
Preface

The book is a collection of contributions concerning the theories, applications and perspectives of Variable Structure Systems (VSS).

Variable Structure Systems have been a major control design methodology for many decades. The term *Variable Structure Systems* was introduced in the late 1950's, and the fundamental concepts were developed for its main branch *Sliding Mode Control* by Russian researchers Emelyanov and Utkin. The 20th Century has seen the formation and consolidation of VSS theory and its applications. It has also seen an emerging trend of cross-fertilization and integration of VSS with other control and non-control techniques such as feedback linearization, flatness, passivity based control, adaptive and learning control, system identification, pulse width modulation, H^∞ geometric and algebraic methods, artificial intelligence, modeling and optimization, neural networks, fuzzy logic, to name just a few. This trend will continue and flourish in the new millennium.

To reflect these major developments in the 20th Century, this book includes 16 specially invited contributions from well-known experts in VSS theory and applications, covering a wide range of topics.

The first chapter, "First State of VSS: People and Events" written by Vadim Utkin, the founder of VSS, oversees and documents the historical developments of VSS in the 20th Century, including many interesting events not known to the West until now.

The second chapter, "An Integrated Learning Variable Structure Control Method" written by Jian-Xin Xu, addresses an important issue regarding control integration between variable structure control and learning control.

The third chapter, "Discrete-time Variable Structure Control" co-authored by Katsuhisa Furata and Yaodong Pan, describes the design and analysis of discrete-time variable structure control.

The fourth chapter, "Higher-Order Sliding Modes for the Output-Feedback Control of Nonlinear Uncertain Systems" written by Giorgio Bartolini, Arie Levant, Alessandro Pisano and Elio Usai, discusses the properties of higher-order sliding mode and its application to output feedback control tasks.

The fifth chapter, "Variable Structure Systems with Terminal Sliding Modes" written by Xinghou Yu and Man Zhihong, suggests the utility of a terminal attractor to improve the convergence performance of VSS in sliding modes.

The sixth chapter, "Adaptive Backstepping Control" written by Ali Jafari Koshkouei, Russell Mills, Allan Zinober, applies the backstepping design method to sliding mode control and combines adaptive mechanism.

The seventh chapter, "Sliding Mode Compensation, Estimation and Optimization Methods in Automotive Control" written by İbrahim Haskara,

Cem Hatipoğlu and Ümit Özgüner, which demonstrates the effectiveness of the sliding mode control and estimation approaches in solving automotive control problems.

The eighth chapter, “On Quasi-optimal Variable Structure Control Approaches” written by Jian-Xin Xu and Jin Zhang, explores the possibility of combining sliding mode control with nonlinear optimal control methods.

The ninth chapter, “Robust Control of Infinite-Dimensional Systems via Sliding Modes” written by Yuri Orlov, deals with an important area: how to construct a sliding mode controller for infinite dimensional systems.

The tenth chapter, “Sliding Modes Applications in Power Electronics and Electrical Drives” written by Asif Šabanović, Karel Jezernik and Nadira Šabanović, summarizes sliding mode applications in power electronics and electrical drives.

The eleventh chapter, “On the Development and Application of Sliding Mode Observers” written by Chris Edwards, Sarah Spurgeon and Chee Pin Tan, illustrates recent developments and applications of sliding mode observers.

The twelfth chapter, “Multivariable Output-Feedback Sliding Mode Control” written by Liu Hsu, José Paulo Vilela Soares da Cunha, Ramon R. Costa and Fernando Lizarralde, examines output feedback issues associated with sliding mode control.

The thirteen chapter, “Sliding Modes, Differential Flatness and Integral Reconstructors” written by Herbertt Sira-Ramírez and Victor Hernández, studies the connections between sliding mode and flatness/integral reconstruction.

The fourteenth chapter, “On Robust VSS Nonlinear Servomechanism Problem” is written by Vadim Utkin, B. Castillo-Toledo, A. Loukianov and O. Espinosa-Guerra, attacks nonlinear servo problems by means of variable structure control method.

The fifteen chapter, “Variable Structure Systems In Computational Intelligence” by Önder Efe, Okyay Kaynak, Xinghou Yu, gives a general introduction on how to make VSS and computational intelligence function in a complementary manner to each other.

The last chapter entitled “Sliding Modes Control for Systems with Fast Actuators: Singularly Perturbed Approach” written by Leonid Fridman, revisits sliding mode control based on singular perturbation approaches for systems with both slow and fast dynamics.

We are sincerely grateful to the many contributors who have given their valuable time and expertise to this project, without whom, this collection of contributions would not have been possible. Our thanks go to Professor M. Thoma for his support of this book proposal and Dr T. Ditzniger for his assistance in the preparation of this book. We would like to particularly thank Mr Noel Patson, who helped a great deal in editing the book, taking care of most painful editorial tasks.

Special thanks to our families, Zhenwei, Iris Hong, Alice, Kettie, and Elizabeth, for their support, devotion and patience.

Australia, Singapore,
September 2001

Xinghuo Yu
Jian-Xin Xu

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First Stage of VSS: People and Events

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Abstract. The objective of paper to present Variable Structure Control in historical perspective from the late fifties along with related research. The author tries to sketch the events that gave birth and were associated with the further development of this interesting research area. In many books, survey papers the authors mention that the research in this area were initiated in the former Soviet Union about 40 years ago and then the sliding mode control methodology has been receiving much more attention from the international control community within the last two decades. In author's opinion this paper is a good chance to describe what happened before "the last two decades", to demonstrate the initial ideas and to mention the researchers actively working at the initial stage. Their names and contributions deserve to be mentioned because the results were published mainly in Russian and practically unknown to the colleagues outside the Soviet Union.

1 Introduction

The term "Variable Structure System" (VSS) first appeared in the late fifties. Naturally at the very beginning several specific control tasks for second-order linear and non-linear systems were tackled and advantages of the new approach were demonstrated. Then the main directions of further research were formulated. In the course of further development the first expectations of such systems were modified, their real potential has been revealed. Some research trends proved to be unpromising while the others, being enriched by new achievements of the control theory and technology, have become milestones in VSS theory.

The paper is not a survey. The idea of preparing a survey seems to the author pointless, since at each stage of development of VSS theory, survey papers were published with vast material had been accumulated in the area by the time of publications.

From the point of the present day, much more interesting than survey information is to make "travel in years" staring from original ideas and hopes to scientific arsenal of modern VSS theory and applications. It is of interest to establish bridges between initial and terminal stations of this travel. Application aspects of VSS is a topic of interest as well, since some of the main VSS theory trends were initiated due to application problems. Mathematical, design and implementation aspects of the above topics constitute the subject of this paper.

2 First Steps

Three paper were published by S. Emel'yanov in late fifties on feedback design for the second-order linear systems ([1] was the first of them). The novelty of the approach was that the feedback gains could take several constant values depending on the system state. Although the term "Variable Structure System" was not introduced in the papers, each of the systems consisted of a set linear structures and was supplied with a switching logic and actually was VSS. The author observed that due to altering the structure in the course of control process the properties could attained which were not inherent in any of the structures. For example the system consisting of two conservative subsystems (Fig.1)

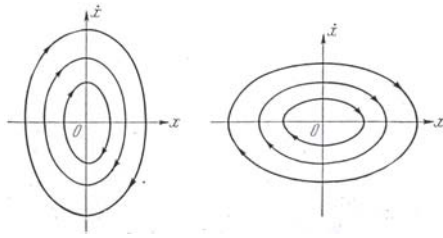


Fig. 1. Two conservative structures

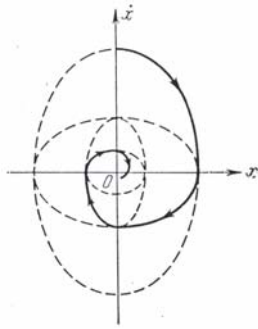


Fig. 2. Asymptotically stable VSS

$$\ddot{x} = -kx \tag{1}$$

$$k = \begin{cases} k_1 & \text{if } x\dot{x} > 0 \\ k_2 & \text{if } x\dot{x} < 0 \end{cases} \quad k_1 > k_2 > 0.$$

becomes asymptotically stable due to varying its structure on coordinate axes (Fig.2). Another way for stabilization is to find a trajectory in a state plane of one of the structures with converging motion. Then the switching logic should be found such that the state reaches this trajectory for any initial conditions and move along it. If in the system

$$\ddot{x} = a_2 \dot{x} - kx, a_2 > 0,$$

there are two structures with $k_1 > 0$ and $k_2 < 0$ (Fig3,4), then such trajectory exists in the second structure (straight line $s = c^*x_1 + x_2 = 0$, $c^* = a_2/2 + \sqrt{a_2^2/4 - k_2}$ in Fig.4). As it can be seen in Fig.5 variable structure system with switching logic

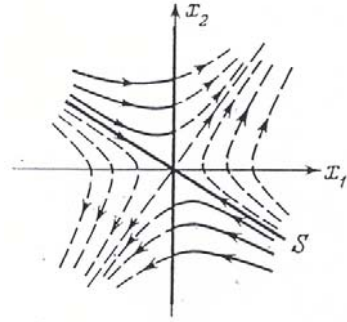


Fig.3. Unstable structure I

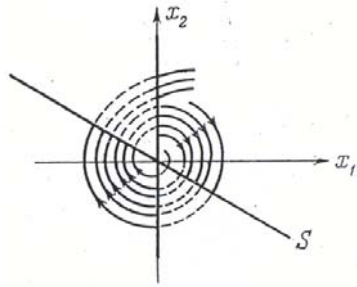


Fig.4. Unstable structure II

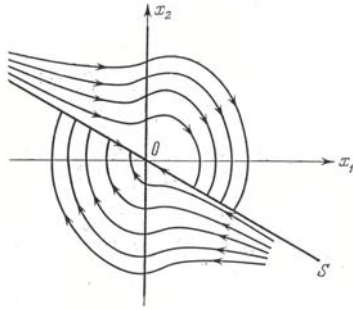


Fig. 5. Asymptotically stable VSS with monotonous processes

$$k = \begin{cases} k_1, & \text{if } xs > 0, \\ k_2, & \text{if } xs < 0. \end{cases} \quad (2)$$

is asymptotically stable with monotonous processes. It is of interest that a similar approach was offered by A. Letov [2], but his approach implied calculation of the switching time as a function of initial conditions. As a result any calculation error led to instability, therefore the system was called “conditionally stable”.

Starting from early sixties term “Variable Structure Control” appeared in titles of the papers by S. Emel’yanov and his colleagues. Interesting attempts were made to stabilize second-order nonlinear plants [3]. The plants under study were unstable with several equilibrium points and could not be stabilized by any linear control which was common for many processes of chemical technology. The universal design recipe could hardly be developed, so for any specific case the authors tried to “cut and glue” different pieces of available structures such that the system turned to be globally asymptotically stable. In Fig.6, Fig.7, Fig.8 the state planes ($x_1 = x$, $x_2 = \dot{x}$) of three structures with linear feedback are shown. Equilibrium points of each of them are unstable. Partitioning the state plane into six parts led to an asymptotically stable variable structure system (Fig.9). Note that some of the state trajectories are oriented towards switching line $s = x_2 + cx_1 = 0$ in the above example (Fig.9). It means that that having reached this line the state trajectory can not leave it and for further motion the state vector will be on this line. This motion is called *sliding mode*. Sliding mode played the dominant role in the further development of VSS theory. Sliding mode may appear in our second example on the switching line, if $0 < c < c^*$ (Fig.10). Since the state trajectory coincides with the switching line in sliding mode its equation may be interpreted as the sliding mode equation

$$\dot{x} + cx = 0 \quad (3)$$

The equation (3) is the ideal model of sliding mode. In reality due small imperfections in a switching device (small delay, hysteresis, time constant)

the control switches at a finite frequency and the state is not confined to the switching line but oscillates in its small vicinity.

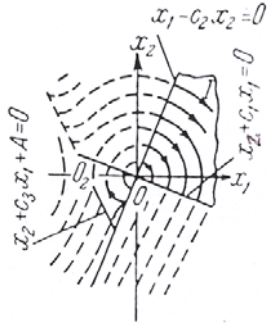


Fig. 6. Unstable nonlinear structure I

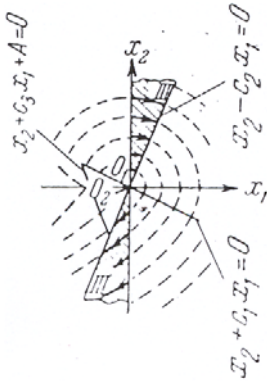


Fig. 7. Unstable nonlinear structure II

Three important facts should be underlined now:

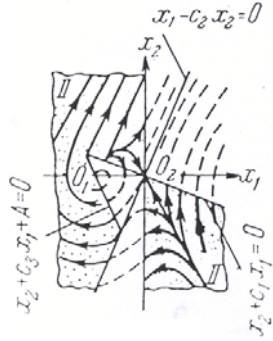


Fig. 8. Unstable nonlinear structure III

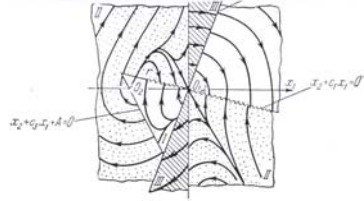


Fig. 9. Asymptotically stable nonlinear VSS

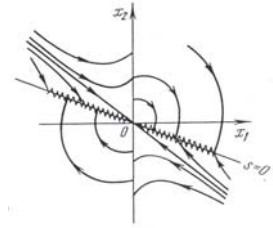


Fig. 10. VSS with sliding mode

1. The original system is governed by a non-linear second order equation, the order of motion equation is reduced after sliding mode starts.
2. The motion equation is linear and homogenous.
3. Sliding mode does not depend on the plant dynamics and determined by parameter c selected by a designer.

The above examples served as a starting point and let us outline various design principles offered at the initial stage of VSS theory development. The first most obvious principle implies taking separate pieces of trajectories of the existing structures and combining them together to get a good (in some sense) trajectory of the motion in a feedback system. The second principle consists in seeking individual trajectories in one of the structures with the desired dynamic properties and designing the switching logic such that starting from

some instant the state moves along one of these trajectories. And finally, design principle based on enforcing sliding modes in the surface where the system structure is varied or control undergoes discontinuities.

Unfortunately, the hopes associated with the first two approaches have not been justified; their applications have been limited to the study of several specific systems of low order. The promising control design principles are based on enforcing sliding modes due to the properties observed in our second-order examples. As a result sliding modes have had, and are still having, an exceptional role in the development of VSS theory. Therefore the term “Sliding Mode Control” is often used in literature on VSS as more adequate to the nature of feedback design approach.

3 VSS in Canonical Space

3.1 Free Motion

The sliding mode control, demonstrated for the second-order system in Section 2, was generalized for linear SISO dynamic systems of an arbitrary order under the strong assumption that their behavior is represented in canonical space - space of output and its time derivatives:

$$\begin{aligned} x_i &= x_{i+1}, \quad i = 1, \dots, n-1 \\ \dot{x}_n &= -\sum_{i=1}^n a_i x_i + bu, \quad a_i, b \text{ are plant parameters, } u \text{ is control input.} \end{aligned} \quad (4)$$

Similarly to the second-order systems control was designed as piece-wise linear function of system output

$$\begin{aligned} u &= -kx_1, \\ k &= \begin{cases} k_1 & \text{if } x_1 s > 0 \\ k_2 & \text{if } x_1 s < 0 \end{cases} \end{aligned} \quad (5)$$

with switching plane

$$s = \sum_{i=1}^n c_i x_i = 0, \quad c_i = \text{const}, \quad c_n = 1.$$

The design method of VSS (4),(5) was developed after V. Taran joined the research team [4]. The methodology was preserved:

- Sliding mode should exist at any point of switching plane, then it is called sliding plane.
- Sliding mode should be stable.
- The state should reach the plane for any initial conditions.

Unfortunately the first and second requirements may be in conflict. On one hand, sliding mode exists if the state trajectories in the vicinity of the switching plane are directed to the plane, or [5]

$$\lim_{s \rightarrow +0} \dot{s} < 0, \quad \lim_{s \rightarrow -0} \dot{s} > 0. \quad (6)$$

These conditions for our system are of form

$$\frac{c_{i-1} - a_i}{c_i} = c_{n-1} - a_n, \quad i = 2, \dots, n-1. \quad (7)$$

On the other hand coefficients c_i in sliding mode equation

$$x_1^{(n-1)} + c_{n-1}x_1^{(n-2)} + \dots + c_1x_1 = 0 \quad (8)$$

should satisfy Hurwitz conditions. The result of [4]: a sliding plane with stable motion exists if and only if there exists $k_0, k_2 < k_0 < k_1$ such that the linear system with control $u = k_0x_1$ has $(n-1)$ eigenvalues with negative real parts.

The result of [6]: for the state to reach a switching plane from any initial position it is necessary and sufficient that the linear system with $u = -k_1x_1$ does not have real positive eigenvalues.

The above results mean that for asymptotical stability of VSS system each of the structures may be unstable. For example the third-order VSS system,

$$\frac{d^3x}{dt^3} = u, \quad u = -kx, \quad k = \begin{cases} 1, & \text{if } xs > 0, \\ -1, & \text{if } xs < 0. \end{cases} \quad s = c_1x + c_2\dot{x} + \ddot{x},$$

consisting of two unstable linear structures is asymptotically stable for $c_1 = c_2^2$ (sliding plane existence condition (7)), $c_1 > 0, c_2 > 0$ (Hurwitz condition for sliding mode). The reaching condition holds as well since the linear system with $k = 1$ does not have real positive eigenvalues.

Again sliding mode equation (8) is of a reduced order, linear, homogeneous, does not depend on plant dynamics and determined by coefficients in switching plane equation. This property looks promising when controlling plants with unknown time-varying parameters. Unfortunately control (5) is not applicable for this purpose because the conditions for sliding plane to exist (7) need knowledge on the parameters a_i .

For the modified version of VSS control

$$u = - \sum_{i=1}^{n-1} k_i x_i, \quad (9)$$

$$k = \begin{cases} k'_i, & \text{if } x_i s > 0, \\ k''_i, & \text{if } x_i s < 0. \end{cases}$$

The plane $s = 0$ is a sliding plane for any values of c_i if

$$bk'_i > \max_{a_i, a_n} (c_{i-1} - a_i - c_{n-1}c_i + a_n c_i), \quad a_0 = 0, \quad i = 1, \dots, n-1. \quad (10)$$

The design procedure of VSS (9) consisting of 2^{n-1} linear structures implies selection of switching plane or sliding mode equation (8) with desired dynamics and then coefficients k'_i and k_i'' (10) such that $s = 0$ is a sliding plane (assuming that the ranges of parameter variations are known). Reaching this plane may be provided by increasing coefficient k'_1 . Sliding mode with the desired properties starts in the VSS after a finite time interval. The time, preceding the sliding mode, may be decreased by increasing the gains in control (9).

Development of special methods is needed if the last equation in (4) depends on time derivatives of control, since trajectories in the canonical space become discontinuous. Two approaches were offered by N.Kostyleva in the framework of the VSS theory: first, designing a switching surface with the part of state variables, and, second, using a pre-filter in controller [7]. For the both cases the conventional sliding mode with the desired properties can be enforced. Traces of the approaches may be found in modern publications.

3.2 Disturbance Rejection

The property of insensitivity of sliding modes to plant dynamics may be utilized to control plants subjected to unknown external disturbances. It is obvious that control (9) does not fit for this purpose. Indeed at the desired state (all x_i are equal to zero) the control is equal to zero as well and unable to keep the plant there at presence of disturbances. We demonstrate how the disturbance rejection problem can be solved using dynamic actuators with variable structure. Let plant and actuator be integrators in the second-order system (Fig.11). An external disturbance $f(t)$ is not accessible for measurement.

The control is designed as a piece-wise linear function not only of the output $x = x_1$ to be reduced to zero but also of actuator output y :

$$\begin{aligned} \dot{x}_1 &= y + f(t) \\ \dot{y} &= u, \\ u &= -kx_1 - k_y y. \end{aligned} \quad \text{or} \quad \begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -kx_1 - k_y x_2 - k_y f + \dot{f}. \end{aligned}$$

For the variable structure system with

$$k = \begin{cases} k_0, & \text{if } x_1 s > 0, \\ -k_0, & \text{if } x_1 s < 0. \end{cases}, \quad k_y = \begin{cases} k_{y0}, & \text{if } y s > 0, \\ -k_{y0}, & \text{if } y s < 0. \end{cases}, \quad s = cx_1 + x_2, \quad k_0, k_{y0}, c \text{ are const.}$$

The state semi-planes $x_1 > 0$ and $x_1 < 0$ are shown in Fig.12. For domain $x_2 < |f(t)|$ the singular points for each semi-plane are located in the opposite one and as a result state trajectories are oriented toward switching line $s = 0$. It means that sliding mode occurs in this line with motion equation $\dot{x} + cx = 0$ and solution tending to zero. The effect of disturbance rejection may be explained easily in structural language. Due to altering sign of local feedback for the actuator its output may be either diverging or converging exponential

function (Fig.13). In sliding mode due to high-frequency switching an average value of the output is equal to the disturbance with an opposite sign. It is clear that the disturbance, which can be rejected, should be between the diverging and converging exponential functions at any time. Similar approach stands behind the disturbance rejection method in VSS of an arbitrary order.

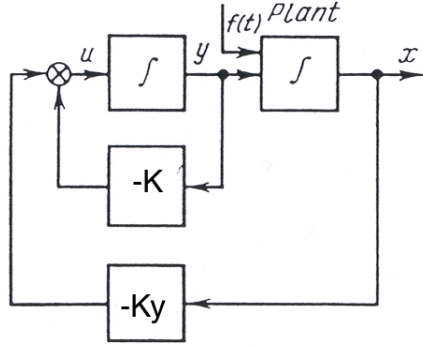


Fig. 11. System with VS actuator

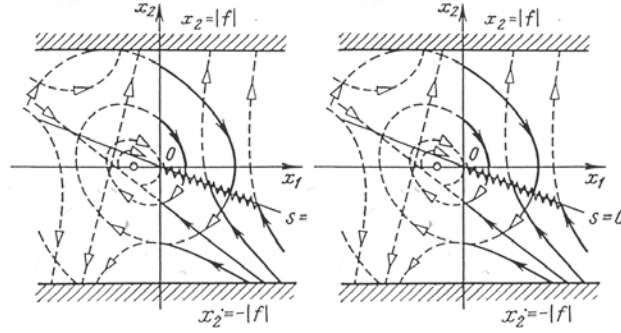


Fig. 12. Phase portrait of system with disturbance

3.3 Problem of Differentiation

Implementation of variable structure control algorithms studied in the previous sections needs information on time derivatives of a system output. Since

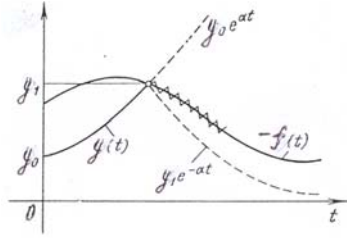


Fig. 13. Disturbance rejection in system with VS actuator

ideal differentiators can not be built, ideal sliding mode can not appear in the systems with real differentiators. V. Taran [8] was the first who analyzed the behavior of such VSS. The ideal derivative in the second-order VSS was replaced by the output of the first-order filter with transfer function $G(p) = \frac{T_1 p + 1}{T_2 p + 1}$, (T_1 and T_2 are constant values):

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -a_1 x_1 - a_2 x_2 + b u, \quad u = -k x_1, \\ k &= \begin{cases} k_1, & \text{if } x_1 s > 0, \\ k_2, & \text{if } x_1 s < 0. \end{cases} \quad s = z_1 - Q x_1, \quad z_1(p) = G(p) x_1(p) \end{aligned} \quad (11)$$

It was shown that the oscillations occurs in the system with a finite frequency (Fig. 14 and 15) instead of ideal sliding mode on the switching line $s = 0$. We believe that it was the first mentioning the phenomenon called lately chattering which became the main obstacle for implementation of sliding modes in variable structure systems. The idea of chattering suppression was also offered in the paper by V. Taran [9]. Again the switching function was selected as a linear combination of the system output and output of a first-order filter but the discontinuous control was an additional input of the filter: the function s in (10) is replaced by $s = c x_1 - z$, $\dot{z} = \frac{1}{T}(z - \alpha u)$, c, T, α are const.

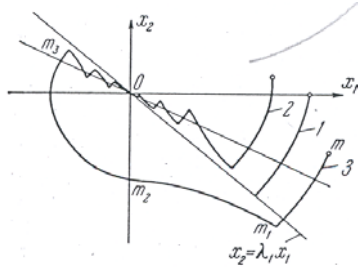


Fig. 14. VSS with real differentiator

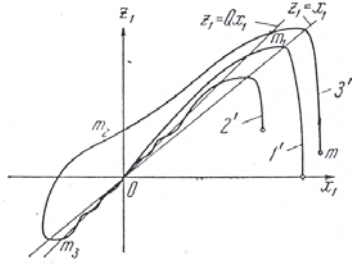


Fig. 15. Motion projection on plane (x_1, z)

Since time-derivative $\dot{s} = cx_2 - \frac{1}{T}(z - \alpha u)$ is a discontinuous function of s , the conditions (6) may be fulfilled and the sliding mode free of chattering may occur in some domain D (Fig.16(a)). This motion is governed by the second-order equations with the desired dynamics (Fig.16(b)) although the ideal derivative is not used in the control algorithm.

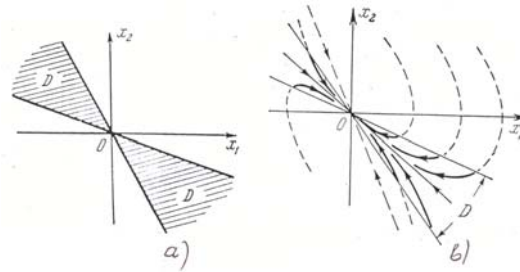


Fig. 16. System with VS filter

Although generalization for arbitrary-order systems met serious analytical difficulties, these publications first drew attention of the researchers in the area to the chattering problem and outlined the approach to solve this problem.

3.4 Adaptive VSS

In all second-order examples, after sliding mode starts, the motion does not depend on plant parameters and decays at the rate determined by the coefficient c in the equation (3) (of course c should be positive). For the system

(4) with $n = 2$, $a_i = 0$ and control (5) the conditions (6) for sliding mode to exist on the switching line $s = cx_1 + x_2 = 0$ are of form $bk_2 < -c^2 < bk_1$. If parameter b is unknown and varies in known range $0 < b_{\min} < b < b_{\max}$ and the feedback gain is bounded $|k| < k_0$ then the line $s^* = c_0x_1 + x_2$ is always a sliding line for $c_0 = \sqrt{k_0 b_{\min}}$ and $k_1 = -k_2 = k_0$. Then after reaching the line at some time t_1 , the system output will decay as exponential function $x_1(t_1)e^{-c_0(t-t_1)}$ for any value of b .

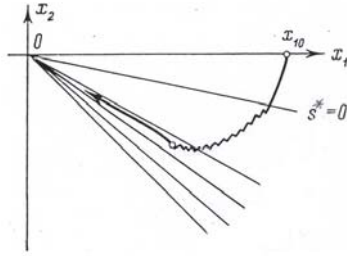


Fig. 17. Adaptive VSS with state dependent switching line

However the rate of decay may be increased if b takes any values different from $b_{\min}$ by increasing gain c . It should be done without measuring b . We describe the adaptation approach offered by E. Dubrovsky [10]. While sliding mode exists the gain c is increased until sliding mode disappears. The fact of sliding mode existence may easily be established by averaging discontinuous coefficient k taking two values k_0 or $-k_0$ and switching at high frequency. If $|k_{av}| < k_0$ then sliding mode exists, if $|k_{av}|$ is approaching k_0 then sliding mode is about to disappear and further increasing c should be terminated. As a result a sliding line with maximal value of c or maximal rate of solution decay for a current unknown value of coefficient b is found (Fig. 17).

3.5 Preliminary Mathematical Remark

The basic design idea of the first stage of VSS theory was to enforce sliding mode in a plane of the canonical space. An equation of a sliding plane depending a system output and its derivatives was interpreted as a sliding mode equation. Formally this interpretation is not legitimate. "To solve differential equation" means to find a time function such that substitution of the solution into the equation makes its right- and left-hand sides equal identically.

In our second-order example with control (2) and $0 < c < c^*$, sliding mode existed on the switching line $s = 0$ and equation (3) was taken as the sliding mode equation. Its solution Ae^{-ct} being substituted into function s

makes it equal to zero. Control (2) is not defined for $s = 0$ and respectively the right-hand side of the system equation (1) is not defined as well. Therefore we cannot answer the question whether the solution to (2) is the solution to the original system. One of the founders of control theory in the USSR academician A.A. Andronov indicated that ambiguity in the system behavior is eliminated if minor non-idealities such as time delay, hysteresis, small time constants are recognized in the system model which results in so-called real sliding mode in a small neighborhood of the discontinuity surface. Ideal sliding motion is regarded as a result of limiting procedure with all non-idealities tending to zero [11]. The examples of such limiting procedure were also given. The relay second-order system was considered with motion equations in the canonical state and a straight line (3) as a set of discontinuity points for control. The behavior of the system was studied under the assumption that a time delay was inherent in the switching device and, consequently, the discontinuity points were isolated in time. It was found that with the delay tending to zero irrespective of the plant parameters and disturbances, the solution of the second-order equation in sliding mode always tended to the solution of the first order equation (3) which depends only on the gain c of the switching line equation. The validity of the equation (8) as the model of sliding mode in the canonical space of an arbitrary-order system may be substantiated in the similar way.

3.6 Comments for VSS in Canonical Space

In the sixties VSS studies were mainly focused on linear (time-invariant and time-varying) systems of an arbitrary order with scalar control and scalar variable to be controlled. These first studies utilized the space of an error coordinate and its time derivatives, or canonical space, while the control was designed as a sum of state components and accessible for measurement disturbances with piece-wise constant gains. When the plant was subjected to unknown external disturbances, local feedback of the actuators was designed in the similar way. As a rule, the discontinuity surface was a plane in the canonical space or in an extended space, including the states of the filters for deriving approximate values of derivatives. In short, most works of this period on VSS treated piece-wise linear systems in the canonical space with scalar control scalar controlled coordinate. The invariance property of sliding modes in the canonical space to plant dynamics was the key idea of all design methods.

The first attempts to apply the results of VSS theory demonstrated that the invariance property of sliding modes in canonical spaces had not been beneficial only. In a sense it decelerated development of the theory. The illusion that any control problems may be easily solved should sliding mode be enforced led to some exaggeration of sliding mode potential. The fact is that the space of derivatives is a mathematical abstraction, and in practice any real differentiators have denominators in transfer functions; so the study

of the system behavior in the canonical space can prove to be unacceptable idealization. Unfortunately the attempts to use filters with variable structure for multiple differentiation have not led to any significant success.

For the just discussed reason, research of the fairly narrow class of VSS mainly carried out at Institute of Control Sciences, Moscow and by a group of mathematicians headed by E. Barbashin at Institute of Mathematics and Mechanics, Sverdlovsk, did not result in wide applications and did not produced significant echo in the scientific press. The results of this first stage of VSS development, i.e. analysis and design of VSS in the canonical space, were summarized in [12] and [5]. In view of the limited field of practical applications in the frame of this approach (VSS with differentiating circuits), another extreme appeared, reflecting certain pessimism about implementation of any VSS with sliding modes.

The second stage of development of VSS theory began roughly in the late sixties and early seventies, when design procedures in the canonical space ceased to be looked as obligatory and studies were focused on systems of general form with arbitrary state components and with vector control and output values. The first attempts to revise VSS methodology in this new environment demonstrated that the pessimism about sliding mode control was unjustified and refusal to utilize the potential of sliding mode was an unreasonable extravagance.

4 Problem Statements

At the second stage, from early seventies to the present, the range of the problems discussed in the context of VSS theory was essentially widened. A transition was made from canonical to more general state space, systems with vector control were under study, the dynamic plant to be controlled became essentially nonlinear, the surfaces where the system structure was altered became nonlinear as well. Development of VSS theory for the new class of problems demanded creation of qualitatively new mathematical and design methods.

We assume that for the class of the systems under study each component of control vector can be equal to one of two continuous functions of state and time and the structure of the system is altered at surfaces in the system state space:

$$\dot{x} = f(x, t, u), \quad (12)$$

where $x \in R^n$ is a state vector, $u \in R^m$ is a control vector. The control is designed as a discontinuous function of the state such that each component undergoes discontinuities in some surface in the system state space:

$$u_i = \begin{cases} u_i^+(x, t), & \text{if } s_i(x) > 0, \\ u_i^-(x, t), & \text{if } s_i(x) < 0. \end{cases}, i = 1, \dots, m \quad (13)$$

where $f(x, t, u)$, $u_i^+(x, t)$, $u_i^-(x, t)$ and $s_i(x)$ are continuous functions of their arguments $u_i^+(x, t) \neq u_i^-(x, t)$.

Similar to the above examples of VSS in the canonical space, state velocity vectors may be directed towards one of the surfaces and sliding mode arises along it (arcs ab and cb in Fig.18). It may arise also along the intersection of two of the surfaces (arc bd). Fig.19 illustrates the sliding mode in the intersection even if it does not exist at each of them taken separately. For general case (12) and (13) sliding mode may exist in the intersection of all discontinuity surfaces $s_i = 0$, or in the manifold $s(x) = 0$, $s^T(x) = [s_1(x), \dots, s_m(x)]$,

$s \in \mathbb{R}^{n-m}$.

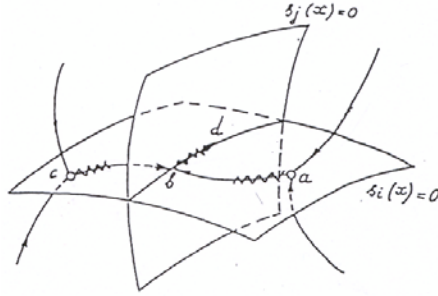


Fig. 18. Two-dimensional sliding mode

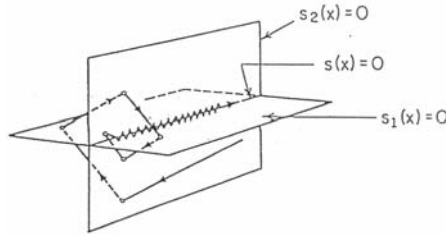


Fig. 19. Sliding mode in intersection of switching surfaces only

Let us discuss what kind of benefit sliding modes may result in, should this motion be enforced in the control system. First, in sliding mode the output s of the element implementing discontinuous control is close to zero, while its

output (exactly speaking its average value u_{av}) takes finite values (Fig.20). Hence the element implements high (theoretically infinite) gain, that is the conventional tool to reject disturbance and other uncertainties in the system behavior. Unlike to systems with continuous controls, this property called invariance is attained using finite control actions. Second, since sliding mode trajectories belong to a manifold of dimension lower than that of the original system, the order of the system is reduced as well. This enables a designer to simplify and decouple the design procedure. Both order reduction and invariance are transparent for the above two second-order systems.

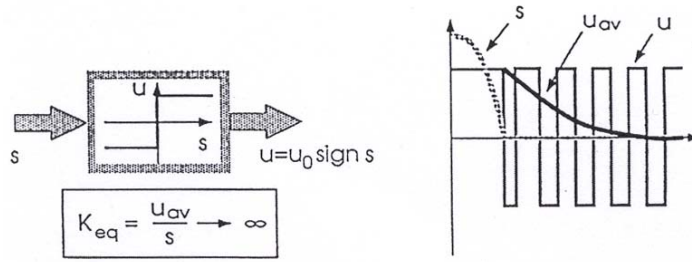


Fig. 20. High gain implementation in sliding mode

In order to justify the above arguments in favor of using sliding modes in control systems, we, first, need mathematical methods for deriving equations of sliding modes in the intersection of discontinuity surfaces and, second, the conditions for the sliding mode to exist should be obtained. Only having these mathematical tools the design methods of sliding mode control for wide range of control problems may be developed.

Sliding mode equations, the existence condition and design are base stones of sliding mode control theory. Of course this research scope should be considered along with application issues.

5 Sliding Mode Equations

The first mathematical problem concerns differential equations of sliding mode. For our second-order examples this equation was obtained using heuristic approach: the equation of the switching line $\dot{x} + cx = 0$ was interpreted as the motion equation. But even for an arbitrary time invariant second-order relay system

$$\begin{aligned} \dot{x}_1 &= a_{11}x_1 + a_{12}x_2 + b_1u \\ \dot{x}_2 &= a_{21}x_1 + a_{22}x_2 + b_2u, \\ u &= -M \operatorname{sign}(s), \quad s = cx_1 + x_2; \quad M, a_{ij}, b_i, c \text{ are const.} \end{aligned} \quad (14)$$

the problem does not look trivial since in sliding mode $s = 0$ is not a motion equation.

The problem of mathematical description of sliding mode is quite a challenge and requires the design of special techniques. It arises due to discontinuities in control, since the relevant motion equations with discontinuous right-hand sides do not satisfy the conventional theorems on existence-uniqueness of solutions (Lipshitz constant does not exist for discontinuous functions).

Uncertainty in behavior of discontinuous systems on the switching surfaces gives freedom on choosing an adequate mathematical model and gave birth to a number of lively discussions. For example, description of sliding mode in system (14) by a linear first-order differential equation, a common practice today, seemed unusual at first glance; this model was offered by Yu. Kornilov [13] and A. Popovski [14] and the approach was generalized for linear systems of an arbitrary order by Yu. Dolgolenko [15] in fifties.

The approach by A. Filippov [16], now recognized as classical, was not accepted by all experts in the area: at the 1st IFAC congress in 1960 Yu. Neimark (author of one more model of sliding mode based on convolution equation) offered an example of a system with two relay elements with a solution different from that of Filippov's method [17]. The discussion at the congress proved to be fruitful from the point of stating a new problem in the theory of sliding modes. The discussion led to the conclusion that two dynamic systems with identical equations outside a discontinuity surface may have different sliding mode equations. Most probably the problem of unambiguous description of sliding modes in discontinuous systems was first brought to light.

In situations when conventional methods are not applicable, the usual approach is to employ regularization or replacing the initial problem by a closely similar one, for which familiar methods can be used. In part, taking into account delay or hysteresis of a switching element, small time constants in an ideal model, replacing a discontinuous function by a continuous approximation are examples of the regularization since discontinuity points (if they exist) are isolated. As we discussed in Section 3.5, such regularization with time delay was employed by Andronov for substantiation of sliding mode equation (3). Similar approach for nonlinear relay system with imperfections of time delay and hysteresis type was developed by J. Andre and P. Seibert [18]; it is interesting that their sliding mode equations coincided with those of Filippov's method and the result may serve as substantiation of Filippov's method. At the same time the question may be asked, whether the method is applicable for the systems with different types of non-idealities. Generally speaking the answer is negative: a continuous approximation of a discontinuous function leads to motion equations different from those of Filippov's method [19]. So we should admit that for systems with a right-hand side as a nonlinear function of discontinuous control (12), (13) sliding mode equations can not be derived unambiguously even for the case of scalar control. The

“point-to-point” technique used in the cited here papers for scalar case is not applicable for the system with vector control and sliding modes in the intersection of a set of discontinuity surfaces.

The universal approach to regularization consists of introducing a boundary layer $\|s\| < \Delta$, $\Delta = \text{const}$ around manifold $s = 0$, where an ideal discontinuous control is replaced by a real one such that the state trajectories are not confined to this manifold but run arbitrarily inside the layer. The nonidealities, resulting in motion in the boundary layer, are not specified, the only assumption for this motion is that the solution exists in the conventional sense. If, with the width of the boundary layer Δ tending to zero, the limit of the solution exists, it is taken as a solution to the system with ideal sliding mode. Otherwise we have to recognize that the equations beyond discontinuity surfaces do not derive unambiguously equations in their intersection, or equations of the sliding mode.

The boundary layer regularization enables substantiation of so-called Equivalent Control Method intended for deriving sliding mode equations in manifold $s = 0$ if the system (12) is linear with respect to control

$$\dot{x} = f(x, t) + B(x, t)u \quad (15)$$

with $B(x, t)$ being $n \times m$ full rank matrix

$$G(x)B(x), G(x) = \partial s / \partial x, \det(GB) \neq 0.$$

Following this method the sliding mode equation with a unique solution may be derived for nonsingular matrix.

