(t)) + (x(t), \dot{x}(t)) = (A(t)x(t), x(t)) + \\ (x(t), A(t)x(t)) = (A(t)x(t), x(t)) + (A^*(t)x(t), x(t)) = \\ ([A(t) + A^*(t)]x(t), x(t)) = 0 \end{gathered}$$

that proves the result. ∎

Lemma 19.11 (Green's formula) *Let $A(t)$, $g(t)$ and $\tilde{g}(t)$ be continuous for $t \in [a,b]$; $x(t)$ be a solution of (19.61) and $z(t)$ be a solution of (19.79). Then for all $t \in [a,b]$*

$$\boxed{\int\limits_{s=a}^{t} \left[g^{\intercal}(s)\, z(s) \text{-} x(s)^{\intercal}\, \tilde{g}(s)\right] ds = x^{\intercal}(t)\, z(t) \text{-} x^{\intercal}(a), z(a)} \tag{19.84}$$

Proof. The relation (19.84) is proved by showing that both sides have the same derivatives, since $(Ay, z) = (y, A^{*}z)$. ■

19.2.4 Index of increment for ODE solutions

Definition 19.4 *A number τ is called a **Lyapunov order number** (or **the index of the increment**) **for a vector function** $x(t)$ defined for $t \geq t_0$, if for every $\varepsilon > 0$ there exists positive constants C_{ε}^{0} and C_{ε} such that*

$$\boxed{\begin{array}{c} \|x(t)\| \leq C_{\varepsilon} e^{(\tau+\varepsilon)t} \text{ for all large } t \\[2ex] \|x(t)\| \leq C_{\varepsilon}^{0} e^{(\tau-\varepsilon)t} \text{ for some arbitrary large } t \end{array}} \tag{19.85}$$

that equivalently can be formulated as

$$\boxed{\tau = \limsup_{t\to\infty} t^{-1} \ln \|x(t)\|} \tag{19.86}$$

Lemma 19.12 *If $x(t)$ is the solution of (19.61), then it has the Lyapunov order number*

$$\boxed{\begin{array}{c} \tau \leq \limsup\limits_{t\to\infty} t^{-1} \ln \left(\|x(t_0)\| + \int\limits_{s=t_0}^{t} \|f(s)\|\, ds \right) \\ + \limsup\limits_{t\to\infty} t^{-1} \int\limits_{s=t_0}^{t} \|A(s)\|\, ds \end{array}} \tag{19.87}$$

Proof. It follows directly from (19.66). ■

19.2.5 Riccati differential equation

Let us introduce the symmetric $n \times n$ matrix function $P(t) = P^{\intercal}(t) \in C^1[0,T]$ which satisfies the following ODE:

$$\boxed{\left.\begin{array}{c} -\dot{P}(t) = P(t)A(t) + A(t)^{\intercal}P(t) \\ -P(t)R(t)P(t) + Q(t) \\ \\ P(T) = G \geq 0 \end{array}\right\}} \tag{19.88}$$

with

$$A(t), Q(t) \in \mathbb{R}^{n\times n}, \; R(t) \in \mathbb{R}^{m\times m} \tag{19.89}$$

Definition 19.5 *We call ODE (19.88) the* ***matrix Riccati differential equation****.*

Theorem 19.9 (on the structure of the solution) *Let $P(t)$ be a symmetric nonnegative solution of (19.88) defined on $[0,T]$. Then there exist two functional $n \times n$ matrices $X(t), Y(t) \in C^1[0,T]$ satisfying the following linear ODE*

$$\boxed{\begin{array}{c} \begin{pmatrix} \dot{X}(t) \\ \dot{Y}(t) \end{pmatrix} = H(t)\begin{pmatrix} X(t) \\ Y(t) \end{pmatrix} \\ \\ X(T) = I, \; Y(T) = P(T) = G \end{array}} \tag{19.90}$$

with

$$\boxed{H(t) = \begin{bmatrix} A(t) & -R(t) \\ -Q(t) & -A^{\intercal}(t) \end{bmatrix}} \tag{19.91}$$

where $A(t)$ and $Q(t)$ are as in (19.88) and such that $P(t)$ may be uniquely represented as

$$\boxed{P(t) = Y(t)X^{-1}(t)} \tag{19.92}$$

for any finite $t \in [0,T]$.

Proof.

a) Notice that the matrices $X(t)$ and $Y(t)$ exist since they are defined by the solution to the ODE (19.90).

b) Show that they satisfy the relation (19.92). Firstly, remark that $X(T) = I$, so $\det X(T) = 1 > 0$. From (19.90) it follows that $X(t)$ is a continuous matrix function and, hence, there exists a time τ such that for all $t \in (T-\tau, T]$ $\det X(t) > 0$. As a result, $X^{-1}(t)$ exists within the small semi-open interval $(T-\tau, T]$. Then, directly using (19.90) and in view of the identities

$$X^{-1}(t)\,X(t) = I,\ \frac{d}{dt}\left[X^{-1}(t)\right]X(t) + X^{-1}(t)\,\dot{X}(t) = 0$$

it follows

$$\begin{gathered}\frac{d}{dt}\left[X^{-1}(t)\right] = -X^{-1}(t)\,\dot{X}(t)\,X^{-1}(t) = \\ -X^{-1}(t)\left[A(t)\,X(t) - R(t)\,Y(t)\right]X^{-1}(t) = \\ -X^{-1}(t)\,A(t) + X^{-1}(t)\,R(t)\,Y(t)\,X^{-1}(t)\end{gathered} \tag{19.93}$$

and, hence, for all $t \in (T-\tau, T]$ in view of (19.88)

$$\begin{gathered}\frac{d}{dt}\left[Y(t)\,X^{-1}(t)\right] = \dot{Y}(t)\,X^{-1}(t) + Y(t)\,\frac{d}{dt}\left[X^{-1}(t)\right] = \\ \left[-Q(t)\,X(t) - A^{\intercal}(t)\,Y(t)\right]X^{-1}(t) + \\ Y(t)\left[-X^{-1}(t)\,A(t) + X^{-1}(t)\,R(t)\,Y(t)\,X^{-1}(t)\right] = \\ -Q(t) - A^{\intercal}(t)\,P(t) - P(t)\,A(t) + P(t)\,R(t)\,P(t) = \dot{P}(t)\end{gathered}$$

that implies $\frac{d}{dt}\left[Y(t)\,X^{-1}(t) - P(t)\right] = 0$, or,

$$Y(t)\,X^{-1}(t) - P(t) = \underset{t\in(T-\tau,T]}{\text{const}}$$

But for $t = T$ we have

$$\underset{t\in(T-\tau,T]}{\text{const}} = Y(T)\,X^{-1}(T) - P(T) = Y(T) - P(T) = 0$$

So, for all $t \in (T-\tau, T]$ it follows $P(t) = Y(t)\,X^{-1}(t)$.

c) Show that $\det X(T-\tau) > 0$. The relations (19.90) and (19.92) lead to the following presentation within $t \in t \in (T-\tau, T]$

$$\dot{X}(t) = A(t) X(t) - R(t) Y(t) = [A(t) - R(t) P(t)] X(t)$$

and, by the Liouville's theorem 19.7, it follows

$$\det X(T-\tau) = \det X(0) \exp \left\{ \int_{t=0}^{T-\tau} tr [A(t) - R(t) P(t)] \, dt \right\}$$

$$1 = \det X(T) = \det X(0) \exp \left\{ \int_{t=0}^{T} tr [A(t) - R(t) P(t)] \, dt \right\}$$

$$\det X(T-\tau) = \exp \left\{ - \int_{t=T-\tau}^{T} tr [A(t) - R(t) P(t)] \, dt \right\} > 0$$

By continuity, again there exists a time $\tau_1 > \tau$ that $\det X(t) > 0$ for any $t \in [T-\tau, T-\tau_1]$. Repeating the same considerations we may conclude that $\det X(t) > 0$ for any $t \in [0, T]$.

d) Show that the matrix $G(t) := Y(t) X^{-1}(t)$ is symmetric. One has

$$\frac{d}{dt} [Y^{\intercal}(t) X(t) - X^{\intercal}(t) Y(t)] = \dot{Y}^{\intercal}(t) X(t) + Y^{\intercal}(t) \frac{d}{dt} [X(t)]$$

$$-\frac{d}{dt} X^{\intercal}(t) Y(t) - X^{\intercal}(t) \dot{Y}(t) = [-Q(t) X(t) - A^{\intercal}(t) Y(t)]^{\intercal} X(t) +$$

$$Y^{\intercal}(t) [A(t) X(t) - R(t) Y(t)] - [A(t) X(t) - R(t) Y(t)]^{\intercal} Y(t)$$

$$-X^{\intercal}(t) [-Q(t) X(t) - A^{\intercal}(t) Y(t)] = 0$$

and $Y(T)^{\intercal} X(T) - [X(T)]^{\intercal} Y(T) = Y^{\intercal}(T) - Y(T) = G^{\intercal} - G = 0$ that implies $Y^{\intercal}(t) X(t) - X^{\intercal}(t) Y(t) = 0$ for any $t \in [0, T]$. So, $Y^{\intercal}(t) = X^{\intercal}(t) Y(t) X^{-1}(t) = X^{\intercal}(t) P(t)$ and, hence, by the transposition operation we get $Y(t) = P^{\intercal}(t) X(t)$ and $P(t) = Y(t) X^{-1}(t) = P^{\intercal}(t)$. The symmetricity of $P(t)$ is proven.

e) The Riccati differential equation (19.88) is uniquely solvable with $P(t) = Y(t) X^{-1}(t) \geq 0$ on $[0, T]$ since the matrices $X(t)$ and $Y(t)$ are uniquely defined by (19.92). ■

19.2.6 Linear first order partial DE

Consider the following *linear first order partial* DE

$$\boxed{\sum_{i=1}^{n} X_i(x,z)\frac{\partial z}{\partial x_i} = Z(x,z)} \tag{19.94}$$

where $x \in \mathbb{R}^n$ is a vector of n independent real variables and $z = z(x)$ is a real-valued function of the class $C^1(\mathcal{X})$, $x \in \mathcal{X} \subseteq \mathbb{R}^n$. Defining

$$X(x,z) := (X_1(x,z), ..., X_n(x,z))^{\intercal}, \quad \frac{\partial z}{\partial x} := \left(\frac{\partial z}{\partial x_1}, ..., \frac{\partial z}{\partial x_n}\right)^{\intercal}$$

the equation (19.94) can be rewritten as follows

$$\boxed{\left(X(x,z), \frac{\partial z}{\partial x}\right) = Z(x,z)} \tag{19.95}$$

Any function $z = z(x) \in C^1(\mathcal{X})$ satisfying (19.95) is its solution. If so, then its full differential dz is

$$\boxed{dz = \left(\frac{\partial z}{\partial x}, dx\right) = \sum_{i=1}^{n} \frac{\partial z}{\partial x_i} dx_i} \tag{19.96}$$

Consider also the following auxiliary system of ODE:

$$\boxed{\frac{dx_1}{X_1(x,z)} = \cdots = \frac{dx_n}{X_n(x,z)} = \frac{dz}{Z(x,z)}} \tag{19.97}$$

or, equivalently,

$$\boxed{X_1^{-1}(x,z)\frac{dx_1}{dz} = \cdots = X_n^{-1}(x,z)\frac{dx_n}{dz} = Z^{-1}(x,z)} \tag{19.98}$$

or,

$$\boxed{\left.\begin{array}{c} \dfrac{dx_1}{dz} = X_1(x,z) Z^{-1}(x,z) \\ \cdots \\ \dfrac{dx_n}{dz} = X_n(x,z) Z^{-1}(x,z) \end{array}\right\}} \tag{19.99}$$

which is called *the system of characteristic ODE* related to (19.95). The following important result, describing the natural connection of (19.95) and (19.98), is given below.