First, the equivalent control should be found as the solution to the equation $\dot{s} = 0$ on the system trajectories (G and $(GB)^{-1}$ are assumed to exist):

$$\dot{s} = Gf + GBu = 0, u_{eq} = -(GB)^{-1}Gf. \quad (16)$$

Then the solution should be substituted into (15) for the control

$$\dot{x} = f - B(GB)^{-1}Gf \quad (17)$$

Equation (17) is the sliding mode equation with initial conditions

$$s(x(0), 0) = 0.$$

Since $s(x) = 0$ in sliding mode m components of the state vector may be found as a function of the rest $(n - m)$ ones: $x_2 = s_0(x_1)$; $x_2, s_0 \in \mathbb{R}^m$; $x_1 \in \mathbb{R}^{n-m}$ and, correspondingly, the order of sliding mode equation may be reduced by m :

$$\dot{x}_1 = f_1[x_1, t, s_0(x_1)], f_1 \in \mathbb{R}^{n-m} \quad (18)$$

The idea of the equivalent control method may be easily explained with the help of geometric consideration. Sliding mode trajectories lie in the manifold $s = 0$ and the equivalent control u_{eq} being a solution to the equation

$\dot{s} = 0$ implies replacing discontinuous control by such continuous one that the state velocity vector lies in the tangential manifold and as a result the state trajectories are in this manifold. It will be important for control design that sliding mode equation (18)

- is of reduced order
- does not depend on control
- depends on the equation of switching surfaces.

6 Sliding Mode Existence Conditions

The second mathematical problem in the analysis of sliding mode as a phenomenon is deriving the conditions for sliding mode to exist. As to the systems with scalar control the conditions may be obtained from geometrical considerations: the deviation from the switching surface s and its time derivative should have opposite signs in the vicinity of a discontinuity surface $s = 0$ (6).

As it was demonstrated in the example in Fig.14, for existence of sliding mode in an intersection of a set of discontinuity surfaces $s_i(x) = 0$, ($i = 1, \dots, m$) it is not necessary to fulfill inequalities (6) for each of them. The trajectories should converge to the manifold $s^T = (s_1, \dots, s_m) = 0$ and reach it after a finite time interval similarly to the systems with scalar control. The term “converge” means that we deal with the problem of stability of the origin in m -dimensional subspace $(s_1, \dots, s_m)$, therefore the existence conditions may be formulated in terms of the stability theory. The non-traditional condition: finite time convergence should take place. This last condition is important to distinguish the systems with sliding modes and the continuous system with state trajectories converging to some manifold asymptotically. For example the state trajectories of the system $\ddot{x} - x = 0$ converge to the manifold $s = \dot{x} - x = 0$ asymptotically since $\dot{s} = -s$, however it would hardly be reasonable to call “sliding mode” the motion in $s = 0$.

Further we will deal with the conditions for sliding mode to exist for affine systems(15). To derive the existence conditions, the stability of the motion projection on subspace s

$$\dot{s} = Gf + GBu \quad (19)$$

should be analyzed. The control (13) may be represented as

$$u(x, t) = u_0(x, t) + U(x, t)\text{sign}(s)$$

with

$$u_0(x) = \frac{u^+(x, t) + u^-(x, t)}{2},$$

diagonal matrix U with elements

$$U_i = \frac{u_i^+(x, t) - u_i^-(x, t)}{2}, \quad i = 1, \dots, m$$

and

$$[sign(s)]^T = [sign(s_1), \dots, sign(s_m)]$$

Then the motion projection on subspace s is governed by

$$\dot{s} = d(x) - D(x)sign(s), \quad (20)$$

with $d = Gf + GBu_0$, $D = -GBU$.

To find the stability conditions of the origin $s = 0$ for nonlinear system (20), or the conditions for sliding mode to exist, we will follow the standard methodology for stability analysis of nonlinear systems – try to find a Lyapunov function.

Definition 1. The set $S(x)$ in the manifold $s(x) = 0$ is the domain of sliding mode if for the motion governed by equation (20) the origin in the subspace s is asymptotically stable with finite convergence time for each x from $S(x)$.

Definition 2. Manifold $s(x) = 0$ is referred to as sliding manifold if sliding mode exists at its each point, or $S(x) = \{x : s(x) = 0\}$.

Theorem 1. $S(x)$ is a sliding manifold for the system with motion projection on subspace s governed by $\dot{s} = -Dsign(s)$, if matrix $D + D^T$ is positive definite.

Theorem 2. $S(x)$ is a sliding manifold for system (20) if $D(x) + D^T(x) > 0$, $\lambda_0 > d_0\sqrt{m}$, $\lambda_{min}(x) > \lambda_0 > 0$, $\|d(x)\| < d_0$, λ_{min} is the minimal eigenvalue of matrix $\frac{D+D^T}{2}$, $\lambda_{min} > 0$.

The statements of the both the theorems may be proven using sum of absolute values of s_i

$$V = s^T sign(s) > 0$$

as a Lyapunov function.

7 Design Principles

7.1 Decoupling and Invariance

The above mathematical results constitute the background of the design methods for sliding mode control involving two independent subproblems of lower dimensions:

- design of the desired dynamics for a system of the $(n - m)^{th}$ order by a proper choice of a sliding manifold $s = 0$;

- enforcing sliding motion in this manifold which is equivalent to stability problem of the m^{th} order system.

All the design methods will be discussed for affine systems (15) which are linear with respect to control but not necessary with respect to state. Since the principle operating mode is in the vicinity of the discontinuity points, the effects inherent in the systems with infinite feedback gains may be obtained with finite control actions. As a result sliding mode control is an efficient tool to control dynamic high-order nonlinear plants operating under uncertainty conditions (e.g. unknown parameter variations and disturbances).

Formally the sliding mode is insensitive, or invariant to “uncertainties” in the systems satisfying the matching conditions $h(x, t) \in \text{range}(B)$ by B. Drazenovic, with vector $h(x, t)$ characterizing all factors in a motion equation

$$\dot{x} = f(x, t) + B(x, t)u + h(x, t)$$

whose influence on the control process should be rejected. Matching condition means that disturbance vector $h(x, t)$ may be represented as a linear combination of columns of matrix B [20].

7.2 Regular Form

The design procedure may be illustrated easily for the systems represented in the Regular Form

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2, t), \quad x_1 \in R^{n-m} \\ \dot{x}_2 &= f_2(x_1, x_2, t) + B_2(x_1, x_2, t)u, \quad x_2 \in R^m, \det(B_2) \neq 0. \end{aligned} \quad (21)$$

The state subvector x_2 is handled as a fictitious control in the first equation of (21) and selected as a function of x_1 to provide the desired dynamics in the first subsystem (the design problem in the system of the $(n - m)^{th}$ order with m -dimensional control):

$$x_2 = -s_0(x_1).$$

Then the discontinuous control should be designed to enforce sliding mode in the manifold

$$s(x_1, x_2) = x_2 + s_0(x_1) = 0 \quad (22)$$

(the design problem of the m^{th} order with m -dimensional control).

After a finite time interval sliding mode in the manifold (22) starts and the system will exhibit the desired behavior governed by

$$\dot{x}_1 = f_1[x_1, -s_0(x_1), t].$$

Note that the motion is of a reduced order and depends neither function $f_2(x_1, x_2, t)$ nor function $B_2(x_1, x_2, t)$ in the second equation of the original system (21).

Since the design procedures for the systems in Regular Form (21) are simpler than for those (16) it of interest to find the class of systems (16) for which a nonlinear coordinate transformation exists such that it reduces the original system (15) to the form (21). By the assumption $\text{rank}(B) = m$ therefore (15) may be represented as

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2, t) + B_1(x_1, x_2, t)u, \quad x_1 \in R^{n-m} \\ \dot{x}_2 &= f_2(x_1, x_2, t) + B_2(x_1, x_2, t)u, \quad x_2 \in R^m, \det(B_2) \neq 0. \end{aligned}$$

For coordinate transformation

$$y_1 = \varphi(x, t), \quad y_2 = x_2, \quad y_1, \varphi \in \mathbb{R}^{n-m},$$

the equation for y_1

$$\dot{y}_1 = \frac{\partial \varphi}{\partial x} f + \frac{\partial \varphi}{\partial x} B u + \frac{\partial \varphi}{\partial t}$$

does not depend on control u , if vector function φ is a solution to matrix partial differential equation

$$\frac{\partial \varphi}{\partial x} B = 0. \quad (23)$$

Then the motion equations with respect to y_1 and x_2 are in the regular form. Necessary and sufficient conditions of solution existence and uniqueness for (23) may be found in terms of Pfaff's forms theory and Frobenius theorem which constitute a well developed branch of mathematical analysis [21].

7.3 Enforcing Sliding Modes

At the second stage of feedback design, discontinuous control should be selected such that sliding mode is enforced in manifold $s = 0$. As shown in Section 6, sliding mode will start at manifold $s = 0$ if the matrix $GB + (GB)^T$ is positive-definite and the control is of the form

$$u = -M(x, t) \text{sign}(s) \quad (\text{component-wise})$$

with a positive scalar function $M(x, t)$ chosen to satisfy the inequality

$$M(x, t) > \lambda^{-1} \|Gf\|,$$

where λ is the lower bound of the eigenvalues of the matrix $GB + (GB)^T$.

Now we demonstrate a method of enforcing sliding mode in manifold $s = 0$ for an arbitrary nonsingular matrix GB . The motion projection equation on subspace s may be written in the form

$$\dot{s} = GB(u - u_{eq}).$$

Remind that the equivalent control u_{eq} is the value of control such that time derivative of vector s is equal to zero.

Let $V = \frac{1}{2}s^T s$ be a Lyapunov function candidate and the control be of form

$$u = -M(x, t) \text{sign}(s^*), s^* = (GB)^T s.$$

Then $\dot{V} = -s^{*T} \text{sign}(s^*) + s^{*T} \frac{u_{eq}}{M(x, t)}$ and $\dot{V}$ is negative definite for $M(x, t) > \|u_{eq}\|$.

It means that sliding mode is enforced in manifold $s^* = 0$ which means its existence in manifold $s = 0$ selected on the first step of the design procedure. It is important that the conditions for sliding mode to exist are inequalities therefore upper estimate of the disturbances is needed rather than precise information on their values.

7.4 Unit Control

The Lyapunov method approach implies design of control based on Lyapunov function selected for a nominal (feedback or open loop) system. The objective is to find the control such that a time-derivative of the Lyapunov function is negative on the trajectories of the system with perturbations caused by uncertainties of a plant operator and environment conditions. The roots of the above approach may be found in papers by G. Leitmann and S. Gutman published in 70's [22] and it leads to discontinuous control and sliding modes.

Now we demonstrate how this approach can be applied to enforce sliding mode in manifold $s = 0$, selected in compliance with some performance criterion at the first stage of feedback design for the system with unknown disturbance vector $h(x, t)$. The control is designed as a discontinuous function of s

$$u = -\rho(x, t) \frac{D^T s(x)}{\|D^T s(x)\|} \quad (24)$$

with $D = \{\partial s / \partial x\} B$, D is assumed to be nonsingular. Note that the norm of control (24) with the gain is equal to 1 for any value of the state vector. It explains the term “unit control“ for control (24).

The equation of a motion projection of the system with disturbance $h(x, t)$ on subspace s is of form

$$\dot{s} = \left\{ \frac{\partial \varphi}{\partial x} \right\} (f + h) + Du.$$

The conditions for the trajectories to converge to the manifold $s(x) = 0$ and sliding mode to exist in this manifold may be derived based on Lyapunov function

$$V = \frac{1}{2}s^T s \quad (25)$$

with time derivative

$$\dot{V} = s^T \left\{ \frac{\partial s}{\partial x} \right\} (f + h) - \rho(x, t) \|D^T s(x)\| < \|D^T s(x)\| \cdot [\|D^{-1} \left\{ \frac{\partial s}{\partial x} \right\} (f + h)\| - \rho(x, t)] . \quad (26)$$

For $\rho(x, t) > \|D^{-1} \left\{ \frac{\partial s}{\partial x} \right\} (f + h)\|$ the value of $\dot{V}$ is negative therefore the state will reach the manifold $s(x) = 0$ after a finite time interval for any initial conditions and then the sliding mode with the desired dynamics occurs. Finiteness of the interval preceding the sliding motion follows from inequality resulting from (25), (26)

$$\dot{V} < -\gamma V^{\frac{1}{2}}, \gamma = \text{const} > 0$$

with the solution

$$V(t) < \left(-\frac{\gamma}{2}t + \sqrt{V_o}\right)^2, V_o = V(0) .$$

Since the solution vanishes after some $t_s < \frac{2}{\gamma}\sqrt{V_o}$, the vector s vanishes as well and the sliding mode starts after a finite time interval.

It is of interest to note the principle difference in motions preceding the sliding mode in $s(x) = 0$ for the conventional component-wise control and unit control design methods. For the conventional method the control undergoes discontinuities should any of the components of vector s changes sign, while the unit control is a continuous state function until the manifold $s(x) = 0$ is reached. By this reason unit control systems with sliding modes would hardly be recognized as VSS.

8 Discrete-Time Sliding Mode Control

Sliding mode control was first developed for continuous time systems governed by finite-dimensional differential equations. Most of modern control systems are usually based on discrete micro-controller implementation. However, discontinuous control designed for a continuous-time system leads to high frequency oscillations referred to as *chattering* since within a sampling interval control is constant and switching frequency cannot exceed that of sampling. Increasing a sampling frequency to decrease the chattering amplitude seems unjustified. We believe that using a computer is adequate to control system dynamics if a sampling frequency corresponds to average, slow system motion rather than to a high frequency component.

The state trajectories in discrete-time systems with discontinuous control are not confined to the switching manifold but to some domain around it. So the first approach is to design control such that qualitatively this motion has the same properties as its continuous-time counterpart. To reduce this domain the control may be designed as a sum of continuous state function and a discontinuous one with lower magnitude.

The term “*sliding mode*” was first introduced for continuous time equations. For discrete-time equations

$$x_{k+1} = F(x_k, u_k), u_k = u(x_k), x_i \in \mathbb{R}^n, u \in \mathbb{R}^m$$

this term should be newly introduced in the spirit of *Definitions of Section 6*.

Definition 3. The set $S(x)$ in the manifold $s(x) = 0$, $s \in \mathbb{R}^m$ is the domain of sliding mode if there exists vicinity ϵ of S such that $s(x_{k+1}) = s[F(x_k, u_k)] = 0$ for $x_k \in \epsilon$.

The solution to the equation in *Definition 3* is called *equivalent control* u_{keq} as well, since, similar to continuous systems, it results in motions with state trajectories in the manifold $s(x) = 0$.

The discrete-time sliding mode control with bounded control actions $\|u\| \leq u_0$ is of form

$$u_k = \begin{cases} u_{keq}, & \text{if } \|u_{keq}\| \leq u_0 \\ \frac{u_{keq}}{\|u_{keq}\|} u_0, & \text{if } \|u_{keq}\| > u_0 \end{cases}$$

Both discrete-time control and the equivalent control are continuous state functions. For example the equivalent control is linear state function $u_{keq} = -(CB)^{-1}CAx_k$ for linear time-invariant discrete-time systems $x_{k+1} = Ax_k + Bu_k$ with a linear sliding manifold $s_k = Cx_k = 0$.

The control u_{keq} can not be found for linear plants with unknown parameters in matrix A , then the modified version should be applied

$$u_k = \begin{cases} -(CB)^{-1}s_k, & \text{if } \|(CB)^{-1}s_k\| \leq u_0 \\ -u_0(CB)^{-1}s_k / \|(CB)^{-1}s_k\|, & \text{if } \|(CB)^{-1}s_k\| > u_0. \end{cases}$$

For the both versions the control system is free of chattering and the motion equation is of a reduced order. The accuracy of the systems operating under uncertainty conditions is of a sampling interval order.

Similar to continuous-time systems, the motion with state trajectories in a manifold

$$s(x) = 0, s \in \mathbb{R}^m$$

and finite time needed to reach the manifold may occur in discrete-time system as well. The fundamental difference is that the control should be a continuous function of the state.

9 Chattering Problem

As we discussed in Section 3.3, replacement of the ideal time derivative in the switching line equation by a first-order filter led to oscillations at a finite frequency in some vicinity of the line. These oscillations called *chattering* may occur in the systems of general type (13), (15) at the presence of unmodeled dynamics such as disregarded in the ideal model small time constants of actuators and sensors. (μ_1 and μ_2 in Fig.16 with a linear plant and z_1 and z_2 as unmodeled-dynamics state vectors).

Chattering is a harmful phenomenon because it leads to low system accuracy, high wear of mechanical moving parts, high heat losses in power electric circuitry. Chattering is considered as the main reason which hinders application of sliding mode control since dynamic mismatch between real plants and its model always exists.

Continuous approximation of discontinuous control in a boundary layer of a sliding manifold is a method of chattering suppression if the unmodeled dynamics is not excited. At the same time chattering may appear in system with unmodeled dynamics and continuous approximation of control if the gain in a boundary layer is too high. Since the values of time constants, neglected in an ideal model, are unknown, the design should be oriented to the worst case and as a result the invariance property of discontinuous control are not utilized to full extent.

Further studies and practical experience showed that the chattering caused by unmodeled dynamics may be eliminated in systems with asymptotic observers (Fig. 21).

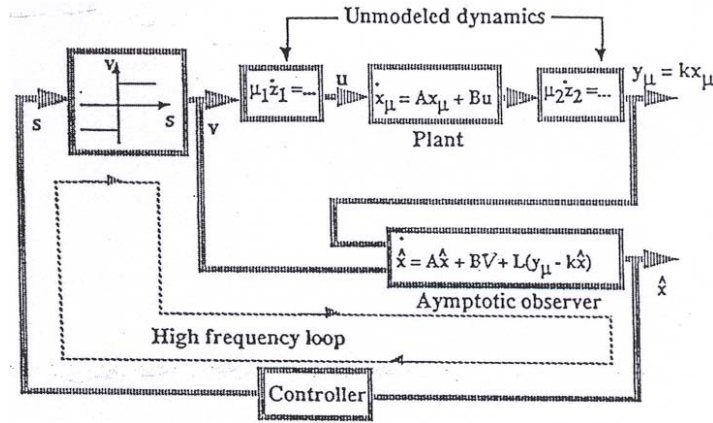


Fig. 21. Chattering suppression in systems with observers

In spite of the presence of unmodeled dynamics, ideal sliding arises, it is described by a singularly perturbed differential equation with solutions free of a high-frequency component and close to those of the ideal system. As shown in Figure 21, the asymptotic observer serves as a bypass for the high-frequency component, therefore the unmodeled dynamics are not excited. Note that a trace of the idea “a bypass for the high-frequency component” may be found in the cited in Section 3.3 publications of the sixties.

To design the observers system parameters are needed, but sliding mode is not destroyed if a *priori* knowledge of parameters differs from their real

values. The sensitivity of state estimation to parameter variations may be reduced in observers with high gains. Preserving sliding modes in systems with asymptotic observers predetermined successful applications of sliding mode control.

10 Application

The experience that has been gained in the applications of sliding mode control validates their efficiency for a wide range of processes in modern technology. As we mentioned in Introduction this paper does not claim to be a survey on VSS theory. The objectives of the paper neither implies to survey the results of applications which have always been associated with theoretical studies. We confine ourselves to one example only – results of the international project of colleagues from Yugoslavia and the Soviet Union in seventieth on sliding mode control of induction motors. The reason is twofold:

- It was the first application of multi-dimensional sliding mode to high-order nonlinear plants with control algorithms given in Section 7 of this paper
- Control of electric drives is the most challenging application of sliding modes; implementation of sliding modes by means of the most common electric power components has turned out to be simple enough, since “on-off” is the only admissible mode for them and discontinuities in control are dictated by the very nature of power converters.

The behavior of an induction motor is described by a nonlinear high-order system of differential equations:

$$\begin{aligned}
 \frac{dn}{dt} &= \frac{1}{J} (M - M_L), \quad M = \frac{x_H}{x_R} (i_\alpha \phi_\beta - i_\beta \phi_\alpha) \\
 \frac{d\phi_\alpha}{dt} &= -\frac{r_R}{x_R} \phi_\alpha - n \phi_\beta + r_R \frac{x_H}{x_R} i_\alpha, \quad \frac{d\phi_\beta}{dt} = -\frac{r_R}{x_R} \phi_\beta + n \phi_\alpha + r_R \frac{x_H}{x_R} i_\beta \\
 \frac{di_\alpha}{dt} &= \frac{x_R}{x_s x_R - x_H^2} \left(-\frac{x_H}{x_R} \frac{d\phi_\alpha}{dt} - r_s i_\alpha + u_\alpha \right) \\
 \frac{di_\beta}{dt} &= \frac{x_R}{x_s x_R - x_H^2} \left(-\frac{x_H}{x_R} \frac{d\phi_\beta}{dt} - r_s i_\beta + u_\beta \right) \\
 u &= (u_\alpha, u_\beta)^T = \frac{2}{3} (e_R, e_S, e_T) (u_R, u_S, u_T)^T
 \end{aligned} \tag{27}$$

where n is a rotor angle velocity, and two-dimensional vectors $\phi^T = (\phi_\alpha, \phi_\beta)$; $i^T = (i_\alpha, i_\beta)$, $u^T = (u_\alpha, u_\beta)$ are rotor flux, stator current and voltage in the fixed coordinate system (α, β) , respectively; M and M_L are a torque developed by a motor and a load torque, u_R, u_S, u_T are phase voltages, which may be made equal either to u_0 or $-u_0$; e_R, e_S, e_T are unit vectors of phase windings, R, S, T ; $J, x_H, x_S, x_R, r_R, r_S$ are motor parameters.

The control goal is to make one of the mechanical coordinates, for example, an angle speed $n(t)$, be equal to a reference input $n_0(t)$ and the magnitude of the rotor flux $\|\phi(t)\|$ be equal to its scalar reference input $\phi_0(t)$. The deviations from the desired motions are described by the function

$s_1 = c_1 [n_0 - n(t)] + \frac{d}{dt} [n_0 - n(t)]$, $s_2 = c_2 [\phi_0 - \|\phi(t)\|] + \frac{d}{dt} [\phi_0 - \|\phi(t)\|]$, where c_1, c_2 are positive. The static inverter forms three independent controls u_R, u_S, u_T so one degree of freedom can be used to satisfy some additional criterion. Let the voltages u_R, u_S, u_T constitute a three-phase balanced system, which means that the equality $s_3 = \int_0^t (u_R + u_S + u_T) dt = 0$ should hold for any t . If all three functions s_1, s_2, s_3 are equal to zero then, in addition to balanced system condition, the speed and flux mismatches decay exponentially since $s_1 = 0, s_2 = 0$ with $c_1, c_2 > 0$ are first-order differential equations. This means that the design of three dimensional control $u^T = (u_R, u_S, u_T)$ is reduced to enforcing the sliding mode in the manifold $s = 0, s^T = (s_1, s_2, s_3)$ in the system (27). Control algorithm of Section 7.3 with a new set of discontinuity surfaces $s^* = 0$ was implemented for an experimental setup with 11kw induction motor.

In the further research sliding mode observers were developed to find a rotor speed and flux simultaneously by measuring stator currents only. Later on different types of electric motors with sliding modes were applied in robotics, metal-cutting machine tools, and electric vehicles.

11 Conclusions

The main attention in this paper was paid to the events and ideas of VSS theory before the last two decades, when the interest to VSS has been increased significantly and the further development has been associated with international cooperation of colleagues from research centers of many countries. It can be easily revealed from intensity of publications in the area and the number of special sessions and issues of the most prestigious conferences and journals on control. "Current state of the art assessment" is beyond the scope of this paper. At the same time brief information on what recently initiated research directions look promising would be helpful for understanding the role of the "old times" in VSS history.

Simplex Control The system with control (13) consists of 2^n continuous structures. However if we give up component-wise design and select the set of control vectors directly, then it is sufficient to have $n + 1$ vectors only for enforcing sliding mode in the intersection of surfaces. Such method is called simplex control design. It eliminates redundancy and ambiguity of chattering or motion in a boundary layer that is important for multi-phase electric machines.

Observers In contrast to the conventional state observers, the input of sliding mode observers is a discontinuous function of an error between a system output and its estimate. As a result the order of error equations is reduced and the estimate does not depend on disturbances under the matching conditions, or if the disturbances are in the observer input space. For discrete-time

observers, deviation from a sliding manifold may be used for estimation both of system state and parameters.

Adaptive Control In addition to the adaptation method described in Section 3.4, the model reference approach associated with sliding modes may be used for designing an adaptive system. Again the adaptation process does not depend on disturbances under the matching conditions.

Output Sliding Mode Control The state vector is not available in many applications. An alternative to systems with observers is designing control as a function of the output only. As to linear systems, the requirements for each structure are weaker than those for a feedback system to be designed and the order of the motion equations is reduced. The similar situation took place for sliding mode control in linear systems studied in the first sections of this paper.

Systems with Delays The concept “sliding mode” for the systems with delays should be revised. It can be done in the spirit of the definition given for discrete-time systems in Section 8. Similar to what was done for discrete-time systems, the control may be designed such that system trajectories are confined to a manifold of a dimension lower than that of the original system.

Infinite-Dimensional Systems The mathematical models of many modern technological processes are partial differential equations, which are particular case of infinite-dimensional systems. The attempts of generalization of VSS methodology to this class showed that even the basic concepts such as sliding mode, discontinuity surface, component-wise design were to be revised. The main obstacle which hinders utilizing sliding mode control is unboundness of the operators in motion equations and the idea of enforcing sliding modes with bounded control is not applicable. The recently developed mathematical and design methods demonstrated that the potential of sliding mode control may be utilized to full extent in infinite-dimensional systems as well.

The above list may be complemented by new design methods: second-order sliding mode control; VSS control of the systems in the form

$$f(\dot{x}, x, u, \dot{u}, \dots, u^{(k)}, t) = 0;$$

sliding modes with finite transient times; integral sliding mode control with no reaching phase preceding sliding mode; robotics oriented methods incorporating dynamics of drives and manipulators in design procedures.

Finally, a little fantasy: The author of Minimum Principle Pontryagin mentioned in 1953 that the systems with “glued trajectories” formally can not be called dynamic systems [23]. The evolution of state $x(t)$ in dynamic systems is represented by shift operator $x(t) = F[x(0), t]$ with unique values for any time t (t may belong to $\mathbb{R}$ or $\mathbb{Z}$ to embrace continuous- and discrete-time systems). Then the one parameter set of operators $F[\bullet, t]$ is

called “group” . However it is not the case for systems with sliding modes for negative time t . Indeed a point in a sliding manifold may be reached in sliding mode or from outside. It means that operator F does not have inverse and the set $F[\bullet, t]$ is called semigroup.

Then a new definition can be offered: the point x of the system $x(t) = F[x, t]$ is called “sliding mode point” if the set $F[x, t]$ constitutes a semigroup. All the definitions offered previously for continuous-, discrete-time, difference equations are particular cases of this definition. *CHALLENGE: to create general sliding mode and sliding mode control theory in terms of shift operators embracing all types of control systems.*

This author hopes that in the very beginning of the new millennium we will have a chance to participate in discussion of the-here-mentioned problems of VSS theory and (what is much more interesting) a set of new ones unknown today.

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An Integrated Learning Variable Structure Control Method

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Subtitle: *For the finite period tracking control task repeatable over iterations, the uniformly bounded learning control is synthesized into variable structure control to approximate the equivalent control and to realize perfect tracking.*

Abstract. In this chapter, we consider repeatable tracking control tasks using a new control approach - Learning Variable Structure Control (LVSC). LVSC synthesizes two main control strategies: Variable Structure Control (VSC) as the robust part and learning control as the intelligent part. The incorporation of the powerful learning function, by virtue of the internal model principle, completely nullifies the tracking error. The switching control mechanism on the other hand, retains the well appreciated properties of VSC, especially the insensitivity to norm-bounded system uncertainties. Through a rigorous proof based on the energy function and functional analysis, we show that the LVSC system achieves the following novel properties: (1) the tracking error sequence converges *uniformly* to zero; (2) the bounded learning control sequence converges to the *equivalent control*, i.e. the desired control profile *almost everywhere*; (3) the system state sequence and VSC control sequence are uniformly continuous. To address important practical considerations, the learning mechanism is implemented by means of *Fourier* series expansions, hence it achieves better tracking performance.

1 Introduction

A. Variable Structure Control (VSC) is well noted for its simplicity in implementation and outstanding robustness to system uncertainties when in sliding mode [1–5]. A switching control with a signum function best caters to the worst case control environment where system perturbations can be structured, unstructured, deterministic, stochastic, nonvanishing, and only upper bounds of system perturbations are available. Replacing the signum function with a continuous function will incur inferior control performance such as less tracking accuracy [6]. Numerous schemes have been developed to improve tracking accuracy by incorporating VSC with other advanced control methods [7–9,6]. It should be noted that a typical VSC with discontinuous gain at the equilibrium is able to generate an infinitely high gain to suppress any bounded disturbances, e.g.

$$u_{vsc} = k \operatorname{sign}(\sigma)$$

where k is a positive gain, σ is a switching quantity, and the signum function $\text{sign}(\sigma) = 1$ when $\sigma > 0$ and $\text{sign}(\sigma) = -1$ when $\sigma < 0$. In the worst case control environment where the process is subject to non-periodic or random perturbations, any advanced and intelligent control schemes will fail to work, and the typical VSC with infinite gain is perhaps the only control scheme still able to achieve perfect trajectory tracking and perfect disturbance rejection. Nevertheless the chattering problem arising in real applications and implementation prevents the use of such an infinite gain. In addition, in the sampled or discrete-time circumstances, VSC will lose its high gain property anyway due to the stability concern. Numerous smoothing schemes have been proposed aiming at eliminating chattering by shaping the control gain, such as using a continuous function to replace the signum function. All these schemes, however, have a side effect of losing tracking accuracy. The smoother the performance, the lower the gain around the equilibrium, and in the sequel the lower the tracking accuracy. The reason is obvious, at the equilibrium $\sigma = 0$, any continuous function of σ will be zero accordingly, nevertheless the process may still demand a non-zero control action in order to sustain the tracking task or cancel the non-vanishing disturbances. To overcome the above inherent limitation of feedback, we have to consider a feedforward control together with feedback. Many feedforward compensation approaches are applicable when the control environment is not the worst. Among them learning control is one such approach particularly effective to the repeatable control environment, i.e. for periodic reference trajectories and periodic disturbances.

B. There are many ways to combine VSC and LC (learning control). Here we show two such topological structures.

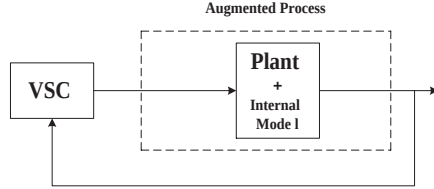


Fig. 1. LVSC Structure I

In Fig.3, it shows that the process is first augmented with an appropriate internal model. Then VSC is designed to guarantee the augmented system to be stable and robust. In ([11]) a repetitive type VSC is made possible for a class of discrete-time LTI systems. As long as the reference signal $r(j)$ and the disturbance $d(j)$ is of period T which is a positive integer, the simplest internal model is $z^T - 1$. We can see that any error signal $e(j)$ of period T ,

once passing through the factor $z^T - 1$, will be zero simply because

$$(z^T - 1)e(j) = e(j + T) - e(j) = e(j) - e(j) = 0.$$

An immediate advantage is, such an augmented system is able to yield the desired internal model which is the same as the reference signal generator and disturbance generator, no matter how complicated they are. All the information needed is to know the period T *a priori*. In other words, the process equipped with the internal model is able to absorb the periodic residue. Such a residue in general cannot be completely eliminated by a typical error feedback such as a VSC with smoothing scheme, because of the limited control gain.

When the process is highly nonlinear and/or the learning mechanism is nonlinear, it is not an easy task to directly “transplant” the internal model into the process. An alternative way is to insert the learning mechanism in between the process and the VSC, as shown in Fig.4.

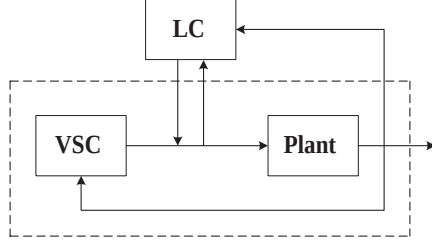


Fig. 2. LVSC Structure II

This leads to a modular strategy. The VSC module will be first designed to ensure the global stability and achieve the uniformly bounded tracking accuracy with an appropriate smoothing scheme. The learning module will be added to further improve the tracking performance whenever the control task repeats. Note that we should not re-design the VSC module in this configuration. In case the learning module is removed, as a stand-alone controller the VSC module still keeps its normal function. The modular approach provides extra flexibility as one can easily add a learning mechanism to an existing VSC system without any modification, hence it is easy to implement.

It is worthy to highlight that, most existing repetitive control or iterative learning control methods are of the typical feedforward class and thus sensitive to any non-periodic factors. By incorporating learning into VSC, the VSC part will “protect” the learning part to certain extent by virtue of its excellent robustness property. Two kinds of control tasks can be easily handled: the first with periodic reference and disturbance, and the second with repeated reference and disturbance. The former is a kind of repetitive VSC, and the latter is a kind of iterative learning VSC. In this chapter we will

give detailed design and analysis of the latter, named as Learning Variable Structure Control (LVSC).

C. Generating the *equivalent control* profile is the ultimate objective of VSC in sliding mode, which assures perfect tracking and complete disturbance rejection. Under the boundedness and *Lipschitz* continuity conditions of the system dynamics, [1] provides an approach to acquire the *equivalent control*: passing the control signal through a first-order low-pass filter. It requires that the system stays strictly on the switching surface and the time constant of the filter approach zero. In other words, it demands an infinite switching frequency and an infinite filter bandwidth in order to deal with the worst case control environment.

In a repeatable control environment, we can make use of learning control approaches. The stringent requirements for obtaining *equivalent control* in [1] through filtering control signals can thus be relaxed. In this chapter we focus on learning for the deterministic control environment which has been well summarized in [12], [13], etc. The non-repeatable factors such as random perturbations are assumed to be much smaller and consequently negligible. Therefore the learning introduced here is more like iterative learning approach instead of a stochastic learning approach such as *Bayesian* learning or learning automaton. Moreover, the robustness of the control system will still be ensured to certain extent by VSC. It is worthy to point out again, the learnability is highly related to the repeatability of the control environment. LVSC can achieve better control performance when the system repeatability increases. By virtue of repeatability, the past control tracking error sequences do reflect the dynamic characteristics of the uncertain system, and reflect the influence from the reference signal as well as system perturbations to the tracking errors. Learning in LVSC aims at extracting useful control knowledge from past control and tracking error sequences so as to approximate *equivalent control*.

As a modular approach, the proposed LVSC has a very simple structure consisting of two components in additive form: a standard switching control mechanism based on the known upper bounds using a continuous smoothing function, and a learning mechanism which simply adds up either a past tracking error sequence or a past VSC sequence. Through rigorous proof based on energy function and functional analysis, we show that LVSC system achieves the following novel properties: (1) the tracking error sequence converges *uniformly* to zero; (2) the learning control sequence which is bounded everywhere converges to the *equivalent control*, i.e. the desired control profile *almost everywhere* (*a.e.*); (3) the system state sequence and VSC control sequence are uniformly continuous.

In this chapter we further address important issues arising from practical considerations and implement the proposed learning variable structure control in the frequency domain by means of *Fourier* series expansion, which

achieves the best functional approximation for periodic functions in the sense of the mean square. In addition, *Fourier* series-based learning can further enhance the robustness property of LVSC and improve tracking performance.

2 Notation and Preliminaries

R^n denotes space of n -tuples of real numbers. $|z|$ denotes the absolute value of a function z . $\|\mathbf{v}\| = (\mathbf{v}^T \mathbf{v})^{\frac{1}{2}}$ denotes the norm of a vector $\mathbf{v} \in R^n$. $\|A\| = \sqrt{\lambda_{\max}(A^T A)}$ denotes the norm of a matrix A . $Z_+ \triangleq \{1, 2, \dots\}$ denotes the set of positive integers. $B(D)$ denotes a space of bounded functions on D . $C(D)$ denotes a space of continuous functions on D . $UC(D)$ denotes a space of uniformly continuous functions on D . $C^n(D)$ denotes a space of n times continuously differentiable functions on D .

$|h|_w$ denotes an extended time-weighted L_2 norm of function h defined by $|h|_w \triangleq \int_0^t e^{-\lambda\tau} h^2(\tau) d\tau$ for a positive constant $\lambda < \infty$. h and 0 are said to be *equivalent* if $|h|_w = 0$, or $h = 0$ *almost everywhere* [14].

Hölder inequality [15] If $p, q \in [1, \infty]$ and $p^{-1} + q^{-1} = 1$, then $f \in L_p, g \in L_q$ imply that $fg \in L_1$ and $\|fg\|_1 \leq \|f\|_p \|g\|_q$. When $p = q = 2$, the *Hölder inequality* becomes the *Schwartz inequality*, i.e., $\|fg\|_1 \leq \|f\|_2 \|g\|_2$.

Bellman-Gronwall Lemma I [15] Let $\lambda(t), g(t), k(t)$ be nonnegative piecewise continuous functions of time t . If the function $y(t)$ satisfies the inequality $y(t) \leq \lambda(t) + g(t) \int_{t_0}^t k(\tau) y(\tau) d\tau, \forall t \geq t_0 \geq 0$, then $y(t) \leq \lambda(t) + g(t) \int_{t_0}^t \lambda(s) k(s) e^{\int_s^t k(\tau) g(\tau) d\tau} ds, \forall t \geq t_0 \geq 0$.

Bellman-Gronwall Lemma II [15] Let $\lambda(t), k(t)$ be nonnegative piecewise continuous function of time t and let $\lambda(t)$ be differentiable. If the function $y(t)$ satisfies the inequality $y(t) \leq \lambda(t) + \int_{t_0}^t k(s) y(s) ds, \forall t \geq t_0 \geq 0$, then $y(t) \leq \lambda(t_0) e^{\int_{t_0}^t k(s) ds} + \int_{t_0}^t \dot{\lambda}(s) e^{\int_s^t k(\tau) d\tau} ds, \forall t \geq t_0 \geq 0$.

Continuity Theorem [16] Let $f \in B(D) \cap C(D)$. Suppose $\phi(t)$ is a solution of $\dot{\mathbf{x}} = f(\mathbf{x}, t)$ where $\mathbf{x} \in R^n$ and $t \in R^1 \forall t \in (a, b)$. Then if $(a, \phi(a^+))$ and $(b, \phi(b^-))$ are in D where $\phi(a^+) = \lim_{t \rightarrow a^+} \phi(t)$ and $\phi(b^-) = \lim_{t \rightarrow b^-} \phi(t)$, then the solution $\phi(t) \in C[a, b]$.

Cantor Theorem [17] Let $f \in C(D)$, if D is a closed nonempty set, then $f \in UC(D)$.

3 Problem Formulation and LVSC Configuration

Consider the n th-order deterministic nonlinear uncertain dynamical system described by

$$\begin{cases} \dot{x}_j = x_{j+1} \\ \dot{x}_n = f(\mathbf{x}, t) + b(\mathbf{x}, t)u \end{cases} \quad (1)$$

$\mathbf{x} = [x_1, \dots, x_n]^T \in X \subseteq R^n$ is the physically measurable state vector. u is the control input. $f(\mathbf{x}, t)$ and $b(\mathbf{x}, t)$ are nonlinear uncertain functions.

A: Repeatable tracking control tasks

Given a finite initial state $\mathbf{x}_i(0)$ and a finite time interval $[0, T_f]$ where i denotes the iteration sequence, the control objective is to design a VSC combined with iterative learning such that, as $i \rightarrow \infty$, the system state $\mathbf{x}_i$ of the nonlinear uncertain system (1) tracks the desired trajectory $\mathbf{x}_d = [x_{d,1} \dots x_{d,n}]^T \in X_D \subseteq R^n$ where $\mathbf{x}_d \in C^1([0, T_f])$ and is generated by the following dynamics over $[0, T_f]$

$$\begin{cases} \dot{x}_{d,j} = x_{d,j+1} \\ \dot{x}_{d,n} = \beta(\mathbf{x}_d, t) + r(t) \end{cases} \quad (2)$$

where $\beta(\mathbf{x}_d, t) \in C(X_D \times [0, T_f])$ is a known function and $r(t) \in C([0, T_f])$ is a reference input. $\mathbf{x}_d \in C^1([0, T_f])$ ensures $\mathbf{x}_d \in C([0, T_f])$, $\dot{\mathbf{x}}_d \in C([0, T_f])$ and therefore $\mathbf{x}_d \in B([0, T_f])$ and $\dot{\mathbf{x}}_d \in B([0, T_f])$ as $[0, T_f]$ is a finite interval. As part of the repeatability condition, the initial states $\mathbf{x}_i(0) = \mathbf{x}_d(0)$, i.e., $\sigma_i(0) = 0$ are available for all trials.

B: Switching surface

The switching surface of a basic VSC is defined as

$$\sigma_i = (x_{d,n} - x_{i,n}) + \sum_{j=1}^{n-1} \alpha_j (x_{d,j} - x_{i,j}) = \boldsymbol{\alpha}^T (\mathbf{x}_d - \mathbf{x}_i) \quad (3)$$

where $\boldsymbol{\alpha}^T = [\boldsymbol{\alpha}_1^T, 1]$, $\boldsymbol{\alpha}_1^T = [\alpha_1, \dots, \alpha_{n-1}]$ and α_j ($j = 1, \dots, n-1$) are chosen coefficients such that the polynomial $s^{n-1} + \alpha_{n-1}s^{n-2} + \alpha_{n-2}s^{n-3} + \dots + \alpha_1$ is *Hurwitz* and s is a *Laplace* operator.

C: Assumption

(A1) $f(\mathbf{x}, t)$ is bounded by a known bounding function $f_{max}(\mathbf{x}, t)$, i.e., $|f(\mathbf{x}, t)| \leq f_{max}(\mathbf{x}, t)$ while $b(\mathbf{x}, t)$ is positive definite which is lower bounded by a known bounding function $b_{min}(\mathbf{x}, t)$ and is also upper bounded, i.e., $0 < \underline{b} \leq b_{min}(\mathbf{x}, t) \leq b(\mathbf{x}, t) \leq \bar{b}$ for some positive constants $\underline{b}$, $\bar{b}$ and $\forall(\mathbf{x}, t) \in X \times [0, T_f]$. It implies that $b^{-1}(\mathbf{x}, t)f(\mathbf{x}, t)$ is also bounded as $|b^{-1}(\mathbf{x}, t)f(\mathbf{x}, t)| \leq b_{min}^{-1}(\mathbf{x}, t)f_{max}(\mathbf{x}, t)$. The bounding functions $f_{max}(\mathbf{x}, t)$ and $b_{min}(\mathbf{x}, t)$ belong to $B(X \times [0, T_f]) \cap C(X \times [0, T_f])$.

(A2) $\forall h \in \{f, b\}$, $h(\mathbf{x}, t) \in C(X \times [0, T_f])$ and $h(\mathbf{x}, t)$ satisfies the *globally Lipschitz* condition, $\|h(\mathbf{x}_1, t) - h(\mathbf{x}_2, t)\| \leq l_h \|\mathbf{x}_1(t) - \mathbf{x}_2(t)\|$, $\forall t \in [0, T_f]$, $\forall \mathbf{x}_1, \mathbf{x}_2 \in X$ and for a positive constant $l_h < \infty$.

D: Equivalent control u_{eq}

Differentiating (3) with respect to time results

$$\dot{\sigma}_i = (\dot{x}_{d,n} - \dot{x}_{i,n}) + \sum_{j=1}^{n-1} \alpha_j (x_{d,j+1} - x_{i,j+1}). \quad (4)$$

Substituting (1) and (2) into (4) gives

$$\begin{aligned}\dot{\sigma}_i &= \sum_{j=1}^{n-1} \alpha_j (x_{d,j+1} - x_{i,j+1}) + \beta(\mathbf{x}_d, t) + r(t) - f_i - b_i u_i \\ &= g_i - f_i - b_i u_i\end{aligned}\quad (5)$$

where $f_i \triangleq f(\mathbf{x}_i, t)$, $b_i \triangleq b(\mathbf{x}_i, t)$, $g_i \triangleq \sum_{j=1}^{n-1} \alpha_j (x_{d,j+1} - x_{i,j+1}) + g_d$, $g_d \triangleq \beta(\mathbf{x}_d, t) + r(t)$. According to (5), making $\dot{\sigma}_i = 0$ obtains $\sigma_i(t) = \sigma_i(0) = 0$, i.e., $\mathbf{x}_i(t) = \mathbf{x}_d(t) \forall t \in [0, T_f]$ and gives the following expression for the *equivalent control*

$$u_{eq} = b_d^{-1} g_d - b_d^{-1} f_d \quad (6)$$

where $b_d \triangleq b(\mathbf{x}_d, t)$, $f_d \triangleq f(\mathbf{x}_d, t)$. Since $\mathbf{x}_d \in B([0, T_f])$, then $g_d \in B([0, T_f])$ as $\beta(\mathbf{x}_d, t) \in C(X_D \times [0, T_f])$, $r(t) \in C([0, T_f])$ and $[0, T_f]$ is a finite interval. According to A1, $b_d^{-1} \leq \underline{b}^{-1}$ and $\mathbf{x}_d \in B([0, T_f])$ ensures $f_d \in B([0, T_f])$, hence $u_{eq} \in B([0, T_f])$.

E: LVSC configuration

The proposed LVSC is given below

$$u_i = u_{v,i} + u^* \text{sat}(u_{l,i-1}, u^*), \quad u_{l,0} = 0, \quad (7)$$

$$u_{v,i} = \zeta \sigma_i + \rho_i \text{sat}(\sigma_i, \varepsilon), \quad \rho_i = b_{min}^{-1}(\mathbf{x}_i, t) [|g_i| + f_{max}(\mathbf{x}_i, t)], \quad (8)$$

$$u_{l,i} = u^* \text{sat}(u_{l,i-1}, u^*) + \iota_i, \quad \iota_i \in \{ \beta_l \sigma_i, u_{v,i} \}, \quad (9)$$

$$\text{sat}(\star, *) \triangleq \begin{cases} \star / * & |\star| \leq * \\ \text{sgn}(\star) & |\star| > * \end{cases} \quad (10)$$

where $\zeta, \varepsilon, \beta_l$ and u^* are positive-definite constants. u^* is a sufficiently large constant such that

$$u^* \geq \sup_{t \in [0, T_f]} |u_{eq}(t)| \quad (11)$$

to ensure learnability. Note that u^* can be either decided from the real limitation of the physical process, or simply chosen to be an arbitrarily large but finite *virtual* bound which does not affect the VSC part. (8) is a VSC law which replaces the signum function $\text{sgn}(\sigma_i)$ with the saturation function $\text{sat}(\sigma_i, \varepsilon)$ to eliminate chattering [6]. Since $u_{v,i}$ is continuous with respect to $\sigma_i, g_i, b_{min}(\mathbf{x}_i, t)$ and $f_{max}(\mathbf{x}_i, t)$, $b_{min}(\mathbf{x}_i, t) \in C(X \times [0, T_f])$, $f_{max}(\mathbf{x}_i, t) \in C(X \times [0, T_f])$ according to A1, $\sigma_i \in C(X \times [0, T_f])$, $g_i \in C(X \times [0, T_f])$ as $\mathbf{x}_d \in C([0, T_f])$, hence $u_{v,i} \in C(X \times [0, T_f])$.

According to (10), the VSC law (8) is equivalent to

$$u_{v,i} = (\zeta + \rho_i k_i) \sigma_i, \quad k_i \triangleq \begin{cases} 1/\varepsilon & |\sigma_i| \leq \varepsilon \\ 1/|\sigma_i| & |\sigma_i| > \varepsilon \end{cases}. \quad (12)$$

The above definition ensures that $k_i \leq 1/\varepsilon$.

Since $b_d \neq 0$, it can be shown by (6) that, if $\beta(\mathbf{x}_d, t) + r(t) - f(\mathbf{x}_d, t) \neq 0$, then $u_{eq} \neq 0$. It is impossible to generate u_{eq} by using the VSC law (8), since when $\mathbf{x}_i \rightarrow \mathbf{x}_d$, $\sigma_i \rightarrow 0$, then $u_{v,i} \rightarrow 0 \neq u_{eq}$. To produce u_{eq} or to achieve perfect tracking using a smooth control input, (9) introduces the simple structure of the proposed learning control $u_{l,i}$ where the item ι_i is for updating learning control which will be discussed in detail in subsequent sections.

To evaluate the learning performance, the following time-weighted L_2 norm of $u_{l,i} - u_{eq}$ is used

$$J_i(t) = |u_{l,i} - u_{eq}|_w = \int_0^t e^{-\lambda\tau} [u_{l,i}(\tau) - u_{eq}(\tau)]^2 d\tau \geq 0 \quad (13)$$

Under saturator (10) and condition (11), the following key inequality holds

$$[u^* \text{sat}(u_{l,i-1}, u^*) - u_{eq}]^2 \leq (u_{l,i-1} - u_{eq})^2, \quad \forall i \in Z_+ \cap (i \geq 2).$$

From (9) and the above, the difference of $J_i(t)$ between two successive trials, $\forall i \in Z_+ \cap (i \geq 2)$ can be derived as

$$\begin{aligned} \Delta J_i(t) &= J_i(t) - J_{i-1}(t) \\ &= \int_0^t e^{-\lambda\tau} [(u_{l,i} - u_{eq})^2 - (u_{l,i-1} - u_{eq})^2] d\tau \\ &\leq \int_0^t e^{-\lambda\tau} \left\{ (u_{l,i} - u_{eq})^2 - [u^* \text{sat}(u_{l,i-1}, u^*) - u_{eq}]^2 \right\} d\tau \\ &= \int_0^t e^{-\lambda\tau} [u_{l,i} - u^* \text{sat}(u_{l,i-1}, u^*)] [u_{l,i} + u^* \text{sat}(u_{l,i-1}, u^*) - 2u_{eq}] d\tau \\ &= \int_0^t e^{-\lambda\tau} \{ \iota_i^2 + 2\iota_i [u^* \text{sat}(u_{l,i-1}, u^*) - u_{eq}] \} d\tau. \end{aligned} \quad (14)$$

From (5), it can be obtained that

$$b_i^{-1} \dot{\sigma}_i = b_i^{-1} g_i - b_i^{-1} f_i - u_i.$$

Substituting the control law (7) into the above yields

$$b_i^{-1} \dot{\sigma}_i = b_i^{-1} g_i - b_i^{-1} f_i - u_{v,i} - u^* \text{sat}(u_{l,i-1}, u^*).$$

Subtracting u_{eq} in (6) from both sides of the above yields

$$\begin{aligned} b_i^{-1} \dot{\sigma}_i - u_{eq} &= (b_i^{-1} g_i - b_i^{-1} f_i) \\ &\quad - (b_d^{-1} g_d - b_d^{-1} f_d) - u_{v,i} - u^* \text{sat}(u_{l,i-1}, u^*) \end{aligned}$$

which can be rewritten into

$$u^* \text{sat}(u_{l,i-1}, u^*) - u_{eq} = -u_{v,i} - b_i^{-1} \dot{\sigma}_i - \gamma_i, \quad (15)$$

where

$$\gamma_i = (b_d^{-1} f_d - b_i^{-1} f_i) - (b_d^{-1} g_d - b_i^{-1} g_i). \quad (16)$$

It can be derived that

$$\begin{aligned} |\gamma_i| &\leq |b_d^{-1}f_d - b_i^{-1}f_d + b_i^{-1}f_d - b_i^{-1}f_i| \\ &\quad + |b_d^{-1}g_d - b_i^{-1}g_d + b_i^{-1}g_d - b_i^{-1}g_i| \\ &\leq b_i^{-1}b_d^{-1}|b_d - b_i| \cdot |f_d| + b_i^{-1}|f_d - f_i| \\ &\quad + b_i^{-1}b_d^{-1}|b_d - b_i| \cdot |g_d| + b_i^{-1}|g_d - g_i|. \end{aligned}$$

Since $g_d - g_i = -\sum_{j=1}^{n-1} \alpha_j (x_{d,j+1} - x_{i,j+1})$, we have

$$|g_d - g_i| \leq (\alpha_1^2 + \dots + \alpha_{n-1}^2)^{\frac{1}{2}} \|\mathbf{x}_d - \mathbf{x}_i\|.$$

Under assumption A1, b_i^{-1} and b_d^{-1} are all bounded by $\underline{b}^{-1}$. Since $f_d \in B([0, T_f])$ and $g_d \in B([0, T_f])$, we denote that $\bar{f}_d = \sup_{t \in [0, T_f]} f_d(t)$ and $\bar{g}_d = \sup_{t \in [0, T_f]} g_d(t)$. Using the *globally Lipschitz* condition described in A2 we can obtain

$$|\gamma_i| \leq c \|\mathbf{x}_d - \mathbf{x}_i\| \quad (17)$$

where $c = \underline{b}^{-1} \left[l_b \underline{b}^{-1} \bar{f}_d + l_f + \underline{b}^{-1} l_b \bar{g}_d + (\alpha_1^2 + \dots + \alpha_{n-1}^2)^{\frac{1}{2}} \right]$ is a finite positive constant. From (9) and (15) it can be obtained that

$$u_{l,i} - u_{eq} = \iota_i - u_{v,i} - b_i^{-1} \dot{\sigma}_i - \gamma_i. \quad (18)$$

F. Property Analysis

To facilitate LVSC analysis, here we give four propositions which reveal the boundedness relationship among the quantities σ_i , $\dot{\sigma}_i$, $\mathbf{x}_i$, $\dot{\mathbf{x}}_i$, γ_i , ρ_i , $u_{v,i}$, $u_{l,i}$, u_i and J_i .

Proposition 1. *For system (1) given the desired trajectory in (2) and switching surface (3), the following hold*

$$\dot{\mathbf{x}}_d - \dot{\mathbf{x}}_i = A(\mathbf{x}_d - \mathbf{x}_i) + \mathbf{b} \dot{\sigma}_i, \quad (19)$$

$$\|\mathbf{x}_d - \mathbf{x}_i\| \leq \|A\| \int_0^t |\sigma_i(\tau)| e^{\|A\|(t-\tau)} d\tau + |\sigma_i| \quad (20)$$

where $A = \begin{bmatrix} \mathbf{0}_{(n-1) \times 1} & I_{(n-1) \times (n-1)} \\ 0 & -\boldsymbol{\alpha}_1^T \end{bmatrix}$, $\mathbf{b} = [\mathbf{0}_{1 \times (n-1)} \ 1]^T$.

Proof. Combining (1) and (2) yields

$$\dot{x}_{d,j} - \dot{x}_{i,j} = x_{d,j+1} - x_{i,j+1}. \quad (21)$$

Rearranging (4) gives

$$\dot{x}_{d,n} - \dot{x}_{i,n} = -\sum_{j=1}^{n-1} \alpha_j (x_{d,j+1} - x_{i,j+1}) + \dot{\sigma}_i. \quad (22)$$

Combining (21) and (22) gives (19). Integrating both sides of (19) and noticing $\sigma_i(0) = 0$ and $\mathbf{x}_i(0) = \mathbf{x}_d(0)$ one obtains

$$\mathbf{x}_d - \mathbf{x}_i = A \int_0^t (\mathbf{x}_d - \mathbf{x}_i) d\tau + \mathbf{b}\sigma_i.$$

Taking the norm of the above and since $\|\mathbf{b}\| = 1$, the following stands

$$\|\mathbf{x}_d - \mathbf{x}_i\| \leq \|A\| \int_0^t \|\mathbf{x}_d - \mathbf{x}_i\| d\tau + |\sigma_i|.$$

Applying the *Bellman-Gronwall Lemma I*, we can obtain (20).

Proposition 2. *For system (1) satisfying assumptions A1, under the control laws (7)-(10), $\sigma_i \in B([0, T_f])$, $\mathbf{x}_i \in B([0, T_f])$, $\dot{\sigma}_i \in B([0, T_f])$, $\dot{\mathbf{x}}_i \in B([0, T_f])$, $\rho_i \in B([0, T_f])$, i.e., $\rho_i \leq \bar{\rho}$ for a positive constant $\bar{\rho} < \infty$, $u_{v,i} \in B([0, T_f])$, $u_{l,i} \in B([0, T_f])$, $u_i \in B([0, T_f])$ and $J_i \in B([0, T_f])$, $\forall t \in [0, T_f]$ and $\forall i \in Z_+$.*

Proof. Define a Lyapunov function as $V_i = \sigma_i^2/2$. Differentiating V_i with respect to t using (5) one obtains

$$\dot{V}_i = -b_i\sigma_i [u_i - (b_i^{-1}g_i - b_i^{-1}f_i)].$$

If $|\sigma_i| \geq \varepsilon$, under (10) and assumptions A1, substituting control laws (7), (8) into the above yields

$$\begin{aligned} \dot{V}_i &= -b_i\zeta\sigma_i^2 - b_i\rho_i|\sigma_i| + b_i\sigma_i(b_i^{-1}g_i - b_i^{-1}f_i) - b_i\sigma_i u^* \text{sat}(u_{l,i-1}, u^*) \\ &\leq -b_i\zeta\sigma_i^2 - b_i|\sigma_i| [\rho_i - (b_i^{-1}g_i - b_i^{-1}f_i) \text{sgn}(\sigma_i)] + b_i u^* |\sigma_i| \\ &\leq -b_i\zeta\sigma_i^2 + b_i u^* |\sigma_i|. \end{aligned}$$

Since $\sigma_i(0) = 0$, $\forall i \in Z_+$, it can be concluded that system stays inside the bound $|\sigma_i(t)| \leq \max\{\varepsilon, \zeta^{-1}u^*\}$, $\forall t \in [0, T_f]$.

According to (20) of *Proposition 1*, $\sigma_i \in B([0, T_f])$ ensures that $\mathbf{x}_i \in B([0, T_f])$ since $\mathbf{x}_d \in B([0, T_f])$. From A1, $f_{max}(\mathbf{x}_i, t)$ and $b_{min}(\mathbf{x}_i, t)$ belong to $B(X \times [0, T_f])$ ensures that $\rho_i \in B(X \times [0, T_f])$ as $\mathbf{x}_d \in B([0, T_f])$, hence $\mathbf{x}_i \in B([0, T_f])$ brings that $\rho_i \in B([0, T_f])$ according to (8). $\rho_i \in B([0, T_f])$, $k_i \leq 1/\varepsilon$ and $\sigma_i \in B([0, T_f])$ lead to $u_{v,i} \in B([0, T_f])$ according to (12). Since $u^* \text{sat}(u_{l,i-1}, u^*) \leq u^*$, then $\sigma_i \in B([0, T_f])$ and $u_{v,i} \in B([0, T_f])$ lead to $u_i \in B([0, T_f])$ and $u_{l,i} \in B([0, T_f])$ according to (7) and (9), hence $\dot{\mathbf{x}}_i \in B([0, T_f])$ according to (1) and $\dot{\sigma}_i \in B([0, T_f])$ according to (5). From (13), $u_{l,i} \in B([0, T_f])$ ensures $J_i \in B([0, T_f])$, $\forall i \in Z_+$.

Proposition 3. *For system (1) given the desired trajectory in (2) and switching surface (3), under assumption A1 and A2, the following stands*

$$\int_0^t e^{-\lambda\tau} |\sigma_i(\tau)| \cdot |\gamma_i(\tau)| d\tau \leq (c + c\|A\|T_f e^{\|A\|T_f}) \int_0^t e^{-\lambda\tau} \sigma_i^2(\tau) d\tau. \quad (23)$$

Proof. Since $0 \leq \nu \leq \tau \leq t \leq T_f$, then $0 \leq \tau - \nu \leq \tau \leq T_f$ and $-\frac{\lambda}{2}\tau \leq -\frac{\lambda}{2}\nu$. Using the *Hölder inequality*, it can be obtained from (17) and (20) under assumption A2 that

$$\begin{aligned}
& \int_0^t e^{-\lambda\tau} |\sigma_i(\tau)| \cdot |\gamma_i(\tau)| d\tau - c \int_0^t e^{-\lambda\tau} \sigma_i^2(\tau) d\tau \\
& \leq c \|A\| \int_0^t e^{-\lambda\tau} |\sigma_i(\tau)| \left[\int_0^\tau e^{\|A\|(\tau-\nu)} |\sigma_i(\nu)| d\nu \right] d\tau \\
& \leq c \|A\| e^{\|A\|T_f} \int_0^t e^{-\frac{\lambda}{2}\tau} |\sigma_i(\tau)| \left[\int_0^\tau e^{-\frac{\lambda}{2}\tau} |\sigma_i(\nu)| d\nu \right] d\tau \\
& \leq c \|A\| e^{\|A\|T_f} \left[\int_0^t e^{-\frac{\lambda}{2}\tau} |\sigma_i(\tau)| d\tau \right]^2 \\
& \leq c \|A\| e^{\|A\|T_f} \left[\int_0^t e^{-\lambda\tau} \sigma_i^2(\tau) d\tau \right] \left[\int_0^t 1^2 d\tau \right] \\
& \leq c \|A\| T_f e^{\|A\|T_f} \int_0^t e^{-\lambda\tau} \sigma_i^2(\tau) d\tau.
\end{aligned}$$

Rearranging the above gives (23).

Since $\iota_i \in \{\beta_l \sigma_i, u_{v,i}\}$ is used in the learning law (9), According to (12), we have

$$\iota_i - u_{v,i} = \kappa_i \sigma_i \quad (24)$$

where $\kappa_i \triangleq \begin{cases} \beta_l - \zeta - \rho_i k_i & \iota_i = \beta_l \sigma_i \\ 0 & \iota_i = u_{v,i} \end{cases}$. Since $\rho_i \leq \bar{\rho}$ according to *Proposition 2* under assumption A1 and $k_i \leq 1/\varepsilon$, we have $|\kappa_i| \leq \bar{\kappa}$ for a positive constant $\bar{\kappa} < \infty$.

Proposition 4. For system (1) given the desired trajectory in (2) and switching surface (3), under assumptions A1 and A2, if the control laws (7), (8) and learning law (9) are applied, then the following stands

$$\|\mathbf{x}_i - \mathbf{x}_d\| \leq \bar{b} e^{lT_f} T_f^{\frac{1}{2}} J_i^{\frac{1}{2}}(T_f), \quad (25)$$

$$|\sigma_i| \leq \bar{b} (\alpha_1^2 + \dots + \alpha_{n-1}^2 + 1)^{\frac{1}{2}} e^{lT_f} T_f^{\frac{1}{2}} J_i^{\frac{1}{2}}(T_f). \quad (26)$$

where $l_1 = \|A\| + \bar{b}c + \bar{b}\bar{\kappa}(\alpha_1^2 + \dots + \alpha_{n-1}^2 + 1)^{\frac{1}{2}}$ and $l \triangleq \max(\lambda, l_1)$.

Proof. From (18) it can be obtained that

$$\dot{\sigma}_i = b_i (\iota_i - u_{v,i}) - b_i \gamma_i - b_i (u_{l,i} - u_{eq}).$$

Substituting the above into (19) yields

$$\dot{\mathbf{x}}_d - \dot{\mathbf{x}}_i = A(\mathbf{x}_d - \mathbf{x}_i) + \mathbf{b} [b_i (\iota_i - u_{v,i}) - b_i \gamma_i - b_i (u_{l,i} - u_{eq})].$$

Since $\mathbf{x}_i(0) = \mathbf{x}_d(0)$, $\|\mathbf{b}\| = 1$, $b_i \leq \bar{b}$, according to (24) and $|\kappa_i| \leq \bar{\kappa}$ under assumption A1, it can be obtained from the above that

$$\begin{aligned} \|\mathbf{x}_d - \mathbf{x}_i\| &\leq \|A\| \int_0^t \|\mathbf{x}_d - \mathbf{x}_i\| d\tau \\ &\quad + \bar{b} \int_0^t |u_{l,i} - u_{eq}| d\tau + \bar{b}\bar{\kappa} \int_0^t |\sigma_i| d\tau + \bar{b} \int_0^t |\gamma_i| d\tau. \end{aligned}$$

Under the assumptions A1 and A2, substituting (17) into the above yields

$$\begin{aligned} \|\mathbf{x}_d - \mathbf{x}_i\| &\leq (\|A\| + \bar{b}c) \int_0^t \|\mathbf{x}_d - \mathbf{x}_i\| d\tau \\ &\quad + \bar{b} \int_0^t |u_{l,i} - u_{eq}| d\tau + \bar{b}\bar{\kappa} \int_0^t |\sigma_i| d\tau. \end{aligned} \quad (27)$$

From (3) we have

$$|\sigma_i| \leq (\alpha_1^2 + \dots + \alpha_{n-1}^2 + 1)^{\frac{1}{2}} \|\mathbf{x}_d - \mathbf{x}_i\| \quad (28)$$

and from which it can be obtained by the *Hölder inequality* and *Bellman-Gronwall Lemma II* that

$$\begin{aligned} \|\mathbf{x}_d - \mathbf{x}_i\| &\leq l_1 \int_0^t \|\mathbf{x}_d - \mathbf{x}_i\| d\tau + \bar{b} \int_0^t |u_{l,i} - u_{eq}| d\tau \\ &\leq \int_0^t \bar{b} e^{l(t-\tau)} |u_{l,i} - u_{eq}| d\tau \leq \bar{b} e^{lT_f} \int_0^{T_f} e^{-l\tau} |u_{l,i} - u_{eq}| d\tau \\ &\leq \bar{b} e^{lT_f} \left[\int_0^{T_f} e^{-2l\tau} (u_{l,i} - u_{eq})^2 d\tau \right]^{\frac{1}{2}} \left[\int_0^{T_f} 1^2 d\tau \right]^{\frac{1}{2}} \\ &\leq \bar{b} e^{lT_f} T_f^{\frac{1}{2}} \left[\int_0^{T_f} e^{-\lambda\tau} (u_{l,i} - u_{eq})^2 d\tau \right]^{\frac{1}{2}} = \bar{b} e^{lT_f} T_f^{\frac{1}{2}} J_i^{\frac{1}{2}}(T_f). \end{aligned}$$

Hence from (28) and the above, we can obtain (26) which completes the proof.

4 LVSC with σ_i -Updating

In this section, we consider the learning law (9) with $\iota_i = \beta_l \sigma_i$, i.e.,

$$u_{l,i} = u^* \text{sat}(u_{l,i-1}, u^*) + \beta_l \sigma_i \quad (29)$$

where β_l is a positive learning gain. Note that this learning updating law, when working in an unsaturated region, is analogous to most P type iterative learning control algorithms. Here the key point is, the above learning law can work concurrently with the VSC law to achieve robust learning control.

Theorem 1. Consider the nonlinear system (1) satisfying assumptions A1 and A2 and giving a desired trajectory $\mathbf{x}_d$ defined by (2). Under the control laws (7), (8) and learning law (29), as $i \rightarrow \infty$, $\sigma_i(t)$ converges uniformly to 0, $\mathbf{x}_i(t)$ converges uniformly to $\mathbf{x}_d(t)$ and $u_{l,i}(t)$ converges to $u_{eq}(t)$, $\forall t \in [0, T_f]$ almost everywhere.

Proof. Substituting (15) into (14) under $\iota_i = \beta_i \sigma_i$ obtains the difference of $J_i(t)$ between two successive trials $\forall i \in Z_+ \cap (i \geq 2)$ as

$$\begin{aligned} \Delta J_i(t) &\triangleq J_i(t) - J_{i-1}(t) \\ &= \int_0^t e^{-\lambda\tau} (\beta_l^2 \sigma_i^2 - 2\beta_l u_{v,i} \sigma_i - 2\beta_l b_i^{-1} \sigma_i \dot{\sigma}_i - 2\beta_l \sigma_i \gamma_i) d\tau. \end{aligned}$$

From (12), it can be found that $u_{v,i} \sigma_i \geq 0$, then

$$\begin{aligned} \Delta J_i(t) &\leq \int_0^t e^{-\lambda\tau} (\beta_l^2 \sigma_i^2 - 2\beta_l b_i^{-1} \sigma_i \dot{\sigma}_i - 2\beta_l \sigma_i \gamma_i) d\tau \\ &\leq -2\beta_l \int_0^t b_i^{-1} e^{-\lambda\tau} \sigma_i \dot{\sigma}_i d\tau + \int_0^t e^{-\lambda\tau} (\beta_l^2 \sigma_i^2 + 2\beta_l |\sigma_i| \cdot |\gamma_i|) d\tau \\ &= -\beta_l \int_0^{\sigma_i^2(t)} b_i^{-1} e^{-\lambda\tau} d\sigma_i^2(\tau) + \int_0^t e^{-\lambda\tau} (\beta_l^2 \sigma_i^2 + 2\beta_l |\sigma_i| \cdot |\gamma_i|) d\tau. \end{aligned}$$

Since $\sigma_i(0) = 0$, using A1, Proposition 3 and taking the integration by parts one obtains

$$\begin{aligned} \Delta J_i(t) &\leq -\beta_l \bar{b}^{-1} \int_0^{\sigma_i^2(t)} e^{-\lambda\tau} d\sigma_i^2(\tau) + \int_0^t e^{-\lambda\tau} (\beta_l^2 \sigma_i^2 + 2\beta_l |\sigma_i| \cdot |\gamma_i|) d\tau \\ &= -\beta_l \bar{b}^{-1} e^{-\lambda t} \sigma_i^2(t) \\ &\quad - \lambda \beta_l \bar{b}^{-1} \int_0^t e^{-\lambda\tau} \sigma_i^2 d\tau + \int_0^t e^{-\lambda\tau} (\beta_l^2 \sigma_i^2 + 2\beta_l |\sigma_i| \cdot |\gamma_i|) d\tau \\ &\leq -\beta_l \bar{b}^{-1} e^{-\lambda t} \sigma_i^2(t) \\ &\quad - \beta_l \bar{b}^{-1} \left[\lambda - \left(\beta_l + 2c + 2c \|A\| T_f e^{\|A\| T_f} \right) \bar{b} \right] \int_0^t e^{-\lambda\tau} \sigma_i^2(\tau) d\tau. \end{aligned}$$

Since $(\beta_l + 2c + 2c \|A\| T_f e^{\|A\| T_f}) \bar{b}$ is a finite positive constant, there exists a sufficiently large λ such that

$$\lambda \geq \left(\beta_l + 2c + 2c \|A\| T_f e^{\|A\| T_f} \right) \bar{b} \quad (30)$$

to ensure that

$$\Delta J_i(t) \leq -\beta_l \bar{b}^{-1} e^{-\lambda t} \sigma_i^2(t). \quad (31)$$

From the above we can see that too small a learning gain β_l slows down the decrease of $J_i(t)$. However, a larger β_l needs a larger λ to ensure the convergence as shown in (30), and also a smaller $e^{-\lambda t}$. Moreover, a smaller

$e^{-\lambda t}$ underestimates the learning convergence $u_{l,i}(t) \rightarrow u_{eq}(t)$ evaluated by (13) more. Hence a moderate β_l should be used to decrease $J_i(t)$ faster, and to speed up the learning convergence.

According to (13), $J_i(t) \geq 0$, then from (31) we have $0 \leq J_i(t) \leq J_{i-1}(t) \leq \dots \leq J_1(t)$. From (31), taking the summation of $\Delta J_j(t)$ over $j = 1$ to i obtains

$$J_i(t) - J_1(t) \leq -\beta_l \bar{b}^{-1} e^{-\lambda t} \sum_{j=1}^i \sigma_j^2(t).$$

As $J_i \geq 0$, we have from the above that $\lim_{i \rightarrow \infty} \sum_{j=1}^i \sigma_j^2(t) \leq \beta_l^{-1} \bar{b} e^{\lambda t} J_1(t)$ which concludes that $\lim_{i \rightarrow \infty} \sigma_i(t) = 0$, $\forall t \in [0, T_f]$ due to that $J_1(t) \in B([0, T_f])$.

Substituting $\iota_i = \beta_l \sigma_i$ into (18) one obtains

$$u_{l,i} - u_{eq} = \beta_l \sigma_i - u_{v,i} - b_i^{-1} \dot{\sigma}_i - \gamma_i. \quad (32)$$

According to (20), $\lim_{i \rightarrow \infty} \sigma_i = 0$ leads to $\lim_{i \rightarrow \infty} \mathbf{x}_i = \mathbf{x}_d$ which brings that $\lim_{i \rightarrow \infty} u_{v,i} = 0$ and $\lim_{i \rightarrow \infty} \gamma_i = 0$ according to (8) and (17). Thus it can be obtained from (13) and (5) that

$$\begin{aligned} \lim_{i \rightarrow \infty} J_i(T_f) &= \lim_{i \rightarrow \infty} \int_0^{T_f} e^{-\lambda \tau} [u_{l,i}(\tau) - u_{eq}(\tau)]^2 d\tau \\ &= \lim_{i \rightarrow \infty} \int_0^{T_f} e^{-\lambda \tau} [\beta_l \sigma_i - u_{v,i} - b_i^{-1} \dot{\sigma}_i - \gamma_i]^2 d\tau \\ &= \lim_{i \rightarrow \infty} \int_0^{T_f} e^{-\lambda \tau} b_i^{-2} \dot{\sigma}_i^2 d\tau \\ &= \lim_{i \rightarrow \infty} \int_0^{\sigma_i(T_f)} e^{-\lambda \tau} b_i^{-2} \dot{\sigma}_i d\sigma_i(\tau). \end{aligned}$$

Since $e^{-\lambda t} \leq 1$, $b_i^{-2} \leq \underline{b}^{-2}$ according to A1 and $\dot{\sigma}_i \in B([0, T_f])$ according to Proposition 2, $\lim_{i \rightarrow \infty} \sigma_i(T_f) = 0$ concludes that

$$\lim_{i \rightarrow \infty} J_i(T_f) = 0 \quad (33)$$

and $u_{l,i}(t)$ converges to $u_{eq}(t)$ almost everywhere, $\forall t \in [0, T_f]$. Using Proposition 4 and (33), we have

$$\lim_{i \rightarrow \infty} \sup_{t \in [0, T_f]} |\sigma_i(t)| = 0, \quad \lim_{i \rightarrow \infty} \sup_{t \in [0, T_f]} \|\mathbf{x}_d(t) - \mathbf{x}_i(t)\| = 0,$$

i.e., σ_i and $\mathbf{x}_i$ are uniformly convergent which completes the proof.

Corollary 1. Under the same conditions as those in Theorem 1, the learning law (9) warrants that $\sigma_i \in UC([0, T_f])$, $\mathbf{x}_i \in UC([0, T_f])$ and $u_{v,i} \in UC([0, T_f])$, $\forall i \in Z_+$.

Proof. Since $\lim_{i \rightarrow \infty} J_i(T_f) = 0$ and $\mathbf{x}_d \in UC([0, T_f])$, then $\forall \epsilon > 0, \exists N(\epsilon) \in \mathbb{Z}_+$ and $\exists \delta(\epsilon) > 0$, such that $|J_i(T_f)| < \epsilon$ and $\|\mathbf{x}_d(t_2) - \mathbf{x}_d(t_1)\| < \epsilon$, whenever $i \geq N(\epsilon)$ and $t_1, t_2 \in [0, T_f] \cap [|t_1 - t_2| < \delta(\epsilon)]$. Hence from (25) of *Proposition 4*, $\forall \epsilon > 0, \exists N(\epsilon) \in \mathbb{Z}_+$ and $\exists \delta(\epsilon) > 0$, such that

$$\begin{aligned} \|\mathbf{x}_i(t_1) - \mathbf{x}_i(t_2)\| &= \|\mathbf{x}_i(t_1) - \mathbf{x}_d(t_1) + \mathbf{x}_d(t_1) - \mathbf{x}_d(t_2) + \mathbf{x}_d(t_2) - \mathbf{x}_i(t_2)\| \\ &\leq \|\mathbf{x}_d(t_1) - \mathbf{x}_i(t_1)\| \\ &\quad + \|\mathbf{x}_d(t_2) - \mathbf{x}_d(t_1)\| + \|\mathbf{x}_d(t_2) - \mathbf{x}_i(t_2)\| \\ &< 2\bar{b}e^{lT_f} T_f^{\frac{1}{2}} J_i^{\frac{1}{2}}(T_f) + \epsilon \end{aligned}$$

whenever $i \geq N(\epsilon)$ and $t_1, t_2 \in [0, T_f] \cap (|t_1 - t_2| < \delta)$, then $\mathbf{x}_i \in UC([0, T_f])$, $\forall i \geq N(\epsilon)$. Since $\sigma_i \in C(X \times [0, T_f])$, $u_{v,i} \in C(X \times [0, T_f])$, then $\mathbf{x}_i \in UC([0, T_f])$ ensures that $\sigma_i \in UC([0, T_f])$ and $u_{v,i} \in UC([0, T_f])$, $\forall i \geq N(\epsilon)$.

When $i < N(\epsilon)$, we prove $\sigma_i \in UC([0, T_f])$, $\mathbf{x}_i \in UC([0, T_f])$ and $u_{v,i} \in UC([0, T_f])$ by induction. Combining (4) and (5) obtains $\dot{\mathbf{x}}_{d,n} - \dot{\mathbf{x}}_{i,n} = g_d - f_i - b_i u_i$. Since $\dot{\mathbf{x}}_{d,j} - \dot{\mathbf{x}}_{i,j} = x_{d,j+1} - x_{i,j+1}$, $j = 1, \dots, n-1$, we have

$$\dot{\mathbf{x}}_d - \dot{\mathbf{x}}_i = A_1(\mathbf{x}_d - \mathbf{x}_i) + \mathbf{b}(g_d - f_i - b_i u_i) \quad (34)$$

where $A_1 \triangleq \begin{bmatrix} \mathbf{0}_{(n-1) \times 1} & I_{(n-1) \times (n-1)} \\ 0 & \mathbf{0}_{1 \times (n-1)} \end{bmatrix}$. Under the control law (7), (34) can be rearranged into the following

$$\dot{\mathbf{x}}_i = \boldsymbol{\chi}_i + \mathbf{b} b_i u^* \text{sat}(u_{l,i-1}, u^*). \quad (35)$$

where $\boldsymbol{\chi}_i = A_1 \mathbf{x}_i + \dot{\mathbf{x}}_d - A_1 \mathbf{x}_d - \mathbf{b} g_d + \mathbf{b} f_i + \mathbf{b} b_i u_{v,i}$.

According to assumption A1, $\mathbf{x}_i \in B([0, T_f])$ ensures that $f_i \in B([0, T_f])$. Hence the right hand side of (35) belongs to $B([0, T_f])$ as $\mathbf{x}_d, \dot{\mathbf{x}}_d, \mathbf{x}_i, \sigma_i, u_{v,i}$ belong to $B([0, T_f])$, $\forall i \in \mathbb{Z}_+$ according to *Proposition 2*.

Since $\mathbf{x}_d \in C^1([0, T_f])$, $\dot{\mathbf{x}}_d \in C([0, T_f])$, $g_d \in C([0, T_f])$, $u_{v,i} \in C(X \times [0, T_f])$ and $f_i \in C(X \times [0, T_f])$, $b_i \in C(X \times [0, T_f])$ according to assumption A2, then $\boldsymbol{\chi}_i \in C(X \times [0, T_f])$.

During the first iteration, since $u_{l,0} = 0$, the right hand side of (35) belongs to $B([0, T_f]) \cap C(X \times [0, T_f])$. As $\mathbf{x}_1 \in B([0, T_f])$, then according to *Continuity Theorem*, $\mathbf{x}_1 \in C([0, T_f])$, hence $\sigma_1 \in C([0, T_f])$ and $u_{l,1} = u_{v,1} \in C([0, T_f])$. Since $\text{sat}(\star, \star)$ is continuous with respect to $(\star)$, hence $u^* \text{sat}(u_{l,1}, u^*) \in C([0, T_f])$ which brings that $\mathbf{x}_2 \in C([0, T_f])$, $\sigma_2 \in C([0, T_f])$, $u_{v,2} \in C([0, T_f])$ and $u_{l,2} = u^* \text{sat}(u_{l,1}, u^*) + \beta_l \sigma_2 \in C([0, T_f])$ owing to the similar arguments.