Lemma 19.13 *If $z = z(x)$ satisfies (19.97), then it satisfies (19.95) too.*

Proof. Indeed, by (19.99) and (19.96) we have

$$dx_i = X_i(x, z) Z^{-1}(x, z) dz$$
$$dz = \sum_{i=1}^{n} \frac{\partial z}{\partial x_i} dx_i = \sum_{i=1}^{n} \frac{\partial z}{\partial x_i} X_i(x, z) Z^{-1}(x, z) dz$$

that implies (19.94). ■

Cauchy's method of characteristics

The method, presented here, permits to convert the solution of a linear first order partial DE the solution of a system of nonlinear ODE.

Suppose that we can solve the system (19.98) of ODE and its solution is

$$\boxed{x_i = x_i(z, c_i),\ i = 1, ..., n} \tag{19.100}$$

where c_i are some constants.

Definition 19.6 *The solutions (19.100) are called* ***the characteristics*** *of (19.94).*

Assume that this solution can be resolved with respect to the constants c_i, namely, there exists functions

$$\boxed{\psi_i = \psi_i(x, z) = c_i \quad (i = 1, ..., n)} \tag{19.101}$$

Since these functions are constants on the solutions of (19.98) they are the first integrals of (19.98). Evidently, any arbitrary function $\Phi : \mathbb{R}^n \to \mathbb{R}$ of constants $c_i \ \ (i = 1, ..., n)$ is a constant too, that is,

$$\boxed{\Phi(c_1, ..., c_n) = \text{const}} \tag{19.102}$$

Without the loss of a generality we can take const $= 0$, so the equation (19.102) becomes

$$\boxed{\Phi(c_1, ..., c_n) = 0} \tag{19.103}$$

Theorem 19.10 (Cauchy's method of characteristics) *If the first integrals (19.74) $\psi_i(x,z)$ of the system (19.98) are* ***independent****, that is,*

$$\boxed{\det\left[\frac{\partial\psi_i(x,z)}{\partial x_j}\right]_{i,j=1,\dots,n} \neq 0} \tag{19.104}$$

then the solution $z = z(x)$ of (19.97) can be found from the algebraic equation

$$\boxed{\Phi\left(\psi_1(x,z),\dots,\psi_n(x,z)\right) = 0} \tag{19.105}$$

where $\Phi(\psi_1,\dots,\psi_n)$ is an arbitrary function of its arguments.

Proof. By Theorem 16.8 on an implicit function, the systems (19.74) can be uniquely resolved with respect to x if (19.104) is fulfilled. So, the obtained functions (19.100) satisfy (19.99) and, hence, by Lemma 19.13 it follows that $z = z(x)$ satisfies (19.95). ■

Example 19.3 *Let us integrate the equation*

$$\sum_{i=1}^{n} x_i \frac{\partial z}{\partial x_i} = pz \text{ } (p \text{ is a constant}) \tag{19.106}$$

The system (19.97)

$$\frac{dx_1}{x_1} = \cdots = \frac{dx_n}{x_n} = \frac{dz}{pz}$$

has the following first integrals

$$x_i^p - z = c_i \quad (i = 1,\dots,n)$$

So, $z = z(x)$ can be found from the algebraic equation

$$\Phi\left(x_1^p - z,\dots,x_n^p - z\right) = 0$$

where $\Phi(\psi_1,\dots,\psi_n)$ is an arbitrary function, for example,

$$\Phi(\psi_1,\dots,\psi_n) := \sum_{i=1}^{n} \lambda_i\psi_i, \; \sum_{i=1}^{n} \lambda_i \neq 0$$

that gives

$$z = \left(\sum_{i=1}^{n} \lambda_i\right)^{-1} \sum_{i=1}^{n} \lambda_i x_i^p$$

19.3 Carathéodory's Type ODE

19.3.1 Main definitions

The differential equation

$$\boxed{\dot{x}(t) = f(t, x(t)),\ t \geq t_0} \tag{19.107}$$

in the regular case (with continuous right-hand side in both variables) is known to be equivalent to the integral equation

$$\boxed{x(t) = x(t_0) + \int_{s=t_0}^{t} f(s, x(s))\, ds} \tag{19.108}$$

Definition 19.7 *If the function $f(t,x)$ is* ***discontinuous*** *in t and* ***continuous*** *in $x \in \mathbb{R}^n$, then the functions $x(t)$, satisfying the integral equation (19.108) where the integral is understood in the Lebesgue sense, is called* ***solutions*** *ODE (19.107).*

The material presented bellow follows (Filippov 1988).

Let us define more exactly the conditions which the function $f(t,x)$ should satisfy.

Condition 19.1 (Carathéodory's conditions) *Let in the domain $\mathcal{D}$ of the (t,x)-space the following conditions be fulfilled:*

1) *the function $f(t,x)$ be defined and continuous in x for almost all t;*
2) *the function $f(t,x)$ be measurable (see (15.97)) in t for each x;*
3) $$\boxed{\|f(t,x)\| \leq m(t)} \tag{19.109}$$
   where the function $m(t)$ is summable (integrable in the Lebesgue sense) on each finite interval (if t is unbounded in the domain $\mathcal{D}$).

Definition 19.8

a) *The equation (19.107), where the function $f(t,x)$ satisfies the conditions 19.1, is called the **Carathéodory's type ODE**.*

b) *A function $x(t)$, defined on an open or closed interval l, is called **a solution** of the Carathéodory's type ODE if*

- *it is **absolutely continuous** on each interval $[\alpha,\beta] \in l$;*

- *it satisfies **almost everywhere** this equation or, which under the conditions 19.1 is the same thing, satisfies the integral equation (19.108).*

19.3.2 Existence and uniqueness theorems

Theorem 19.11 ((Filippov 1988)) *For $t \in [t_0, t_0 + a]$ and $x : \|x - x_0\| \leq b$ let the function $f(t,x)$ satisfies the Carathéodory's conditions 19.1. Then on a closed interval $[t_0, t_0 + d]$ **there exists a solution** of the Cauchy's problem*

$$\boxed{\dot{x}(t) = f(t, x(t)),\, x(t_0) = x_0} \tag{19.110}$$

In this case one can take an arbitrary number d such that

$$\boxed{0 < d \leq a,\ \varphi(t_0 + d) \leq b \text{ where } \varphi(t) := \int_{s=t_0}^{t} m(s)\, ds} \tag{19.111}$$

($m(t)$ is from (19.109)).

Proof. For integer $k \geq 1$ define $h := d/k$, and on the intervals $[t_0 + ih, t_0 + (i+1)h]$ $(i = 1, 2, ..., k)$ construct iteratively an approximate solution $x_k(t)$ as

$$x_k(t) := x_0 + \int_{s=t_0}^{t} f(s, x_{k-1}(s))\, ds \ \ (t_0 < t \leq t_0 + d) \tag{19.112}$$

(for any initial approximation $x_0(s)$, for example, $x_0(s) = \text{const}$). Remember that if $f(t,x)$ satisfies the Carathéodory's conditions 19.1

then and $x(t)$ is measurable on $[a, b]$, then the composite function $f(t, x(t))$ is summable (integrable in the Lebesgue sense) on $[a, b]$. In view of this and by the condition (19.111) we obtain $\|x_k(t) - x_0\| \leq b$. Moreover, for any $\alpha, \beta : t_0 \leq \alpha < \beta \leq t_0 + d$

$$\|x_k(\beta) - x_k(\alpha)\| \leq \int_{s=t_0}^{t} m(s)\, ds = \varphi(\beta) - \varphi(\alpha) \tag{19.113}$$

The function $\varphi(t)$ is continuous on the closed interval $[t_0, t_0 + d]$ and therefore uniformly continuous. Hence, for any $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon)$ such that for all $|\beta - \alpha| < \delta$ the right-hand side of (19.113) is less than ε. Therefore, the functions $x_k(t)$ are equicontinuous (see (14.18)) and uniformly bounded (see (14.17)). Let us choose (by the Arzelà's theorem 14.16) from them a uniformly convergent subsequence having a limit $x(t)$. Since

$$\|x_k(s-h) - x(s)\| \leq \|x_k(s-h) - x_k(s)\| + \|x_k(s) - x(s)\|$$

and the first term on the right-hand side is less then ε for $h = d/k < \delta$, it follows that $x_k(s-h)$ tends to $x(s)$, by the chosen subsequence. In view of continuity of $f(t, x)$ in x, and the estimate $\|f(t, x)\| \leq m(t)$ (19.109) one can pass to the limit under the integral sign in (19.112). Therefore, we conclude that the limiting function $x(t)$ satisfies the equation (19.108) and, hence, it is a solution of the problem (19.110). Theorem is proven. ■

Corollary 19.12 *If the Carathéodory's conditions 19.1 are satisfied for $t_0 - a \leq t \leq t_0$ and $\|x - x_0\| \leq b$, then a solution exists on the closed interval $[t_0 - d, t_0]$ where d satisfies (19.111).*

Proof. The case $t \leq t_0$ is reduced to the case $t \geq t_0$ by the simple substitution of $(-t)$ for t. ■

Corollary 19.13 *Let $(t_0, x_0) \in \mathcal{D} \subseteq \mathbb{R}^{1+n}$ and let there exists a summable function $l(t)$ (in fact, this is a Lipschitz constant) such that for any two points (t, x) and (t, y) of $\mathcal{D}$*

$$\boxed{\|f(t, x) - f(t, y)\| \leq l(t)\, \|x - y\|} \tag{19.114}$$

*Then in the domain $\mathcal{D}$ there exists **at most one solution** of the problem (19.110).*

Proof. Using (19.114) it is sufficient to check the Carathéodory's conditions 19.1. ∎

Theorem 19.12 (on the uniqueness) *If in Corollary 19.13 instead of (19.114) there is fulfilled the inequality*

$$\boxed{(f(t,x) - f(t,y), x - y) \leq l(t) \|x - y\|^2} \tag{19.115}$$

then in the domain $\mathcal{D}$ there exists the ***unique solution*** *of the problem (19.110).*

Proof. Let $x(t)$ and $y(t)$ be two solutions of (19.110). Define for $t_0 \leq t \leq t_1$ the function $z(t) := x(t) - y(t)$ for which it follows

$$\frac{d}{dt}\|z\|^2 = 2\left(z, \frac{d}{dt}z\right) = 2(f(t,x) - f(t,y), x - y)$$

almost everywhere. By (19.115) we obtain $\frac{d}{dt}\|z\|^2 \leq l(t)\|z\|^2$ and, hence, $\frac{d}{dt}\left(e^{-L(t)}\|z\|^2\right) \leq 0$ where $L(t) = \int\limits_{s=t_0}^{t} l(s)\,ds$. Thus, the absolutely continuous function (i.e., it is a Lebesgue integral of some other function) $e^{-L(t)}\|z\|^2$ does not increase, and it follows from $z(t_0) = 0$ that $z(t) = 0$ for any $t \geq t_0$. So, the uniqueness is proven. ∎