In general, suppose $\sigma_1, \dots, \sigma_{i-1}$ belong to $C([0, T_f])$, hence $\mathbf{x}_1, \dots, \mathbf{x}_{i-1}$ and $u_{v,1}, \dots, u_{v,i-1}$ belong to $C([0, T_f])$. From (9),

$$\begin{aligned} u_{l,i-1} &= \beta_l \sigma_{i-1} + u^* \text{sat}(u_{l,i-2}, u^*) \\ &= \beta_l \sigma_{i-1} + u^* \text{sat}(\beta_l \sigma_{i-2} + u^* \text{sat}(u_{l,i-3}, u^*), u^*) \\ &= \dots \\ &= \beta_l \sigma_{i-1} + u^* \text{sat}(\beta_l \sigma_{i-2} \\ &\quad + u^* \text{sat}(\beta_l \sigma_{i-3} + \dots + u^* \text{sat}(\beta_l \sigma_1, u^*) \dots, u^*), u^*) \end{aligned} \quad (36)$$

belongs to $C([0, T_f])$. Then $u^* \text{sat}(u_{l,i-1}, u^*) \in C([0, T_f])$ and the right hand side of (35) belongs to $B([0, T_f]) \cap C(X \times [0, T_f])$. According to the *Continuity Theorem*, $\mathbf{x}_i \in B([0, T_f])$ leads to $\mathbf{x}_i \in C([0, T_f])$, hence $\sigma_i \in C([0, T_f])$ and $u_{v,i} \in C([0, T_f])$. As $[0, T_f]$ is a closed interval, according to the *Cantor Theorem*, $\sigma_i \in UC([0, T_f])$, $\mathbf{x}_i \in UC([0, T_f])$ and $u_{v,i} \in UC([0, T_f])$, $\forall i < N(\epsilon)$. Combining this conclusion with $\mathbf{x}_i(t) \in UC([0, T_f])$, $\sigma_i(t) \in UC([0, T_f])$ and $u_{v,i}(t) \in UC([0, T_f])$, $\forall i \geq N(\epsilon)$ completes the proof.

5 LVSC with $u_{v,i}$ -Updating

Generally speaking, the VSC part $u_{v,i}$ may better reflect the demand from the tracking control task in comparison with a simple feedback by $\beta_l \sigma_i$. Therefore in this section we consider learning with $u_{v,i}$ -updating, namely in the learning law (9) $\iota_i = u_{v,i}$ is used

$$u_{l,i} = u^* \text{sat}(u_{l,i-1}, u^*) + u_{v,i}. \quad (37)$$

Theorem 2. *Consider the nonlinear system (1) satisfying assumptions A1, A2 and giving a desired trajectory $\mathbf{x}_d$ defined by (2). Under the control laws (7), (8) and learning law (37), $\sigma_i \in UC([0, T_f])$, $\mathbf{x}_i \in UC([0, T_f])$, $u_{v,i} \in UC([0, T_f])$, $\forall i \in Z_+$. As $i \rightarrow \infty$, $\sigma_i(t)$ converges uniformly to 0, $\mathbf{x}_i(t)$ converges uniformly to $\mathbf{x}_d(t)$ and $u_{l,i}(t)$ converges to $u_{eq}(t)$, $\forall t \in [0, T_f]$ almost everywhere.*

Proof. Substituting (15) into (14) using $\iota_i = u_{v,i}$ and (12), the difference of $J_i(t)$ between two successive trials $\forall i \in Z_+ \cap (i \geq 2)$ is

$$\begin{aligned} \Delta J_i(t) &= \int_0^t e^{-\lambda\tau} [u_{v,i}^2 - 2u_{v,i} (u_{v,i} + b_i^{-1} \dot{\sigma}_i + \gamma_i)] d\tau \\ &= \int_0^t e^{-\lambda\tau} (-u_{v,i}^2 - 2b_i^{-1} u_{v,i} \dot{\sigma}_i - 2u_{v,i} \gamma_i) d\tau \\ &\leq -2 \int_0^t e^{-\lambda\tau} b_i^{-1} u_{v,i} \dot{\sigma}_i d\tau - 2 \int_0^t e^{-\lambda\tau} u_{v,i} \gamma_i d\tau \\ &= -2 \int_0^t e^{-\lambda\tau} b_i^{-1} (\zeta + \rho_i k_i) \sigma_i \dot{\sigma}_i d\tau - 2 \int_0^t e^{-\lambda\tau} (\zeta + \rho_i k_i) \sigma_i \gamma_i d\tau. \end{aligned}$$

Since $\sigma_i(0) = 0$, $\rho_i \leq \bar{\rho}$ and $k_i \leq 1/\epsilon$, using A1, Proposition 3 and taking the integration by parts one obtains

$$\begin{aligned} \Delta J_i(t) &\leq -\bar{b}^{-1} \zeta \int_0^{\sigma_i^2(t)} e^{-\lambda\tau} d\sigma_i^2(\tau) + 2(\zeta + \bar{\rho}/\epsilon) \int_0^t e^{-\lambda\tau} |\sigma_i| \cdot |\gamma_i| d\tau \\ &= -\bar{b}^{-1} \zeta e^{-\lambda t} \sigma_i^2(t) \\ &\quad - \lambda \bar{b}^{-1} \zeta \int_0^t e^{-\lambda\tau} \sigma_i^2 d\tau + 2(\zeta + \bar{\rho}/\epsilon) \int_0^t e^{-\lambda\tau} |\sigma_i| \cdot |\gamma_i| d\tau \end{aligned}$$

$$\begin{aligned} &\leq -\bar{b}^{-1}\zeta e^{-\lambda t}\sigma_i^2(t) \\ &\quad -\bar{b}^{-1}\zeta \left[\lambda - 2(\zeta + \bar{\rho}/\varepsilon) \left(c + c\|A\|T_f e^{\|A\|T_f} \right) \bar{b}\zeta^{-1} \right] \int_0^t e^{-\lambda\tau} \sigma_i^2 d\tau. \end{aligned}$$

Hence there exists a sufficiently large λ such that

$$\lambda \geq 2(\zeta + \bar{\rho}/\varepsilon) \left(c + c\|A\|T_f e^{\|A\|T_f} \right) \bar{b}\zeta^{-1}$$

to ensure that $\Delta J_i(t) \leq -\bar{b}^{-1}\zeta e^{-\lambda t}\sigma_i^2(t)$. Following the same procedure in the proof of *Theorem 1*, we have $\lim_{i \rightarrow \infty} \sigma_i = 0$. Substituting $\iota_i = u_{v,i}$ into (18) yields $u_{l,i} - u_{eq} = -b_i^{-1}\dot{\sigma}_i - \gamma_i$. As $\lim_{i \rightarrow \infty} \sigma_i = 0$, we have $\lim_{i \rightarrow \infty} \gamma_i = 0$ according to (16) and (20). Thus it can be obtained from (13) that

$$\begin{aligned} \lim_{i \rightarrow \infty} J_i(T_f) &= \lim_{i \rightarrow \infty} \int_0^{T_f} e^{-\lambda\tau} [u_{l,i}(\tau) - u_{eq}(\tau)]^2 d\tau \\ &= \lim_{i \rightarrow \infty} \int_0^{T_f} e^{-\lambda\tau} b_i^{-2} \dot{\sigma}_i^2 d\tau. \end{aligned}$$

Following the same procedure in the proof of *Theorem 1*, $\lim_{i \rightarrow \infty} J_i(T_f) = 0$, $u_{l,i}(t)$ converges to u_{eq} *almost everywhere*, $\sigma_i(t)$ converges uniformly to 0 and $\mathbf{x}_i(t)$ converges uniformly to $\mathbf{x}_d(t)$, $\forall t \in [0, T_f]$.

Under the learning law (37), (36) is changed to be the following form:

$$\begin{aligned} u_{l,i-1} &= u_{v,i-1} + u^* \text{sat}(u_{l,i-2}, u^*) \\ &= u_{v,i-1} + u^* \text{sat}(u_{v,i-2} + u^* \text{sat}(u_{l,i-3}, u^*), u^*) \\ &= \dots \\ &= u_{v,i-1} \\ &\quad + u^* \text{sat}(u_{v,i-2} + u^* \text{sat}(u_{v,i-3} + \dots + u^* \text{sat}(u_{v,1}, u^*) \dots, u^*), u^*). \end{aligned}$$

Following the same procedure in the proof of *Corollary 15*, we can prove that $\sigma_i \in UC([0, T_f])$, $\mathbf{x}_i \in UC([0, T_f])$ and $u_{v,i} \in UC([0, T_f])$ which ends the proof.

Remark 1. The only direct treatment for obtaining *equivalent control* appears in [1] which uses a first-order low-pass filter. It requires $\Delta/\tau \rightarrow 0$ and $\tau \rightarrow 0$ where $|\sigma| \leq \Delta$ and τ is the time constant of the filter. This shows the difficulty in achieving *equivalent control* in general: it demands infinite switching frequency to produce an infinitesimal bound Δ of σ and an infinite filter bandwidth due to the worst case control environment. These two stringent requirements can be relaxed where repeatable control tasks are concerned.

Remark 2. From practical point of view, a potential problem with $u_{v,i}$ -updating is that $u_{v,i}$ is in general less smooth than $\beta_l \sigma_i$. Due to the “integral” nature of learning updating, high frequency components in u_i may grow up. Fourier series based iterative learning can well overcome this problem.

6 Fourier Series Based Iterative Learning

In this section we further address important issues arising from practical considerations and implement learning control in the frequency domain by means of the *Fourier* series expansion. First, it should be noted that the repeatability of the control environment implies only countable integer frequencies being involved in the *equivalent control* profile, this ensures the implementability of constructing componentwise learning in the frequency domain. Second, nowadays all advanced control approaches (including LVSC) have to be implemented using microprocessing technology. According to sampling theory, LVSC needs only to learn a finite number of frequencies limited by one half of the sampling frequency $\omega_s = 2\pi/T_s$ where T_s is the sampling interval. Any attempt to learn and manipulate frequencies above that limit will be completely meaningless. Third, most real systems can be characterized as low-pass filter because their bandwidth is much lower than the sampling frequency. It is sufficient for LVSC to take into account only a small portion of the *Fourier* series in such cases.

Therefore, the spectrum of u_{eq} , σ_i , $u_{v,i}$, $u_{l,i}$ and $u^* \text{sat}(u_{l,i-1}, u^*)$ is located in a relatively low frequency band compared to ω_s . Assume that system (1) has a limited bandwidth $[0, \omega_b]$ rad/sec where $\omega_b < \omega_s/2$, u_{eq} , σ_i , $u_{v,i}$, $u_{l,i}$ and $u^* \text{sat}(u_{l,i-1}, u^*)$ can be expressed in the following truncated *Fourier* Linear Combiners form

$$u_{eq} = \psi^T \nu, \quad \sigma_i = \psi^T \eta_i, \quad u_{v,i} = \psi^T \theta_i, \quad (38)$$

$$u_{l,i} = \psi^T \nu_i, \quad u^* \text{sat}(u_{l,i-1}, u^*) = \psi^T \hat{\nu}_{i-1} \quad (39)$$

where $N \geq \omega_b/\omega$, $\omega = 2\pi/T_f$,

$$\psi = [0.5 \quad \cos \omega t \quad \cos 2\omega t \quad \cdots \quad \cos N\omega t \quad \sin \omega t \quad \sin 2\omega t \quad \cdots \quad \sin N\omega t]^T,$$

$$\psi_1 = [1 \quad \cos \omega t \quad \cos 2\omega t \quad \cdots \quad \cos N\omega t \quad \sin \omega t \quad \sin 2\omega t \quad \cdots \quad \sin N\omega t]^T,$$

and

$$\eta_i = \frac{2}{T_f} \int_0^{T_f} \sigma_i \psi_1 d\tau, \quad \theta_i = \frac{2}{T_f} \int_0^{T_f} u_{v,i} \psi_1 d\tau, \quad (40)$$

$$\hat{\nu}_{i-1} = \frac{2}{T_f} \int_0^{T_f} u^* \text{sat}(u_{l,i-1}, u^*) \psi_1 d\tau. \quad (41)$$

Substituting (39) into the learning law with σ_i -updating (29), we have $\psi^T \nu_i = \psi^T (\hat{\nu}_{i-1} + \beta_l \eta_i)$. Since the basis of the *Fourier* space is orthogonal, we have the following frequency domain learning law with σ_i -updating

$$\nu_i = \hat{\nu}_{i-1} + \beta_l \eta_i. \quad (42)$$

Similarly, the frequency domain learning law with $u_{v,i}$ -updating can be obtained from (37) and (39) as

$$\nu_i = \hat{\nu}_{i-1} + \theta_i. \quad (43)$$

Since u_{eq} is invariant over every iteration, each element of $\boldsymbol{\nu}$ is constant. The frequency domain learning laws (42) and (43) produce estimate $\boldsymbol{\nu}_i$ which converges to $\boldsymbol{\nu}$ as $i \rightarrow \infty$.

The main advantage of the *Fourier* series based learning is the enhancement of LVSC robustness and the improvement of tracking performance. Note that there always exists system noise or other small non-repeatable factors even in a repeatability dominant control environment. Accumulation of these tiny components contained in control sequence $u_{v,i}$ and tracking error sequence σ_i may degrade the approximation precision of the learning control sequence $u_{l,i}$. The *Fourier* series based learning mechanism, on the other hand, updates coefficients $\boldsymbol{\nu}_i$, $\hat{\boldsymbol{\nu}}_{i-1}$, $\boldsymbol{\eta}_i$ and $\boldsymbol{\theta}_i$ of the learned frequency components $\boldsymbol{\psi}$ and those coefficients are calculated according to (40) which takes the integration of the control sequence $u_{v,i}$ and tracking error sequence σ_i over the entire control interval $[0, T_f]$. In the sequel, the integration processes (40), (41) play the role of an averaging operation on the two noisy sequences $u_{v,i}$ and σ_i and are able to remove the majority of those high frequency components. This averaging operation is especially important to $u_{v,i}$ -updating which usually contains more high frequency components.

Remark 3. In practical implementation, both existing VSC and iterative learning schemes can only effectively control the system in the frequency band $[0, \omega_s/2]$. Based on the same reason, in this chapter, the *Fourier* series of u_{eq} , σ_i , $u_{v,i}$, $u_{l,i}$ and $\hat{\boldsymbol{\nu}}_{i-1}$ are truncated into summations from 0 to N instead of from 0 to ∞ .

7 Illustrative Example

The nonlinear dynamical system described below was chosen for simulation

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = f(x_1, x_2) + b(x_1, t)u \end{cases}$$

where $f(x_1, x_2) = x_1 \sin(x_2) + 10 \sin(10\pi t)$ and $b(x_1, t) = 2 + \cos(x_1)$. The desired trajectory is $x_{1,d} = 0.2 - 0.2 \cos(2\pi t)$. The switching surface is defined by $\sigma = (\dot{x}_{1,d} - \dot{x}_1) + (x_{1,d} - x_1)$. The initial values are $x_1(0) = 0$, $\dot{x}_1(0) = 0$ and $\sigma(0) = 0$ is satisfied. The sampling interval is $T_s = 1 \text{ msec}$. It is assumed that $f_{max} = 10 + |x_1|$ and $b_{min} = 1$ are known *a priori*. It can be seen that $f(x_1, x_2)$ and $b(x_1, t)$ are *Lipschitzian* which satisfies A2. According to (8), a typical VSC is implemented as $u_v = b_{min}^{-1} [f_{max} + |\ddot{x}_{1,d} + \dot{x}_{1,d} - \dot{x}_1| + 0.01] \text{sat}(\sigma, 0.2)$.

Fig.3 shows the steady-state tracking error is bounded by 9×10^{-3} . To improve tracking accuracy, the learning algorithms developed in Sections IV, V and VI were individually tested. The learning rate is $\beta_l = 30$ for learning with σ_i -updating and $N = 20$ is chosen for frequency domain learning. For comparison, Fig.4 and Fig.5 plot together the maximum tracking error over iterations of σ_i - and $u_{v,i}$ -updating for the learning in the time

domain and frequency domain respectively. It is shown that the developed learning algorithms effectively improve the tracking accuracy. $u_{v,i}$ -updating achieves faster convergence than σ_i -updating as $u_{v,i}$ better reflects the demand from the control tasks. Frequency domain learning further improves tracking accuracy since the integration process (40) nullifies the majority of high frequency components caused by quantization error and other non-ideal factors due to limited sampling frequency. Learning in the frequency domain with $u_{v,i}$ -updating obtains the fastest convergence. It is shown in Fig.6 that the tracking error has been reduced to below 12×10^{-7} during the 10-th iteration and Fig.7 shows the control profile is fairly smooth.

8 Conclusion

In this chapter, a new control approach - Learning Variable Structure Control (LVSC) is proposed for repeatable tracking control tasks. LVSC is constructed by simply adding an iterative learning mechanism to VSC with a smoothing function and the implementation is easy. By rigorous proof we show that the LVSC scheme makes the tracking error converge *uniformly* to zero, system states converge *uniformly* to the desired trajectories, and bounded learning control converge to the *equivalent control almost everywhere*. The LVSC scheme also retains the insensitivity property to system uncertainties. Implementation of the proposed learning mechanism by means of *Fourier* series expansion enhances the robust property of LVSC and improves tracking performance. Simulation results show the effectiveness of the proposed LVSC approaches.

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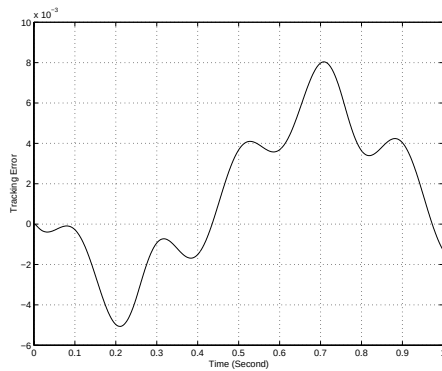


Fig. 3. $x_{1,d} - x_1$, tracking error for a typical VSC

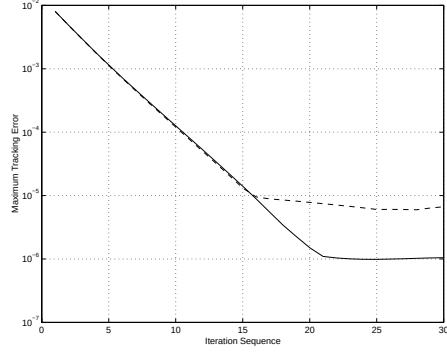


Fig. 4. Maximum tracking error over each iteration for σ_i -updating, dashed line – time domain learning; solid line – frequency domain learning.

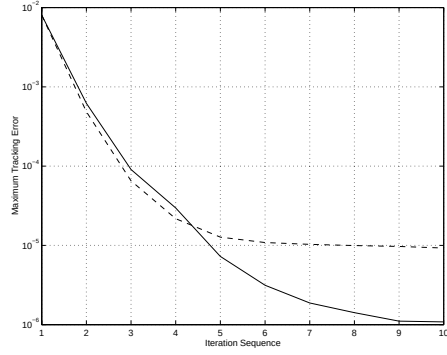


Fig. 5. Maximum tracking error over each iteration for $u_{v,i}$ -updating, dashed line – time domain learning; solid line – frequency domain learning.

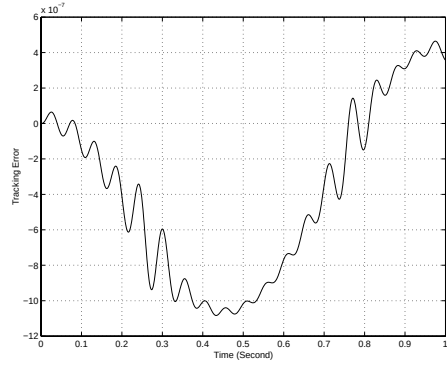


Fig. 6. $x_{1,d} - x_1$, tracking error during the 10-th iteration under frequency domain learning with $u_{v,i}$ -updating.

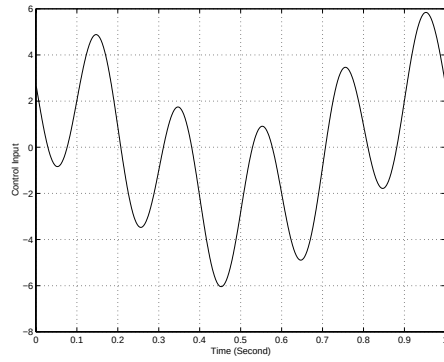


Fig. 7. Control profile during the 10-th iteration under frequency domain learning with $u_{v,i}$ -updating.

Discrete-time Variable Structure Control

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Abstract. This chapter presents some discrete-time variable structure (VS) control design algorithms with sliding sectors and then proposes a discrete-time VS controller with an invariant sliding sector. The invariant sliding sector is an invariant subset of the state space determined by a linear and a quadratic functions on the state variable. To ensure the invariance, a VS control law is implemented. Inside the sector with the VS control law, a Lyapunov function keeps decreasing. If the state is inside the subset in some time instance, then the VS control input will let the state remain inside. A discrete-time VS controller with the invariant sliding sector for discrete-time systems is designed such that the state is moved into the sector in finite steps and stays there from then on. The resultant VS control system is quadratically stable as there exists a Lyapunov function which decreases in the state space. Simulation results are given to show the efficiency of the proposed design algorithm of the discrete-time VS controller.

1 Introduction

The Variable Structure Control (VSC) system has been mainly considered for continuous-time systems in the form of sliding mode [25]. When it is implemented in practical systems by digital controllers, not only may the chattering be generated around the sliding mode because of the finite switching frequency, but the stable sliding mode designed for continuous-time systems may also become unstable after discretizing [8]. Therefore it is important and also necessary to investigate the properties of the VS controller with sliding mode after discretization. A number of research papers that appear in recent literature have been devoted to the implementation of continuous-time VS control by computer or via discretization and to the design of discrete-time VS controllers.

Milosavljevic[16] proposed a concept of **quasi-sliding mode** and gave a necessary reaching condition

$$(s_{k+1} - s_k)s_k < 0$$

to ensure the existence of the quasi-sliding mode for discrete-time VS control systems, which is similar to the reaching condition

$$\dot{s}(x)s(x) < 0$$

applied to continuous-time VS control systems. Sarpurk et al[21][15] gave a sufficient reaching condition

$$|s_{k+1}| < |s_k|$$

and pointed out that the control input must have upper and lower bounds depending on the distance of the system state from the hyperplane. An equivalent reaching condition, i.e.

$$|s_{k+1}s_k| < s_k^2$$

may be found in Sira-Ramirez's paper[22].

Furuta[4] proposed another equivalent reaching condition:

$$s_k(s_{k+1} - s_k) < -\frac{1}{2}(s_{k+1} - s_k)^2$$

and defined a **sliding sector** as

$$|s_k| \leq \frac{1}{2}|G\Gamma|f_0 \sum_{i=1}^n |x_i(k)|$$

where h is a positive constant and $x_i(k)$ ($i = 1, 2, \dots, n$) is the i -th element of the state variable x_k . The equivalent control in [4] is obtained by setting

$$s_{k+1} = s_k$$

while in the β equivalent control approach[6], the equivalent control is found with

$$s_{k+1} = \beta s_k$$

where $|\beta| < 1$. The state will move into the sliding sector because the proposed discrete-time VS control input satisfies the above reaching condition outside the sliding sector. Inside the sliding sector, the equivalent control law ensures the stability of the closed-loop system. With the consideration of parameter uncertainties, a robust discrete-time VS controller was also designed in [4] where the sliding sector was modified as

$$|s_k| \leq \frac{1}{2}|G\Gamma|(f_0 + d) \sum_{i=1}^n |x_i(k)|$$

where the positive constant d is the upper bound of the parameter uncertainty. Furuta [5] also proposed a discrete-time VS controller with sliding sector for those systems described by a transfer function, where the self-tuning control is used to ensure the robust stability.

Drakunov and Utkin[3] suggested for sampled-data systems that some sliding manifold must be reached in a finite time interval and the system representative point after reaching the sliding manifold must be confined to that region to ensure the existence of discrete-time sliding mode. Yu and

Potts[27] investigated the **pseudo-sliding mode** behavior of the discrete-time VS control system where a class of n -th order linear systems with a VS controller was discretized.

Similar to the sliding sector proposed by Furuta [4], Wang et al [26] defined a **switching region**, which is determined by

$$-(\delta + \bar{d}) \leq \frac{2s_k}{C_2 B_2 \sum_{i=1}^n |x_i(k)|} \leq (\delta + \bar{d})$$

and designed a discrete-time VS controller based on the region such that not only is the discrete-time uncertain system stabilized robustly but the chattering along the sliding mode is also reduced explicitly.

Gao et al [10] defined a **quasi-sliding mode band**

$$|s(k)| < \frac{\epsilon T}{1 - qT}$$

where T is the sampling interval and ϵ and q are positive constants, which satisfy

$$0 < 1 - qT < 1.$$

It is pointed out that the plant states oscillate between the limited band with the so-called reaching law

$$s(k+1) = (1 - qT)s(k) - \epsilon T \text{sgn}(s).$$

which is obtained by approximating the following continuous-time reach law.

$$\dot{s}(t) = -\epsilon \text{sgn}(s(t)) - qs(t),$$

Koshkouei and Zinober proposed discrete-time sliding mode controllers for signal input systems[13] and multi-input systems[14] with a **lattice-wise hyperplane**, on which there is a countable set of points comprising a so-called lattice. They separated the sequence $\{s(k)\}_{k=0}^{\infty}$ into two subsequences $\{s^+(k)\}_{k=0}^{\infty}$ and $\{s^-(k)\}_{k=0}^{\infty}$, which are the positive and negative subsequences, respectively. Then sufficient conditions for the existence of the sliding mode were given by

$$\begin{aligned} s^+(k+1) &< s^+(k) \\ s^-(k+1) &> s^-(k) \end{aligned}$$

Misawa [18] suggested to use a **boundary layer** and proposed a discrete-time sliding mode control, which ensures the attractiveness and invariance of the boundary layer. Some discrete-time VS controllers were designed with observer[17][24] or adaptive control algorithms[1] or uncertain estimation[12]. And some others were designed for sampled-data systems by discretizing continuous-time VS controllers, for example, [2][23][11]. It is pointed out that the use of a discontinuous control law in a sampled-data system will result

in chattering phenomenon in a boundary layer with thickness $O(T^2)$ where T is the sampling interval [23] and the robustness of the VS controller can be improved by decreasing the sampling interval T [11].

As an alternate design algorithm of the Variable Structure Control (VSC), a ***PR-sliding sector*** has been proposed in the design of VSC for a chattering free controller and for the implementation in discrete-time control systems [9]. The *PR-sliding sector* in [9] is defined as

$$|s_k| < \delta_k = \sqrt{x_k^T Q x_k},$$

which is a subset on the state space where Q is a positive semi-definite matrix. Inside the sector, P -norm of state defined as

$$\|x_k\|_P = \sqrt{x_k^T P x_k}$$

decreases with zero control and satisfies

$$\|x_{k+1}\|_P^2 - \|x_k\|_P^2 < -\|x_k\|_R^2$$

where P and R are positive definite matrices, which implies why it is called the *PR-sliding sector*. The corresponding VS controller was designed such that the state moves from the outside to the inside of the sector with some VS control law, the control input is zero inside the sector, and some Lyapunov function keeps decreasing in the state space with specified negativity of its derivative.

The proposed continuous-time *PR-sliding sector* theoretically is an invariant subset. But inside the *PR-sliding sector*, there exists a potential possibility for the state to move out of the sector with zero control input. Therefore the invariance may be lost if the switching frequency is finite although it does not affect the stability of the proposed continuous-time VSC system. The discrete-time *PR-sliding sector* is not an invariant subset as the switching frequency is finite in a discrete-time VS control system.

The **invariant sliding sectors** for continuous-time [19] and discrete-time [20] systems have been proposed. The invariant sliding sector is a *PR-sliding sector* and an invariant subset of the state space. Inside the invariant sliding sector, some norm of state decreases because it is a *PR-sliding sector*. And especially inside the sector, the state will not move out of the subset with some control law. The proposed VS controllers with the invariant *PR-sliding sector* ensures that the state moves from the outside to the inside of the sector in a finite time, stays inside it forever after being moved into it, and some Lyapunov function keeps decreasing in the state space with some VS control law. The resulted VS control system thus is quadratically stable and without chattering. Such invariant sliding sector for continuous-time systems remains invariant even if the switching frequency is finite. Therefore it is possible to be implemented in digital systems.

In [28], Yu and Yu also discussed the existence of an invariant sliding sector for a second order discrete-time system and conditions to guarantee the existence were derived.

In this chapter, definitions and design algorithms of the discrete-time sliding sector and PR -sliding sector will be presented at first. Then the discrete-time VS control algorithm using transfer function will be discussed. Finally an invariant discrete-time sliding sector will be defined. A VS controller with the proposed invariant sliding sector for discrete-time systems will be proposed.

The organization of the paper is as follows. Section 2 presents the discrete-time VS control with sliding sector[4]; Section 3 presents the discrete-time VS control using transfer function[7]; Section 4 presents the discrete-time VS control with PR -sliding sector[9]; Section 5 defines the invariant discrete-time sliding sector, gives the design algorithm and proposes the discrete-time VS controller with the invariant sliding sector; and Section 6 gives the simulation result.

2 VS Control with Sliding Sector

A single-input discrete-time plant described by the following state equation is considered in this chapter except Section 3 where transfer function is used.

$$x_{k+1} = \Phi x_k + \Gamma u_k \quad (1)$$

where $x_k \in R^n$ and $u_k \in R^1$ are state and input vectors, respectively; Φ and Γ are constant matrices of appropriate dimensions; and the pair (Φ, Γ) is controllable.

Similar to the continuous-time sliding mode control, a function s_k is defined as

$$s_k = Gx_k. \quad (2)$$

where G is designed so that the state staying on $s_k = 0$ for all k is stable as shown in the following lemma and theorem.

Lemma 1. [4] *The equivalent control law for the system (1) such that the state rests on the hyperplane of $s_k = s_{k+1}$ for all k is given by*

$$u_k = F_{eq}x_k \quad (3)$$

where

$$F_{eq} = -(G\Gamma)^{-1}G(\Phi - I). \quad (4)$$

Theorem 1. [4] *The hyperplane, equivalently G , should be determined so that the system*

$$\begin{aligned} x_{k+1} &= [\Phi - \Gamma(G\Gamma)^{-1}G(\Phi - I)]x_k, \\ Gx_k &= 0, \end{aligned}$$

is stable.

As the second step, the control law to transfer the state on the hyperplane should be determined. Let V_k and Δs_k be respectively defined as

$$V_k = \frac{1}{2}s_k^2$$

$$\Delta s_{k+1} = s_{k+1} - s_k$$

Then the control law to decrease V_k is given by the following lemma.

Lemma 2. [4] *If the control satisfies*

$$s_k \Delta s_{k+1} < -\frac{1}{2}(\Delta s_{k+1})^2, \quad \text{for } s_k \neq 0 \quad (5)$$

then

$$V_{k+1} < V_k$$

In this section, the following type of control law is considered:

$$u_k = (F_{eq} + F_D)x_k, \quad (6)$$

where F_{eq} is given by (4) and F_D is a discontinuous control law. Then the variable structure control law to make the system stable is given by the following theorem.

Theorem 2. [4] *If the sliding mode satisfies the conditions of Theorem 1, and the control is chosen by (6), where the absolute value of the i -th element of F_D , f_i is constant for all time and the same for all i , i.e.,*

$$|f_i^+| = |f_i^-| = f_0, \quad i = 1, 2, \dots, n,$$

then the control law

$$f_i = \begin{cases} f_0 & \text{for } (GF)s_k x_{ki} < -\delta_i, \\ 0 & \text{for } -\delta_i \leq (GF)s_k x_{ki} \leq \delta_i, \\ -f_0 & \text{for } (GF)s_k x_{ki} > \delta_i, \end{cases} \quad (7)$$

makes the system stable, where x_{ki} is the i -th element of x_k and δ_i is defined as

$$\delta_i = \frac{1}{2}(GF)^2 \sum_{j=1}^n |x_{ki}| |x_{kj}| f_0$$

with the amplitude of f_0 limited by

$$0 < f_0 < \left| \frac{2}{GF \sum_{j=1}^n |t_{1j}|} \right|,$$

t_{1i} being the i -th element of t_1 satisfying $Gt_1 = 1$, $Gt_i = 0$, $t_i \perp t_j$ ($i \neq j$).

With the consideration of the parameter uncertainty, represent the actual system matrix Φ as

$$\Phi = \Phi_0 + \Delta\Phi$$

where Φ_0 is known and $\Delta\Phi$ is the uncertainty of the system matrix, which is represented by

$$\Delta\Phi = \Gamma D, \quad (8)$$

$$D = [d_1 \ d_2 \ \cdots \ d_n], \quad |d_i| < \bar{d} (i = 1, 2, \dots, n). \quad (9)$$

The known system is represented by

$$x_{k+1} = \Phi_0 x_k + \Gamma u_k. \quad (10)$$

It is assumed that both (1) and (10) are stabilized on $\{s_k\} = 0$. Then a robust discrete-time VS controller is designed by the following theorem.

Theorem 3. [4] *If the plant system matrix Φ has the uncertainty $\Delta\Phi$ satisfying (8) and (9), and on $s_k = 0$, the plant is stable with the equivalent control, then the i -th element f_i of the control law F_D satisfying*

$$f_i = \begin{cases} f_0 & \text{for } (G\Gamma)s_k x_{ki} < -\delta'_i, \\ 0 & \text{for } -\delta'_i \leq (G\Gamma)s_k x_{ki} \leq \delta'_i, \\ -f_0 & \text{for } (G\Gamma)s_k x_{ki} > \delta'_i, \end{cases} \quad (11)$$

makes the system stable, where

$$\delta'_i = \frac{1}{2}(G\Gamma)^2 \sum_{j=1}^n |x_{ki}| |x_{kj}| (f_0 + \bar{d})$$

with the amplitude of f_0 limited by

$$\bar{d} < f_0 < \left| \frac{2}{G\Gamma \sum_{j=1}^n |t_{1j}|} \right| - \bar{d}.$$

3 VSC for Discrete-time Input-Output System

This section considers a single input and single output system. The following discrete relation represents the controlled plant with input u_k , disturbance w_k and output y_k ,

$$A(q^{-1})y_k = q^{-d}B(q^{-1})u_k + D(q^{-1})w_k, \quad (12)$$

where $A(q^{-1})$ and $B(q^{-1})$ have no common factors, q denotes the time shift operator defined by

$$q^{-t}y_k = y_{k-t}$$

and q^{-d} is the pure time delay of the system, $d(d \geq 1)$ is an integer. $A(q^{-1})$ and $B(q^{-1})$ are assumed known and representing

$$\begin{aligned} A(q^{-1}) &= 1 + a_1 q^{-1} + a_2 q^{-2} + \cdots + a_n q^{-n}, \\ B(q^{-1}) &= b_0 + b_1 q^{-1} + b_2 q^{-2} + \cdots + b_m q^{-m} \quad (b_0 \neq 0), \end{aligned}$$

The objective of the control is that the output y_k tracks the reference r_k in the presence of the disturbance. The polynomial models of the reference and the disturbance are assumed to be

$$\Psi_r(q^{-1})r_k = 0, \quad \Psi_w(q^{-1})w_k = 0.$$

Let $\Psi(q^{-1})$ be the least common multiple of $\Psi_r(q^{-1})$ and $\Psi_w(q^{-1})$ then

$$\Psi(q^{-1})r_k = 0, \quad \Psi(q^{-1})w_k = 0,$$

where $\Psi(q^{-1})$ is coprime with $A(q^{-1})$, $B(q^{-1})$. Subtract $A(q^{-1})r_k$ from (12) and multiplying $\Psi(q^{-1})$ to both sides, the following relation can be obtained

$$A(q^{-1})\Psi(q^{-1})e_k = q^{-d}B(q^{-1})\Psi(q^{-1})u_k, \quad (13)$$

where the error will be defined as

$$e_k = y_k - r_k.$$

3.1 Servo Control of Generalized Minimum Variance Control

The objective of the control in this subsection is to minimize the generalized variance of the controlled variables s_{k+d} , that is, in the deterministic case, to give the control input satisfying

$$s_{k+d} = C(q^{-1})e_{k+d} + Q(q^{-1})\Psi(q^{-1})u_k = 0, \quad (14)$$

where real polynomials $C(q^{-1})$ and $Q(q^{-1})$

$$\begin{aligned} C(q^{-1}) &= 1 + c_1 q^{-1} + c_2 q^{-2} + \cdots + c_n q^{-n}, \\ Q(q^{-1}) &= Q_0(1 + q_1 q^{-1} + q_2 q^{-2} + \cdots + q_{m-1} q^{-m+1}), \end{aligned}$$

are determined so that the error vanishes if the above (14) is satisfied. At first, the generalized minimum variance control without using VSS will be discussed. The equation (14) is rewritten as

$$\begin{aligned} s_{k+d} &= (E(q^{-1})B(q^{-1}) + Q(q^{-1}))\Psi(q^{-1})u_k + F(q^{-1})e_k \\ &= G(q^{-1})\Psi(q^{-1})u_k + F(q^{-1})e_k, \end{aligned} \quad (15)$$

where $E(q^{-1})$ and $F(q^{-1})$ are polynomials determined to satisfy

$$C(q^{-1}) = A(q^{-1})\Psi(q^{-1})E(q^{-1}) + q^{-d}F(q^{-1}) \quad (16)$$

and $G(q^{-1})$ is defined as

$$G(q^{-1}) = \{E(q^{-1})B(q^{-1}) + Q(q^{-1})\}. \quad (17)$$

The control input to make $s_{k+d} = 0$ is, therefore, given by

$$u_k = -[G(q^{-1})\Psi(q^{-1})]^{-1}[F(q^{-1})e_k]. \quad (18)$$

This subsection considers the use of variable structure control in addition to the above conventional generalized minimum variance control so that the output satisfies $s_k = 0$. $C(q^{-1})$ and $Q(q^{-1})$ should be chosen to satisfy the following lemma.

Lemma 3. [5] *The necessary and sufficient condition that the output with zero reference making $s_{k+d} = 0$ stable is that all zeros of*

$$A(q^{-1})Q(q^{-1})\Psi(q^{-1}) + B(q^{-1})C(q^{-1}) = 0 \quad (19)$$

are inside the unit disk.

Instead of (18), the following input is considered to be used,

$$u_k = -[G(q^{-1})\Psi(q^{-1})]^{-1}[F(q^{-1})e_k - \beta s_k - v_k], \quad (20)$$

where

$$0 < \beta \leq 1.$$

Substituting (20) into (12) yields

$$s_{k+d} = v_k + \beta s_k. \quad (21)$$

The auxiliary control input v_k is chosen as the state feedback with the variable coefficients.

$$v_k = h_0 e_k + h_1 e_{k-1} + \cdots + h_{n-1} e_{k-n+1}. \quad (22)$$

The control input with $v_k = 0$ is called the β equivalent control where $s_{k+d} = \beta s_k$. The control law given as follows gives a stable system.

Theorem 4. [5] *For the plant (12), if the coefficients of the feedback control law are chosen*

$$h_i = \begin{cases} h & s_k e_{k-i} < -\delta_i \\ 0 & |s_k e_{k-i}| \leq \delta_i \\ -h & s_k e_{k-i} > \delta_i \end{cases} \quad (i = 0, 1, \dots, n-1), \quad (23)$$

then the control system becomes stable, where

$$\delta_i = \eta \sum_{j=0}^{n-1} |e_{k-i}| |e_{k-j}| h \quad (24)$$

and

$$\eta \geq \frac{\alpha}{2(\alpha\beta - \alpha + 1)} \quad (25)$$

and it is assumed that $\alpha \geq 1$.