Remark 19.7 *The uniqueness of the solution of the problem (19.110) implies that if there exists two solutions of this problem, the graphs of which lie in the domain $\mathcal{D}$, then these solutions coincide on thew common part of their interval of existence.*

Remark 19.8 *Since the condition (19.114) implies the inequality (19.115) (this follows from the Cauchy-Bounyakoski-Schwartz inequality), thus the uniqueness may be considered to be proven for $t \geq t_0$ also under the condition (19.114).*

19.3.3 Variable structure and singular perturbed ODE

Variable structure ODE

In fact, if by the *structure* of ODE (19.107) $\dot{x}(t) = f(t, x(t))$ we will understand the function $f(t, x)$, then evidently any nonstationary system may be considered as a dynamic system with a *variable structure,* since for different $t_1 \neq t_2$ we will have $f(t_1, x) \neq f(t_2, x)$. From this point of view such treatment seems to be naive and having no correct mathematical sense. But if we consider the special class of ODE (19.107) given by

$$\boxed{\dot{x}(t) = f(t, x(t)) := \sum_{i=1}^{N} \chi(t \in [t_{i-1}, t_i)) f^i(x(t))} \tag{19.116}$$

where $\chi(\cdot)$ is the characteristic function of the corresponding event, namely,

$$\boxed{\chi(t \in [t_{i-1}, t_i)) := \begin{cases} 1 & \text{if} \quad t \in [t_{i-1}, t_i) \\ 0 & \text{if} \quad t \notin [t_{i-1}, t_i) \end{cases}, \ t_{i-1} < t_i} \tag{19.117}$$

then ODE (19.116) can be also treated as ODE with "jumping" parameters (coefficients). Evidently, that if $f^i(x)$ are continuous on a compact $\mathcal{D}$, and, hence, are bounded, that is,

$$\max_{i=1,...,N} \max_{x \in \mathcal{D}} \left\| f^i(x) \right\| \leq M \tag{19.118}$$

then the third Carathéodory's condition (19.109) will be fulfilled on the time interval $[\alpha, \beta]$, since

$$\boxed{m(t) = M \sum_{i=1}^{N} \chi(t \in [t_{i-1}, t_i)) = MN < \infty} \tag{19.119}$$

Therefore, such ODE equation (19.116) has *at most one solution.* If, in the addition, for each $i = 1, ..., N$ the Lipschitz condition holds, i.e.,

$$\boxed{\left(f^i(x) - f^i(y), x - y\right) \leq l_i \|x - y\|^2}$$

then, as it follows from Theorem 19.12, this equation has a *unique solution.*

Singular perturbed ODE

Consider the following ODE containing a *singular-type* of perturbations:

$$\boxed{\dot{x}(t) = f(x(t)) + \sum_{i=1}^{N} \mu_i \delta(t - t_i), \ t, t_i \geq t_0} \tag{19.120}$$

where $\delta(t - t_i)$ is the "Dirac delta-function" (15.128), μ_i is a real constant and f is a continuous function. The ODE (19.120) must be understand as the integral equation

$$\boxed{x(t) = x(t_0) + \int_{s=t_0}^{t} f(x(s))\, dt + \sum_{i=1}^{N} \mu_i \int_{s=t_0}^{t} \delta(t - t_i)\, dt} \tag{19.121}$$

The last term, by the property (15.134), can be represented as

$$\sum_{i=1}^{N} \mu_i \int_{s=t_0}^{t} d\chi(s > t_i) = \sum_{i=1}^{N} \mu_i \chi(t > t_i)$$

where $\chi(t \geq t_i)$ is the "*Heavyside's (step) function*" defined by (19.117). Let is apply the following state transformation:

$$\tilde{x}(t) := x(t) + \sum_{i=1}^{N} \mu_i \chi(t > t_i)$$

New variable $\tilde{x}(t)$ satisfies (with $\mu_0 := 0$) the following ODE:

$$\boxed{\begin{aligned} \frac{d}{dt}\tilde{x}(t) &= f\left(\tilde{x}(t) - \sum_{i=1}^{N} \mu_i \chi(t > t_i)\right) \\ &= \sum_{i=1}^{N} \chi(t > t_i) f\left(x(t) - \sum_{s=1}^{i} \mu_i\right) \\ &= \sum_{i=1}^{N} \chi(t > t_i) \tilde{f}^i(x(t)) \end{aligned}} \tag{19.122}$$

where

$$\tilde{f}^i(x(t)) := f\left(x(t) - \sum_{s=1}^{i} \mu_i\right)$$

Claim 19.1 *This exactly means that* ***the perturbed ODE (19.120) are equivalent to a variable structure ODE*** *(19.116).*

19.4 ODE with DRHS

In this chapter we will follow (Utkin 1992)), (Filippov 1988) and (Gelig et al. 1978).

19.4.1 Why ODE with DRHS are important in Control Theory

Here we will present some motivating consideration justifying our further study of ODE withy DRHS. Let us start with the simplest scalar case dealing with the following standard ODE which is *affine* (linear) on control:

$$\boxed{\dot{x}(t) = f(x(t)) + u(t),\ x(t) = x_0 \text{ is given}} \tag{19.123}$$

where $x(t), u(t) \in \mathbb{R}$ are interpreted here as the *state* of the system (19.123) and, respectively, the *control action* applied to it at time $t \in [0, T]$. The function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a Lipschitz function satisfying the, so-called, Lipschitz condition, that is, for any $x, x' \in \mathbb{R}$

$$|f(x) - f(x')| \leq L |x - x'|,\ 0 \leq L < \infty \tag{19.124}$$

Problem 19.1 *Let us try to stabilize this system at the point* $x^* = 0$ *using the, so-called, feedback control*

$$\boxed{u(t) := u(x(t))} \tag{19.125}$$

considering the following informative situations

- *the* ***complete information*** *case when the function* $f(x)$ *is exactly known;*
- *the* ***incomplete information*** *case when it is only known that the function* $f(x)$ *is bounded as*

$$|f(x)| \leq f_0 + f^+ |x|,\ f_0 < \infty,\ f^+ < \infty \tag{19.126}$$

(this inequality is assumed to be valid for any $x \in \mathbb{R}$*).*

There are two possibilities to do that:

1. use any **continuous control**, namely, take $u : \mathbb{R} \longrightarrow \mathbb{R}$ as a continuous function, i.e., $u \in C$;

2. use **a discontinuous control** which will be defined below.

The complete information case

Evidently that at the stationary point $x^* = 0$ any continuous control $u(t) := u(x(t))$ should satisfy the following identity

$$f(0) + u(0) = 0 \tag{19.127}$$

For example, this property may be fulfilled if use the control $u(x)$ containing the nonlinear compensating term

$$u_{comp}(x) := -f(x)$$

and the linear correction term

$$u_{cor}(x) := -kx, \ \ k > 0$$

that is, if

$$u(x) = u_{comp}(x) + u_{cor}(x) = -f(x) - kx \tag{19.128}$$

The application of this control (19.128) to the system (19.123) implies that

$$\dot{x}(t) = -kx(t)$$

and, as the result, one gets

$$x(t) = x_0 \exp(-kt) \underset{t \to 0}{\longrightarrow} 0$$

So, this *continuous control (19.128) in the complete information case solves the stabilization problem* (19.1).

The incomplete information case

Several informative situations may be considered.

1. $f(x)$ ***is unknown, but a priory it is known that*** $f(0) = 0$. In this situation the condition the Lipschitz condition (19.124) is transformed in to

$$|f(x)| = |f(x) - f(0)| \le L|x|$$

that for the Lyapunov function candidate $V(x) = x^2/2$ implies

$$\begin{array}{c} \dot{V}(x(t)) = x(t)\dot{x}(t) = \\ x(t)[f(x(t)) + u(x(t))] \le |x(t)||f(x(t))| \\ + x(t)u(x(t)) \le L|x(t)|^2 + x(t)u(x(t)) \end{array} \tag{19.129}$$

Since $f(x)$ is unknown let us select $u(x)$ in (19.128) as

$$\begin{array}{c} u(x) = u_{cor}(x) = -kx \\ u_{comp}(x) := 0 \end{array} \tag{19.130}$$

The use of (19.130) in (19.129) leads to the following identity:

$$\begin{array}{c} \dot{V}(x(t)) \le Lx^2(t) + x(t)u(x(t)) = \\ (L-k)x^2(t) = -2(k-L)V(x(t)) \end{array}$$

Selecting k big enough (this method is known as the "*high-gain control*") we get

$$\begin{array}{c} \dot{V}(x(t)) \le -2(k-L)V(x(t)) \le 0 \\ V(x(t)) \le V(x_0)\exp(-2[k-L]t) \underset{t\to 0}{\to} 0 \end{array}$$

This means that in the considered informative situation *the "high-gain control" solves the stabilization problem.*

2. $f(x)$ ***is unknown and it is admissible that*** $f(0) \ne 0$. In this situation the condition (19.127) never can be fulfilled since we do not know exactly the value $f(0)$ and, hence, neither the control (19.128) no the control (19.130) can be applied. Let us try to apply a discontinuous control, namely, let us take $u(x)$ in the form of the, so-called, *sliding-mode(or relay) control*:

$$\boxed{u(x) = -k_t\,\mathrm{sign}(x),\, k_t > 0} \tag{19.131}$$

where

$$\boxed{\operatorname{sign}(x) := \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \\ \in [-1, 1] & \text{if } x = 0 \end{cases}} \tag{19.132}$$

(see Fig.19.1). Staring from some $x_0 \neq 0$, analogously to (19.129)

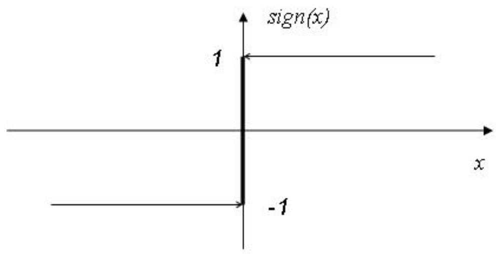

Figure 19.1: The signum function.

and using (19.126), we have

$$\dot{V}(x(t)) = x(t)\,\dot{x}(t) = x(t)\left[f(x(t)) + u(x(t))\right] \leq$$

$$|x(t)|\,|f(x(t))| + x(t)\,u(x(t)) \leq |x(t)|\,(f_0 + f^+\,|x(t)|)$$

$$-k_t x(t)\operatorname{sign}(x(t)) = |x(t)|\,f_0 + f^+\,|x(t)|^2 - k_t\,|x(t)|$$

Taking

$$\boxed{\begin{array}{c} k_t = k(x(t)) := k^0 + k^1\,|x(t)| \\ \\ k^0 > f_0,\ k^1 > f^+ \end{array}} \tag{19.133}$$

one has

$$\dot{V}(x(t)) \leq -\,|x(t)|\,(k^0 - f_0) - (k^1 - f^+)\,|x(t)|^2 \leq$$

$$-\,|x(t)|\,(k^0 - f_0) = -\sqrt{2}\,(k^0 - f_0)\,\sqrt{V(x(t))} \leq 0$$

Hence,

$$\frac{dV(x(t))}{\sqrt{V(x(t))}} \leq -\sqrt{2}\left(k^0 - f_0\right)dt$$

that leads to the following identity

$$2\left(\sqrt{V(x(t))}-\sqrt{V(x_0)}\right)\leq-\sqrt{2}\left(k^0-f_0\right)t$$

or, equivalently,

$$\sqrt{V(x(t))}\leq\sqrt{V(x_0)}-\frac{k^0-f_0}{\sqrt{2}}t$$

This means that the, so-called, "*reaching phase*", during which the system (19.123) controlled by the sliding-mode algorithm (19.131)-(19.133) reaches the origin, is equal to

$$\boxed{t^*=\frac{\sqrt{2V(x_0)}}{k^0-f_0}} \tag{19.134}$$

Conclusion 19.1 *As it follows from the considerations above,* ***the discontinuous (in this case, sliding-mode) control*** *(19.131)-(19.133)* ***can stabilize the class of the dynamic systems*** *(19.123), (19.124), (19.126)* ***in finite time*** *(19.134) without the exact knowledge of its model. Besides,* ***the reaching phase may be done as small as you wish*** *by the simple selection of the gain parameter k^0 in (19.134). In other words, the discontinuous control (19.131)-(19.133) is* ***robust with respect to the presence of unmodelled dynamics*** *in (19.123) that means that it is capable to stabilize a wide class of "black/grey-box" systems.*

Remark 19.9 *Evidently, that using such discontinuous control, the trajectories of the controlled system can not stay in the stationary point $x^*=0$ since it arrives to it in finite time but with a nonzero rate, namely, with $\dot{x}(t)$ such that*

$$\dot{x}(t)=\begin{cases} f(0)+k^0 & \text{if } x(t)\longrightarrow+0 \\ f(0)-k^0 & \text{if } x(t)\longrightarrow-0 \end{cases}$$

*that provokes the, so-called, "**chattering effect**" (see Fig.19.2). Simple engineering considerations show that some sort of smoothing (or, low-pass filtering) should be applied to keep dynamics close to the stationary point $x^*=0$.*

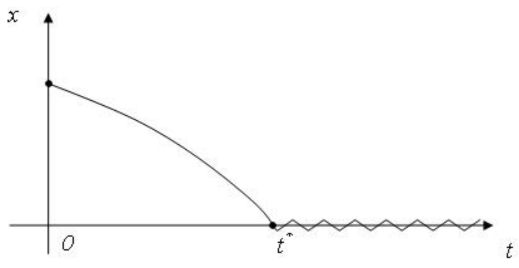

Figure 19.2: The chattering effect.

Remark 19.10 *Notice that when $x(t) = x^* = 0$ we only know that*

$$\boxed{\dot{x}(t) \in \left[f(0) - k^0,\ f(0) + k^0\right]} \tag{19.135}$$

This means that we deal with a ***differential inclusion*** *(not an equation) (19.135). So, we need to define what does it mean mathematically correctly a solution of a differential inclusion and what is it itself.*

All these questions, arising in the remarks above, will be considered below in details and be illustrated by the corresponding examples and figures.

19.4.2 ODE with DRHS and differential inclusions

General requirements to a solution

As it is well known, a solution of the differential equation

$$\dot{x}(t) = f(t, x(t)) \tag{19.136}$$

with a continuous right-hand side is a function $x(t)$ which has a derivative and satisfies (19.136) everywhere on a given interval. This definition is not, however, valid for DE with DRHS since in some points of discontinuity the derivative of $x(t)$ does not exists. That's why the consideration of DE with DRHS requires a generalization of the concept of a solution. Anyway, such generalized concept should necessarily meet the following requirements:

- For differential equations with a continuous right-hand side the definition of a solution must be equivalent to the usual (standard) one.

- For the equation $\dot{x}(t) = f(t)$ the solution should be the functions $x(t) = \int f(t)\,dt + c$ only.

- For any initial data $x(t_0) = x_{init}$ within a given region the solution $x(t)$ should exist (at least, locally) for any $t > t_0$ and admit the possibility to be continued up to the boundary of this region or up to infinity (when $(t, x) \to \infty$).

- The limit of a uniformly convergent sequences of solutions should be a solution too.

- Under the commonly used changes of variables a solution must be transformed into a solution.

The definition of a solution

Definition 19.9 *A vector-valued function $f(t, x)$, defined by a mapping $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^p$, is said to be* ***piecewise continuous*** *in a finite domain $\mathcal{G} \subseteq \mathbb{R}^{n+1}$ if G consists of a finite numbers of a domains $\mathcal{G}_i$ $(i = 1, ..., l)$, i.e.,*

$$\mathcal{G} = \bigcup_{i=1}^{l} \mathcal{G}_i$$

such that in each of them the function $f(t, x)$ is continuous up to the boundary

$$\mathcal{M}_i := \bar{\mathcal{G}}_i \backslash \mathcal{G}_i \ \ (i = 1, ..., l) \tag{19.137}$$

of a measure zero.

The most frequent case is the one where the set

$$\mathcal{M} = \bigcup_{i=1}^{l} \mathcal{M}_i$$

of all discontinuity points consists of a finite number of hypersurfaces

$$0 = S_k(x) \in C^1, \ \ k = 1, ..., m$$

where $S_k(x)$ is a smooth function.

Definition 19.10 *The set* $\mathcal{M}$ *defined as*

$$\boxed{\mathcal{M} = \{x \in \mathbb{R}^n \mid S(x) = (S_1(x), ..., S_m(x))^{\intercal} = 0\}} \qquad (19.138)$$

is called a ***manifold*** *in* $\mathbb{R}^n$*. It is referred to as* ***a smooth manifold*** *if* $S_k(x) \in C^1$, $k = 1, ..., m$.

Now we are ready to formulate the main definition of this section.

Definition 19.11 (A solution in the Filippov's sense) *A solution* $x(t)$ *on a time interval* $[t_0, t_f]$ *of ODE* $\dot{x}(t) = f(t, x(t))$ *with DRHS* ***in the Filippov's sense*** *is called a solution of the* ***differential inclusion***

$$\boxed{\dot{x}(t) \in \mathcal{F}(t, x(t))} \qquad (19.139)$$

that is, an ***absolutely continuous*** *on* $[t_0, t_f]$ *function* $x(t)$ *(which can be represented as a Lebesgue integral of another function) satisfying (19.139) almost everywhere on* $[t_0, t_f]$*, where the set* $\mathcal{F}(t, x)$ *is* ***the smallest convex closed set containing all limit values of the vector-function*** $f(t, x^*)$ *for* $(t, x^*) \notin \mathcal{M}$, $x^* \to x$, $t = \text{const}$.

Remark 19.11 *The set* $\mathcal{F}(t, x)$

1) consists of one point $f(t, x)$ *at points of continuity of the function* $f(t, x)$*;*

2) is a segment (a convex polygon, or polyhedron), which in the case when $(t, x) \in \mathcal{M}_i$ *(19.137) has the vertices*

$$f_i(t, x) := \lim_{(t, x^*) \in \mathcal{G}_i,\ x^* \to x} f(t, x^*) \qquad (19.140)$$

All points $f_i(t, x)$ *are contained in* $\mathcal{F}(t, x)$*, but it is not obligatory that all of them are vertices.*

Example 19.4 *For the scalar differential inclusion*

$$\dot{x}(t) \in -\text{sign}(x(t))$$

the set $\mathcal{F}(t, x)$ *is as follows (see Fig.19.3):*

1. $\mathcal{F}(t, x) = -1$ *if* $x > 0$;

2. $\mathcal{F}(t, x) = 1$ *if* $x < 0$;

3. $\mathcal{F}(t, x) = [-1, 1]$ *if* $x = 0$.

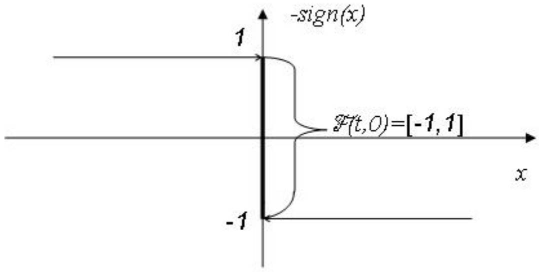

Figure 19.3: The right-hand side of the differential inclusion $\dot{x}_t = -sign\,(x_t)$.

Semi-continuous sets as functions

Definition 19.12 *A multi-valued function (or, a set)* $\mathcal{F} = \mathcal{F}(t, x)$ $(t \in \mathbb{R},\ x \in \mathbb{R}^n)$ *is said to be*

- *a* ***semi-continuous*** *in the point* (t_0, x_0) *if for any* $\varepsilon > 0$ *there exists* $\delta = \delta(t_0, x_0, \varepsilon)$ *such that the inclusion*

$$\boxed{(t, x) \in \{z \mid \|z - (t_0, x_0)\| \leq \delta\}} \tag{19.141}$$

implies

$$\boxed{\mathcal{F}(t, x) \in \{f \mid \|f - f(t_0, x_0)\| \leq \varepsilon\}} \tag{19.142}$$

- *a* ***continuous*** *in the point* (t_0, x_0) *if it is a semi-continuous and, additionally, for any* $\varepsilon > 0$ *there exists* $\delta = \delta(t_0, x_0, \varepsilon)$ *such that the inclusion*

$$\boxed{(t_0, x_0) \in \{z \mid \|z - (t', x')\| \leq \delta\}} \tag{19.143}$$

implies

$$\boxed{\mathcal{F}(t_0, x_0) \in \{f \mid \|f - f(t', x')\| \leq \varepsilon\}} \tag{19.144}$$

Example 19.5 *Consider the multi-valued functions* $\mathcal{F}(t, x)$ *depicted at Fig.19.4.*

Here the functions (sets) $\mathcal{F}(t, x)$*, corresponding the plots 1)-4), are semi-continuous.*

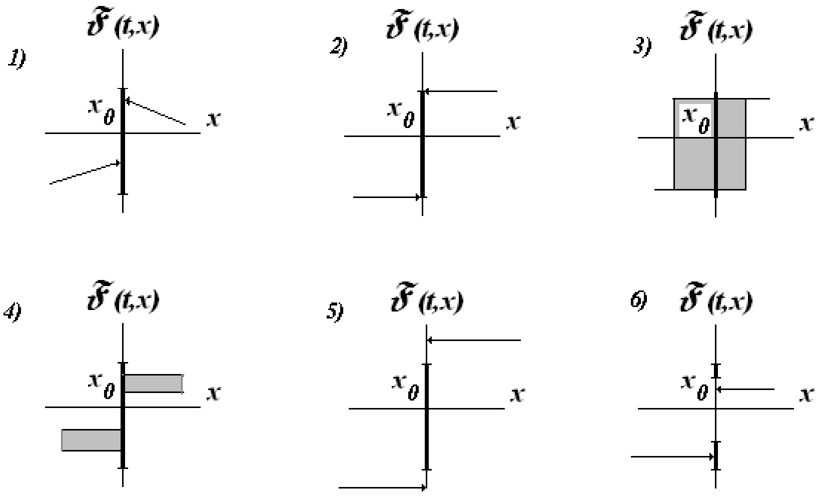

Figure 19.4: Multivalued functions.

Theorem on the local existence of solution

First, let us formulate some useful result which will be applied in the following considerations.