3.2 Self-Tuning Servo Control based on VSS

Now it is assumed that the plant (12) has parameter uncertainty. The given plant is considered to have the known delay d in the plant. Parameters $\{a_i, b_i\}$ of $A(q^{-1})$ and $B(q^{-1})$ are, however, assumed unknown except b_0 ($\neq 0$). In this subsection, the control algorithm is determined based on generalized minimum variance control. Different control laws and parameter identification methods are employed inside and outside the sector. The sector is defined as

$$\mathcal{S}_k = \{s_k \mid |s_k| \leq \left(\frac{(1-\beta)p + \sqrt{(1-\beta)^2 p^2 + 2p^2 \gamma}}{\gamma} \right) (|\phi_k|)\}, \quad (26)$$

where

$$\gamma = \frac{2}{\alpha}(1-\beta) - (1-\beta)^2,$$

$$|\phi_k| = \sum_{j=0}^{n-1} |e_{k-j}| + \sum_{j=1}^{m+d-1} |\Psi(q^{-1})u_{k-j}|$$

and p will be defined later in (34). In the outside of the sector, control and parameter identification are done simultaneously based on the Lyapunov function. The polynomial s_k is defined by (14). When parameters $A(q^{-1})$, $B(q^{-1})$ are known, s_k is given by (15). When $A(q^{-1})$ and $B(q^{-1})$ are unknown, $G(q^{-1})$ and $F(q^{-1})$ cannot be obtained exactly. In this case, the control input u_k is determined by using the estimate of $G(q^{-1})$ and $F(q^{-1})$, denoted by $\hat{G}(q^{-1})$ and $\hat{F}(q^{-1})$, as follows. For the outside of the sector,

$$u_k = -\{\hat{G}_k(q^{-1})\Psi(q^{-1})\}^{-1} \left[\hat{F}_k(q^{-1})e_k - \beta s_k - \sum_{j=0}^{n-1} h_j e_{k-j} - \sum_{j=1}^{m+d-1} w_j \Psi(q^{-1})u_{k-j} \right], \quad (27)$$

where $\{h_j\}$, and $\{w_j\}$ are nonlinear bang-bang type functions depending on the state outside the sector and take values out of h , 0 , $-h$. Since b_0 is assumed known, g_0 can be given. Let the estimate of $G(q^{-1})$ and $F(q^{-1})$ be

$$\hat{\theta}_k = \hat{\theta}_{k-d} + \Gamma^{-1} \phi_k s_k \quad (\Gamma > 0), \quad (28)$$

θ , ϕ_k are defined as

$$\theta = [f_0, f_1, \dots, f_{n-1}, g_1, \dots, g_{m+d-1}]^T$$

$$\phi_k = [e_k, e_{k-1}, \dots, \Psi(q^{-1})u_{k-1}, \dots, \Psi(q^{-1})u_{k-m-d+1}]^T$$

For the inside of the sector;

$$u_k = -\{\hat{G}_k(q^{-1})\Psi(q^{-1})\}^{-1} \left[\hat{F}_k(q^{-1})e_k \right]. \quad (29)$$

The following main theorem establishes the stability of the closed loop system.

Theorem 5. [5] *The control system, which is employed for the outside of the sector $\mathcal{S}_k$ where the control and the simultaneous parameter estimation are determined by (27) and (28) respectively, brings either the system into the sector or the error to zero. The coefficients of (27), $\{h_i\}$ and $\{w_i\}$ are given by the following relations:*

$$h_i = \begin{cases} h & s_k e_{k-i} < -\delta_i \\ 0 & |s_k e_{k-i}| \leq \delta_i \\ -h & s_k e_{k-i} > \delta_i \end{cases} \quad (30)$$

$(i = 0, 1, \dots, n-1)$

$$w_i = \begin{cases} h & s_k \Psi(q^{-1})u_{k-i} < -\sigma_i \\ 0 & |s_k \Psi(q^{-1})u_{k-i}| \leq \sigma_i \\ -h & s_k \Psi(q^{-1})u_{k-i} > \sigma_i \end{cases} \quad (31)$$

$(i = 1, 2, \dots, m+d-1)$

where

$$\delta_i = \eta_a |e_{k-i}| \left(\sum_{j=0}^{n-1} |e_{k-j}| + \sum_{j=1}^{m+d-1} |\Psi(q^{-1})u_{k-j}| + \sum_{j=0}^{n-1} |r_{k-j}| \right) h$$

$(i = 0, 1, \dots, n-1),$ (32)

$$\sigma_i = \eta_a |\Psi(q^{-1})u_{k-i}| \left(\sum_{j=0}^{n-1} |e_{k-j}| + \sum_{j=1}^{m+d-1} |\Psi(q^{-1})u_{k-j}| + \sum_{j=0}^{n-1} |r_{k-j}| \right) h$$

$(i = 0, 1, \dots, m+d-1),$ (33)

where

$$\eta_a \geq \frac{\alpha}{1 - \alpha + \alpha\beta},$$

p is the upper bound of the uncertainty of the parameters defined as

$$\max_i |\theta_i - \hat{\theta}_{ki}| < p, \quad (34)$$

where $\hat{\theta}_{ki}$ denotes the i -th element of $\hat{\theta}_k$ and $\alpha > 1$. The following Lyapunov function

$$V_k = \frac{1}{2} s_k^2 + \frac{1}{2} \tilde{\theta}_{k-d}^T \Gamma \tilde{\theta}_{k-d} \quad (35)$$

decreases outside the sector. By using the control (29), inside the sector, the stability of the closed loop system is assured.

4 VS Control with PR - Sliding Sector

4.1 PR - Sliding Sector

Consider the discrete-time system described by the state equation (1).

Definition 1. [9] The P -Norm $\|\cdot\|_P$ of the discrete-time system state is defined as

$$\|x_k\|_P = (x_k^T P x_k)^{\frac{1}{2}}, \quad x_k \in R^n \quad (36)$$

where P is a $n \times n$ positive definite symmetric matrix.

The square of the P -norm is denoted as

$$L_k = \|x_k\|_P^2 = x_k^T P x_k > 0, \quad \forall x_k \in R^n, x_k \neq 0 \quad (37)$$

where the positive definite matrix P will be chosen later by Theorem 1.

If the autonomous system given by equation (1) is quadratically stable, then there exists a positive definite symmetric matrix P and a positive semi-definite symmetric matrix $R = C^T C$ such that

$$\Delta L_k = L_{k+1} - L_k = x_k^T (\Phi^T P \Phi - P) x_k \leq -x_k^T R x_k, \quad \forall x_k \in R^n$$

where $P \in R^{n \times n}$, $R \in R^{n \times n}$, $C \in R^{l \times n}$, $l \geq 1$, and (C, Φ) is an observable pair. For an unstable system, this inequality does not hold. It is possible to decompose the state space into two parts such that one part satisfies the condition

$$\Delta L_k > -x_k^T R x_k$$

for some element $x_k \in R^n$, and the other part satisfies the condition

$$\Delta L_k \leq -x_k^T R x_k$$

for some other element $x_k \in R^n$. The latter elements form a special subset in which the P -norm $\|x_k\|_P$ decreases with zero input. Accordingly, this special subset is defined as a PR -sliding sector.

Definition 2. [9] A *Discrete-time PR -Sliding Sector* is defined on the state space R^n as

$$\mathcal{S} = \{x_k \mid |s_k| \leq \delta_k, x_k \in R^n\} \quad (38)$$

inside which the P -norm decreases with zero input and the difference of the candidate Lyapunov function L_k (37) satisfies the condition:

$$\Delta L_k = L_{k+1} - L_k = s_k^2 - \delta_k^2 - x_k^T R x_k \leq -x_k^T R x_k, \quad \forall x_k \in \mathcal{S}.$$

where P is a $n \times n$ positive definite symmetric matrix, R is a $n \times n$ positive semi-definite symmetric matrix, $R = C^T C$, $C \in R^{l \times n}$, $l \geq 1$, and (C, Φ) is an observable pair. The linear functional s_k and the square root δ_k of the quadratic function δ_k^2 are respectively determined in the form as follows:

$$s_k = S x_k, S \in R^{1 \times n}, \quad (39)$$

$$\delta_k = \sqrt{x_k^T D x_k}, D(\geq 0) \in R^{n \times n} (D \neq 0). \quad (40)$$

Such sliding sector is a subset around a hyperplane $s_k = 0$ and between two surfaces determined by $|s_k| = \pm\delta_k$. The way to choose S and D using Riccati Equation

$$P = Q + \Phi^T P \Phi - \Phi^T P \Gamma (1 + \Gamma^T P \Gamma)^{-1} \Gamma^T P \Phi \quad (41)$$

will be presented as follows:

Theorem 6. [9] *If the positive definite symmetric solution P of the discrete-time Riccati equation (41) is used to define the P -norm and the positive semi-definite symmetric matrix $R \in R^{n \times n}$ is chosen so that $D = Q - R \geq 0$ and $D \neq 0$, then the parameters of the PR -sliding sector (38) are determined by*

$$\begin{aligned} s_k &= Sx_k, \quad S = \Gamma^T P \Phi / \sqrt{1 + \Gamma^T P \Gamma} \\ \delta_k &= \sqrt{x_k^T D x_k}, \quad D = Q - R \end{aligned}$$

where Q is the positive definite symmetric matrix in the discrete-time Riccati equation (41).

Remark 1. For simplicity, the parameter matrices Q , P , R , and D of the PR -sliding sector (38) may be chosen using the following steps:

1. Choose a $n \times n$ positive definite symmetric matrix as Q ,
2. Solve the discrete-time Riccati equation (41) for the positive definite symmetric solution P ,
3. Choose a positive constant r ($0 < r < 1$) and let $D = rQ$ and $R = (1 - r)Q$.

4.2 Lazy Control with PR - Sliding Sector

Based on the PR -sliding sector (38), a discrete-time VS controller can be designed by the following theorem.

Theorem 7. [9] *For any controllable discrete-time plant described by (1), the discrete-time VS control law*

$$u_k = \begin{cases} 0 & x_k \in \mathcal{S}, \\ -b^{-1}(Fx_k + K \operatorname{sgn}(bs_k)\delta_k) & x_k \notin \mathcal{S} \end{cases} \quad (42)$$

enables the system to be quadratically stable if b is invertible, and

$$\begin{aligned} 0 < K &\leq \min\left\{1, \frac{\sqrt{1+h}}{h}|b|\right\} \\ K^2 R &> F^T F \end{aligned}$$

where the PR -sliding sector $\mathcal{S}$ is defined in (38) by using the discrete-time Riccati equation with $D = rQ$ and $R = (1 - r)Q$ for some positive constant r ($0 < r < 1$), and

$$b = S\Gamma, \quad F = S\Phi, \quad h = \Gamma^T P \Gamma.$$

With the VS control law (42), it can be ensured that

1. the system state is moved from the outside to the inside of the PR -sliding sector, and
2. the following Lyapunov function

$$V_k = x_k^T P x_k + s_k^2 \quad (43)$$

keeps decreasing in the state space, where $P \in R^{n \times n}$ is the solution of the discrete-time Riccati equation (41), and s_k is the linear functional used in the definition of the PR - sliding sector $\mathcal{S}$ (38).

Therefore the resultant discrete-time VS control system with the PR -sliding sector is quadratically stable. And especially as the control input inside the PR -sliding sector is zero, i.e., no control effect is needed inside the sector, we named this kind of discrete-time VS control algorithm **Lazy Control**.

5 VS Control with Invariant $\tilde{P}\tilde{R}$ - Sliding Sector

5.1 Invariant $\tilde{P}\tilde{R}$ - Sliding Sector

The PR -sliding sector presented in the last section is a subset of the state space R^n around a hyperplane $s_k = 0$ and is bounded by two surfaces $s_k = \pm\delta_k$. Inside it P -norm decreases with zero control input to ensure the stability of the VS control system. Therefore, the followings hold inside the PR -sliding sector with zero control input:

$$\begin{aligned} |s_k| &\leq \delta_k \\ \Delta L_k &\leq -x_k^T R x_k \end{aligned}$$

which determine the form of the sector and the convergence inside the sector, respectively.

An invariant $\tilde{P}\tilde{R}$ -sliding sector is defined to be a PR -sliding sector at first and then to be invariant. To ensure the invariance, some control law inside the sector is necessary. Thus the invariant $\tilde{P}\tilde{R}$ -sliding sector

1. has the same form as the PR -sliding sector,
2. is a convergent subset, i.e. inside it the $\tilde{P}$ -norm of the state decreases with some VS control law, and
3. is an invariant subset of the state space, i.e., if the state is inside or moves into the sector in some time instant, then the state will stay inside the sector since then.

Definition 3. An Invariant $\tilde{P}\tilde{R}$ -Sliding Sector $\tilde{\mathcal{S}}$ for the discrete-time system (1) with some control input is defined as

$$\tilde{\mathcal{S}} = \{x_k | s_k^2 \leq \delta_k^2, \Delta s_k^2 \leq \Delta \delta_k^2, \Delta \tilde{L}_k \leq -x_k^T \tilde{R} x_k \quad x_k \in R^n\} \quad (44)$$

where the discrete-time Lyapunov function candidate $\tilde{L}_k$ is defined as

$$\tilde{L}_k = x_k^T \tilde{P} x_k > 0, \quad \forall x_k \in R^n \text{ and } x_k \neq 0, \quad (45)$$

s_k is a linear function on x_k and δ_k is the square root of a quadratic function on x_k as:

$$\begin{aligned} s_k &= S x_k, \quad S \in R^{1 \times n} \\ \delta_k &= \sqrt{x_k^T \tilde{D} x_k}, \end{aligned}$$

$\tilde{P}$, $\tilde{R}$, and $\tilde{D}$ are some $n \times n$ positive definite symmetric matrices.

Inside the discrete-time $\tilde{P}\tilde{R}$ -invariant sliding sector defined in (44), the following holds:

$$s_k^2 \leq \delta_k^2, \quad \forall x_k \in \tilde{\mathcal{S}} \quad (46)$$

$$s_{k+1}^2 \leq \delta_{k+1}^2, \quad \forall x_k \in \tilde{\mathcal{S}} \quad (47)$$

$$\Delta \tilde{L}_k \leq -x_k^T \tilde{R} x_k, \quad \forall x_k \in \tilde{\mathcal{S}} \quad (48)$$

where the first inequality determines the form of the sector, the second one ensures the invariance of the sector, and the third one guarantees the stability of the sector, i.e. the state inside the sector will converge to the origin.

As the forms of the PR -sliding sector and the invariant $\tilde{P}\tilde{R}$ -sliding sector are the same, it is possible to design the invariant $\tilde{P}\tilde{R}$ -sliding sector in the following steps:

1. Design a PR -sliding sector using the discrete-time Riccati equation as presented in the last section and let the hyperplane $s_k = 0$ of the invariant $\tilde{P}\tilde{R}$ -sliding sector be the one of the PR -sliding sector;
2. Design a VS control law such that both the absolute value of the generalized function s_k and the $\tilde{P}$ -norm decrease for some positive definite matrices $\tilde{P}$ and $\tilde{R}$.
3. Modify the sector such that the conditions in (46), (47), and (48) are satisfied. Thus as the result, an invariant $\tilde{P}\tilde{R}$ -sliding sector is designed.

Based on the PR -sliding sector, to ensure the decreasing of the function $|s_k|$, the following VS control law is often used:

$$u_k = -(S\Gamma)^{-1}(S\Phi x_k - \beta s_k) \quad (49)$$

with which s_k^2 decreases as

$$s_{k+1}^2 = \beta^2 s_k^2 < s_k^2$$

and the closed-loop system of the plant (1) with the VS control law (49) is given by:

$$x_{k+1} = \tilde{\Phi} x_k \quad (50)$$

where

$$\tilde{\Phi} = \Phi - (S\Gamma)^{-1}\Gamma S(\Phi - \beta I_n)$$

β ($|\beta| < 1$) is a constant and I_n is the $n \times n$ identity matrix. It is obvious that one of the eigenvalues of the above closed-loop system is equal to the constant β and others are equal to the eigenvalues of the plant (1) in the hyperplane $s_k = 0$, i.e.

$$\begin{cases} x_{k+1} = \Phi x_k + \Gamma u_k \\ s_k = Sx_k = 0 \end{cases} \quad (51)$$

Therefore if the eigenvalues of the plant (1) in the hyperplane $s_k = 0$ are all inside the unity circle, then the eigenvalues of the closed-loop system (50) are all inside the unity circle. Thus it is possible to find two $n \times n$ positive definite matrices $\tilde{P}$ and $\tilde{R}$ such that

$$\tilde{\Phi}^T \tilde{P} \tilde{\Phi} - \tilde{P} = -\tilde{R}, \quad (52)$$

i.e., with the VS control law (49) the $\tilde{P}$ -norm or the discrete-time Lyapunov function decreases as

$$\Delta \tilde{L}_k = x_k^T \tilde{R} x_k \leq 0.$$

Moreover if the constant β is chosen to satisfy:

$$|\beta| \leq |\lambda_i(\tilde{\Phi})|, \quad (i = 1, 2, \dots, n)$$

where $\lambda_i(\tilde{\Phi})$ ($i = 1, 2, \dots, n$) are the eigenvalues of $\tilde{\Phi}$, then the following inequality can be easily shown to be true for some positive definite matrices $\tilde{P}$ and $\tilde{R}$ satisfying (52):

$$(1 - \beta^2)\tilde{P} - \tilde{R} \geq 0 \quad (53)$$

i.e.

$$\tilde{\Phi} \tilde{P} \tilde{\Phi} - \beta^2 \tilde{P} \geq 0$$

According to the definitions of the PR - and the invariant $\tilde{P}\tilde{R}$ -sliding sectors, the hyperplanes inside the sectors are the same but the boundaries of the sectors, i.e. the parameter matrices D and $\tilde{D}$ of the quadratic functions δ_k^2 's are different. Let the invariant $\tilde{P}\tilde{R}$ -sliding sector be a subset of the PR -sliding sector by defining the quadratic function δ_k^2 as

$$\tilde{D} = \gamma \tilde{P} \leq D \quad (54)$$

where D is the parameter used to determine the PR -sliding sector (38) and γ is a positive constant which enables the above inequality hold. Then inside the invariant sector (44) the condition in (47) holds as:

$$\begin{aligned} s_{k+1}^2 - \delta_{k+1}^2 &= (s_{k+1}^2 - \delta_k^2) - \Delta \delta_k^2 \\ &= \beta^2 s_k^2 - \delta_k^2 + \gamma x_k^T \tilde{R} x_k \\ &= \beta^2 (s_k^2 - \delta_k^2) - \gamma x_k^T ((1 - \beta^2)\tilde{P} - \tilde{R}) x_k \\ &\leq \beta^2 (s_k^2 - \delta_k^2) \leq 0 \end{aligned} \quad (55)$$

The above discussion results in the following theorem.

Theorem 8. *A subset of the discrete-time PR-sliding sector in (38) designed by the discrete-time Riccati equation is a discrete-time PR-invariant sliding sector with a VS control law:*

$$u_k = -(S\Gamma)^{-1}(S\Phi x_k - \beta s_k) \quad (56)$$

if all of eigenvalues of the following matrix

$$\tilde{\Phi} = \Phi - (S\Gamma)^{-1}\Gamma S(\Phi - \beta I_n)$$

are inside the unit circle and the parameters of the sector are chosen as

$$\tilde{D} = \gamma \tilde{P} < D$$

where γ is a positive constant, β ($|\beta| < 1$) is a constant satisfying

$$(1 - \beta^2)\tilde{P} - \tilde{R} \geq 0$$

and $\tilde{P}$ and $\tilde{R}$ are positive definite matrices satisfying

$$\tilde{\Phi}^T \tilde{P} \tilde{\Phi} - \tilde{P} = \tilde{R}$$

In fact, an invariant $\tilde{P}\tilde{R}$ -sliding sector can also be designed directly. As the conclusion of this section, the steps to design an invariant discrete-time $\tilde{P}\tilde{R}$ are given as follows:

1. Choose a $1 \times n$ matrix S such that the reduced-order system

$$\begin{cases} x_{k+1} = \Phi x_k + \Gamma u_k \\ s_k = Sx_k = 0 \end{cases}$$

in the hyperplane $s_k = 0$ is stable;

2. Find positive definite matrices $\tilde{R}$ and $\tilde{P}$ and a constant β ($|\beta| < 1$) satisfying:

$$\begin{aligned} \tilde{\Phi}^T \tilde{P} \tilde{\Phi} - \tilde{P} &= \tilde{R} \\ (1 - \beta^2)\tilde{P} - \tilde{R} &\geq 0 \end{aligned}$$

where $\tilde{\Phi}$ is determined by

$$\tilde{\Phi} = \Phi - (S\Gamma)^{-1}\Gamma S(\Phi - \beta I_n).$$

3. Define the parameter $\tilde{D}$ as

$$\tilde{D} = \gamma \tilde{P}$$

where γ is a positive constant, which should be determined with the consideration of the parameter uncertainty and external disturbance.

Then the sector designed by the above steps satisfies the conditions (46), (47), and (48) of the definition of the invariant $\tilde{P}\tilde{R}$ -sliding sector.

5.2 Variable Structure Controller

With the invariant $\tilde{P}\tilde{R}$ -sliding sector proposed in the last section, a VS controller should be designed such that the state will converge to the inside of the sector. In this chapter, we propose a VS control law as follows:

$$u_k = -(S\Gamma)^{-1}(S\Phi x_k - \alpha s_k) \quad (57)$$

where α is a constant which should be chosen to satisfy:

$$|\alpha| < |\beta|$$

Considering the VS control law used inside the sector, the VS control law with the invariant $\tilde{P}\tilde{R}$ -sliding sector is given as:

$$\begin{cases} -(S\Gamma)^{-1}(S\Phi x_k - \beta s_k), & x_k \in \tilde{\mathcal{S}} \\ -(S\Gamma)^{-1}(S\Phi x_k - \alpha s_k), & x_k \notin \tilde{\mathcal{S}}. \end{cases} \quad (58)$$

Theorem 9. *With the above VS control law the state will converge to the inside of the invariant $\tilde{P}\tilde{R}$ -sliding sector in finite steps if the absolute value of the parameter α is chosen to be small enough. And the resultant VS control system is quadratically stable.*

Proof. Outside the invariant $\tilde{P}\tilde{R}$ -sliding sector $\tilde{\mathcal{S}}$, the VS control input is given by (57), i.e.

$$u_k = -(S\Gamma)^{-1}(S\Phi x_k - \alpha s_k).$$

Thus the following holds:

$$\begin{aligned} s_{k+1}^2 - \delta_{k+1}^2 &= (s_{k+1}^2 - \delta_k^2) - \Delta\delta_k^2 \\ &= \alpha^2 s_k^2 - \delta_k^2 + \gamma x_k^T \tilde{R} x_k \\ &= \alpha^2 s_k^2 - \gamma x_k^T (\tilde{P} - \tilde{R}) x_k \\ &\leq \alpha^2 s_k^2 - \beta^2 \delta_k^2. \end{aligned} \quad (59)$$

It is assumed that the state is outside the sector in time instant k_0 , i.e.

$$s_{k_0}^2 > \delta_{k_0}^2$$

if we choose:

$$|\alpha| \leq |\beta| \frac{|\delta_{k_0}|}{|s_{k_0}|}$$

then the state will converge into the sector by one step, i.e.

$$s_{k_0+1}^2 - \delta_{k_0+1}^2 \leq \alpha^2 s_{k_0}^2 - \beta^2 \delta_{k_0}^2 \leq 0$$

After converging into the sector, it has been shown in the last section that the state will remain inside the sector as the sector is designed to be an invariant one with the Lyapunov function $\tilde{L}_k$ decreasing as:

$$\Delta\tilde{L}_k = x_k^T \tilde{R} x_k \leq 0.$$

Therefore the resultant VS control system is quadratically stable.

6 Simulation

Consider a second order continuous-time plant.

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t). \quad (60)$$

Discretizing it with the sampling interval $\tau = 0.01$ second gives the sampled-data state equation as:

$$x_{k+1} = \begin{bmatrix} 1 & 0.01 \\ 0 & 1 \end{bmatrix} x_k + \begin{bmatrix} 0.00005 \\ 0.01 \end{bmatrix} u_k. \quad (61)$$

Choose the parameter matrix S of the hyperplane be

$$S = [1 \ 0.9]$$

which determines the eigenvalue λ of the discrete-time system (61) in the hyperplane $s_k = 0$ as:

$$\lambda = 0.98895$$

Choose the positive definite matrices $\tilde{P}$ and $\tilde{R}$ as

$$\tilde{P} = \begin{bmatrix} 4.35017939 & 1.66671425 \\ 1.66671425 & 1.28644034 \end{bmatrix}$$

$$\tilde{R} = \begin{bmatrix} 0.35722967 & 0.27209179 \\ 0.27209179 & 0.24018819 \end{bmatrix}$$

which satisfy the equation (52). Choose the positive constant γ as

$$\gamma = 0.1$$

then the designed invariant $\tilde{P}\tilde{R}$ -sliding sector for the discrete-time system (61) with the above parameters is shown in Figure 1.

Other parameters of the VS controller used in the simulation are chosen as

$$\alpha = 0.5$$

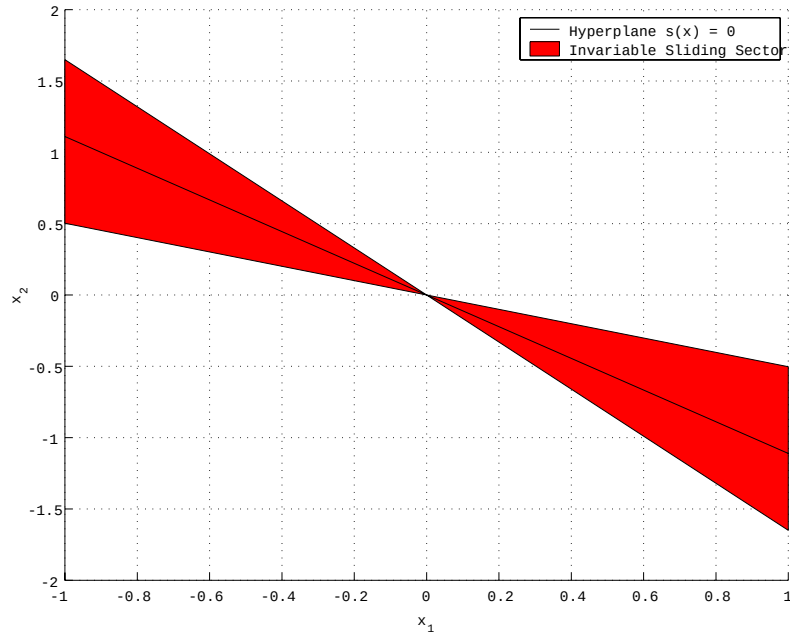
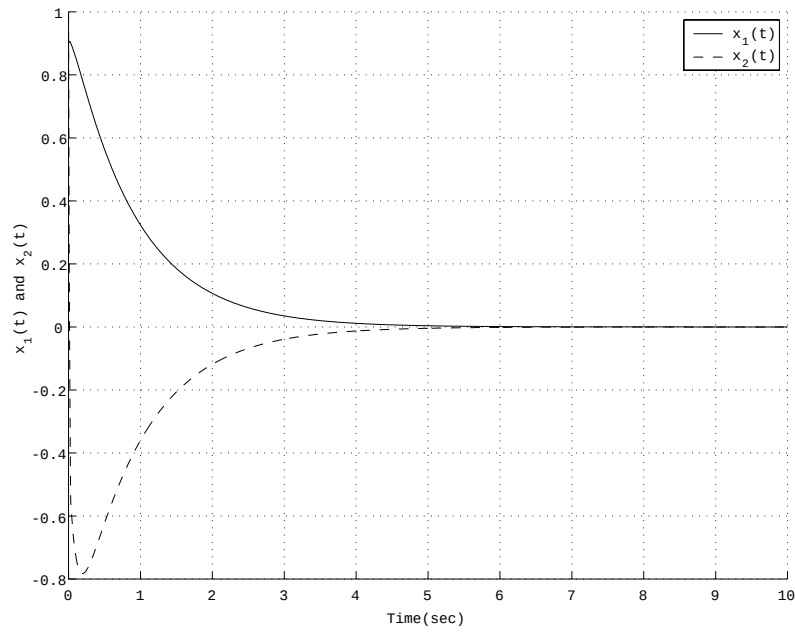
$$\beta = 0.9$$

The simulation results are given in Figure 2, 3, and 4, which show that the state converges into the invariant $\tilde{P}\tilde{R}$ -sliding sector in a finite time and converges to the origin inside the sector.

Figure 5, 6, and 7 are the simulation results for the plant with a parameter uncertainty of

$$A = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1.1 \end{bmatrix}$$

which show the robustness of the proposed VS controller.

Fig. 1. Invariant $\tilde{P}\tilde{R}$ -Sliding SectorFig. 2. Evolution of State Variables $x_1(t)$ and $x_2(t)$

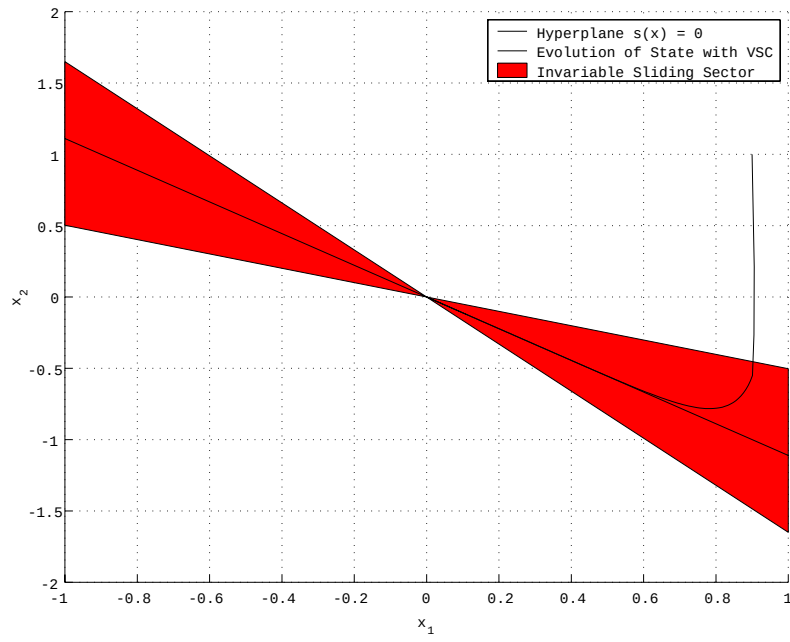


Fig. 3. Phase Plane Diagram of $x_2(t)$ versus $x_1(t)$

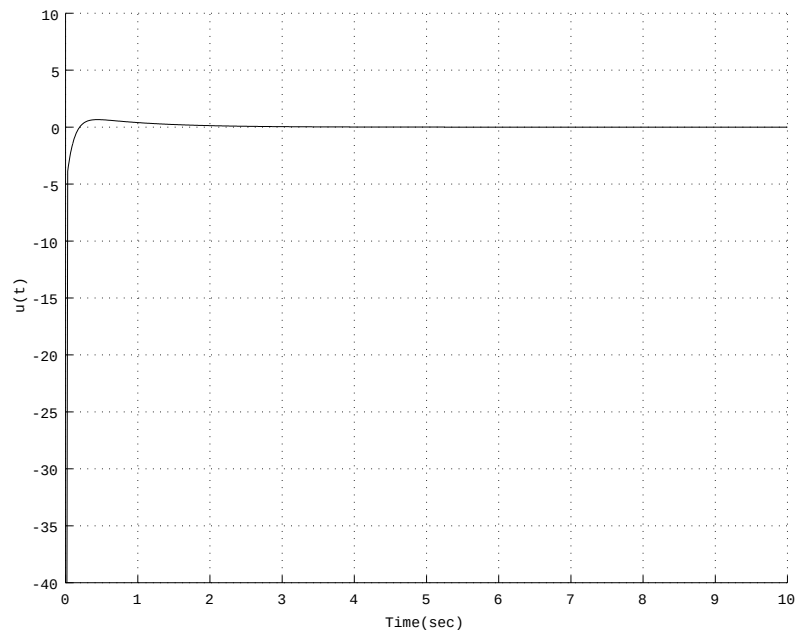


Fig. 4. Evolution of Input $u(t)$

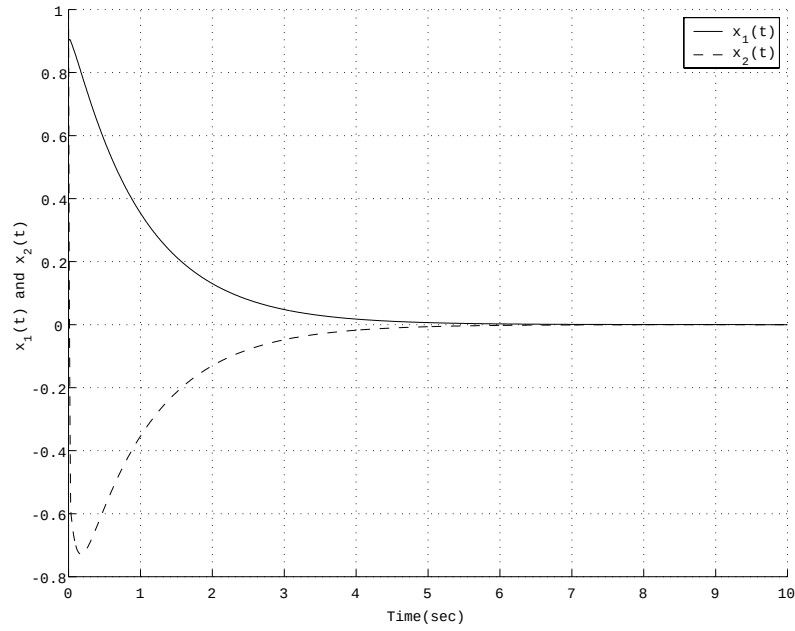


Fig. 5. Evolution of State Variables $x_1(t)$ and $x_2(t)$ with Parameter Uncertainty

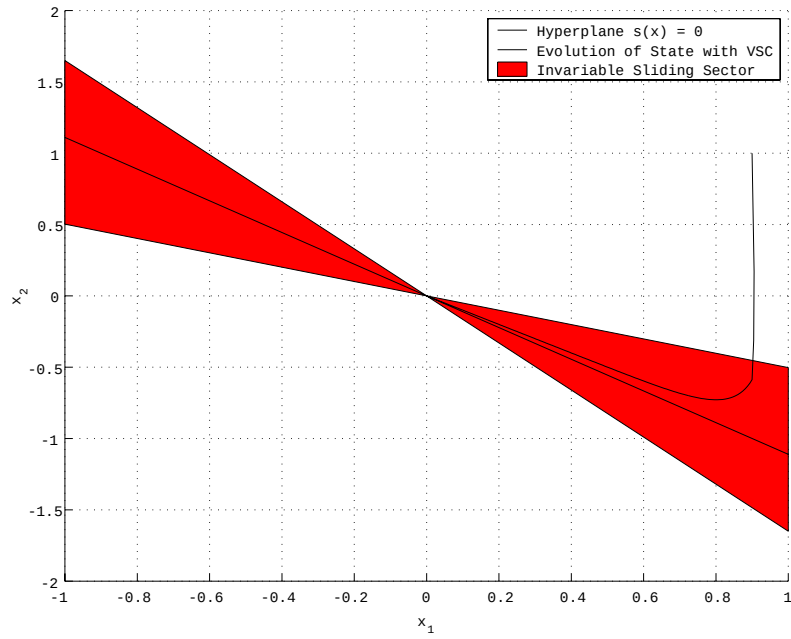


Fig. 6. Phase Plane Diagram of $x_2(t)$ versus $x_1(t)$ with Parameter Uncertainty

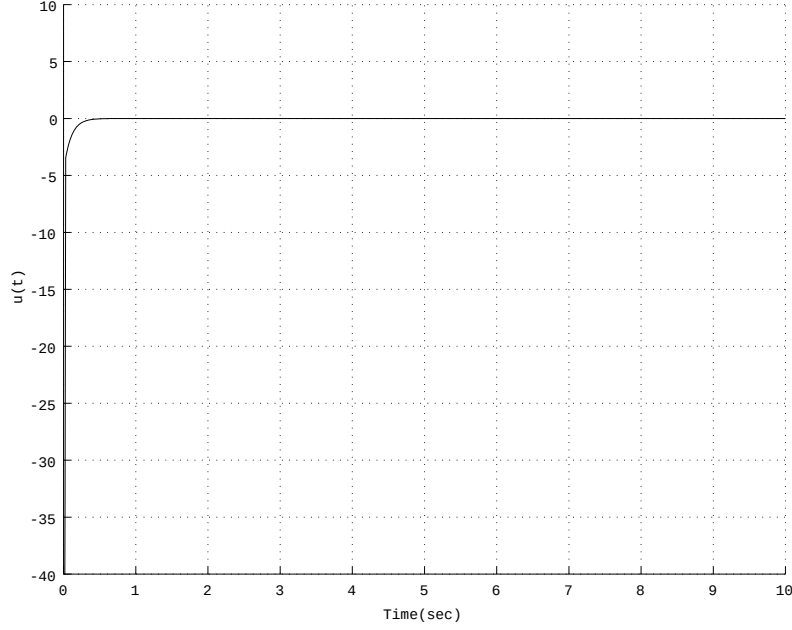


Fig. 7. Evolution of Input $u(t)$ with Parameter Uncertainty

7 Conclusion

In this chapter, at first we presented some discrete-time VS control design algorithms using sliding sectors or PR -sliding sector for systems described by state space equation [4][9] or transfer function[7]. Then we proposed an invariant $P\hat{R}$ -sliding sector and the corresponding VS controller for discrete-time systems, with which the state converges into the sector in a finite time and stays inside it from then on as the designed sector is an invariant subset of the state space R^n . The resultant VS control system is quadratically stable and chattering-free.

The simulation result shows that the proposed sector is really an invariant one and the corresponding discrete-time VS controller has good control performance even if there exists parameter uncertainty.

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Higher-Order Sliding Modes for the Output-Feedback Control of Nonlinear Uncertain Systems

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Abstract. This chapter examines some aspects of the output-feedback control problem for nonlinear uncertain plants, with special emphasis on possible applications of recent results about higher order sliding modes (HOSMs). This regime is established when the simultaneous, finite-time, zeroing of an output quantity (the sliding quantity), and of a certain number of its derivatives, is ensured. In this work, for any step of an output feedback variable structure control design, namely, the definition of the sliding variable, the synthesis of the control law, and the state estimation, a survey of proposals characterized by a finite-time convergence transient is presented. Some different types of sliding surfaces in the state space, such that the associated constrained motion is characterized by a finite-time converging dynamics, are recalled. The use of a discontinuous control to make them attractive and invariant is then analyzed. Finally, real-time differentiators based on HOSMs for estimating the output derivatives are considered. The twofold objective of the present chapter is to survey the most recent results on HOSMs and to highlight their possible role in improving existing approaches, to motivate and to draw possible lines for future research.

1 Introduction

In recent years various systematic procedures and methods to design output-feedback nonlinear control systems have been presented in the literature [2,13,14,20,21,38-46].

In real life, an obvious task is to cope with uncertainties, and an important contribution to this topic was given by the combined introduction of saturated inputs and high-gain observers (HGOs), for estimating the output derivatives, performed by Khalil and co-workers [2,21], improving previous results presented in Tornambé [45,46]. The very idea of combining saturating control and high-gain observers proved to be essential for a number of successive developments.

By using, as feedback signals, the output and a certain number of its derivatives, possibly estimated, the evaluation of the diffeomorphic transfor-

mation required to put the system in normal form [19] could be avoided. In particular, all the uncertainties appearing in the system equation are reduced to satisfy the matching condition [19,21,47].

Outstanding results regarding the semiglobal stabilization of nonlinear uncertain systems using only output measurements were presented by Teel and Praly in [43,44], exploiting a complete uniform observability property. They presented two general backstepping lemmas that can be applied to rather general classes of systems in order to achieve semiglobal stabilization.

In a recent work by Atassi and Khalil [2], under the hypothesis that a stabilizing globally bounded (possibly dynamic) state-feedback control is available, it was demonstrated that using a high gain observer one can recover the performance achieved under state feedback; thus a rather general nonlinear separation principle was established.

Few results are available regarding the output-feedback control of non-minimum phase systems [46], which seems to be one of the most challenging problems arising from the status of the art of nonlinear control theory. Recently, Isidori presented a quite general framework for nonlinear robust output-feedback control design [20], and his result encompasses systems with unstable zero-dynamics.

We focus the present chapter on the finite-time output-feedback control of systems having globally-defined relative degree and stable zero dynamics; in particular, a way to output feedback provided by the sliding-mode control approach is described [47,40,35,8].

Our aim is to contribute to put the basis for possible enhancements of the recently developed theory, in order to cope with problems that are still partially unsolved: the presence of non-parameteric uncertainties and/or unmodeled dynamic actuators, the robustness against measurement noise (dramatic when multiple output differentiation is required), the transient peaking, the counteraction of finite escape-time, etc.