Lemma 19.14 *If $x(t)$ is absolutely continuous on the interval $t \in [\alpha, \beta]$ and within this interval $\|\dot{x}(t)\| \leq c$, then*

$$\boxed{\frac{1}{\beta - \alpha}(x_\beta - x_\alpha) \subset \underset{a.a.\ t\in[\alpha,\beta]}{\mathsf{Conv}} \cup \dot{x}(t)} \tag{19.145}$$

where Conv *is a convex closed set containing $\cup\, \dot{x}(t)$ for almost all $t \in [\alpha, \beta]$.*

Proof. By the definition of the Lebesgue integral

$$\frac{1}{\beta - \alpha}(x_\beta - x_\alpha) = \frac{1}{\beta - \alpha}\int_{t=\alpha}^{\beta} \dot{x}(t)\, dt = \lim_{k\to\infty} s_k$$

where

$$s_k = \sum_{i=1}^{k} \frac{\mu_i}{|\beta - \alpha|} \dot{x}(t_i), \ \mu_i \geq 0, \ \sum_{i=1}^{k} \mu_i = |\beta - \alpha|$$

are Lebesgue sums of the integral above. But $s_k \in \underset{\text{a.a. } t \in [\alpha,\beta]}{\mathsf{Conv}} \cup \dot{x}(t)$. Hence, the same fact is valid for the limit vectors $\lim_{k \to \infty} s_k$ that proves the lemma. ■

Theorem 19.13 (on the local existence) *Suppose that*

A1) a multi-valued function (set) $\mathcal{F}(t,x)$ is a semi-continuous at each point

$$(t,x) \in D_{\gamma,\rho}(t_0,x_0) := \{(t,x) \mid \|x - x_0\| \leq \gamma, |t - t_0| \leq \rho\}$$

A2) the set $\mathcal{F}(t,x)$ is a convex compact and $\sup \|y\| = c$ *whenever*

$$y \in \mathcal{F}(t,x) \ \text{ and } \ (t,x) \in D_{\gamma,\rho}(t_0,x_0)$$

Then for any t such that $|t - t_0| \leq \tau := \rho/c$ there exists an absolutely continuous function $x(t)$ (may be, not unique) such that

$$\dot{x}(t) \in \mathcal{F}(t,x), \ x(t_0) = x_0$$

*that is, the ODE $\dot{x}(t) = f(t,x(t))$ with DRHS **has a local solution** in the Filippov's sense (see Definition 19.11).*

Proof. Divide the interval $[t_0 - \tau, t_0 + \tau]$ into $2m$-parts $t_i^{(m)} := t_0 + j\frac{\tau}{m}$ $(i = 0, \pm 1, .. \pm m)$ and construct the, so-called, partially linear *Euler's curves*

$$x_m(t) := x_m\left(t_i^{(m)}\right) + \left(t - t_i^{(m)}\right) \hat{f}_i^{(m)}\left(t_i^{(m)}\right), \ t \in \left[t_i^{(m)}, t_i^{(m+1)}\right]$$

$$x_m\left(t_0^{(m)}\right) = x_0, \ \hat{f}_i^{(m)}\left(t_i^{(m)}\right) \in \mathcal{F}\left(t_i^{(m)}, x_m\left(t_i^{(m)}\right)\right)$$

By the assumption A2) it follows that $x_m(t)$ is uniformly bounded and continuous on $D_{\gamma,\rho}(t_0,x_0)$. Then, by the Arzelà's theorem 14.16 there exists a subsequence $\{x_{m_k}(t)\}$ which uniformly converges to some vector function $x(t)$. This limit evidently has a Lipschitz constant on

$D_{\gamma,\rho}(t_0, x_0)$ and satisfies the initial condition $x(t_0) = x_0$. In view of Lemma 19.14, for any $h > 0$ one has

$$h^{-1}[x_{m_k}(t+h) - x_{m_k}(t)] \subset \underset{\text{a.a.}[t_0-\tau, t_0+\tau]}{\mathsf{Conv}} \bigcup_{i=-m_k}^{m_k} \hat{f}_i^{(m_k)}$$

$$\subset \underset{\text{a.a. } \lambda \in \left[t_0 - \frac{\tau}{m_k}, t_0 + \frac{\tau}{m_k} + h\right]}{\mathsf{Conv}} \bigcup_{i=-m_k}^{m_k} \hat{f}_i^{(m_k)}(\lambda) := A_k$$

Since $\mathcal{F}(t,x)$ is semi-continuous, it follows that $\sup_{x \in A_k} \inf_{y \in A} \|x - y\| \to 0$ whenever $k \to \infty$ (here $A := \underset{\text{a.a. } \lambda \in [t, t+h]}{\mathsf{Conv}} \bigcup_{i=-m_l}^{m_l} f(\lambda, x_\lambda))$. The convexity of $\mathcal{F}(t,x)$ implies also that $\sup_{x \in A} \inf_{y \in \mathcal{F}(t,x)} \|x - y\| \to 0$ when $h \to 0$ that, together with previous property, proves the theorem. ■

Remark 19.12 *By the same reasons as for the case of regular ODE, we may conclude that the solution of the differential inclusion (if it exists) is continuously dependent on* t_0 *and* x_0.

19.4.3 Sliding mode control

Sliding mode surface

Consider the special case where the function $f(t,x)$ is discontinuous on a smooth surface S given by the equation

$$\boxed{s(x) = 0, \ s : \mathbb{R}^n \to \mathbb{R}, \ s(\cdot) \in C^1} \tag{19.146}$$

The surface separates its neighborhood (in $\mathbb{R}^n$) into domains $\mathcal{G}^+$ and $\mathcal{G}^-$. For $t = \text{const}$ and for the point x^* approaching the point $x \in S$ from the domains $\mathcal{G}^+$ and $\mathcal{G}^-$ let us suppose that the function $f(t, x^*)$ has the following limits:

$$\begin{aligned} \lim_{(t,x^*) \in \mathcal{G}^-, \ x^* \to x} f(t, x^*) &= f^-(t,x) \\ \lim_{(t,x^*) \in \mathcal{G}^+, \ x^* \to x} f(t, x^*) &= f^+(t,x) \end{aligned} \tag{19.147}$$

Then by the Filippov's definition, $\mathcal{F}(t,x)$ is a linear segment joining the endpoints of the vectors $f^-(t,x)$ and $f^+(t,x)$. Two situations are possible.

- If for $t \in (t_1, t_2)$ this segment lies on one side of the plane P tangent to the surface S at the point x, the solutions for these t pass from one side of the surface S to the other one (see Fig.19.5 depicted at the point $x = 0$);

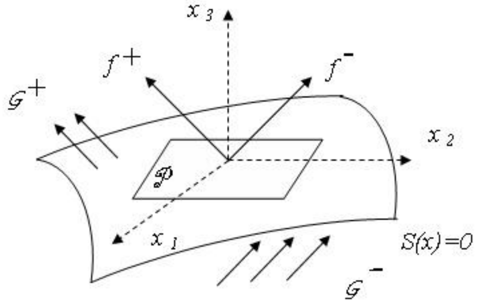

Figure 19.5: The sliding surface and the rate vector field at the point $x = 0$.

- If this segment intersects the plane P, the intersection point is the endpoint of the vector $f^0(t, x)$ which defines the velocity of the motion

$$\boxed{\dot{x}(t) = f^0(t, x(t))} \tag{19.148}$$

along the surface S in $\mathbb{R}^n$ (see Fig.19.6 depicted at the point $x = 0$). Such a solution, lying on S for all $t \in (t_1, t_2)$, is often called a **sliding motion** *(*or*, **mode**)*. Defining the projections of the vectors $f^-(t, x)$ and $f^+(t, x)$ to the surface S ($\nabla s(x) \neq 0$) as

$$p^-(t, x) := \frac{(\nabla s(x), f^-(t, x))}{\|\nabla s(x)\|}, \quad p^+(t, x) := \frac{(\nabla s(x), f^+(t, x))}{\|\nabla s(x)\|}$$

one can find that when $p^-(t, x) < 0$ and $p^+(t, x) > 0$

$$f^0(t, x) = \alpha f^-(t, x) + (1 - \alpha) f^+(t, x)$$

Here α can be easily found from the equation

$$\left(\nabla s(x), f^0(t, x)\right) = 0$$

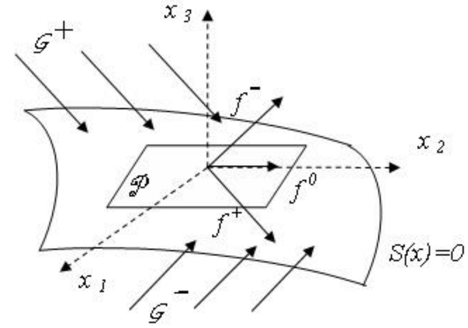

Figure 19.6: The velocity of the motion.

or, equivalently,

$$0 = (\nabla s(x), \alpha f^{-}(t,x) + (1-\alpha) f^{+}(t,x)) =$$

$$\alpha p^{-}(t,x) + (1-\alpha) p^{+}(t,x)$$

that implies

$$\alpha = \frac{p^{+}(t,x)}{p^{+}(t,x) - p^{-}(t,x)}$$

Finally, we obtain that

$$\boxed{\begin{array}{c} f^{0}(t,x) = \dfrac{p^{+}(t,x)}{p^{+}(t,x) - p^{-}(t,x)} f^{-}(t,x) + \\ \left(1 - \dfrac{p^{+}(t,x)}{p^{+}(t,x) - p^{-}(t,x)}\right) f^{+}(t,x) \end{array}} \tag{19.149}$$

Sliding mode surface as a desired dynamics

Let us consider in this subsection several examples demonstrating that a desired dynamic behavior of a controlled system may be expressed not only in the traditional manner, using some cost (or payoff) functionals as possible performance indices, but also representing a nominal (desired) dynamics in the form of a surface (or, manifold) in a space of coordinates.

First-order tracking system Consider a first-order system given by the following ODE:

$$\boxed{\dot{x}(t) = f(t, x(t)) + u(t)} \tag{19.150}$$

where $u(t)$ is a control action and $f : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is supposed to be bounded, that is,

$$|f(t, x(t))| \leq f^{+} < \infty$$

Assume that the desired dynamics (signal), which should be tracked, is given by a smooth function $r(t)$ ($|\dot{r}(t)| \leq \rho$), such that the tracking error e_t is (see Fig.19.7)

$$e(t) := x(t) - r(t)$$

Select a desired surface s as follows

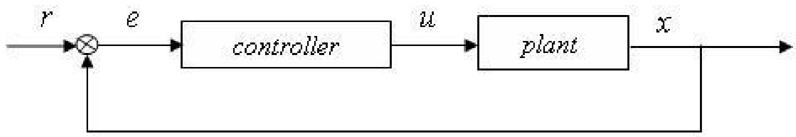

Figure 19.7: A tracking system.

$$\boxed{s(e) = e = 0} \tag{19.151}$$

that exactly corresponds to an "ideal tracking" process. Then, designing the control $u(t)$ as

$$\boxed{u(t) := -k\text{sign}(e(t))}$$

we derive that

$$\dot{e}(t) = f(t, x(t)) - \dot{r}(t) - k\text{sign}(e(t))$$

and for $V(e) = e^2/2$ one has

$$\dot{V}(e(t)) = e(t)\dot{e}(t) = e(t)[f(t, x(t)) - \dot{r}(t) - k\text{sign}(e(t))]$$

$$= e(t)[f(t, x(t)) - \dot{r}(t)] - k|e(t)| \leq |e(t)|[f^{+} + \rho] - k|e(t)|$$

$$= |e(t)|[f^{+} + \rho - k] = -\sqrt{2}[k - f^{+} - \rho]\sqrt{V(e(t))}$$

and, hence,

$$\sqrt{V\left(e_t\right)} \leq \sqrt{V\left(e_0\right)} - \frac{1}{\sqrt{2}}\left[k - f^+ - \rho\right] t$$

So, taking $k > f^+ + \rho$ implies the finite time convergence of e_t (with the reaching phase $t_f = \frac{\sqrt{2V\left(e_0\right)}}{k - f^+ - \rho}$) to the surface (19.151) (see Fig. 19.8, and Fig. 19.9).

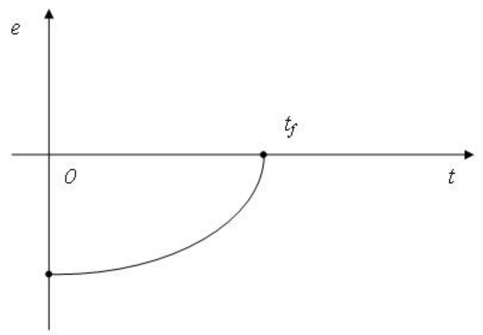

Figure 19.8: The finite time error cancellation.

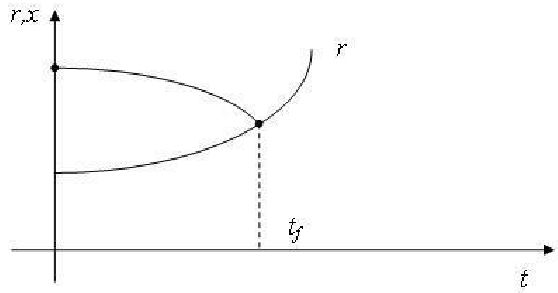

Figure 19.9: The finite time tracking.

Stabilization of a second order relay-system Let us consider a second order relay-system given by the following ODE

$$\boxed{\begin{array}{c} \ddot{x}(t) + a_2\dot{x}(t) + a_1 x(t) = u(t) + \xi(t) \\ u(t) = -\,k\text{sign}(\tilde{s}(t)) \text{ - the relay-control} \\ \tilde{s}(t) := \dot{x}(t) + cx(t),\ c > 0 \\ |\xi(t)| \le \xi^+ \text{ - a bounded unknown disturbance} \end{array}} \tag{19.152}$$

One may rewrite the dynamic ($x_1 := x$) as

$$\begin{array}{c} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = -\,a_1 x_1(t) - a_2 x_2(t) + u(t) + \xi(t) \\ u(t) = -\,k\text{sign}(x_2(t) + cx_1(t)) \end{array} \tag{19.153}$$

Here the *sliding surface* is

$$\boxed{s(x) = x_2 + cx_1}$$

So, the sliding motion, corresponding the dynamics $\tilde{s}(t) := \dot{x}(t) + cx(t) = 0$, is given by (see Fig.19.10)

$$x(t) = x_0 e^{-ct}$$

Let us introduce the following Lyapunov function candidate:

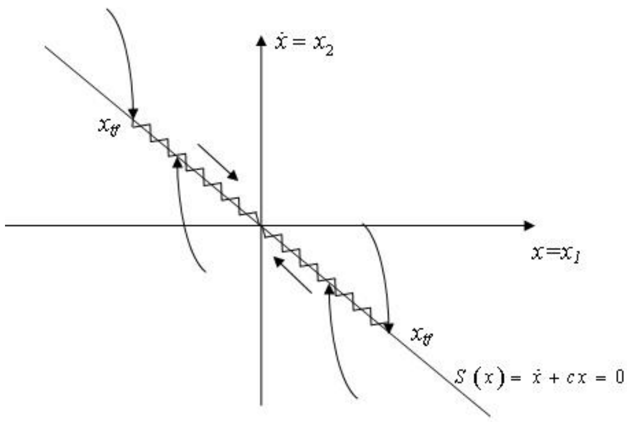

Figure 19.10: The sliding motion on the sliding surface $s(x) = x_2 + cx_1$.

$$\boxed{V(s) = s^2/2}$$

for which the following property holds:

$$\dot{V}(s) = s\dot{s} = s(x(t))\left[\frac{\partial s(x(t))}{\partial x_1}\dot{x}_1(t) + \frac{\partial s(x(t))}{\partial x_2}\dot{x}_2(t)\right] =$$

$$s(x(t))\left[cx_2(t) - a_1x_1(t) - a_2x_2(t) + u(t) + \xi(t)\right] \leq$$

$$|s(x(t))|\left[|a_1||x_1(t)| + (c+|a_2|)|x_2(t)| + \xi^+\right] \text{-} ks(x(t))\,\text{sign}(s(x(t)))$$

$$= -\left[k - |a_1||x_1(t)| - (c+|a_2|)|x_2(t)| - \xi^+\right]|s(x(t))| \leq 0$$

if take

$$\boxed{k = |a_1||x_1(t)| + (c+|a_2|)|x_2(t)| + \xi^+ + \rho,\ \rho > 0} \tag{19.154}$$

This implies $\dot{V}(s) \leq -\rho\sqrt{2V(s)}$, and, hence, the reaching time t_f (see Fig.19.9) is

$$\boxed{t_f = \frac{\sqrt{2V(s_0)}}{\rho} = \frac{|\dot{x}_0 + cx_0|}{\rho}} \tag{19.155}$$

Sliding surface and a related LQ-problem Consider a linear multi-dimensional plant given by the following ODE

$$\boxed{\begin{array}{l} \dot{x}(t) = A(t)x(t) + B(t)u(t) + \xi(t) \\ x_0 \text{ is given, } x(t) \in \mathbb{R}^n,\ u(t) \in \mathbb{R}^r \\ B^{\intercal}(t)B(t) > 0 \text{ rank}[B(t)] = r \text{ for any } t \in [t_s, t_1] \\ \xi(t)\ \text{ is known external perturbation} \end{array}} \tag{19.156}$$

A sliding mode is said to be taking place in this system (19.156) if there exists a finite reaching time t_s, such that the solution $x(t)$ satisfies

$$\boxed{\sigma(x,t) = 0 \text{ for all } t \geq t_s} \tag{19.157}$$

where $\sigma(x,t) : R^n \times R_+ \to R^r$ is *a sliding function* and (21.65) defines *a sliding surface* in R^{n+1}. For each $t_1 > 0$ the quality of the system (19.156) motion in the sliding surface (21.65) is characterized by *the performance index* (Utkin 1992)

$$\boxed{J_{t_s,t_1} = \frac{1}{2}\int_{t_s}^{t_1}(x(t), Qx(t))\,dt, Q = Q^{\intercal} \geq 0} \tag{19.158}$$

Below we will show that the system motion in the sliding surface (21.65) does not depend on the control function u, that's why (19.158) is a functional of x and $\sigma(x,t)$ only. Let us try to solve the following problem.

Problem formulation: for the given linear system (19.156) and $t_1 > 0$ define the optimal sliding function $\sigma = \sigma(x,t)$ (21.65) providing the optimization in the sense of (19.158) in the sliding mode, that is,

$$\boxed{J_{t_s,t_1} \to \inf_{\sigma \in \Xi}} \tag{19.159}$$

where Ξ is the set of the admissible smooth (differentiable on all arguments) sliding functions $\sigma = \sigma(x,,t)$. So, we wish to minimize the performance index (19.158) varying (optimizing) the sliding surface $\sigma \in \Xi$.

Introduce new state vector z defined by

$$z = T(t)\,x \tag{19.160}$$

where the linear nonsingular transformations $T(t)$ are given by

$$T(t) := \begin{bmatrix} I_{(n-r)\times(n-r)} & -B_1(t)\,(B_2(t))^{-1} \\ 0 & (B_2(t))^{-1} \end{bmatrix} \tag{19.161}$$

Here $B_1^{(n-r)\times(n-r)}(t) \in \mathbb{R}^{r\times r}$ and $B_2(t) \in \mathbb{R}^{r\times r}$ represent the matrices $B(t)$ in the form

$$B(t) = \begin{bmatrix} B_1(t) \\ B_2(t) \end{bmatrix}, \ \det[B_2(t)] \neq 0 \ \forall\, t \geq 0 \tag{19.162}$$

Applying (19.162) to the system (19.156), we obtain (below we will omit the time-dependence)

$$\dot{z} = \begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} \tilde{A}_{11}z_1 + \tilde{A}_{12}z_2 \\ \tilde{A}_{21}z_1 + \tilde{A}_{22}z_2 \end{pmatrix} + \begin{pmatrix} 0 \\ u \end{pmatrix} + \begin{pmatrix} \tilde{\xi}_1 \\ \tilde{\xi}_2 \end{pmatrix} \tag{19.163}$$

where $z_1 \in R^{n-r}$, $z_2 \in R^r$ and

$$\tilde{A} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix} = TAT^{-1} + \dot{T}T^{-1}, \ \begin{pmatrix} \tilde{\xi}_1(t) \\ \tilde{\xi}_2(t) \end{pmatrix} = T\xi(t) \tag{19.164}$$

Using the operator T^{-1}, it follows $x = T^{-1}z$ and, hence, the performance index (19.158) in new variables z may be rewritten as

$$\boxed{\begin{array}{c} J_{t_s,t_1} = \frac{1}{2}\int\limits_{t_s}^{t_1} (x, Qx)\, dt = \frac{1}{2}\int\limits_{t_s}^{t_1} \left(z, \tilde{Q}^{\alpha} z\right) dt = \\ \frac{1}{2}\int\limits_{t_s}^{t_1} \left[\left(z_1, \tilde{Q}_{11}^{\alpha} z_1\right) + 2\left(z_1, \tilde{Q}_{12}^{\alpha} z_2\right) + \left(z_2, \tilde{Q}_{22}^{\alpha} z_2\right)\right] dt \\ \tilde{Q} := \left(T^{-1}\right)^{\intercal} Q T^{-1} = \begin{bmatrix} \tilde{Q}_{11} & \tilde{Q}_{12} \\ \tilde{Q}_{21} & \tilde{Q}_{22} \end{bmatrix} \end{array}} \tag{19.165}$$

and the sliding function $\sigma = \sigma(x, t)$ becomes

$$\sigma = \sigma\left(T^{-1}z, t\right) := \tilde{\sigma}(z, t) \tag{19.166}$$

Remark 19.13 *The matrices $\tilde{Q}_{11}, \tilde{Q}_{12}, \tilde{Q}_{21}$ and $\tilde{Q}_{22}$ are supposed to be symmetric. Otherwise, they can be symmetrized as follows:*

$$\begin{array}{c} J_{t_s,t_1} = \frac{1}{2}\int\limits_{t_s}^{t_1} \left[\left(z_1, \bar{Q}_{11} z_1\right) + 2\left(z_1, \bar{Q}_{12} z_2\right) + \left(z_2, \bar{Q}_{22} z_2\right)\right] dt \\ \bar{Q}_{11}^{\alpha} := \left(\tilde{Q}_{11} + \tilde{Q}_{11}^{\intercal}\right)/2,\ \bar{Q}_{22} := \left(\tilde{Q}_{22} + \tilde{Q}_{22}^{\intercal}\right)/2 \\ \bar{Q}_{12} = \left(\tilde{Q}_{12} + \tilde{Q}_{12}^{\intercal} + \tilde{Q}_{21} + \tilde{Q}_{21}^{\intercal}\right)/2 \end{array} \tag{19.167}$$