In this context we highlight the role of higher-order sliding modes (HOSMs) as a possible alternative, or a complement, to existing methods, in order to deal with the challenging problem of designing output-feedback control schemes for nonlinear systems of high relative-degree affected by heavy model uncertainties.

The chapter is organized as follows: in next Section 2 a formal statement of the problem, and the main standing assumptions, are given. Then, in Section 3, some simulation examples, which identify the presence of a “cost” associated with nonlinear output-feedback schemes analogous to that of the linear case [18], are discussed. In Section 4 the basic concepts and definitions regarding higher order sliding modes are recalled for the readers’ convenience, while in Section 5-7 recent results regarding sliding constraint, controller, and state-observer design are discussed, respectively. Simulations are reported throughout the paper.

2 Problem Statement

Consider the nonlinear SISO system

$$\begin{aligned}\dot{x} &= f(x) + g(x)u \\ y &= h(x)\end{aligned}\tag{1}$$

with unavailable state vector $x \in R^n$, control variable $u \in R$ and measurable output $y \in R$. Let f , g and h be unknown smooth vector-fields of proper dimension.

Systematic approaches for nonlinear output feedback control design in presence of various type of uncertainties can be found in [19,47,21,23,37].

Actually, the heavy uncertainty of the problem prevents immediate reduction of (1) to any standard form by means of standard approaches based on the knowledge of f , g and h . The drift and control vector fields f and g , and the output map h , are unknown, but they satisfy proper growth conditions to be specified.

If system (1) possesses a globally-defined relative degree r [19], then the input-output dynamics turn out to be expressed as

$$y^{(r)} = L_f^r h(x) + L_g L_f^{r-1} h(x)u\tag{2}$$

where L_g , L_f are Lie derivatives, and condition $L_g L_f^{r-1} h(x) \neq 0$ holds, globally, by assumption (see [19]). Put $\xi = [y, \dot{y}, \dots, y^{(r-1)}]$.

It is always possible to define a set η of $n - r$ variables, such that the map

$$x = \Phi(\xi, \eta)\tag{3}$$

is a diffeomorphism on R^n , and the dynamics of $\eta \in R^{n-r}$, which is referred to as the “internal dynamics” [19], can be expressed in the following form

$$\dot{\eta} = q(\xi, \eta)\tag{4}$$

Note that if $r = n$ there are no internal dynamics, and the system is said to be “fully linearizable” [19].

Let y_R be the desired output response, and consider the error dynamics

$$e^{(r)} = L_f^r h(x) - y_R^{(r)} + L_g L_f^{r-1} h(x)u\tag{5}$$

where $e = \xi - \xi_R$ and $\xi_R = [y_R, \dot{y}_R, \dots, y_R^{(n-r)}]$.

Let us make the following assumption:

Assumption: *The internal dynamics*

$$\dot{\eta} = q(\xi, \eta)\tag{6}$$

is input-to-state stable (ISS) (i.e., for any bounded $\xi(t)$ the internal state $\eta(t)$ remains bounded irrespectively of the initial conditions.)

Let the control task be to solve the finite-time output-feedback tracking control problem.

As usual in the context of the sliding mode approach, a solution to this problem is generally characterized by a two-step procedure:

STEP 1. Identify a function $s = s(e, \dot{e}, \dots, e^{(p)})$, $p \leq r - 1$, called *constraint (or sliding) quantity* such that any motion of the system on the manifold $s = 0$ is characterized by the finite-time zeroing of $e, \dot{e}, \dots, e^{(p)}$.

Possible choices for the sliding quantity s are mentioned in Sect. 5.1 - 5.3.

STEP 2. Find a control u which stabilizes the nonlinear uncertain dynamics (differential inclusion)

$$\begin{aligned} s^{(r-p)} &= \frac{\partial s^{(r-p-1)}}{\partial e^{(r-1)}} [L_f^r h(x) - y_R^{(r)} + L_g L_f^{r-1} h(x)u] = \\ &= \varphi(\xi, \xi_R, \eta) + \gamma(\xi, \eta)u \end{aligned} \quad (7)$$

on the basis of suitable known upperbounds to the uncertainties appearing in (7) without requiring knowledge (or estimation) of $\dot{s}, \ddot{s}, \dots, s^{(r-p-1)}$

We assume that the uncertainties φ and γ globally satisfy the boundedness conditions

$$\begin{aligned} |\varphi(\xi, \xi_R, \eta)| &\leq F(\xi, \xi_R) \\ 0 < I_1 &\leq \gamma(\xi, \eta) \leq I_2 \end{aligned} \quad (8)$$

where $F(\cdot)$ is a known positive function and I_1, I_2 are known constants.

The second step involves the problem of finite-time stabilization of an uncertain differential inclusion of order $(r - p)$ (see Sect. 5, 6).

It must be stressed that while the simple discontinuous control on $s = 0$ is effective if $r - p = 1$, it is possibly unstable if $r - p = 2$ and always unstable if $r - p > 2$ [1]. Therefore, very complex switching logic must be adopted (e.g. the time optimal bang-bang control) considering, furthermore, the difficulties of predicting the behaviour of nonlinear systems of high order. Actually, step 2 represents a formidable research task, especially if $r - p > 2$.

But what are the pros of incrementing $r - p$? In particular, in what sense $r - p = 2$ is better than $r - p = 1$?

In the following section a preliminary attempt to give an answer to these questions is presented through simulations.

3 Motivating examples

The actual research state seems to suggest that the output-feedback control problem has been satisfactorily solved for wide classes of systems of practical

relevance. The key tool is to implement an HGO of order $p = r - 1$ and, then, to use it in conjunction with proper control techniques for relative-degree one systems.

An obvious question which should be addressed is: what is the *cost* of high-gain observers ?

The parameter affecting the performance of these systems is the observer gain, which positively affects two basic requirements: the precision and the possibility of recovering the region of attraction achieved under state feedback [2]. The general effect of increasing the gain is: the higher the gain, the better the overall performance.

While the peaking phenomenon affecting the observer transient can be easily managed by saturating the system input outside a suitable domain of interest, the only cost to pay, as the differentiation order increases, appears to be the geometric growth of the coefficients of the observer state equation, with the associated stability problems arising from the discrete-time implementation.

These facts, seemingly, prevent any attempt to make research relevant to the control of systems with relative degree higher than one; any system, indeed, could be reduced to a relative degree one system thanks to the use of an high gain observer of suitable order.

However, it is well known that a *feedback cost* can be associated to the effect of the measurement errors on the plant control input [18]. The resulting phase-distortion and amplification of the system control input strongly affect the system behaviour.

Anyone can observe that, using an HGO, the frequency response of the p -th order estimated derivative increases with rate $6p$ dB/octave in a frequency range which goes larger and larger as the observer gain increases, with the risk of including the noise spectrum range. Therefore, both the actual gain and the order of the observer strongly affect the discrepancy between the “ideal” controller and the real one, worsening precision, transient performance and basin of attraction.

As an example, consider the output-feedback stabilization problem for the following open-loop unstable plant

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= -2x_1 - x_2 + x_3 + 5 + \sin(4t) + u = \varphi(x, t) + u\end{aligned}\tag{9}$$

with output $y = x_1$.

Since the input-output relative-degree equals the system order, there are no internal dynamics. No a-priori information about the drift term $\varphi(x, t)$ is available, besides its upper boundedness by a known function with linear growth w.r.t. the state norm

$$|\varphi(x, t)| \leq N + M\|x\|_1 \quad N = 6 \quad M = 2\tag{10}$$

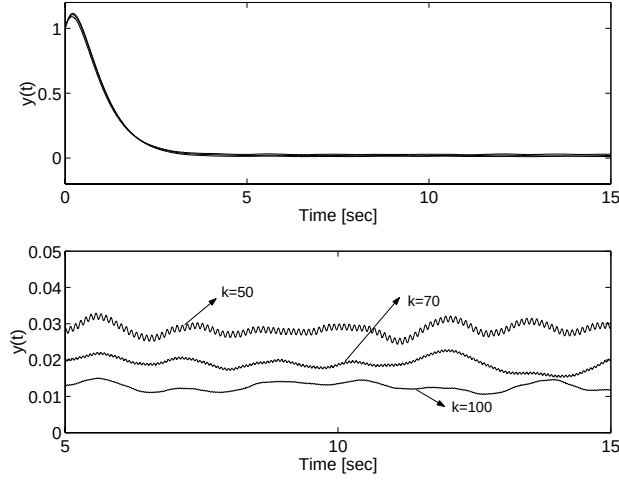


Fig. 1. Noise-free output measurement. The output behaviour

The following relative-degree-one sliding output can be defined

$$s = \ddot{y} + 4\dot{y} + 4y \quad (11)$$

whose coefficients assign a twin pole in -2 to the reduced-order sliding mode dynamics.

The first and second output derivatives have been estimated by means of an HGO with triple pole in $-k$, k being the observer gain. Then, the conventional relay-type first-order SMC has been implemented.

In Fig. 1 the output responses for $k = 50$, $k = 70$ and $k = 100$ are depicted. It can be seen that the steady-state accuracy goes greater and greater as the gain k increases.

This qualitative behaviour reverses in the presence of noise. Let the measured output be corrupted by additive high frequency noise

$$N(t) = 0.002\sin(200t) \quad (12)$$

Now, the larger the gain the worse the accuracy (Fig. 2), since the effect of noise amplification predominates on the accuracy improvement due to the increase of the observer gain.

Remark: Note that low-pass filtering the output before the differentiation attenuates the effect of noise but does not change the qualitative dependence of the overall accuracy on gain k . Moreover, stability problems may occur in the closed loop, especially for unstable plants, due to the increase of relative degree.

On the basis of the above considerations, the order of the observer appears to be a good candidate to represent the *cost* associated with the use of high gain observers.

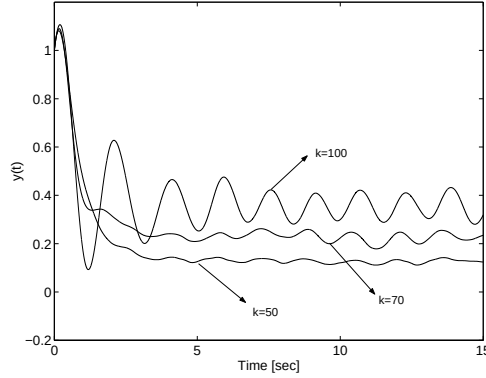


Fig. 2. The effect of high-frequency noise affecting the output

A sensible way to reduce the number of output derivatives required for feedback (i.e. the observer order) is to use a different type of control algorithm, especially devoted to deal with systems with relative degree greater than one.

The use of a second-order sliding mode control (2-SMC) algorithm [4,24] is a way for steering to zero the output of a relative-degree-two system by discontinuously modifying its second derivative with *no information demand* about the first derivative. This latter property is crucial, since it means that the order of the observer can be reduced by one.

To be more specific, as far as the stabilization problem for system (9) is concerned, the use of a 2-SMC algorithm allows us to define a relative-degree two sliding output of the type

$$s = \dot{y} + cy \quad c > 0 \quad (13)$$

with the apparent benefit that only $\dot{y}$ needs to be estimated, thus, a first-order high-gain observer can be actually implemented.

Further improvements can be attained if other types of real-time differentiation units are considered. Indeed, an important application of 2-SM algorithms is the possibility of designing real-time robust differentiators [7,26,28] with a more profitable compromise between accuracy and noise-immunity, as compared with conventional 1-SM based differentiators [51] and HGOs.

Let us introduce a compact terminology :

k -SMC: k -th order sliding-mode controller

HGO- p : HGO for estimating output derivatives up to the p -th order one

In Fig. 3 the results obtained in the three cases:

- 1-SMC combined with HGO-2 (sliding quantity (11)).
- 2-SMC combined with HGO-1 (sliding quantity (13)).

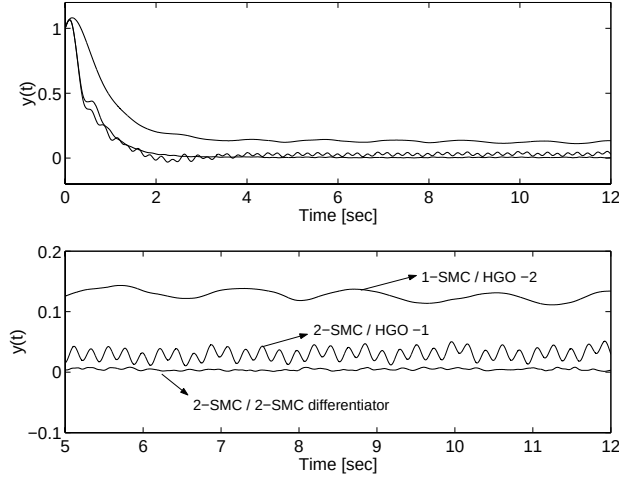


Fig. 3. Comparison between 1-SMC and 2-SMC based schemes

- 2-SMC combined with 2-SMC differentiator (sliding quantity (13)).

are depicted

The improvement obtained through the combined use of a 2-SMC and an HGO-1, first, and of a 2-SMC and a 2-SMC differentiator is apparent (Fig. 3). A detailed comparative analysis of different solutions and approaches is left to future research; the aim of the present chapter is mainly pointing out that there are important unsolved problems in the output feedback control of nonlinear uncertain systems, and that higher-order sliding modes may constitute an effective tool to satisfactorily address such problems.

4 Higher order sliding modes: basic definitions

Let us first remember that according to the definition by Filippov [15] any discontinuous differential equation $\dot{x} = v(x)$, where $x \in R^n$ and $v(\cdot)$ is a locally bounded measurable vector function, is replaced by an equivalent *differential inclusion* $\dot{x} \in V(x)$.

In the simplest case, when $v(\cdot)$ is continuous almost everywhere, $V(x)$ is the convex closure of the set of all possible limits of $v(y)$ as $y \rightarrow x$, while $\{y\}$ are continuity points of $v(\cdot)$. Any solution of the equation is defined as an absolutely continuous function $x(t)$ satisfying the differential inclusion almost everywhere.

For simplicity we restrict ourselves to sliding modes with respect to scalar constraint functions. General definitions of HOSMs with respect to vector-functions, and of HOSMs on manifolds (see [24,25]) are not considered here.

Let a constraint be given by an equation $s(x) = 0$, where $s : R^n \rightarrow R$ is a sufficiently smooth constraint function. It is supposed that total time derivatives $s, \dot{s}, \ddot{s}, \dots, s^{(k-1)}$ along the system trajectories exist and are single-valued functions of x , which is not trivial for discontinuous dynamic systems. In other words, discontinuity does not appear in the first $k-1$ total derivatives of the constraint function s .

Then, the k -th order sliding set (k -sliding set) is determined by the equalities

$$s = \dot{s} = \ddot{s} = \dots = s^{(k-1)} = 0 \quad (14)$$

forming a k -dimensional condition on the state of the dynamic system.

Definition 1. Let the k -sliding set (14) be non-empty and assume that it is locally an integral set in Filippov's sense (i.e. it consists of Filippov's trajectories of the discontinuous dynamic system). Then the corresponding motion satisfying (14) is called **k -sliding mode** (k -SM) with respect to the constraint function s .

A sliding mode is called stable if the corresponding integral sliding set is stable. A typical trajectory when approaching a 2-SM is shown in Fig. 4.

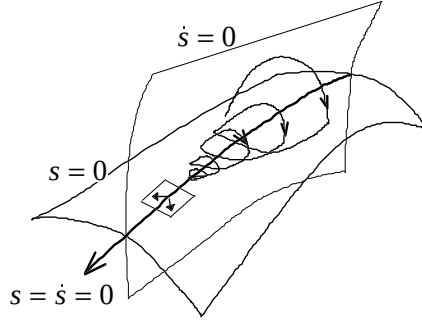


Fig. 4. 2-sliding mode trajectories: the reaching phase

Under the assumption that $s, \dot{s}, \ddot{s}, \dots, s^{(k-1)}$ are differentiable functions of x , the additional k -sliding regularity condition

$$\text{rank} [\nabla s, \nabla \dot{s}, \dots, \nabla s^{(k-1)}] = k \quad (15)$$

where ∇ represents the gradient operator, implies that the k -sliding set is a differentiable manifold and that $s, \dot{s}, \ddot{s}, \dots, s^{(k-1)}$ may be supplemented up to new local coordinates.

Remarks.

1. It is frequently heard that higher order sliding modes differ in the number of successive total derivatives of s which vanish in the sliding mode $s \equiv 0$. Nevertheless, that number cannot be considered as a characteristic of the mode, since formally derivatives of any order are nullified. Sliding modes should be distinguished by system properties outside of them. The most natural characteristic of a sliding mode is the number of successive *continuous* total derivatives of s in a vicinity of the mode. In other words the number k is taken corresponding to the first derivative $s^{(k)}$ which is discontinuous or does not exist due to some reason, like trajectory nonuniqueness. k is called the *sliding order*. It characterizes the system motion with respect to the given sliding output s , and may change if another output is considered with the same sliding motion.

2. The above definitions are easily extended to include non-autonomous differential equations (by introduction of the fictitious equation $\dot{t} = 1$) and to the case of the closed-loop controlled system $\dot{x} = f(t, x, u)$, $u = U(t, x)$, with discontinuous $U(\cdot)$ and smooth $f(\cdot)$, $s(\cdot)$.

4.1 Real sliding

Only ideal sliding modes were considered in Def. 1, keeping exactly the k -sliding mode condition (14). However, ideal sliding is achieved by means of a control signal commuting at infinite frequency, which cannot be attained in real plants due to switching imperfections (the simplest switching imperfection is the switching delay caused by discrete measurements). It was proved [24] that the best possible sliding accuracy attainable with discrete switching in $s^{(k)}$ is

$$|s|^{(j)} \approx T^{k-j} \quad j = 0, 1, \dots, k \quad (16)$$

where $T > 0$ is the minimal switching time interval and $s^{(0)} = s$.

Thus, in order to achieve the k -th order of accuracy in discrete realization, the sliding order has to be at least k . It is known that standard sliding modes provide for first-order real sliding only. Real sliding of higher orders are achieved by discrete switching modifications of higher order sliding modes with finite-time convergence [31]. Real sliding of second and third orders are obtained by special discrete switching algorithms (see Sect. 6.2) [10,9,24,42].

5 Higher Order Sliding modes with finite-time convergence

Actually, HOSMs definitions in the previous Section leave out the problem of *how* such kinds of motion are established, simply defining them as a special type of sliding mode that satisfies the set of constraints (14).

Note that asymptotically stable, or unstable, HOSMs may appear in variable structure systems with fast actuators, as it was pointed out in [25,22]. In that case, stable HOSMs reveal themselves by spontaneous disappearance of the chattering effect. Dynamical sliding mode control [41,40] leads to asymptotically stable higher-order sliding modes, and has to be specially mentioned here.

5.1 Terminal sliding modes

An interesting family of sliding surfaces featuring finite-time convergence is used to achieve the so-called “*terminal sliding modes*” [50].

Terminal sliding surface corresponds to the vanishing of a sliding variable s_{t_r} which is defined by means of an iterative procedure:

$$s_{t_0} = e \quad (17)$$

$$s_{t_i} = \dot{s}_{t_{i-1}} + c_i s_{t_{i-1}}^{p_i/q_i} \quad i = 1, 2, \dots, r-1 \quad (18)$$

$$s_{t_r} = \dot{s}_{t_{r-1}} + c_{r-1} s_{t_{r-1}}^{p_r/q_r} \quad (19)$$

Here p_i, q_i ($p_i < q_i$) are suitable integer odd coefficients [50]. Sufficient conditions for the stability of a discontinuous feedback

$$u = -U \text{sign}(s_{t_r}) \quad (20)$$

are derived in [50] for a given class of plants, assuming full-state availability for the real-time feedback calculation.

Unfortunately, the resulting closed-loop system may have an unbounded right-hand side (for some initial conditions) which prevents the very implementation of the Filippov theory for this situation. The corresponding control is formally bounded along each switching manifold, but may take infinite values in some vicinity of the sliding manifold [52].

5.2 Arbitrary-Order Sliding Controllers

Another class of sliding controllers is based on arbitrary-order sliding modes with finite time convergence [31].

Let us recall that r is the relative degree of system (1), and let m be any positive number, the common choice being the least common multiple of $1, 2, \dots, r$.

Define the following quantities

$$N_{1,r} = |e|^{(r-1)/r} \quad (21)$$

$$N_{i,r} = \left(|e|^{m/r} + |\dot{e}|^{m/(r-1)} + \dots + |e^{(i-1)}|^{m/(r-i+1)} \right)^{(r-i)/m} \quad (22)$$

$$i = 1, \dots, r-2$$

$$N_{r-1,r} = \left(|e|^{m/r} + |\dot{e}|^{m/(r-1)} + \dots + |e^{(r-2)}|^{m/2} \right)^{1/m} \quad (23)$$

and consider also

$$\phi_{0,r} = e \quad (24)$$

$$\phi_{1,r} = \dot{e} + \beta_1 N_{1,r} \operatorname{sign}(\phi_{0,r}) \quad (25)$$

$$\phi_{i,r} = e^{(i)} + \beta_i N_{i,r} \operatorname{sign}(\phi_{i-1,r}) \quad i = 2, \dots, r-1 \quad (26)$$

The following Theorem was proved in [31]:

Theorem 1. *Consider system (1), (8) with $F(\cdot) \equiv F = \text{const.}$ Assume that trajectories of system (1) are infinitely extendible in time for any Lebesgue-measurable bounded feedback control. Then, with properly chosen positive parameters $\beta_1, \beta_1, \dots, \beta_{r-1}, \alpha$, the controller*

$$u = -\alpha \operatorname{sign}(\phi_{r-1,r}(e, \dot{e}, \dots, e^{(r-1)})) \quad (27)$$

where $\Phi_{r-1,r}(\cdot)$ is defined in (21)–(26) leads to the establishment of a finite-time converging r -sliding mode on the manifold $e = 0$, and the resulting closed-loop system is finite-time output-stable.

The above Theorem determines a controller family applicable to all systems of the type (1) with relative degree r , satisfying (8), with $F(\cdot) \equiv F$, for some constants F, Γ_1, Γ_2 . Parameters $\beta_1, \dots, \beta_{r-1}$ affect the reaching time and are to be chosen sufficiently large in index order. While the number of choices of β_i is certainly infinite, it is possible to take some predefined values chosen for each r in advance. Parameter $\alpha > 0$ must be chosen specifically for any fixed F, Γ_1, Γ_2 . As the value of α is determined by the upper bounds of the input-output dynamics, the controller performance is insensitive to any system perturbation preserving these bounds.

Following are a few examples of sliding controllers with β_i tested for $r \leq 3, m$ being the least common multiple of $1, 2, \dots, r$.

1. $u = -\alpha \operatorname{sign} e$
 2. $u = -\alpha \operatorname{sign}(\dot{e} + |e|^{1/2} \operatorname{sign} e)$
 3. $u = -\alpha \operatorname{sign}(\ddot{e} + 2(|\dot{e}|^3 + |e|^2)^{1/6} \operatorname{sign}(\dot{e} + |e|^{2/3} \operatorname{sign} e))$
 4. $u = -\alpha \operatorname{sign}\{e^{(3)} + 3(\ddot{s}^6 + \dot{e}^4 + |e|^3)^{1/12} \operatorname{sign}[\ddot{s} + (\dot{e}^4 + |e|^3)^{1/6} \operatorname{sign}(\dot{e} + 0.5|e|^{3/4} \operatorname{sign} e)]\}$
- (28)

The rationale of the controller is that a 1-sliding mode is established in the continuity points of the discontinuity set

$$\Gamma : \phi_{r-1,r} = 0 \quad (29)$$

Such a sliding mode is described by the differential equation $\phi_{r-1,r} = 0$ providing in its turn for the existence of a 1-sliding mode $\phi_{r-2,r} = 0$. Note that the primary sliding mode disappears when the secondary one is to appear.

The resulting movement takes place in some vicinity of the subset of Γ satisfying $\psi_{r-2,r} = 0$, then it transfers in finite time into some vicinity of the subset satisfying $\psi_{r-3,r} = 0$ and so on. While the trajectory approaches the r -sliding set, set Γ contracts to the origin in the coordinates $e, \dot{e}, \dots, e^{(r-1)}$.

Controller (27) requires the availability of $e, \dot{e}, \dots, e^{(r-1)}$. Note that $e^{(r)}$ is bounded, which provides for the possibility of using an $(r-1)$ -th order exact differentiator in the feedback (see Sect. 7). The information demand may also be lowered by use of a suitable discretized controller [31]:

Theorem 2. [31] *Consider system (1) satisfying the same assumptions and definitions as in Theorem 1, and let the measurements be carried out at the sampling instants t_i , with constant step $T > 0$. Then, with properly chosen positive parameters $\beta_0, \beta_1, \dots, \beta_{r-1}$, α the controller*

$$u(t) = u_i = -\alpha \text{sign} \left(\Delta_i e^{(r-2)} + \beta_i T N_{r-1,r} \text{sign} \left(\phi_{r-2,r} \left(e_i, \dot{e}_i, \dots, e_i^{(r-2)} \right) \right) \right) \quad (30)$$

$$t_i \leq t < t_{i+1}$$

where $e_i = e(t_i)$, $\dot{e}_i = \dot{e}(t_i)$, $\dots$, $e_i^{(r-2)} = e^{(r-2)}(t_i)$, and $\Delta_i e^{(r-2)} = e_i^{(r-2)} - e_{i-1}^{(r-2)}$, provides for the finite-time fulfillment of inequalities

$$|e| \leq a_0 T^r, \quad |\dot{e}| \leq a_1 T^{r-1}, \quad \dots, \quad |e^{(r-1)}| \leq a_{r-1} T \quad (31)$$

where a_j , $j = 0, 2, \dots, r-1$ are some positive constants and the convergence time is a locally-bounded function of initial conditions.

That is the best possible accuracy attainable with discontinuous $e^{(r)}$ separated from zero [24]. The discontinuous manifold $\phi_{r-1,r}$ may be replaced by its smooth approximation by means of some regularization procedure [27]. A model example, and relevant computer simulations, can be found in [31,27,28].

Some additional remarks on the use of the proposed controllers follow.

Remarks

1. *Implementation of r -sliding controller when the relative degree is less than r (chattering avoidance).*

Introducing successive time derivatives $u, \dot{u}, \dots, u^{(r-k-1)}$ as new auxiliary variables, and $u^{(r-k)}$ as a new control (dynamical extension), one achieves different modifications of each r -sliding controller devoted to control systems with relative degrees $k = 1, 2, \dots, r$. The resulting control input, obtained at the output of a chain of $(r-k)$ integrators, turns out to be an $(r-k-1)$ -smooth function of time with $k < r$, a Lipschitz function with $k = r-1$, and a bounded “infinite-frequency switching” function with $k = r$. Using the same trick the chattering effect can be removed.

2. *Controlling systems not affine on control.*

If the system is nonlinear on control, one can differentiate the input-output dynamics until a system affine in the higher-order control derivative is obtained. The problem may be solved, having introduced as before new auxiliary state variables $u, \dot{u}, \dots$, using the higher-order control time derivative as the new control variable.

5.3 Time-Optimal Sliding Surfaces

The time optimal bang-bang control law represents, historically, the first example of higher order sliding mode with finite time convergence. The strategy developed for double and triple integrator can be expressed in the form of a discontinuous control across a switching surface [34,12].

Any uncertain system of the same order (2 or 3) forced to evolve on the associated time-optimal surface, e.g. by means of first-order sliding mode control, results in being characterized by the same finite-time convergence property [34,12].

Sliding outputs defining surfaces of order 2 and 3 are

$$s_{TO_2} = e + k\dot{e} \quad k > 0 \quad (32)$$

$$s_{TO_3} = e + \frac{1}{3}\ddot{e} - \left[\ddot{e} + \left(\frac{1}{2}\dot{e}^2 + \dot{e} \operatorname{sign} \left(\dot{e} + \frac{1}{2}\ddot{e}|\ddot{e}| \right) \right) \right] \operatorname{sign} \left(\dot{e} + \frac{1}{2}\ddot{e}|\ddot{e}| \right) \quad (33)$$

Note that with a first order sliding mode controller the system can be forced to perform a higher order sliding motion.

6 Recent results on 2-SMC design

Previous considerations regarding the positive implications of taking $r - p$ (the relative degree between the constraint variable and the discontinuous control) as high as possible (see Section 2) can justify the research activity on higher-order sliding controllers. While the previous Section 5 was devoted to illustrate some possible choices for sliding manifolds with finite-time convergence, the aim of the present Section is to give an overview about recently-developed methodologies to enforce a second-order sliding mode on the chosen sliding manifold.

Up to now only few 2-sliding controllers have been proposed [4]. The so-called “super-twisting” algorithm is conceptually different from the others, for two reasons: first, it depends only on the actual value of the sliding constraint, while the others have more information demands. Second, it is effective only for anti-chattering purposes as far as relative-degree one constraint variables are dealt with.

On the contrary, both the “twisting” and the “sub-optimal” algorithm [4] can deal with relative-degree two constraint variables. They are, both, special cases of the general algorithm

$$u(t) = -\alpha(s, s_M)U_M \text{sign}(s - \beta s_M)$$

$$\alpha(s, s_M) = \begin{cases} 1 & \text{if } (s - \beta s_M)s_M \leq 0 \\ \alpha^* & \text{otherwise} \end{cases} \quad (34)$$

where s_M is the *last* (meaning the *most recent*) point with a zero derivative (“singular point”) of s , $U_M > 0$, and $\alpha, \beta \in (0, 1]$, are suitable coefficients to be set, according to the uncertainty bounds, in order to guarantee the existence and stability of the 2-SM. Note that $\beta = 0$ and $\beta = 1/2$ for the “twisting” and the “sub-optimal” algorithms, respectively.

The recently proposed 2-SMC with global convergence features [11], for which $\alpha = 1$ and β is adaptively adjusted on-line, belongs to the above class of sliding controllers.

The distinguishing feature of such class of controllers is that the control law depends on the current value of the sliding constraint and on its past history, represented by the last occurred singular point s_M .

While these algorithms are actually implemented looking at the first difference of s (whose sign approximates the sign of $\dot{s}$), a different controller structure, which stores and processes the past time history of s with no attempt to estimate the sign of $\dot{s}$, is also possible.

The accuracy of these algorithms, both in their continuous and discrete-time implementation, can be improved by means of proper learning and adaptation procedures, which counteract the chattering effect as well [6,3,10,9].

6.1 Second-Order Sliding Mode Controllers with Global Convergence

As far as uncertain systems confined in a known bounded domain are concerned, it is possible to relate *a-priori* the control amplitude and the switching logic to the magnitude of the uncertainties, such that the transient is stable and finite-time converging.

In a recent work this assumption has been dispensed with, and a globally-converging algorithm has been presented [11]. To this end, a crucial role has been played by one of the parameters affecting the transient of second order discontinuous differential equations, i.e. the so-called “anticipating factor” β in (34) [4].

The following Theorem 3 illustrates the proposed controller; then a simple simulation example is provided in order to highlight its good features.

Theorem 3. Consider system (1) with $r = n$ and satisfying (8) with $F(\xi, \xi_R) \leq \bar{F}(s, \dot{s})$, $\bar{F}(s, \dot{s})$ being a known function. Let $s(t) = s(e, \dot{e}, \dots, e^{(r-2)})$ be the scalar constraint quantity. Then, the control law

$$u(t) = \begin{cases} -\frac{1}{\Gamma_1} [F[\xi(0)] + \chi] \text{sign}(s(0)) & t = 0 \\ -\frac{1}{\Gamma_1} [F[\xi(t)] + \chi] \text{sign}(s(t) - s(0)) & 0 < t \leq t_{M_1} \end{cases} \quad \chi > 0 \quad (35)$$

ensures the finite-time reaching of a first singular point s_{M_1} . From this point on, the control law

$$u(t) = -U_{M_k} \text{sign}[s(t) - \beta_k s_{M_k}] \quad t_{M_k} < t \leq t_{M_{k+1}} \quad k = 1, 2, \dots \quad (36)$$

guarantees the global finite-time vanishing of the constraint variable s and of its unmeasurable derivative $\dot{s}$, provided that the controller parameters are chosen according to

$$U_{M_k} = \frac{\alpha}{\Gamma_1} \left[\bar{F}(s_{M_k}, \eta \sqrt{|s_{M_k}|}) + \frac{1}{3} \eta^2 \right] \quad \alpha > 1 \quad (37)$$

$$\beta_k = \max \left\{ \frac{1}{2}, 1 - \frac{\eta^2}{2[\bar{F}(s_{M_k}, \eta \sqrt{|s_{M_k}|}) + \Gamma_2 U_{M_k}]} \right\} \quad (38)$$

where η is a positive constant, t_{M_k} are the time instants at which $\dot{s}$ is zero, and $s_{M_k} = s(t_{M_k})$.

An interesting feature of the above algorithm is that, due to the adaptive switching rule, it allows for counteracting the transient peaking and it is effective also when the controlled system may exhibit finite escape time.

As an example, consider the output stabilization problem for the system

$$\dot{y} = ay^2 + u \quad |a| \leq 4 \quad (39)$$

with unmodelled actuator

$$\tau \dot{u} = -u + v \quad \tau > 0 \quad (40)$$

The input-output relative degree is two, and $\dot{y}$ is not measurable. The HGO-based 1-SMC scheme, with $s = \hat{y} + y$, and the above presented global 2-SMC scheme, with $s = y$, are compared. Note that using 2-SMC no output differentiation is required. The significant counteraction of the peaking phenomenon is apparent from Fig. 5.

6.2 Second-Order Sliding Mode Control for sampled-data systems

Consider system (1) under the same assumptions as in Theorem 1, and let s_i be the sequence of sampled values of the sliding quantity, $s_i = s(iT)$

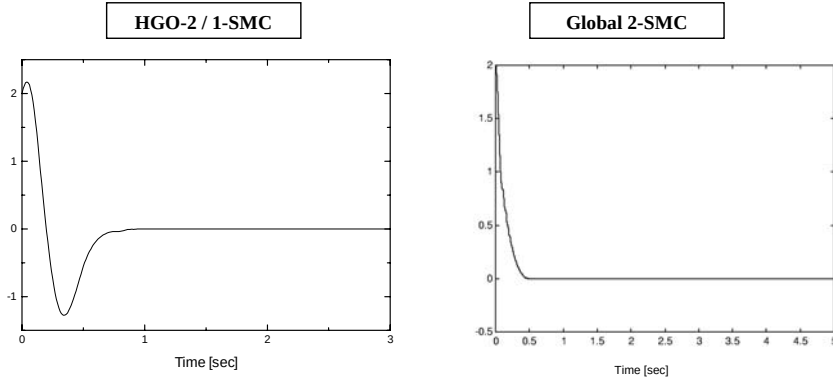


Fig. 5. Combined HGO-1/1-SMC scheme and global 2-SMC scheme. A comparison of the output response.

($i = 0, 1, \dots$), T being the sampling period. The plant input is piecewise-constant (ZOH device), i.e., $u(t) = u_i$, $t \in [i T, (i + 1) T)$.

The discrete-time version of the sub-optimal 2-SMC algorithm can be derived by direct discretization of the continuous-time algorithm, provided that the control amplitude is properly set [4], that is

$$u_i = -\alpha_i U_M \operatorname{sign} \left[s_i - \frac{1}{2} \hat{s}_{M_j} \right] \quad (41)$$

$$U_M \in \left(\frac{F}{\alpha^* \Gamma_1}, \infty \right) \cap \left(\frac{4F}{3\Gamma_1 - \alpha^* \Gamma_2} + \theta_1 T, \theta_2 T^{-2} \right) \quad (42)$$

where θ_1, θ_2 are proper constants, $\hat{s}_{M_j}$ is an estimate of the last singular value of s (see (34)), obtained by the following approximate digital peak-detector:

$$\begin{aligned} & \text{set } s_{-1} = s(0) ; \quad s_{-2} = 0 \quad j = -1 \\ & \text{set } \Lambda_i = (s_i - s_{i-1})(s_{i-1} - s_{i-2}) \\ & \text{If } (\Lambda_i \leq 0) \quad \text{then } \begin{cases} j = j + 1 \\ \hat{s}_{M_j} = s_{i-1} \end{cases} \end{aligned} \quad (43)$$

and α_i is adjusted according to

$$\alpha_i = \begin{cases} 1 & \text{if } \{s_i - \frac{1}{2} \hat{s}_{M_j}\} \hat{s}_{M_j} < 0 \\ \alpha^* & \text{otherwise} \end{cases} \quad (44)$$

with the constant $\alpha^* \in (0, 1) \cap \left(0, \frac{3\Gamma_1}{\Gamma_2} \right)$.

It has been proven [5] that the above digital control strategy guarantees the reaching of the boundary layer (16) of the 2-sliding set $s = \dot{s} = 0$ after a finite number of sampling instants. However, controller (41)–(44) reveals an almost ringing behaviour within the boundary layer.

Nevertheless, assuming that the uncertainties affecting the sliding variable dynamics (7) are locally Lipschitz, it was shown in [5] that the sliding variable dynamics can be represented, with an $O(T^3)$ approximation, by the following discrete model:

$$s_{i+1} = 2s_i - s_{i-1} + \dot{\varphi}_{i-1}T^2 + \frac{1}{2}(\gamma_i u_i + \gamma_{i-1} u_{i-1})T^2 \quad (45)$$

By resorting to the extension of the equivalent control concept to the discrete-time setting [47,48], the corresponding discrete-time equivalent control (DTEC) can be defined as

$$u_{\text{eqi}}^d = -\frac{1}{\gamma_i} \left(\gamma_{i-1} u_{i-1} + d_i + 2 \frac{2s_i - s_{i-1}}{T^2} \right) \quad (46)$$

where d_i is unknown due to system uncertainties. Using a one step delay estimate of d_i , computed by means of discrete model (45), the DTEC can be approximated as follows

$$\hat{u}_{\text{eqi}}^d = \frac{1}{\gamma_n} \left(\gamma_n u_{i-2} - 2 \frac{3s_i - 3s_{i-1} + s_{i-2}}{T^2} \right) \quad (47)$$

where $\gamma_n = \sqrt{T_1 T_2}$ is a reasonable estimate of the uncertain control gain.

The main problem in using the DTEC method is that the amplitude of the DTEC is proportional to $s[k]/T$ as far as systems with relative degree one are dealt with [42], and to $s[k]/T^2$ in the considered case [5]. Therefore, in the case under investigation, a $\mathcal{O}(T^2)$ -vicinity of the sliding manifold must be achieved first. Next application of control law (47) provides the finite-time attainment of a $\mathcal{O}(T^3)$ sliding accuracy, which is the highest achievable one when a piecewise-constant control signal forces to zero a relative-degree-two output variable. That is the same accuracy of real 3-SM.

The presence of uncertainties in control gain makes stability analysis rather involved [42]. Suitable assumptions regarding the uncertainties are needed to ensure that the system trajectory reaches, and does not leave, the $\mathcal{O}(T^3)$ boundary layer of $s = 0$. A qualitative requirement is that the uncertainty in the control gain is “sufficiently small”.

The following Theorem was proved in [9]:

Theorem 4. *Consider system (1) with the same assumptions as in Theorem 1. Assume the directly discretized control (41)–(44) has driven the system in an $\mathcal{O}(T^2)$ vicinity of $s = 0$. The successive application of the control law*

$$u_i = \hat{u}_{\text{eqi}}^d \quad (48)$$

where $\hat{u}_{\text{eqi}}^d$ is defined in (47), guarantees that the accuracy is improved up to

$$\begin{aligned} |s(t)| &\leq O(T^3) \\ |\dot{s}(t)| &\leq O(T^2) \end{aligned} \quad t \geq T^* \quad (49)$$

T^* being a finite transient.

7 Real-time output differentiation via sliding modes

In this section a sliding-mode solution to the multiple differentiation problem is reported, and a particular emphasis is devoted to the recently presented arbitrary-order finite-time converging robust differentiator ([28]).

Let a Lebesgue-measurable locally bounded input signal $h(t)$ be defined on $[0, \infty)$ and let it consist of an unknown smooth base signal $h_0(t)$ and an unknown-but-bounded Lebesgue-measurable noise $N(t)$ with $\sup |N(t)| = \varepsilon$, ε being unknown as well. Let the p -th derivative of $h_0(t)$ have a known Lipschitz constant C_p .