Assumption (A1). We will look for the sliding function (19.166) in the form

$$\boxed{\tilde{\sigma}(z, t) := z_2 + \tilde{\sigma}_0(z_1, t)} \tag{19.168}$$

If the sliding mode exists for the system (19.163) in the sliding surface $\tilde{\sigma}(z, t) = 0$ under the assumption A1, then for all $t \geq t_s$ the corresponding sliding mode dynamics, driven by the unmatched disturbance $\tilde{\xi}_1(t)$, are given by

$$\begin{array}{l} \dot{z}_1 = \tilde{A}_{11} z_1 + \tilde{A}_{12} z_2 + \tilde{\xi}_1 \\ z_2 = -\tilde{\sigma}_0(z_1, t) \end{array} \tag{19.169}$$

with the initial conditions $z_1(t_s) = (Tx(t_s))_1$. Defining z_2 as a virtual control, that is,

$$v := z_2 = -\tilde{\sigma}_0(z_1, t) \tag{19.170}$$

the system (19.169) may be rewritten as

$$\dot{z}_1 = \tilde{A}_{11} z_1 + \tilde{A}_{12} v + \tilde{\xi}_1 \tag{19.171}$$

and the performance index (19.165) becomes

$$J_{t_s,t_1} = \frac{1}{2} \int_{t_s}^{t_1} \left[\left(z_1, \tilde{Q}_{11}^{\alpha} z_1 \right) + 2 \left(z_1, \tilde{Q}_{12}^{\alpha} v \right) + \left(v, \tilde{Q}_{22}^{\alpha} v \right) \right] dt \tag{19.172}$$

In view of (19.171) and (19.172), the sliding surface design problem (19.159) is reduced to the following one:

$$\boxed{J_{t_s,t_1} \to \inf_{v \in R^r}} \tag{19.173}$$

But this is the standard LQ-optimal control problem. This means that the optimal control $v_t^* = v^* (z_{1,t}, t)$, optimizing the cost functional (19.172), defines the optimal sliding surface $\sigma^* (x, t)$ (see (19.171) and (19.168)) by the following manner:

$$\begin{gathered} v^* (z_{1,t}, t) = -\tilde{\sigma}_0 (z_1, t) \\ \tilde{\sigma} (z, t) = z_2 - v^* (z_{1,t}, t) = 0 \end{gathered}$$

or, equivalently,

$$\boxed{\sigma^* (x, t) = (Tx)_2 - v^* ((Tx)_1, t) = 0} \tag{19.174}$$

Equivalent control method

Equivalent control construction Here a formal procedure will be described to obtain sliding equations along the intersection of sets of discontinuity for a nonlinear system given by

$$\boxed{\begin{gathered} \dot{x} (t) = f (t, x (t), u (t)) \\ x_0 \text{ is given} \\ x (t) \in \mathbb{R}^n, \ u (t) \in \mathbb{R}^r \end{gathered}} \tag{19.175}$$

and the manifold $\mathcal{M}$ (19.138) defined as

$$\boxed{S (x) = (S_1 (x), ..., S_m (x))^{\intercal} = 0} \tag{19.176}$$

representing an intersection of m submanifolds $S_i (x)$ $(i = 1, ..., m)$.

Definition 19.13 *Hereinafter the control* $u(t)$ *will be referred to (according to V. Utkin) as the* ***equivalent control*** $u^{(eq)}(t)$ *in the system (19.175) if it satisfies the equation*

$$\boxed{\begin{array}{c} \dot{S}(x(t)) = G(x(t))\dot{x}(t) = G(x(t)) f(t, x(t), u(t)) = 0 \\ \\ G(x(t)) \in \mathbb{R}^{m \times n}, \; G(x(t)) = \dfrac{\partial}{\partial x} S(x(t)) \end{array}} \tag{19.177}$$

It is quite obvious that, by virtue of the condition (19.177), a motion starting at $S(x(t_0)) = 0$ in time t_0 will proceed along the trajectories

$$\boxed{\dot{x}(t) = f\left(t, x(t), u^{(eq)}(t)\right)} \tag{19.178}$$

which lies on the manifold $S(x) = 0$.

Definition 19.14 *The above procedure is called the* ***equivalent control method*** *(Utkin 1992), (Utkin, Guldner & Shi 1999) and the equation (19.178), obtained as a result of applying this method, will be regarded as the* ***sliding mode equation*** *describing the motion on the manifold* $S(x) = 0$.

From the geometric viewpoint, the equivalent control method implies a replacement of the undefined discontinued control on the discontinuity boundary with a continuous control which directs the velocity vector in the system state space along the discontinuity surface intersection. In other words, it exactly realizes the velocity $f^0\left(t, x(t), u^{(eq)}(t)\right)$ (19.149) corresponding to the Filippov's definition of the differential inclusion in the point $x = x(t)$.

Consider now the equivalent control procedure for an important particular case of a nonlinear system which is affine on u, the right-hand side of whose differential equation is a linear function of the control, that is,

$$\boxed{\dot{x}(t) = f(t, x(t)) + B(t, x(t))\, u(t)} \tag{19.179}$$

where $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ and $B : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^{n \times r}$ are all argument continuous vector and matrix, respectively, and $u(t) \in \mathbb{R}^r$ is a control action. The corresponding equivalent control should satisfies (19.177), namely,

$$\begin{aligned}\dot{S}(x(t)) &= G(x(t))\,\dot{x}(t) = G(x(t))\,f(t, x(t), u(t)) \\ &= G(x(t))\,f(t, x(t)) + G(x(t))\,B(t, x(t))\,u(t) = 0\end{aligned} \tag{19.180}$$

Assuming that the matrix $G(x(t))\,B(t, x(t))$ is nonsingular for all $x(t)$ and t, one can find the equivalent control from (19.180) as

$$\boxed{u^{(eq)}(t) = -\left[G(x(t))\,B(t, x(t))\right]^{-1} G(x(t))\,f(t, x(t))} \tag{19.181}$$

Substitution of this control into (19.179) yields the following ODE:

$$\begin{gathered}\dot{x}(t) = f(t, x(t)) - \\ B(t, x(t))\left[G(x(t))\,B(t, x(t))\right]^{-1} G(x(t))\,f(t, x(t))\end{gathered} \tag{19.182}$$

which describes the sliding mode motion on the manifold $S(x) = 0$. Below the corresponding trajectories in (19.182) will be referred to as $x(t) = x^{(sl)}(t)$.

Remark 19.14 *If we deal with an uncertain dynamic model (19.175) or, particularly, with (19.179), then the equivalent control $u^{(eq)}(t)$* ***is not physically realizable****.*

Below we will show that $u^{(eq)}(t)$ may be successfully approximated (in some sense) by the output of the first order low-pass filter with the input equal to the corresponding sliding mode control.

Sliding mode control design Let us try to stabilize the system (19.179) applying sliding mode approach. For the Lyapunov function $V(x) := \|S(x)\|^2/2$, considered on the trajectories of the controlled system (19.179), one has

$$\dot{V}(x(t)) = \left(S(x(t)), \dot{S}(x(t))\right) =$$

$$(S(x(t)), G(x(t))\,f(t, x(t)) + G(x(t))\,B(t, x(t))\,u(t)) =$$

$$(S(x(t)), G(x(t))\,f(t, x(t))) + (S(x(t)), G(x(t))\,B(t, x(t))\,u(t)) \leq$$

$$\|S(x(t))\|\,\|G(x(t))\,f(t, x(t))\| + (S(x(t)), G(x(t))\,B(t, x(t))\,u(t))$$

Taking $u(t)$ as a *sliding mode control*, i.e.,

$$\boxed{\begin{array}{c} u(t) = u^{(sl)}(t) \\ u^{(sl)}(t) := -k_t \left[G(x(t)) B(t, x(t)) \right]^{-1} \mathrm{SIGN}(S(x(t))) \\ k_t > 0, \ \mathrm{SIGN}(S(x)) := (\mathrm{sign}(S_1(x)), ..., \mathrm{sign}(S_m(x)))^{\intercal} \end{array}} \tag{19.183}$$

we obtain

$$\dot{V}(x(t)) \leq \|S(x)\| \, \|G(x(t)) f(t, x(t))\| - k_t \sum_{i=1}^{m} |S_i(x(t))|$$

that, in view of the inequality, $\|S\| \geq \sum_{i=1}^{m} |S_i|$, implies

$$\dot{V}(x(t)) \leq - \|S(x)\| \left(k_t - \|G(x(t)) f(t, x(t))\| \right)$$

Selecting

$$\boxed{k_t = \|G(x(t)) f(t, x(t))\| + \rho, \ \rho > 0} \tag{19.184}$$

gives $\dot{V}(x(t)) \leq -\rho \|S(x)\| = -\rho\sqrt{2V(x(t))}$ that provides the reaching phase in time

$$\boxed{t_f = \frac{\sqrt{2V(x_0)}}{\rho} = \frac{\|S(x_0)\|}{\rho}} \tag{19.185}$$

Remark 19.15 *If the sliding motion on the manifold $S(x) = 0$ is stable then there exists a constant $k^0 \in (0, \infty)$ such that*

$$\|G(x(t)) f(t, x(t))\| \leq k^0$$

and, hence, k_t (19.184) may be selected as a constant

$$\boxed{k_t := k = k^0 + \rho} \tag{19.186}$$

Low-pass filtering To minimize the influence of the chattering effect arising after the reaching phase let us consider the property of the signal obtained as an output of a low-pass filter with the input equal to the sliding mode control, that is,

$$\boxed{\mu\dot{u}^{(av)}(t) + u^{(av)}(t) = u^{(sl)}(t),\ u_0^{(av)} = 0, \mu > 0} \tag{19.187}$$

where $u^{(sl)}(t)$ is given by (19.183). The next simple lemma states the relation between the, so-called, averaged control $u^{(av)}(t)$, which is the filtered output, and the input signal $u^{(sl)}(t)$.

Lemma 19.15 *If*

$$\begin{gathered} g^{+} \geq \|GB(t, x(t))\| := \lambda_{\max}^{1/2}\left(\left[B^{\intercal}(t, x(t))\,G^{\intercal}\right]\left[GB(t, x(t))\right]\right) \\ \geq \lambda_{\min}^{1/2}\left(\left[B^{\intercal}(t, x(t))\,G^{\intercal}\right]\left[GB(t, x(t))\right]\right) \geq \varkappa I_{r\times r},\ \ \varkappa > 0 \end{gathered} \tag{19.188}$$

then for the low-pass filter (19.187) the following properties hold:

1. *The difference between the input and output signals are bounded, i.e.,*

$$\boxed{\begin{gathered} u^{(av)}(t) = u^{(sl)}(t) + \zeta(t) \\ \|\zeta(t)\| \leq 2c,\ \ c := (g^{+} + \rho)\, m/\varkappa \\ \left\|\dot{u}^{(av)}(t)\right\| \leq 2c/\mu \end{gathered}} \tag{19.189}$$

2. *The amplitude-frequency characteristic $A(\omega)$ of the filter is*

$$\boxed{A(\omega) = \frac{1}{\sqrt{1 + (\mu\omega)^2}},\ \omega \in [0, \infty)} \tag{19.190}$$

   whose plot is depicted at Fig.19.11 for $\mu = 0.01$, where $y = A(\omega)$ and $x = \omega$.