7.1 First-order SM differentiators

The first problem is to find an estimation of $\dot{h}_0(t)$, robust in presence of $N(t)$ and exact when $N(t) = 0$.

The conventional differentiator based on first-order sliding modes takes the form:

$$\begin{aligned} \dot{\zeta} &= v \\ v &= -\mu \operatorname{sign}(\zeta - h(t)) \\ \tau \dot{v}_{av} + v_{av} &= v \end{aligned} \quad (50)$$

where $\mu > C_1$, $\tau > 0$, $C_1 = \max |\dot{h}_0|$. According to [47,51], v_{av} is a $O(\tau)$ -estimate of the derivative of h . Cascade implementation is feasible, but each stage increases noise sensitivity.

Under the above smoothness assumptions, it has been shown in [24] that no differentiator can provide for differentiation accuracy of the j -th derivative ($j = 0, 1, \dots, p$) better than $C_p^{j/p} \varepsilon^{(p-j)/p}$. It has been also proved in [24] that there is a differentiator providing for the accuracy proportional to the just reported one, but, unfortunately, that was only a pure existence theorem. Note that if additional restrictions are imposed on h_0 the adduced performance may be, in principle, improved.

It should be stressed that, due to the averaging, differentiator (50) has an intrinsic error also in the absence of noise, thus it is robust but not exact.

To overcome the need to known a Lipschitz constant of the signal to differentiate, a combined adaptive-variable structure scheme has been proposed

in [49]. A dead-zone based adaptation mechanism give the proposed scheme robustness against the measurement noise. Similarly to the conventional 1-SMC differentiator (50), the averaging of the output makes the scheme in [49] robust but not exact.

On the contrary, an exact and robust 2-SM first-order differentiator can be based on the “super-twisting” algorithm [26]:

$$\begin{aligned}\dot{\zeta}(t) &= v(t) \\ v(t) &= \theta(t) - \lambda|\zeta(t) - h(t)|^{1/2} \operatorname{sign}(\zeta(t) - h(t)) \\ \dot{\theta}(t) &= -\mu \operatorname{sign}(\zeta - h(t))\end{aligned}\tag{51}$$

Here $\lambda, \mu > 0$, and both $v(t)$ and $\theta(t)$ may be considered as the output of the differentiator. Solutions of the system are understood in the Filippov sense.

Parameters may be chosen, for example, in the form $\mu = 1.1C_2, \lambda = 1.5C_2^{1/2}$, where $C_2 = \max |\ddot{h}_0|$. Other possible criteria are given in [26]. Differentiator (51) provides for finite-time convergence to the exact derivative of $h_0(t)$ if $N(t) = 0$. Otherwise, it provides for accuracy proportional to $C_2^{1/2}\varepsilon^{1/2}$, which is the best possible precision in the considered case [26].

In case of a p -stage cascade implementation, differentiator (51) will provide for p -th order differentiation accuracy of the order of ε^{2-p} . All necessary conditions for successive p -th order differentiation, and implementation of higher-order sliding controller (Sect. 5.2), are fulfilled, at least locally, for smooth dynamic systems with relative degree p . Thus, full local real-time robust control of output variables is possible, using only output variable measurements and knowledge of the relative degree. Nevertheless, the cascade of differentiation stages is not satisfactory for multiple differentiation purposes, due to excessive noise propagation. It is more effective to implement higher-order differentiators, specially designed for the multiple differentiation task, which are described in the following subsection.

7.2 Arbitrary-order, exact, finite-time converging, robust differentiation

The aim is now to find real-time robust estimates of $h_0(t), \dot{h}_0(t), \dots, h_0^{(p)}(t)$, exact in the absence of measurement noise and continuously depending on its magnitude (e.g. robust). A recursive design scheme is proposed [32].

Let a $(p-1)$ -th-order differentiator $D_{p-1}(h(t), C_{p-1})$ produce outputs D_{p-1}^i ($i = 0, 1, \dots, p-1$) which are estimates of $h_0, \dot{h}_0, \dots, h_0^{(p-1)}$ for any input signal h with $h_0^{(p-1)}$ having Lipschitz constant $C_p > 0$.

Then, the p -th order differentiator has the outputs $z_i = D_p^i$, $i = 0, 1, \dots, p$, defined as follows:

$$\begin{aligned} \dot{z}_0 &= \nu, \quad \nu = -\lambda |z_0 - h(t)|^{\frac{p}{p+1}} \text{sign}(z_0 - h(t)) + z_1, \\ z_1 &= D_{p-1}^0(\nu, C_p), \quad \dots, \quad z_p = D_{p-1}^{p-1}(\nu, C_p) \end{aligned} \quad (52)$$

Here $D_0(h(t), C_p)$ is a simple nonlinear filter

$$D_0 : \quad \dot{z} = -\lambda \text{sign}(z - h(t)), \quad \lambda > C_p. \quad (53)$$

In other words it has the form

$$\begin{aligned} \dot{z}_0 &= \nu_0, \quad \nu_0 = -\lambda_0 |z_0 - h(t)|^{\frac{p}{p+1}} \text{sign}(z_0 - h(t)) + z_1, \\ &\dots \\ \dot{z}_i &= \nu_i, \quad \nu_i = -\lambda_i |z_i - \nu_{i-1}|^{\frac{p-i}{p-i+1}} \text{sign}(z_i - \nu_{i-1}) + z_{i+1}, \\ &\dots \\ \dot{z}_p &= -\lambda_p \text{sign}(z_p - \nu_{p-1}) \end{aligned} \quad (54)$$

It is easy to check that the above-presented p -th order differentiator can be expressed in the following non-recursive form:

$$\begin{aligned} \dot{z}_0 &= z_1 - \kappa_0 |z_0 - h(t)|^{\frac{p}{p+1}} \text{sign}(z_0 - h(t)) \\ \dot{z}_1 &= z_2 - \kappa_1 |z_0 - h(t)|^{\frac{p-1}{p+1}} \text{sign}(z_0 - h(t)) \\ &\dots \\ \dot{z}_i &= z_i - \kappa_i |z_0 - h(t)|^{\frac{p-i}{p+1}} \text{sign}(z_0 - h(t)) \\ &\dots \\ \dot{z}_p &= -\kappa_p \text{sign}(z_0 - h(t)) \end{aligned} \quad (55)$$

for suitable positive constant coefficients κ_i .

Admissible values for the coefficients in (54) are easier to find than those in (55). In fact, in the first case, the values $\lambda_0, \dots, \lambda_{p-1}$ defining the $(p-1)$ -th order differentiator still apply for the p -th order one, and, therefore, one more parameter needs to be evaluated. See [32] for more details on the parameters design procedure.

Finite-time convergence and Lyapunov stability as well are proved in [32] on the basis of the concept of homogeneous vector fields [36]. In [32] a different procedure for building another class of differentiators enjoying similar convergence properties has been also proposed.

In the following Theorems 5 and 6 the performance of the proposed differentiator class in presence of bounded measurement noise are investigated, together with discrete-time implementation, respectively.

Theorem 5. *Let the input noise satisfy the inequality $|h(t) - h_0(t)| \leq \varepsilon$. Then the following inequalities are fulfilled in finite time for some positive constants μ_i, ν_i depending only on the parameters of differentiator (55)*

$$\begin{aligned} |z_i - h_0^{(i)}(t)| &\leq \mu_i \varepsilon^{\frac{(p-i+1)}{(p+1)}} & i = 0, \dots, p \\ |v_i - h_0^{(i+1)}(t)| &\leq \nu_i \varepsilon^{\frac{(p-i)}{(p+1)}} & i = 0, \dots, p-1 \end{aligned} \quad (56)$$

Using recursive high-order differentiators, noise propagation is better counteracted than with the cascade implementation of first-order differentiators (Sect. 7.1).

Consider the discrete-sampling case, where $z_0 - h(t)$ is replaced by $z_0(t_j) - h(t_j)$, with $t_j \leq t < t_{j+1}$, $t_{j+1} - t_j = T > 0$.

Theorem 6. *Let differentiator (55) be implemented via Eulero discretization with the sampling period $T > 0$, and let the measurements of h be free of noise. Then the following inequalities are fulfilled in finite time for some positive constants μ_i, ν_i depending exclusively on the parameters of the differentiator*

$$\begin{aligned} |z_i - h_0^{(i)}(t)| &\leq \mu_i T^{p-i+1}, \quad i = 0, \dots, p \\ |v_i - h_0^{(i+1)}(t)| &\leq \nu_i T^{p-i}, \quad i = 0, \dots, p-1 \end{aligned} \quad (57)$$

Combining controller (27), with $p = r - 1$, and the $(r - 1)$ -th order exact differentiator (54), the tracking problem is ideally solved in finite time under the conditions in Theorem 1. A model example, and simulation results, are reported in [27,28].

7.2.1 Simulation examples of arbitrary-order differentiation

Differentiator (54) of order 5 with $C_5 = 1$ and coefficients $\lambda_0 = 50$, $\lambda_1 = 30$, $\lambda_2 = 16$, $\lambda_3 = 8$, $\lambda_4 = 4$, $\lambda_5 = 2.2$ has been tested. These parameters can be easily changed, since the differentiator is not very sensitive to these values. The tradeoffs are as follows: the larger the parameters, the faster the convergence and the higher the sensitivity to input noise and discretization. The estimation of the i -th derivative achieved by means of the k -th order differentiator is denoted as $D_k^i(t)$.

Third-order differentiation of the noise-free input signal

$$f(t) = f_0(t) = \sin 0.5t + \cos t. \quad (58)$$

was first considered. Mutual graphics of $\dot{f}(t)$ and D_3^1 , $\ddot{f}(t)$ and D_3^2 , and, finally, $f^{(3)}(t)$ and D_3^3 with the measurement step $\tau = 5 \cdot 10^{-5}$ are shown in Fig. 6.a. As for the fifth-order differentiator, pictures of $h^{(j)}(t)$ and D_3^j , $j = 1, 2, \dots, 5$ are depicted in Fig. 6.b.

The attained accuracies are $1.46 \cdot 10^{-13}$, $7.16 \cdot 10^{-10}$, $9.86 \cdot 10^{-7}$, $3.76 \cdot 10^{-4}$, 0.0306 and 0.449 for tracking the signal and the first, second, third, fourth and fifth derivatives, respectively. Reducing τ accuracy is improved according to Theorem 6.

Third-order differentiation results are considered in Fig. 6c in the presence of a periodic non-differentiable high-frequency noise signal with maximum magnitude 10^{-4} .

Estimates of the first and second derivatives produced by the 5th order differentiator, with noise magnitude 0.01 and frequency about 10000 Hz, are

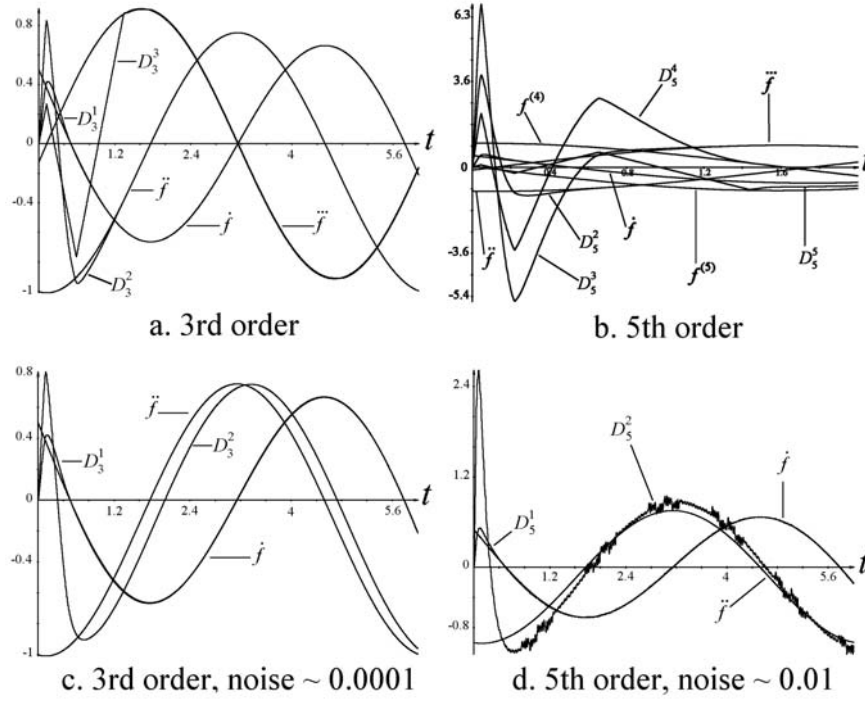


Fig. 6. High-order differentiation

shown in Fig. 6.d. The highest derivative has large phase-deviation, but do not chatter much. The accuracies of estimations D_5^1 , D_5^2 , D_5^3 , D_5^4 and D_5^5 are 0.0002, 0.0136, 0.184, 0.649, and 0.740, respectively. Changing noise magnitude is consistent with the statement of Theorem 6.

8 Conclusions

In this chapter, following the sliding-mode approach, we have addressed the output-feedback control problem of nonlinear uncertain plants, which is one of the main topics of modern nonlinear control theory and has a strong impact in many control applications. In this context, recent developments of the theory of higher order sliding modes appear to furnish promising alternatives, and possible complement, to other existing approaches.

While presenting the most recent results of the theory, the authors aim at identifying the role of this particular constrained motion in the framework of an output feedback design process. In particular, the three elements which characterize the sliding mode approach (the sliding constraint design, the synthesis of the discontinuous control law, and the estimation of the required

number of output derivatives) can be endowed with finite-time transient and higher accuracy when higher-order sliding modes are implemented.

Just to give a perspective to the work done in this chapter, the crucial problem of the counteraction of the effect of measurement errors has been introduced, which could be a reasonable paradigm for the comparison between the various existing solutions. A systematic analysis of the relevant properties of control schemes with higher order sliding modes represents a challenging topic for future investigations.

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Variable Structure Systems with Terminal Sliding Modes

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Abstract. In this chapter, we discuss recent developments in a special topic of Variable Structure Systems, namely, the terminal sliding mode control. Dynamic properties of the terminal sliding mode control systems are explored and their applications in single input single output (SISO) systems and multi input multi output (MIMO) systems are presented. Further improvements of the particular sliding mode control strategy are suggested.

1 Introduction

This chapter discusses recent developments in a special topic of Variable Structure Systems (VSS), that is, the terminal sliding mode (TSM) control. The effectiveness of the VSS approaches has been well documented in the abundant VSS literature as well as the chapters in this book. The robustness and simplicity for implementation are the trademarks of VSS. The key concept in the VSS is the so called sliding mode behavior, which is attained by designing the control laws which drive the system state to reach and remain on the intersection of a set of prescribed switching manifolds. When in the sliding mode, the system exhibits invariance properties, such as robustness to certain internal parameter variations and external disturbances.

The dynamic performance of a VSS system is determined by the prescribed switching manifolds upon which the control structure is switched. Most commonly used switching manifolds are the linear hyperplanes, which guarantee the asymptotic stability of the system motion in the sliding mode. That is, the system state will reach the equilibrium in infinite time. This asymptotical stability feature is, in general, sufficient for delivering effective control but there are cases which may not be dealt with properly without imposing substantial control efforts. For example, when high precision and stringent reaching time are required, controllers enabling asymptotical stability may not perform well when the system state is close to the equilibrium. One alternative to overcome this problem is by means of specially tailored nonlinear switching manifolds. The recently developed finite time mechanism, the terminal sliding mode, is such a useful nonlinear switching manifold which can improve the transient performance substantially. The concept of TSM

was originated from the notion, terminal attractors [28], which was used in studying the content addressable memory, associative memory, and pattern recognition in artificial neural networks. The TSM design was first used in [12] for finite time sliding mode control design for robotic manipulators. It was then extended to several control problems of SISO systems and MIMO systems including robotics [24–26, 13–15, 4, ?]. This chapter aims to examine the current developments in this special topic and makes some useful extensions of existing results for control analysis and design using TSM.

2 The Terminal Sliding Mode Concept

The TSM concept can be described by the following first order dynamics [24]

$$s = \dot{x}_1 + \beta x_1^{q/p} = 0 \quad (1)$$

where $x_1 \in R^1$ is a scalar variable, $\beta > 0$, and p, q ($p > q$) are positive integers. Note that the parameter p must be an odd integer and only real solution is considered so that for any real number x_1 , $x_1^{q/p}$ is always a real number. The equation (24) then becomes

$$\dot{x}_1 = -\beta x_1^{q/p} \quad (2)$$

Given an initial state $x_1(0) \neq 0$ the dynamics (24) will reach $x_1 = 0$ in finite time. The time taken from the initial state $x_1(0)$ to 0, t^s , is determined by

$$t^s = \frac{p}{\beta(p-q)} |x_1(0)|^{(p-q)/p} \quad (3)$$

It can also be proved that the equilibrium 0 is an attractor, i.e. when the state x_1 reaches zero, it will stay at zero forever. This can be demonstrated by taking a Lyapunov function $v = \frac{1}{2}x_1^2$. The time derivative of v along (24) is

$$\dot{v} = x_1 \dot{x}_1 = -\beta x_1 \dot{x}_1^{q/p} = -\beta x_1^{(p+q)/p}$$

since $(p+q)$ is even, then $\dot{v}$ is negative definite, so $x_1 = 0$ is “terminally” stable (not necessarily asymptotically stable).

The reaching time t^s which is determined by (3) depends on the parameters p, q, β , and the initial value $x_1(0)$. As $x_1(0)$ is fixed or in a known bounded region, one can choose β such that t^s is very small.

It is interesting to note that when $p = q$, then

$$s = \dot{x}_1 + \beta x_1 = 0 \quad \beta > 0 \quad (4)$$

which becomes a linear dynamics. The introduction of the nonlinearity term $x_1^{q/p}$ improves the convergence toward the equilibrium. The closer to the

equilibrium, the faster the convergence rate, resulting in finite time convergence. This can be demonstrated as follows. Consider the Jacobian around the equilibrium $x_1 = 0$, i.e.

$$J = \frac{\partial \dot{x}_1}{\partial x_1} = -\frac{\beta q}{p x_1^{\frac{p-q}{q}}}$$

If one consider J as λ , an eigenvalue of the first order approximation matrix J , then it means

$$J \rightarrow -\infty \quad \text{when} \quad x_1 \rightarrow 0$$

which indicates that at the equilibrium the “eigenvalue” tends to negative infinity, and of course, the system trajectory with such an “infinitely” negative eigenvalue, will converge to the equilibrium with an “infinitely” large speed which results in finite time reachability.

Note that here the Lipschitz condition does not hold, i.e. $|J| < \infty$ does not hold and J is singular at $x_1 = 0$. This situation introduces a violation to the Lipschitz condition for the existence and uniqueness of solutions of differential equations. With the Lipschitz condition, a transient solution cannot intersect the corresponding constant solution, therefore the theoretical time of approaching the equilibrium is always infinite. Violating the Lipschitz condition may give rise to the solutions which may reach a so called “terminal trajectory” and solutions may intersect each other [28].

It should be noted that there is a close relationship between the first order TSM (24) and the time optimal control. It is well established that the time optimal control for the double integrator system [16]

$$\dot{x}_1 = x_2; \quad \dot{x}_2 = u; \quad |u| \leq 1$$

can be described as

$$u = \text{sgn}[\Xi(x)]$$

where $x = (x_1, x_2)^T$ and

$$\Xi(x) = \begin{cases} \xi(x) = x_1 + \frac{1}{2}x_2|x_2|; & \xi(x) \neq 0 \\ x_2; & \xi(x) = 0 \end{cases}$$

Function $\xi(x) = 0$ forms the arc on which the system trajectory will reach zero. One can easily see that if we take very large odd numbers p and q such that $q/p \approx 1/2$ and $\beta = 2^{p/q}$, then the first order dynamics (24)

$$s(x) = 2^{p/q} x_1^{q/p} + x_2 = 0$$

is equivalent to

$$x_1 + \frac{1}{2}x_2^{p/q} \approx x_1 + \frac{1}{2}x_2|x_2| = \xi(x) = 0.$$

This means that the TSM model can approximate the time optimal arc for the double integrator system with any accuracy regardless of the sign of x_2 because it is smooth hence differentiable. It should be noted that this TSM model is also the optimal switching function for second order systems with positive real eigenvalues in simple ratio $\lambda_1/\lambda_2 = 2$ [16].

One can clearly see that when x_1 is far away from the equilibrium, the TSM model (24) does not prevail over its linear counterpart (setting $p = q$) since the term $x_1^{q/p}$ tends to reduce the magnitude of convergence rate at a distance from the equilibrium. One immediate solution is to introduce the following so called fast TSM model,

$$s = \dot{x}_1 + \alpha x_1 + \beta x_1^{q/p} = 0 \quad (5)$$

where $x_1 \in R^1$, $\alpha, \beta > 0$. By doing so, we have

$$\dot{x}_1 = -\alpha x_1 - \beta x_1^{q/p} \quad (6)$$

For properly chosen q, p, α, β , given an initial state $x_1(0) \neq 0$ the dynamics (6) will reach $x_1 = 0$ in finite time. The physical interpretation is: When x_1 is far away from zero, the approximate dynamics becomes $\dot{x}_1 = -\alpha x_1$ whose fast convergence when far away from zero is well understood. When close to $x_1 = 0$, the approximate dynamics becomes $\dot{x}_1 = -\beta x_1^{q/p}$ which is a terminal attractor [28].

More precisely, we can solve the differential equation (6) analytically. The exact time to reach zero, t^s , is determined by

$$t^s = \frac{p}{\alpha(p-q)} \ln \frac{\alpha x_1(0)^{(p-q)/p} + \beta}{\beta} \quad (7)$$

and the equilibrium 0 is a terminal attractor.

The fast convergence performance of the fast TSM in comparison with the conventional linear sliding mode can be demonstrated by the following example (using Matlab). Consider $\alpha = 1$ and $\beta = 1$ and initial condition $x_1(0) = 1$. First let assume $p = 3$ and $q = 1$. From (7) one can easily find that the time to reach zero is $t^s = 1.03972077083992$. We now compare the above with the situation where p and q are set to 1. The simulation suggests that at approximate $t^s = 1.039699999999990$, for the case of $p = 3$ and $q = 1$, $x_1(t^s) = 0.00000009178540$ and for the case of $p = 1$ and $q = 1$, $x_1(t^s) = 0.12500519281775$. It is evident that the convergence rate of the fast TSM is far better than its linear counter part. The obvious reason is when close to the equilibrium, the convergence rate of the linear sliding mode exponentially slows down while the convergence rate of the fast sliding mode accelerates exponentially.

In the following sections, we will illustrate how to use the TSM concept for control design for SISO systems and MIMO systems respectively.

3 Fast TSM Control of SISO Systems

We consider the general nonlinear smooth single-input single-output (SISO) system

$$\begin{aligned}\dot{z} &= f(z) + g(z)u \\ y &= h(z)\end{aligned}\tag{8}$$

where $z \in R^n$, f and g are the smooth vector fields on $\mathcal{R}^n$, h is the scalar smooth field on $\mathcal{R}^n$, and $u \in \mathcal{R}^1$.

Before we proceed, we introduce several notations. The Lie derivative of h with respect to f is defined as the directional derivative $\mathcal{L}_f h = \nabla h$ where $\nabla h = (\partial h / \partial z)$ representing the gradient of h . Higher order Lie derivatives can be defined recursively as $\mathcal{L}_f^i h = \nabla(\mathcal{L}_f^{i-1} h)f$ for $i = 1, 2, \dots$ with $\mathcal{L}_f^0 h = h$. see [8,18].

Without loss of generality, we make the following assumption. However, the results to be obtained can easily extended to those systems with relative degree less than n . *Assumption* The SISO system (8) has relative degree n in a region $\Sigma \in \mathcal{R}^n$, i.e.

$$\mathcal{L}_g \mathcal{L}_f^i h = 0 \quad 0 \leq i < n-1 \tag{9}$$

$$\mathcal{L}_g \mathcal{L}_f^{n-1} h \neq 0 \tag{10}$$

With Assumption 1, in Σ , the nonlinear system (8) can be transformed to the following normal form

$$\begin{aligned}\dot{x}_i &= x_{i+1} \quad i = 1, \dots, n-1 \\ \dot{x}_n &= a(x) + b(x)u\end{aligned}\tag{11}$$

where $x_i = \mathcal{L}_f^{i-1} h(z)$ for $i = 1, \dots, n$, $a(x) = \mathcal{L}_f^n h(z)$ and $b(x) = \mathcal{L}_g \mathcal{L}_f^{n-1} h(z)$. The transformation matrix is easily seen (see [8], p157)

$$x = \Phi(z) = [h(z), \mathcal{L}_f h(z), \dots, \mathcal{L}_f^{n-1} h(z)]^T.$$

The dynamics (11) can also be written as

$$\dot{x} = A(x) + B(x)u \tag{12}$$

where $A(x) = [x_2, \dots, x_{n-1}, a(x)]^T$ and $B(x) = [0, \dots, 0, b(x)]^T$.

For this class of system in 'nested' canonical form, the following hierarchical TSM structure can be used:

$$s_1 = \dot{s}_0 + \alpha_0 s_0 + \beta_0 s_0^{q_0/p_0} \tag{13}$$

$$s_2 = \dot{s}_1 + \alpha_1 s_1 + \beta_1 s_1^{q_1/p_1} \tag{14}$$

$$\vdots$$

$$s_{n-1} = \dot{s}_{n-2} + \alpha_{n-2} s_{n-2} + \beta_{n-2} s_{n-2}^{q_{n-2}/p_{n-2}} \tag{15}$$

where $s_0 = x_1$, $\alpha_i > 0$, $\beta_i > 0$, and q_i , p_i ($i = 0, \dots, n-2$) are positive odd numbers. Same analogue applies that once $s_{n-1} = 0$ is reached, s_{n-2} will reach zero in finite time, so will $s_{n-3}, \dots, s_0$. It is easy to establish that the time to reach the equilibrium is

$$T = \sum_{i=1}^n t_i \quad (16)$$

where t_n is the time to reach the terminal sliding mode $s_{n-1} = 0$ and

$$t_i = \frac{p_{i-1}}{\alpha_{i-1}(p_{i-1} - q_{i-1})} \ln \frac{\alpha_{i-1}s_{i-1}(t_i)^{(p_{i-1}-q_{i-1})/p_{i-1}} + \beta_{i-1}}{\beta_{i-1}}$$

for $i = n-1, \dots, 1$ is the time from $s_i(t_i) \neq 0$ to $s_i(t_i + t_{i-1}) = 0$.

Remark 1. Note that the procedure (13)–(15) actually defines a path for the state x to converge to the equilibrium. Indeed, if $s_{n-1} = 0$ is considered as $n-1$ dimensional flow in the state space, $s_{n-2} = 0$ can be considered as a subspace with dimension shrunk by unity. So $s_0 = 0$ will be the result of the $n-1$ dimensional dynamic space shrunk by $n-1$ times (dimensions). Letting $p_i = q_i$ ($i = 1, \dots, n-1$), simple computation and transformation using the Laplace operator p lead to

$$s_{n-1} = (p + \alpha_{n-1} + \beta_{n-1}) \cdots (p + \alpha_1 + \beta_1)x.$$

Therefore s_{n-1} becomes a conventional linear hyperplane based sliding mode with real eigenvalues. We can also interpret the convergence towards $x = 0$ as a shrinking process the same as that for the TSMs. However here, because of asymptotic convergence, exact reduction of dimension is not possible.

The sliding mode control u can be designed so that

$$s_{n-1}\dot{s}_{n-1} \leq -K|s_{n-1}| \quad K > 0$$

which is a sufficient condition. Therefore $s_{n-1} = 0$ can be reached in finite time, and $x = 0$ can be reached in finite time as well. Following are some results on the fast TSM control of SISO systems.

Theorem 1. *For the system (8), if the control u is designed as*

$$u = u_{eq} + u_d \quad (17)$$

where

$$u_{eq} = -b^{-1}(x)(a(x) + \sum_{k=0}^{n-2} (\alpha_k \mathcal{L}_{A+Bu}^{n-k-1} s_k + \beta_k \mathcal{L}_{A+Bu}^{n-k-1} s_k^{q_k/p_k})),$$

$$u_d = -b^{-1}(x)K \operatorname{sgn}(s_{n-1}) \quad K > 0$$

where q and p ($p > q$) are positive odd integers defined above with $K > 0$ being a constant, then the system state will reach the sliding mode $s_{n-1} = 0$ in finite time.

Proof. To ensure the finite time reachability of the sliding mode $s_{n-1} = 0$, the condition $s_{n-1} \dot{s}_{n-1} < -K|s_{n-1}|$ should be satisfied. Take the first order derivative of s_{n-1} , one has

$$\dot{s}_{n-1} = \ddot{s}_{n-2} + \alpha_{n-2}\dot{s}_{n-2} + \beta_{n-2}\mathcal{L}_{A+Bu}s_{n-2}^{q_{n-2}/p_{n-2}} \quad (18)$$

since $s_i = \dot{s}_{i-1} + \alpha_{i-1}s_{i-1} + \beta_{i-1}s_{i-1}^{q_{i-1}/p_{i-1}}$, for $i = n-1, n-2, \dots, 1$, and the l th order derivative of s_i is

$$\mathcal{L}_{A+Bu}^l s_i = \mathcal{L}_{A+Bu}^{l+1} \dot{s}_{i-1} + \alpha_{i-1}\mathcal{L}_{A+Bu}^l s_{i-1} + \beta_{i-1}\mathcal{L}_{A+Bu}^l (s_{i-1}^{q_{i-1}/p_{i-1}})$$

then it can be easily calculated that

$$\begin{aligned} \dot{s}_{n-1} &= \mathcal{L}_{A+Bu}^n s_0 + \sum_{k=0}^{n-2} \beta_k \mathcal{L}_{A+Bu}^{n-k-1} s_k + \sum_{k=0}^{n-2} \beta_k \mathcal{L}_{A+Bu}^{n-k-1} s_k^{q_k/p_k} \\ &= \dot{z}_n + \sum_{k=0}^{n-2} \alpha_k \mathcal{L}_{A+Bu}^{n-k-1} s_k + \sum_{k=0}^{n-2} \beta_k \mathcal{L}_{A+Bu}^{n-k-1} s_k^{q_k/p_k} \end{aligned} \quad (19)$$

Substituting the control (17) into (19) yields $s_{n-1}\dot{s}_{n-1} = -K|s_{n-1}|$ which means that the sliding mode $s_{n-1} = 0$ will then be reached in finite time. In fact, following Section 2, from any initial state at $t_0 = 0$, the time to reach zero is $t_n = \frac{|s_{n-1}(0)|}{K}$. Hence the proof of the theorem is completed. $\square$

The control (17) involve calculation of the terms $\mathcal{L}_{A+Bu}^{n-k-1} s_k^{q_{k+1}/p_{k+1}}$, $k = 0, \dots, n-2$ that is lengthy and trivial. Here we present a qualitative result for the calculation of these terms and also show that these terms are independent of the control u .

Theorem 2. For any $k \in \{0, 1, \dots, n-2\}$,

$$\mathcal{L}_{A+Bu}^{n-k-1} s_k^{q_k/p_k} = f_k(x) \quad (20)$$

where f_k is a continuous nonlinear function.

Proof. We prove this proposition using the mathematical induction approach. Let $k = 0$, then apparently $\mathcal{L}_{A+Bu}^{n-1} s_0^{q_0/p_0} = \mathcal{L}_{A+Bu}^{n-1} x_1^{q_0/p_0} = f_0(x)$. Let $k = 1$, then $\mathcal{L}_{A+Bu}^{n-2} s_1^{q_1/p_1} = \mathcal{L}_{A+Bu}^{n-2} (x_2 + \alpha_0 x_1 + \beta_0 x_1^{q_0/p_0})^{q_1/p_1}$ which can be apparently expressed as $f_1(x)$. Assume for $k = k_0$, $\mathcal{L}_{A+Bu}^{n-k_0-1} s_{k_0}^{q_{k_0}/p_{k_0}} = f_{k_0}(x)$. Let us examine the case of $k = k_0 + 1$. Since

$$\begin{aligned} s_{k_0} &= f_{k_0}(\dot{s}_{k_0-1}, s_{k_0-1}) \\ &= f_{k_0}(\ddot{s}_{k_0-2}, \dot{s}_{k_0-2}, s_{k_0-2}) \\ &= \vdots \\ &= f_{k_0}(s_0^{(k_0)}, s_0^{(k_0-1)}, \dots, s_0) \end{aligned}$$

and $s_0 = x_1$, then $s_{k_0}^{(l)}$ is a function of $x_{k_0+l+1}, x_{k_0+l}, \dots, x_1$. Therefore

$$\mathcal{L}_{A+Bu}^{n-k_0-2} s_{k_0+1}^{q_{k_0+1}/p_{k_0+1}}$$

is a function of x . $\square$

The parameters q_k, p_k must be chosen carefully in order to avoid the singularity because there are terms in $\frac{d^{n-k-1}}{dt^{n-k-1}} s_k^{q_k/p_k}$ which may contain negative powers so that when $s_{k-1} \rightarrow 0, u \rightarrow \infty$. This problem can be remedied by the following theorem.

Theorem 3. *If*

$$q_k/p_k > \frac{n-k-1}{n-k}$$

then when $s_k \rightarrow 0$ sequentially from $k = n-2$ to $k = 0$, u is bounded.

Proof. From the rule for the n th derivative of a composite function [9], we have that for function $F(s)$,

$$\frac{d^n}{dt^n} F(s) = \sum \frac{n!}{i_1! i_2! \dots i_l!} \frac{d^m F}{ds^m} \left(\frac{\dot{s}}{1!} \right)^{i_1} \left(\frac{\ddot{s}}{2!} \right)^{i_2} \left(\frac{s^{(3)}}{3!} \right)^{i_3} \dots \left(\frac{s^{(l)}}{l!} \right)^{i_l} \quad (21)$$

over all solutions in non negative integers of the equation $i_1 + 2i_2 + 3i_3 + \dots + li_l = n$ and $m = i_1 + i_2 + i_3 + \dots + i_l$. For simplicity, let $r = q_k/p_k$ and drop the index k . Since in the sliding mode $s_{k+1} = 0$,

$$s_{k+1} = \dot{s}_k + \alpha_k s_k + \beta_k s_k^{q_k/p_k} = \dot{s} + \alpha s + \beta s^r,$$

then $\dot{s} = \mathcal{O}(s^r)$ when $s \rightarrow 0$, where $\mathcal{O}$ is a complexity function. We therefore have $\frac{d^m F}{ds^m} = \mathcal{O}(s^{r-m})$ and $s^{(d)} = (\dot{s})^{(d-1)} = \mathcal{O}(s^{dr-(d-1)})$. So

$$\begin{aligned} \frac{d^n}{dt^n} F(s) &= \sum \mathcal{O}(s^{r-m}) \mathcal{O}(s^{i_1 r}) \mathcal{O}(s^{i_2 (2r-1)}) \dots \mathcal{O}(s^{i_l (lr-(l-1))}) \\ &= \sum \mathcal{O}(s^{r-m}) \times \\ &\quad \mathcal{O}(s^{r(i_1+2i_2+3i_3+\dots+li_l)+(i_1+i_2+i_3+\dots+i_l)-(i_1+2i_2+3i_3+\dots+li_l)}) \\ &= \sum \mathcal{O}(s^{r-m}) \mathcal{O}(s^{nr+m-n}) = \mathcal{O}(s^{(n+1)r-n}) \end{aligned} \quad (22)$$

Hence when $s \rightarrow 0$, i.e. $s_k \rightarrow 0$, $(n+1)r-n = (n+1)q_k/p_k - n > 0$ will ensure that (22) is bounded. Also from the above analysis, we have $\frac{d^n}{dt^n} s = (\dot{s})^{(n-1)} = \mathcal{O}(s^r)^{(n-1)} = \mathcal{O}(s^{nr-n+1})$. Hence when $s \rightarrow 0$, $nr-n+1 = nq_k/p_k - n+1 > 0$ will ensure that $\frac{d^n}{dt^n} s$ is bounded. With the above expressions in mind, the control (17) can be rewritten as

$$\begin{aligned} u &= -b^{-1}(z)(a(z) + \sum_{k=0}^{n-2} (\mathcal{O}(s_k^{(n-k-1)q_k/p_k - (n-k-1)+1}) + \\ &\quad \mathcal{O}(s_k^{(n-k)q_k/p_k - (n-k-1)})) + K \operatorname{sgn}(s_{n-1})) \end{aligned} \quad (23)$$

For the second term of (23) to be bounded while $s_k \rightarrow 0$, it is sufficient that $(n-k)q_k/p_k - (n-k-1) > 0$ for $s_k \rightarrow 0$ sequentially from $k = n-2$ to $k = 0$ so that the control u is bounded. $\square$

Another kind of singularity is that during the transient process towards the fast TSMs, the control may become singular if some s_i becomes zero. For example, for the second order SISO system

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= a(x) + b(x)u\end{aligned}\quad (24)$$

from Theorem 1, the controller can be derived as

$$u = -b^{-1}(x)(a(x) + \alpha_0 x_2 + \frac{\beta_0 q}{p} x_1^{(q-p)/p} x_2 + K \operatorname{sgn}(s)) \quad (25)$$

from which one can see when initially $x_2 \neq 0$ while $x_1 = 0$, the control u becomes singular.

We now analyze the dynamic characteristics of the fast TSM in relation to the singular problem.

In order to overcome the singularity problem, an effective means is to prohibit the trajectory from reaching the switching surface $s_i = 0$ ($i = 0, 1, \dots, j-1$) before $s_j = 0$ is reached (for some j). The idea is to find a region in the state space such that any trajectory starting from this region will not incur the singularity problem and the set of switching manifolds $s_{n-1} = 0, \dots, s_1 = 0, s_0 = 0$ are reached sequentially. For this purpose, we define a set as

$$\Omega = \{x : s_0 > 0\} \cap \{x : s_1 > 0\} \cap \dots \cap \{x : s_{n-1} > 0\}. \quad (26)$$

We shall prove in the following that the trajectory starting from Ω will result in switching manifolds $s_{n-1} = 0, \dots, s_1 = 0, s_0 = 0$ being reached sequentially and the singularity problem will not occur.

Lemma 1. *The set Ω is unempty open set, that is $\Omega \neq \emptyset$.*

Proof. From the definition of s_j ($j = 1, \dots, n-1$) in (13–15), we know $s_0 = x_1$. Select any nonzero initial values $s_0(0) = x_1(0)$ and $x_2(0)$ such that

$$x_2(0) + \alpha_0 s_0 + \beta_0 s_0^{q_0/p_0} > 0,$$

then $s_1(0) = x_2(0) + \alpha_0 s_0 + \beta_0 s_0^{q_0/p_0} > 0$. In general, for s_j ($j = 1, \dots, n-1$), from its definition in (13–15) we get

$$\begin{aligned}s_j &= \dot{s}_{j-1} + \alpha_{j-1} s_{j-1} + \beta_{j-1} s_{j-1}^{q_{j-1}/p_{j-1}} \\ &= \sum_{i=1}^j \left(\frac{\partial s_{j-1}}{\partial x_i} x_{i+1} \right) + \alpha_{j-1} s_{j-1} + \beta_{j-1} s_{j-1}^{q_{j-1}/p_{j-1}} \\ &= \sum_{i=1}^{j-1} \left(\frac{\partial s_{j-1}}{\partial x_i} x_{i+1} \right) + \alpha_{j-1} s_{j-1} + \beta_{j-1} s_{j-1}^{q_{j-1}/p_{j-1}} + \frac{\partial s_{j-1}}{\partial x_j} x_{j+1}.\end{aligned}$$

According to the recursive structure (13–15) and the system (12), it follows that s_{j-1} is independent of $x_{j+1}, \dots, x_n$ and $\frac{\partial s_{j-1}}{\partial x_j} = 1$. Hence,

$$s_j = \sum_{i=1}^{j-1} \left(\frac{\partial s_{j-1}}{\partial x_i} x_{i+1} \right) + \alpha_{j-1} s_{j-1} + \beta_{j-1} s_{j-1}^{q_{j-1}/p_{j-1}} + x_{j+1}.$$

This guarantees that when the $x_i(0)$ ($i = 1, \dots, j$) are given, one can always find a $x_{j+1}(0)$ such that $s_j(0) > 0$. Doing it in order, we can always find initial values $x_1(0), x_2(0), \dots, x_n(0)$ such that $s_i(0) > 0$ ($i = 0, \dots, n-1$). The proof of Ω being an unempty open set is easily followed by the fact that intersection of a set of open sets is also an open set since each set $\{x : s_j > 0\}$ is an open set.