Proof. 1) The solution of the ODE (19.183) and its derivative are as follows:

$$\begin{gathered} \left\|k_t\left[GB(t, x_t)\right]^{-1}\right\| \leq \left(\|Gf(t, x_t)\| + \rho\right)\left\|\left[GB(t, x_t)\right]^{-1}\right\| \\ = \frac{\left(\|Gf(t, x_t)\| + \rho\right)}{\lambda_{\min}^{1/2}\left(\left[B^{\intercal}(t, x_t)\,G^{\intercal}\right]\left[GB(t, x_t)\right]\right)} \leq \varkappa^{-1}\left(g^{+} + \rho\right) \end{gathered}$$

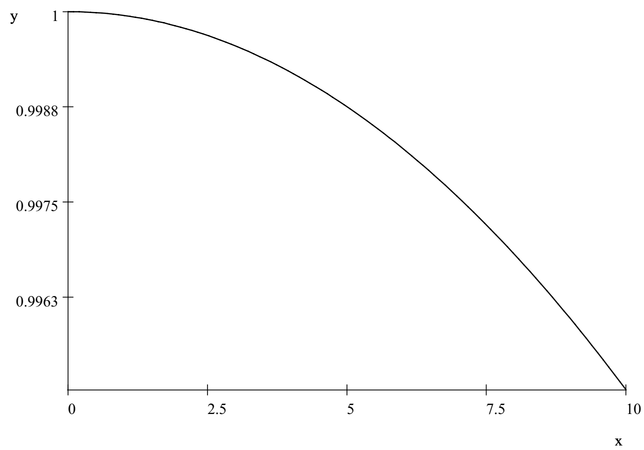

Figure 19.11: The amplitude-phase characteristic of the low-pass filter.

$$\left\| u_t^{(sl)} \right\| \leq \varkappa^{-1} \left(g^{+} + \rho \right) m := c$$

and by (19.187)

$$\begin{array}{c} u_t^{(av)} = \dfrac{1}{\mu} \int\limits_{s=0}^{t} e^{-(t-s)/\mu} u_s^{(sl)} ds \\ \dot{u}_t^{(av)} = \dfrac{1}{\mu} \left[u_t^{(sl)} - \dfrac{1}{\mu} \int\limits_{s=0}^{t} e^{-(t-s)/\mu} u_s^{(sl)} ds \right] \end{array} \tag{19.191}$$

that implies

$$\begin{array}{c} \mu \left\| \dot{u}_t^{(av)} \right\| \leq \left\| u_t^{(sl)} \right\| + \dfrac{1}{\mu} \int\limits_{s=0}^{t} e^{-(t-s)/\mu} \left\| u_s^{(sl)} \right\| ds \leq \\ c + \dfrac{c}{\mu} \int\limits_{s=0}^{t} e^{-(t-s)/\mu} ds = c + c \int\limits_{s=0}^{t} e^{-(t-s)/\mu} d\left(s/\mu\right) = \\ c + c \int\limits_{\tilde{s}=0}^{t/\mu} e^{-(t/\mu - \tilde{s})} d\tilde{s} = c + c \left(1 - e^{-t/\mu}\right) \leq 2c \end{array}$$

Hence, (19.189) holds.

2) Applying the Fourie transformation to (19.187) leads to the following identity:

$$\mu j\omega U^{(av)}\left(j\omega\right) + U^{(av)}\left(j\omega\right) = U^{(sl)}\left(j\omega\right)$$

or, equivalently,

$$U^{(av)}(j\omega) = \frac{1}{1+\mu j\omega} U^{(sl)}(j\omega) = \frac{1-\mu j\omega}{1+(\mu\omega)^2} U^{(sl)}(j\omega)$$

So, the amplitude-frequency characteristic

$$A(\omega) := \sqrt{\left[\operatorname{Re} U^{(av)}(j\omega)\right]^2 + \left[\operatorname{Im} U^{(av)}(j\omega)\right]^2}$$

of the filter (19.187) is as in (19.190). Lemma is proven. ∎

The realizable approximation of the equivalent control

By (19.191) $u_t^{(av)}$ may be represented as $u_t^{(av)} = \int_{s=0}^{t} u_s^{(sl)} d\left(e^{-(t-s)/\mu}\right)$. Consider the dynamics $x_t^{(av)}$ of the system (19.175) controlled by $u_t^{(av)}$ (19.191) at two time intervals: during the *reaching phase* and during the *sliding mode regime*.

1. **Reaching phase** ($t \in [0, t_f]$). Here the integration by part implies

$$u_t^{(av)} = \int_{s=0}^{t} u_s^{(sl)} d\left(e^{-(t-s)/\mu}\right) = u_t^{(sl)} - u_0^{(sl)} e^{-t/\mu} - \int_{s=0}^{t} \dot{u}_s^{(sl)} e^{-(t-s)/\mu} ds$$

Supposing that $u_t^{(sl)}$ (19.183) is bounded almost everywhere, i.e., $\left\|\dot{u}_t^{(sl)}\right\| \leq d$. The above identity leads to the following estimation:

$$\left\|u_t^{(av)} - u_t^{(sl)}\right\| \leq \left\|u_0^{(sl)}\right\| e^{-t/\mu} + d \int_{s=0}^{t} e^{-(t-s)/\mu} ds = \left\|u_0^{(sl)}\right\| e^{-t/\mu} + \mu d \int_{s=0}^{t} e^{-(t-s)/\mu} d(s/\mu) = \left\|u_0^{(sl)}\right\| e^{-t/\mu} + \mu d \int_{\tilde{s}=0}^{t/\mu} e^{-(t/\mu - \tilde{s})} d\tilde{s} = \left\|u_0^{(sl)}\right\| e^{-t/\mu} + \mu d \left(1 - e^{-t/\mu}\right) = \mu d + O\left(e^{-t/\mu}\right)$$

So, $u_t^{(av)}$ may be represented as

$$\boxed{u_t^{(av)} = u_t^{(sl)} + \xi_t} \tag{19.192}$$

where ξ_t may be done as small as you wish taking μ tending to zero, since

$$\|\xi_t\| \leq \mu d + O\left(e^{-t/\mu}\right)$$

As a result, the trajectories $x_t^{(sl)}$ and $x_t^{(av)}$ will differ a little bit. Indeed,

$$\dot{x}_t^{(sl)} = f\left(t, x_t^{(sl)}\right) - B\left(t, x_t^{(sl)}\right) u_t^{(sl)}$$
$$\dot{x}_t^{(av)} = f\left(t, x_t^{(av)}\right) - B\left(t, x_t^{(av)}\right) u_t^{(av)}$$

Defining

$$\tilde{B} = B\left(t, x_t^{(av)}\right), \tilde{G} = G\left(x_t^{(av)}\right), \ \tilde{f} = f\left(t, x_t^{(av)}\right)$$

and omitting the arguments for the simplicity, the last equation may be represented as

$$\dot{x}_t^{(sl)} = f - Bu_t^{(sl)}, \ \dot{x}_t^{(av)} = \tilde{f} - \tilde{B}u_t^{(av)}$$

Hence by (19.192), the difference $\Delta_t := x_t^{(sl)} - x_t^{(av)}$ satisfies

$$\Delta_t = \Delta_0 - \int_{s=0}^{t} \left[\left(f - \tilde{f}\right) - Bu_s^{(sl)} + \tilde{B}u_s^{(av)}\right] ds =$$
$$\Delta_0 - \int_{s=0}^{t} \left[\left(f - \tilde{f}\right) - Bu_s^{(sl)} + \tilde{B}\left(u_s^{(sl)} + \xi_s\right)\right] ds$$

Taking into account that $\Delta_0 = 0$ (the system starts with the same initial conditions independently on an applied control) and that $f(t,x)$ and $B(x)$ are Lipschitz (with the constant L_f and L_B) on x it follows

$$\|\Delta_t\| \leq \int_{s=0}^{t} \left[\left\|f - \tilde{f}\right\| + \left\|\left(\tilde{B} - B\right) u_s^{(sl)} + \tilde{B}\xi_s\right\|\right] ds$$
$$\leq \int_{s=0}^{t} \left[L_f \|\Delta_s\| + L_B \|\Delta_s\| \left\|u_s^{(sl)}\right\| + \left\|\tilde{B}\right\| \|\xi_s\|\right] ds \leq$$
$$\int_{s=0}^{t} \left[\left(L_f + L_B \left\|u_s^{(sl)}\right\|\right) \|\Delta_s\| + \left\|\tilde{B}\right\| \left(\mu d + O\left(e^{-t/\mu}\right)\right)\right] ds$$

Since $O\left(e^{-t/\mu}\right) = \mu O\left(\frac{1}{\mu}e^{-t/\mu}\right) = \mu o(1) \leq \mu\varepsilon$ and

$$\left\|u_s^{(sl)}\right\| \leq u_+^{(sl)} < \infty, \ \left\|\tilde{B}\right\| \leq B^+ < \infty$$

we finally have

$$\begin{aligned}\|\Delta_t\| &\leq \int_{s=0}^{t} \left[\left(L_f + L_B u_+^{(sl)}\right)\|\Delta_s\| + B^+\mu\,(d+\varepsilon)\right] ds \\ &\leq B^+\mu\,(d+\varepsilon)\,t_f + \int_{s=0}^{t}\left(L_f + L_B u_+^{(sl)}\right)\|\Delta_s\|\,ds\end{aligned}$$

Now let us apply the Gronwall lemma which says that if $v(t)$ and $\xi(t)$ are nonnegative continuous functions on $[t_0, \infty)$ verifying

$$v(t) \leq c + \int_{s=t_0}^{t} \xi(s)\,v(s)\,ds \tag{19.193}$$

then for any $t \in [t_0, \infty)$ the following inequality holds:

$$v(t) \leq c \exp\left(\int_{s=t_0}^{t} \xi(s)\,ds\right) \tag{19.194}$$

This results remains true if $c = 0$. In our case

$$v(t) = \|\Delta_t\|,\ c = B^+\mu\,(d+\varepsilon)\,t_f,\ \xi(s) = L_f + L_B u_+^{(sl)}$$

for any $s \in [0, t_f)$. So,

$$\boxed{\|\Delta_t\| \leq \delta := B^+\mu\,(d+\varepsilon)\,t_f \exp\left(\left(L_f + L_B u_+^{(sl)}\right)t_f\right)} \tag{19.195}$$

Claim. *For any finite reaching time t_f and any small value $\delta > 0$ there exists a small enough μ such that $\|\Delta_t\|$ is less than δ.*

2. **Sliding mode phase** $(t > t_f)$. During the sliding mode phase we have

$$S\left(x_t^{(sl)}\right) = \dot{S}\left(x_t^{(sl)}\right) = G\,\left(f - Bu_t^{(eq)}\right) = 0 \tag{19.196}$$

if $u_t = u_t^{(eq)}$ for all $t > t_f$. Applying $u_t = u_t^{(av)}$ we can not guarantee (19.196) already. Indeed,

$$S\left(x_t^{(av)}\right) = S\left(x_{t_f}^{(av)}\right) + \int_{s=t_f}^{t} \dot{S}\left(x_s^{(av)}\right) ds$$

and, by (19.195),

$$\left\| S\left(x_{t_f}^{(av)}\right) \right\| = \left\| S\left(x_{t_f}^{(av)}\right) \text{-} S\left(x_{t_f}^{(sl)}\right) \right\| \leq \left\| G\left(_{t_f}^{(sl)}\right) \Delta_{t_f} \right\| \leq O(\mu)$$

Hence, in view of (19.196), $\left\| S\left(x_t^{(av)}\right) \right\| = O(\mu)$.

Claim 19.2 *During the sliding-mode phase*

$$\boxed{\left\| S\left(x_t^{(av)}\right) \right\| = O(\mu)} \tag{19.197}$$