Lemma 2. *For the system (12) with control law (17), if the initial condition $x(0) \in \Omega$, then trajectory $x(t)$ will first reach $s_{n-1} = 0$ and then $s_{n-2} = 0, \dots, s_0 = 0$ sequentially.*

Proof. From the recursive TSM structure (13–15), it follows that

$$\dot{s}_j = s_{j+1} - \alpha_j s_j - \beta_j s_j^{q_j/p_j}, \quad j = 1, \dots, n \quad (27)$$

Since $x(0)$ satisfies $s_j > 0$ ($j = 1, \dots, n-1$), from the dynamics (27) one can see that if $s_{n-2}(t)$ reaches zero firstly while $s_{n-1}(t) > 0$ we obtain at the time that $s_{n-2} = 0$ but $\dot{s}_{n-2} = s_{n-1} > 0$. This contradicts to the fact that since initially $s_{n-2} > 0$, $s_{n-1} > 0$, then $\dot{s}_{n-2} > 0$ which means $s_{n-2}(t)$ will never decrease before $s_{n-1}(t)$ becomes zero. This implies that the trajectory starting from $x(0) \in \Omega$ must reach switching surface $s_{n-1} = 0$ firstly before reaching $s_{n-2} = 0$. When $s_{n-1} = 0$, from (27), it follows that

$$s_{n-2} \dot{s}_{n-2} = -\alpha_{n-2} - \beta_{n-2} s_{n-2}^{(q_{n-2}+p_{n-2})/p_{n-2}} < 0$$

due to $p_{n-2} + q_{n-2} > 0$ and $\alpha_{n-2}, \beta_{n-2} > 0$. This shows that, once the state trajectory reaches $s_{n-2} = 0$, it will be confined to this manifold. Same analogy applies to other cases and we can conclude that s_i , $i = n-1, \dots, 1$ reach zero sequentially. $\square$

Theorem 4. *For the system (12), if the recursive TSM structure is selected as (13)–(15), the control law is (17), and Lemma 2 holds, then for $x(0) \in \Omega$, the state $x(t)$ will reach zero in finite time and the control $u(t)$ is bounded.*

Proof. Since the switching manifolds $s_i = 0$ ($i = n-1, \dots, 1$) are reached sequentially, we obtain that in finite time $s_0(t) = x_1(t)$ will approach zero. From the TSM structure $x_2(t), \dots, x_n(t)$ will all approach zero in finite time. Since

$$\frac{d}{dt}(s_i^{q_i/p_i}) = \frac{q_i}{p_i} s_i^{(q_i-p_i)/p_i} \dot{s}_i = \frac{q_i}{p_i} s_i^{(2q_i-p_i)/p_i}, \quad (28)$$

and the trajectory reaches $s_{i+1} = 0$ earlier than $s_i = 0$, and Lemma 2 holds, then the term $\frac{d}{dt}(s_i^{q_i/p_i})$ is bounded. For high order derivatives, since the conditions in Lemma 2 hold, the same analogy applies. Therefore, we conclude that $u_{eq}(t)$ as well as $u(t)$ is bounded for $x(0) \in \Omega$. $\square$

Remark 2. The analysis shown above shows that the system trajectory will reach zero in finite time if $x(0) \in \Omega$. However if an $x(0) \notin \Omega$ the singularity problem may occur. In this case this problem can be overcome by the two phase control proposed in [20].

Remark 3. For the nonlinear system (8) with external disturbances and noises v ,

$$\dot{x} = f(x) + g(x)u + d(x)v, \quad y = h(x),$$

if $d(x) \in \text{range}(g(x))$, i.e. the matching condition is satisfied [17], then a slight modification of control (17) will guarantee the stability, the reachability of the sliding modes and finite time reachability of the system equilibrium. Our recent study on robot control has shown the fast TSM control has a better control precision and robustness than its linear counterpart. In fact, fast TSM control is a high gain control when near the equilibrium.

Remark 4. The mechanism can be easily extended to the design of output tracking problem if z is replaced by $e = z_d - z$, where z_d is the desired output signals (for the problem statement, see [8,18]).

4 TSM Control of MIMO Systems

The TSM concept can be used for the control of MIMO systems [13]. Consider the MIMO system represented as follows:

$$\dot{x}_1 = A_{11}x_1 + A_{12}x_2 \quad (29)$$

$$\dot{x}_2 = A_{21}x_1 + A_{22}x_2 + B_2u \quad (30)$$

where $x_1 = (x_{1,1}, \dots, x_{1,n-m})^T \in R^{n-m}$, $x_2 = (x_{2,1}, \dots, x_{2,m})^T \in R^m$ are system states, $A_{11}, A_{12}, A_{21}, A_{22}$ are $(n-m) \times (n-m)$, $(n-m) \times m$, $m \times (n-m)$ and $m \times m$ matrices respectively. The pair (A_{11}, A_{12}) is controllable, B_2 is nonsingular and $n \leq 2m$. The FTSM vector is chosen as

$$s = C_1x_1 + C_2x_2 + C_3x_1^{q/p} \quad (31)$$

where $s \in R^m$, and C_1, C_2, C_3 are constant $m \times (n-m)$, $m \times m$, $m \times (n-m)$ matrices, respectively. $x_1^{q/p}$ represents a equal sized vector whose entries are (q/p) th power of the entries of x_1 .

Theorem 5. For the MIMO system (29) and (30), if the control is designed as

$$u = \begin{cases} -(C_2 B_2)^{-1}[(C_1 A_{11} + C_2 A_{21})x_1 \\ + \frac{q}{p} C_3 \text{diag}(x_1^{(q-p)/p})(A_{11}x_1 + A_{21}x_2) \\ - (C_1 A_{12} + C_2 A_{22})x_2 - K_1 \frac{s}{\|s\|}] & \text{for } s \neq 0 \text{ \& all } x_{1,i} \neq 0 \\ B_2^{-1}(-A_{21}x_1 - A_{22}x_2 - K_2 x_2) & \text{for } s = 0 \text{ \& any } x_{1,i} = 0 \end{cases} \quad (32)$$

with $K_1, K_2 > 0$ and $i = 1, \dots, n - m$, then $s = 0$ will be reached in finite time.

Proof. Substituting the control (32) into the time derivative of the Lyapunov function $V = (1/2)s^T s$ yields $\dot{V} = -K_1 \|s\| < 0$ for $s \neq 0$ and all $x_{1,i} \neq 0$ ($i = 1, \dots, n - m$) which is the sufficient condition for the reachability of $s = 0$. Details see [13]. $\square$

Remark 5. It can be seen from the Theorem 5, at the points such that $s = 0$ and any $x_{1,i} = 0$, $s^T s$ may be singular (unbounded). The second part of the control in (32) is to avoid these singularities. Indeed, when $s = 0$ and any $x_{1,i} = 0$, with the control, we have

$$\dot{x}_2 = -K_2 x_2$$

which solution is $x_2(t) = M \exp(-K_2 t)$. Therefore,

$$\dot{x}_1 = A_{11}x_1 + A_{12}M \exp(-K_2 t)$$

The second term on the right hand side of the equation plays the role of driving x_1 away from zero.

Theorem 6. For the system (29) and (30) with the TSM control (32), if the TSM parameter matrices are designed as

$$A_{11} - A_{12}C_2^{-1}C_1 = 0 \quad (33)$$

$$A_{12}C_2^{-1}C_3 = \text{diag}(\rho_1, \dots, \rho_{n-m}) \quad (34)$$

where $\rho_i > 0, i = 1, \dots, n - m$, then the system states will reach zero in finite time in the TSM.

Proof. See [13]. $\square$

When in the sliding mode, the condition $2q > p > q$ has to be satisfied in order to avoid the singularity problem, similar to the SISO case [13].

The condition (33) may be too strong. We now look at an alternative. When the sliding mode $s = 0$ is reached, i.e.

$$x_2 = -C_2^{-1}C_1x_1 - C_2^{-1}C_3x_1 \quad (35)$$

Substituting (35) into (29) leads to

$$\dot{x}_1 = (A_{11} - A_{12}C_2^{-1}C_1)x_1 - A_{12}C_2^{-1}C_3x_1^{q/p} \quad (36)$$

With the conditions (33) and (34), (36) becomes

$$\dot{x}_1 = -\text{diag}(\rho_1, \dots, \rho_{n-m})x_1 \quad (37)$$

Since (37) is in diagonal form, it can be treated in the same way as for the scalar TSM.

However the conditions (33) can be relaxed. By a properly chosen C_1 and C_2 , the matrix $A_{11} - A_{12}C_2^{-1}C_1$ can have all its eigenvalues on the right hand side of the complex plane[29].

Qualitatively, when the system state is far away from the equilibrium $x_1 = 0$, then in terms of magnitude of power, the dynamics is dominated by

$$\dot{x}_1 = (A_{11} - A_{12}C_2^{-1}C_1)x_1 \quad (38)$$

which is asymptotically stable. The state will reach a neighborhood of $x_1 = 0$ in finite time. When the system state is near the equilibrium $x_1 = 0$, using the same reasoning as above, the dynamics is dominated by

$$\dot{x}_1 = -A_{12}C_2^{-1}C_3x_1^{q/p} \quad (39)$$

which will give rise to a finite time reachability when near the equilibrium, because $A_{12}C_2^{-1}C_3$ is diagonal and each entry of x_1 reaches the zero in finite time so x_1 will reach the equilibrium in finite time.

We will now look at the stability of the system at the equilibrium. For convenience, denote $L_1 = A_{11} - A_{12}C_2^{-1}C_1$, $L_2 = -A_{12}C_2^{-1}C_3$. Consider the Lyapunov function candidate

$$V = x_1^T x_1$$

Differentiating it along the dynamics (38) leads to

$$\begin{aligned} \dot{V} &= \dot{x}_1^T x_1 + x_1^T \dot{x}_1 \\ &= x_1^T (L_1 + L_1^T)x_1 + (x_1^{q/p})^T L_2^T x_1 + x_1^T L_2 x_1^{q/p} \\ &= x_1^T (L_1 + L_1^T)x_1 - \sum_{i=1}^{n-m} \rho_i x_{1,i}^{(p+q)/p} \end{aligned} \quad (40)$$

where $x_{1,i}$ represents the i th entry of vector x_1 . If the following condition holds,

$$\text{Re}\{\lambda(L_1^T + L_1)\} < 0 \quad (41)$$

i.e. the real parts of the eigenvalues of $L_1^T + L_1$ are negative, since $p + q > 0$ is even, then $\dot{V} < 0$, which means the equilibrium is stable. Since we have C_1 and C_2 which have excessive entries to be used for tuning, condition (41) is not hard to fulfil. In summary, if the following conditions are satisfied

1. The matrices C_1 and C_2 are chosen such that

$$\operatorname{Re}\{\lambda(A_{11} - A_{12}C_2^{-1}C_1)\} < 0$$

and

$$\lambda(A_{11} - A_{12}C_2^{-1}C_1 + (A_{11} - A_{12}C_2^{-1}C_1)^T) < 0$$

2. The matrices C_2 and C_3 are chosen such that

$$A_{12}C_2^{-1}C_3 = \operatorname{diag}(\rho_1, \dots, \rho_{n-m})$$

for some $\rho_i > 0$, $i = 1, \dots, n - m$,

then the equilibrium is globally stable and will be reached in finite time.

5 Continuous Approximation of TSM Control

In general, when the TSM control strategies are implemented, due to the switchings involved, chattering may occur which must be eliminated so the controller can perform properly. This can be achieved by smoothing out the control discontinuity in a thin boundary layer neighboring the switching manifold. A common practice is to replace the switching function sgn by a continuous saturation function

$$\operatorname{sat}(s, \phi) = \begin{cases} \frac{s}{\phi} & |s(t)| < \phi \\ \operatorname{sgn}(s(t)) & |s(t)| \geq \phi \end{cases} \quad (42)$$

where ϕ is the width of the layer $S_\phi = \{x, |s(t)| < \phi\}$. It is clear that outside S_ϕ , the convergence toward $s = 0$ is maintained. The problem of interest is the dynamic behaviors inside S_ϕ .

For the first order TSM (24), we have $s(t) = \dot{x}_1 + \beta x_1^{q/p}$ subject to $|s| < \phi$. Define a Lyapunov function $V = |x_1|$. Then its time derivative along the dynamics $s(t)$ becomes

$$\begin{aligned} \dot{V} &= \operatorname{sgn}(x_1)\dot{x}_1 \\ &= \operatorname{sgn}(x_1)(s(t) - \beta x_1^{q/p}) \\ &= \operatorname{sgn}(x_1)s(t) - \beta|x_1|^{q/p} \\ &< |s| - \beta|x_1|^{q/p} < \phi - \beta|x_1|^{q/p} \end{aligned} \quad (43)$$

which indicates that x_1 will converge to the region defined by

$$\beta|x_1|^{q/p} < \phi$$

which leads to

$$|x_1| < \left(\frac{\phi}{\beta}\right)^{p/q} \quad \text{subject to } |s| < \phi \quad (44)$$

Inequality (44) indicates a significant improvement of steady state error in comparison with the linear switching manifold in that because $q < p$, if $\phi < \beta$, then the power of $p/q (> 1)$ would significantly reduce the steady state error. This is one of the advantage of using TSM (24). For the fast TSM (5) analytic expression similar to (44) is not obvious but one can obtain an approximate inequality

$$|x_1| < \left(\frac{\phi}{\alpha + \beta} \right)^{p/q}$$

for $|x_1| < 1$.

For the general SISO case, suppose the width of the boundary layer for s_{n-1} is $\phi > 0$. We can easily conclude that for the hierarchical structure (13)–(15) with $\alpha_i = 0$ ($i = 0, \dots, n-2$), we have

$$|s_{n-2}(t)| < \left(\frac{\phi}{\beta_{n-2}} \right)^{p_{n-2}/q_{n-2}} \quad \text{subject to } |s_{n-1}| < \phi$$

It is then easily obtained that for $s_0(t) = x_1$, we have

$$|x_1| < \frac{\phi \prod_{i=0}^{n-2} \frac{p_i}{q_i}}{\prod_{i=0}^{n-2} \beta_i \prod_{j=0}^i \frac{p_j}{q_j}} \quad \text{subject to } |s_{n-1}| < \phi$$

6 Nonsingular TSM Control

One of the problems of using TSM control is its singularity. Although methods to avoid the singularity problem have been discussed in the above sections, it would be ideal to have a finite time control mechanism that does not incur the singularity problem. In this section, we discuss a simple nonsingular TSM control.

The simple nonsingular TSM control is based on the following modified TSM model [4]

$$s = \beta x_1 + x_2^{p/q} \quad (45)$$

where $x = (x_1, x_2)^T$ are the state of the second order system (24) and p, q are defined as before, and $\beta > 0$ is a design parameter. The key point of using (45) is that when differentiating s , it does not result in terms with negative powers.

We now develop a controller for the second order system (24) to see how effective the TSM model (45) is in terms of removing the singularity problem.

For the switching function (45), its time derivative along the dynamics (24) is

$$\begin{aligned} \dot{s} &= \beta \dot{x}_1 + \frac{p}{q} x_2^{p/q-1} \dot{x}_2 \\ &= \beta x_2 + \frac{p}{q} x_2^{p/q-1} (f(x) + b(x)u) \end{aligned} \quad (46)$$

If the control u is taken as

$$u = -b^{-1}(x)(f(x) + \beta \frac{q}{p} x_2^{(2q-p)/q} + K \operatorname{sgn}(s)) \quad (47)$$

then we have

$$s\dot{s} = -K|x_2|^{(p-q)/q}|s| < 0$$

One question is if the TSM $s = 0$ will be reached in finite time. The answer is yes. Indeed, substituting the control (47) into second equation of the second order system (24) yields

$$\dot{x}_2 = -\frac{q}{p} x_2^{(2q-p)/q} - K \operatorname{sgn}(s) \quad (48)$$

It can be easily seen that if $x_2 = 0$, then (48) becomes

$$\dot{x}_2(t) = -K \operatorname{sgn}(s)$$

which suggests that $x_2 = 0$ while $x_1 \neq 0$ is not an attractor. For cases of $s > 0$ and $s < 0$, we obtain $\dot{x}_2 < -K$ and $\dot{x}_2 > K$ respectively. It means there exists an vicinity of $x_2 = 0$, $|x_2(t)| < \delta$ for a small $\delta > 0$, we have $\dot{x}_2 < -K$ for $s > 0$ and $\dot{x}_2 > K$ for $s < 0$ respectively. Therefore the crossing of trajectory from one boundary of the vicinity $x_2 = \delta$ to the other boundary $x_2 = -\delta$ for $s > 0$ and from $x_2 = -\delta$ to $x_2 = \delta$ for $s < 0$ is finite time. For region outside the $|x_2(t)| < \delta$, since the conventional sliding mode control structure is employed and s will not change sign before $s = 0$ is reached, the time to reach the boundaries of the vicinity is finite. For the region outside the $|x_2(t)| < \delta$, the time to reach the boundaries of the vicinity is finite. Indeed, we can easily show that

$$s\dot{s} < -\delta K|s|,$$

meaning the finite time reachability of the boundaries. Therefore we can conclude that the TSM $s = 0$ will be reached from anywhere in the state space in finite time.

The nonsingular TSM model can be used for the control design of a class of nonlinear dynamical systems such as

$$\dot{\mathbf{x}}_1 = \mathbf{f}_1(\mathbf{x}_1, \mathbf{x}_2) \quad (49)$$

$$\dot{\mathbf{x}}_2 = \mathbf{f}_2(\mathbf{x}_1, \mathbf{x}_2) + g(\mathbf{x}_1, \mathbf{x}_2) + \mathbf{b}(\mathbf{x}_1, \mathbf{x}_2)\mathbf{u} \quad (50)$$

where $\mathbf{x}_1 \in R^n$, $\mathbf{x}_2 \in R^n$, $\mathbf{f}_1$ and $\mathbf{f}_2$ are smooth vector functions and g represents the uncertainties and disturbances satisfying $\|g(\mathbf{x}_1, \mathbf{x}_2)\| < l_g$ where $l_g > 0$, $\mathbf{b}$ is a nonsingular matrix and $\mathbf{u} \in R^n$ is the control vector. Such systems can be found in mechanical systems such as the robot control systems. If we have that $(\mathbf{x}_1, \mathbf{x}_2) = (0, 0)$ if and only if $(\mathbf{x}_1, \dot{\mathbf{x}}_1) = (0, 0)$, we can use the following nonsingular TSM

$$s = \Lambda \mathbf{x}_1 + \dot{\mathbf{x}}_1^T \quad (51)$$

where $\Lambda = \text{diag}(\beta_1, \dots, \beta_n)$, $(\beta_i > 0)$, $\Gamma = \text{diag}(\gamma_1, \dots, \gamma_n)$ with $\gamma_i = p_i/q_i$ ($1 < \gamma_i < 2$) for $i = 1, \dots, n$, and $\mathbf{x}_1^\Gamma$ is represented as

$$\mathbf{x}_1^\Gamma = (x_{11}^{\gamma_1}, \dots, x_{1n}^{\gamma_n})^T$$

We also adopt the notion that

$$\dot{\mathbf{x}}_1^\Gamma = \Gamma \text{diag}(x_{11}^{\gamma_1-1}, \dots, x_{1n}^{\gamma_n-1}) \dot{\mathbf{x}}_1 \quad (52)$$

which can be easily verified. Similar control as in (47) can be designed for (49) and (50). Consider the Lyapunov function

$$V = \frac{1}{2} s^T s \quad (53)$$

If the control $\mathbf{u}$ is chosen as

$$\begin{aligned} \mathbf{u} = & - \left(\frac{\partial \mathbf{f}_1}{\partial \mathbf{x}_2} B(\mathbf{x}_1, \mathbf{x}_2) \right)^{-1} \left(K \frac{s}{\|s\|} + \frac{\partial \mathbf{f}_1}{\partial \mathbf{x}_2} \mathbf{f}_2(\mathbf{x}_1, \mathbf{x}_2) + \frac{\partial \mathbf{f}_1}{\partial \mathbf{x}_1} \mathbf{f}_1(\mathbf{x}_1, \mathbf{x}_2) + \right. \\ & \left. \Gamma^{-1} \Lambda \text{diag}(x_{11}^{2-\beta_1}, \dots, x_{1n}^{2-\beta_n}) \right) \end{aligned} \quad (54)$$

then the time derivative of the Lyapunov function (53) becomes

$$\dot{V} \leq -K \min_i (|x_{1i}|^{\gamma_i-1}) \|s\|$$

which indicates the finite time convergence of $s = 0$.

One can easily see that the nonsingular TSM control (54) does not involve any terms which has negative powers. When in the sliding mode $s = 0$, we have

$$\beta_i x_{1i} + \dot{x}_{1i}^{\gamma_i} = 0$$

which is equivalent to

$$\dot{x}_{1i} + \beta_i^{1/\gamma_i} x_{1i}^{1/\gamma_i} = 0$$

which finite time convergence is well documented in the previous sections. Hence we can claim the the nonsingular TSM control can deliver finite time convergence without any singularity.

7 Discussions and Conclusions

We have examined the dynamic properties of several TSM models and their applications in the control design. Because the TSM does not satisfy the Lipschitz condition, usual sense of Lyapunov stability does not apply. The well known existence and uniqueness of solutions of differential equations does not apply either. The solutions may intersect each other. Although TSMs introduce certain degree of mathematical difficulties in analyzing their behaviors, such introduction of a special form of nonlinearity does improve the system dynamical convergence. Future work can be pursued in applications

of TSM to control problems that require high precision and integration with other control (or non control) methods such as optimal control. Preliminary studies along this line have been conducted in [23], where the TSM is combined with the frequency shaping optimal sliding mode control to give rise to a better dynamical performance, and in [27] where the TSM is used for derivative signal estimators. There is a close relationship between the second order sliding modes and higher order sliding modes [1,11] and the TSM which is worth pursuing as well.

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Adaptive Backstepping Control

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Abstract. Adaptive backstepping algorithms for a class of nonlinear continuous uncertain processes with disturbances are considered. Sliding mode control using a combined adaptive backstepping sliding mode control algorithm is also studied. The algorithms follow a systematic procedure for the design of adaptive control laws for the output of observable minimum phase nonlinear systems. This class of systems may include unmatched uncertainty including disturbances and unmodelled dynamics. The design methods are based upon (i) the backstepping approach, and (ii) a combination of sliding and backstepping.

1 Introduction

The backstepping procedure is a systematic design technique for globally stable and asymptotically adaptive tracking controllers for a class of nonlinear systems. The backstepping approach has been developed in the last decade [4]-[13] for the systematic design of controllers for nonlinear systems, both with and without unmatched parametric uncertainty. Various backstepping control design algorithms compiled in [8], provide a systematic framework for the design of tracking and regulation strategies suitable for large classes of nonlinear systems.

Adaptive backstepping algorithms have been applied to systems which can be transformed into a triangular form, in particular, the parametric pure feedback (PPF) form and the parametric strict feedback (PSF) form [4]. This method has been studied widely in recent years [4], [9], [15]-[17]. When plants include uncertainty with lack of information about the bounds of unknown parameters, adaptive control is more convenient; whilst, if some information about the uncertainty, e.g. bounds, is available, robust control is usually employed. Sliding mode control (SMC) is a robust control method, and backstepping can be considered to be a method of adaptive control. The combination of these methods yields benefits from both approaches. A systematic design procedure has been proposed to combine adaptive control and SMC for nonlinear systems with relative degree one [20].

An algorithm for the synthesis of dynamical adaptive backstepping controllers (Ríos-Bolívar [11] and Ríos-Bolívar *et al* [12]) yields a *dynamical* adaptive backstepping algorithm overcoming the limitations of the *static* adaptive backstepping algorithm designs of Krstić *et al* [9]. This algorithm allows one

to apply the backstepping approach to a larger class of uncertain nonlinear systems, satisfying observability and minimum phase conditions, which may be in either triangular or nontriangular canonical forms, i.e. not in PPF or PSF forms [15]-[17]. A symbolic algebra toolbox allows straightforward design [13] of *dynamical* backstepping control.

The adaptive sliding backstepping control of parametric semi-strict feedback systems (PSSF) with disturbances has been studied by Koshkouei and Zinober [5], [7]. The method ensures that the error state trajectories move on a sliding hyperplane. The sufficient condition for the existence of the sliding mode (in Ríos-Bolívar and Zinober [13],[14]) is not needed. The plant may contain unmodelled terms and unmeasurable external disturbances, bounded by known functions. The classical backstepping method has been extended to this class of systems [6] to achieve the output tracking of a dynamical reference signal.

A dynamic adaptive backstepping controller is presented in Section 2. It is made more robust by including SMC (Section 3) and Second-Order SMC (Section 4). An improved adaptive backstepping method for PSSF systems is studied in Section 5 while sliding backstepping is considered in Section 5.2. These approaches are extended to non-triangular systems in Sections 7 and 8. Illustrative examples are given in Sections 6 and 9 and some conclusions are presented in Section 10.

2 Dynamical Adaptive Backstepping

The Dynamical Adaptive Backstepping (DAB) algorithm was developed by Ríos-Bolívar [11], [12]. It combines the backstepping approach with tuning functions, developed by Krstić [9], with a dynamic input-output linearization. It works for nonlinear uncertain systems satisfying observability and minimum phase conditions, which may be in either triangular or nontriangular forms. It generates an adaptive nonlinear control for regulation or tracking in an iterative and systematic fashion. An interlaced tuning function parameter estimate update law compensates for uncertain parameters. Consider

$$\begin{aligned}\dot{x} &= f_0(x) + \Phi(x)\theta + \left(g_0(x) + \Psi(x)\theta\right)u \\ y &= h(x)\end{aligned}\tag{1}$$

where $x \in \mathbb{R}^n$ is the state; $u, y \in \mathbb{R}$ the input and output respectively; and $\theta = [\theta_1, \dots, \theta_p]^T$ is a vector of unknown parameters. f_0 , g_0 and the columns of the matrices $\Phi, \Psi \in \mathbb{R}^{n \times p}$ are smooth vector fields in a neighbourhood R_0 of the origin $x = 0$ with $f_0(0) = 0$, $g_0(0) \neq 0$; and h is a smooth scalar function also defined in R_0 . It is assumed the relative degree of (1) with respect to u is $1 \leq \rho \leq n$.

The steps leading to the design of the dynamical adaptive compensator, follow an input-output linearization procedure in which, at each step, a

control dependent nonlinear mapping and a tuning function are constructed. The parameter update law and the dynamical adaptive control law which stabilize the controlled plant, are designed at the final step. This method is applicable to a class of nonlinear systems which satisfying observable and minimum phase conditions. Therefore the following conditions should be assumed.

Assumption 1 *System (1) is locally observable in a subspace $R_1 \subset R_0 \subset \mathbb{R}^n$.*

Assumption 2 *System (1) is minimum phase in $R_1 \subset R_0 \subset \mathbb{R}^n$.*

A full proof is in [12]. The algorithm is as follows:

DAB Algorithm

Coordinate transformation

$$\begin{aligned} z_1 &= y - y_r(t) = h^{(0)}(x) - y_r(t) \\ z_k &= \hat{h}^{(k-1)}(\cdot) - y_r^{(k-1)}(t) + \alpha_{k-1}(\cdot), \quad 2 \leq k \leq n \end{aligned} \quad (2)$$

with

$$\begin{aligned} \hat{h}^{(k)} &= \frac{\partial \hat{h}^{(k-1)}}{\partial \hat{\theta}} \tau_k + \frac{\partial \hat{h}^{(k-1)}}{\partial x} \left[f_0 + \Phi \hat{\theta} + (g_0 + \Psi \hat{\theta}) v_1 \right] \\ &\quad + \sum_{i=1}^{k-\rho-1} \frac{\partial \hat{h}^{(k-1)}}{\partial v_i} v_{i+1} + \frac{\partial \hat{h}^{(k-1)}}{\partial t} \end{aligned} \quad (3)$$

$$\omega_k = \left(\frac{\partial \hat{h}^{(k-1)}}{\partial x} + \frac{\partial \alpha_{k-1}}{\partial x} \right) \left(\Phi(x) + v_1 \Psi(x) \right) \quad (4)$$

$$\begin{aligned} \alpha_k &= z_{k-1} + \left(\sum_{i=2}^{k-1} z_i \frac{\partial \hat{h}^{(i-1)}}{\partial \hat{\theta}} + \sum_{i=3}^{k-1} z_i \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}} \right) \Gamma \omega_k^T \\ &\quad + \sum_{i=1}^{k-\rho-1} \frac{\partial \alpha_{k-1}}{\partial v_i} v_{i+1} + \frac{\partial \alpha_{k-1}}{\partial \hat{\theta}} \tau_k + \frac{\partial \alpha_{k-1}}{\partial t} \\ &\quad + \frac{\partial \alpha_{k-1}}{\partial x} \left[f_0 + \Phi \hat{\theta} + (g_0 + \Psi \hat{\theta}) v_1 \right] + c_k z_k \end{aligned} \quad (5)$$

$$\tau_k = \Gamma \sum_{i=1}^k \omega_k^T z_k \quad (6)$$

Parameter update law

$$\dot{\hat{\theta}} = \tau_n = \Gamma W^T z = \Gamma \begin{bmatrix} \omega_1^T & \omega_2^T & \dots & \omega_n^T \end{bmatrix} z \quad (7)$$

Dynamical adaptive compensator

$$\begin{aligned}
\dot{v}_1 &= v_2 \\
\dot{v}_2 &= v_3 \\
&\vdots \\
\dot{v}_{n-\rho} &= \frac{1}{\left(\frac{\partial \hat{h}^{(n-1)}}{\partial v_{n-\rho}} + \frac{\partial \alpha_{n-1}}{\partial v_{n-\rho}}\right)} \left[-z_{n-1} + y_r^{(n)} - \frac{\partial \hat{h}^{(n-1)}}{\partial t} - \frac{\partial \alpha_{n-1}}{\partial t} \right. \\
&\quad \left. - \left(\frac{\partial \hat{h}^{(n-1)}}{\partial x} + \frac{\partial \alpha_{n-1}}{\partial x} \right) \left[f_0 + \Phi \hat{\theta} + (g_0 + \Psi \hat{\theta}) v_1 \right] \right. \\
&\quad \left. + \left(\sum_{i=2}^{n-1} z_i \frac{\partial \hat{h}^{(i-1)}}{\partial \hat{\theta}} + \sum_{i=3}^{n-1} z_i \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}} \right) \Gamma \omega_n^T \right. \\
&\quad \left. - \left(\frac{\partial \hat{h}^{(n-1)}}{\partial \hat{\theta}} + \frac{\partial \alpha_{n-1}}{\partial \hat{\theta}} \right) \tau_n - c_n z_n \right. \\
&\quad \left. - \sum_{i=1}^{n-\rho-1} \left(\frac{\partial \hat{h}^{(n-1)}}{\partial v_i} + \frac{\partial \alpha_{n-1}}{\partial v_i} \right) v_{i+1} \right]
\end{aligned} \tag{8}$$

with $v_1 = u$, the c_i 's constant design parameters and $\Gamma = \Gamma^T > 0$ the adaptation gain matrix. The control u is obtained implicitly as the solution of the nonlinear time-varying differential equation (8).

3 Dynamical Adaptive Backstepping with Sliding

The robust combination of sliding mode and adaptive control methods has been studied in recent years. To provide robustness, the adaptive backstepping algorithm can be modified to obtain adaptive *sliding* output tracking controllers. The modification is carried out at the final step of the algorithm by incorporating the sliding surface, defined in terms of the error coordinates

$$\sigma = k_1 z_1 + \dots + k_{n-1} z_{n-1} + z_n = 0 \tag{9}$$

where the scalar coefficients $k_i > 0$, $i = 1, \dots, n-1$, are chosen in such a manner that the polynomial

$$p(s) = k_1 + k_2 s + \dots + k_{n-1} s^{n-2} + s^{n-1} \tag{10}$$

in the complex variable s is Hurwitz. The update law is

$$\begin{aligned}
\dot{\hat{\theta}} &= \tau_n = \tau_{n-1} + \Gamma \sigma \left(\omega_n + \sum_{i=1}^{n-1} k_i \omega_i \right) \\
&= \Gamma \left(\sum_{i=1}^{n-1} z_i \omega_i + \sigma \left(\omega_n + \sum_{i=1}^{n-1} k_i \omega_i \right) \right)
\end{aligned} \tag{11}$$

and the dynamical adaptive sliding mode output tracking controller is found via

$$\begin{aligned}
& z_{n-1} + \hat{h}^{(n)}(\cdot) - y_r^{(n)}(t) + \alpha_n(\cdot) \\
& + \left(\sum_{i=2}^{n-1} z_i \frac{\partial \hat{h}^{(i-1)}}{\partial \hat{\theta}} + \sum_{i=3}^{n-1} z_i \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}} \right) \Gamma \left(\omega_n + \sum_{i=1}^{n-1} k_i \omega_i \right) \\
& - \sum_{i=1}^{n-1} k_i \left(\sum_{j=2}^{i-1} z_j \frac{\partial \hat{h}^{(j-1)}}{\partial \hat{\theta}} + \sum_{j=3}^{i-1} z_j \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}} \right) \Gamma \omega_i \\
& + \sum_{i=1}^{n-1} k_i \left(\frac{\partial \hat{h}^{(i-1)}}{\partial \hat{\theta}} + \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}} \right) (\tau_n - \tau_i) \\
& + \sum_{i=1}^{n-1} k_i (-z_{i-1} - c_i z_i + z_{i+1}) \\
& = -\kappa \sigma - \beta \text{sign}(\sigma)
\end{aligned} \tag{12}$$

with $\kappa > 0$, $\beta > 0$ and α_n defined by

$$\begin{aligned}
\alpha_n(\cdot) &= \frac{\partial \alpha_{n-1}}{\partial \hat{\theta}} \tau_n + \frac{\partial \alpha_{n-1}}{\partial x} \left[f_0 + \Phi \hat{\theta} + (g_0 + \Psi \hat{\theta}) u \right] \\
&+ \sum_{i=1}^{n-\rho-1} \frac{\partial \alpha_{n-1}}{\partial u^{(i-1)}} u^{(i)} + \frac{\partial \alpha_{n-1}}{\partial t}
\end{aligned} \tag{13}$$

4 Dynamical Adaptive Backstepping with Higher-Order Sliding

The main characteristic of sliding mode control is the theoretically infinite switching of the control law, which provides robustness against measurement errors and unmodelled dynamics. In practice, however, the switching occurs at a finite rate. This can lead to the chattering effect, high frequency vibration of the controlled plant.

Higher order sliding modes (HOSM) may be used to eliminate the chattering and provide improved accuracy in the steady state [10]. Sliding modes $\sigma \equiv 0$ may be classified by the number r of the first total derivative $\sigma^{(r)}$ which is not a continuous function of the state variables. This number r is called the *sliding order*. The r -th order sliding mode is determined by the equalities

$$\sigma = \dot{\sigma} = \ddot{\sigma} = \dots = \sigma^{(r-1)} = 0$$

The standard sliding mode is of the first order. The motion in the r -th order sliding mode is the same as an ideal 1-sliding mode, while the switching is

moved to the higher derivatives of the sliding surface, removing the chattering.

The precision of the standard sliding mode is proportional to the time interval between measurements τ , while r -th order sliding modes may provide up to the r -th order of sliding precision i.e. $\sup|\sigma| \leq C\tau^r$.

We present the DAB procedure with a second-order sliding mode controller (SOSMC) used at the final step. This improves the accuracy of the original DAB algorithm and greatly simplifies the control. A previous DAB-SOSMC algorithm used a law devised by Bartolini [1],[18]. This new method replaces that law with one by Levant [10], known as the twisting algorithm. This generates a simpler control structure.

4.1 Twisting Algorithm

The twisting algorithm is

$$\dot{u} = \begin{cases} -u & |u| > 1 \\ -\alpha_m \text{sign}(\sigma) & |u| \leq 1, \sigma\dot{\sigma} \leq 0 \\ -\alpha_M \text{sign}(\sigma) & |u| \leq 1, \sigma\dot{\sigma} > 0 \end{cases} \quad (14)$$

with $0 < \alpha_m < \alpha_M$. This has a finite time convergence to the manifold $\sigma = \dot{\sigma} = 0$ and is valid even with large disturbances. For calculation purposes, the first difference of σ , $\Delta\sigma = \sigma(t_{i+1}) - \sigma(t_i)$, can be used instead of the derivative. The algorithm can also be rearranged to

$$\dot{u} = \begin{cases} -u & |u| > 1 \\ -\frac{1}{2}((\alpha_m + \alpha_M) \text{sign}(\sigma) + (\alpha_M - \alpha_m) \text{sign}(\Delta\sigma)) & |u| \leq 1 \end{cases} \quad (15)$$

The region $|u| < 1$ is obviously attractive. If we define

$$L_u(\cdot) = \frac{\partial}{\partial x}(\cdot)\dot{x} + \frac{\partial}{\partial t}(\cdot)$$

as the total derivative with respect to time, considering u as constant, then

$$\begin{aligned} \ddot{\sigma} &= \frac{d}{dt}L_u(\sigma) \\ &= L_u(L_u(\sigma)) + \frac{\partial}{\partial u}L_u(\sigma)\dot{u} \end{aligned}$$

Assuming that

$$\begin{aligned} |L_u L_u \sigma| &\leq C \\ 0 < K_1 &< \frac{\partial}{\partial u}L_u(\sigma) < K_2 \end{aligned}$$

then we can say that

$$\ddot{\sigma} \in [-C, C] + [K_1, K_2]\dot{u}$$

Thus the trajectory in the $\sigma, \dot{\sigma}$ plane is bounded by the majorant curves [2]

$$\begin{aligned} |\sigma| + \frac{1}{2} \frac{\dot{\sigma}^2}{K_1 \alpha_M - C} &= \text{constant}, \quad \sigma \dot{\sigma} > 0 \\ |\sigma| + \frac{1}{2} \frac{\dot{\sigma}^2}{K_2 \alpha_m + C} &= \text{constant}, \quad \sigma \dot{\sigma} \leq 0 \end{aligned}$$

and ‘twists’ into the origin inside these curves (Fig. 1). If the intersections with the $\sigma = 0$ axis are denoted by $\dot{\sigma}_i$, ($i = 0, 1, 2, \dots$), then

$$\left| \frac{\dot{\sigma}_{i+1}}{\dot{\sigma}_i} \right| < \sqrt{\frac{K_2 \alpha_m + C}{K_1 \alpha_M - C}} < 1 \quad (16)$$

The origin is reached in a finite time, which can be estimated by a geometric series.

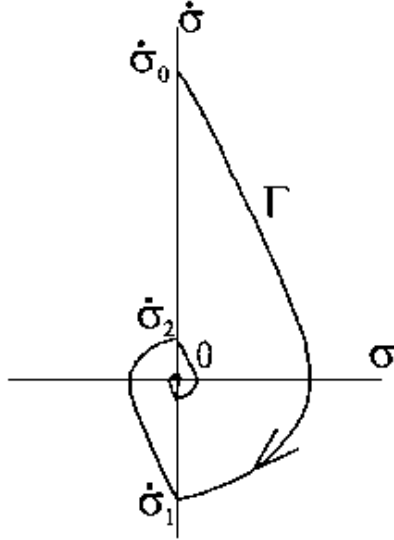


Fig. 1. Bounding curve for twisting algorithm

4.2 DAB-TWISTING algorithm

By linking the DAB and twisting algorithms, we can create a second order backstepping sliding mode controller. Stopping the DAB process at the n th stage, we have the n error variables given by (2). We then generate the sliding

surface (9) (an improved sliding function compared with [18]) and use the Lyapunov function

$$V_n = \frac{1}{2} \sum_{i=1}^{n-1} z_i^2 + \frac{1}{2} \sigma^2 + \frac{1}{2} (\theta - \hat{\theta})^T \Gamma^{-1} (\theta - \hat{\theta}) \quad (17)$$

Taking the derivative along the manifold $\sigma = 0$ i.e. $z_n = -\sum_{i=1}^{n-1} k_i z_i$, we have

$$\begin{aligned} \dot{V}_n = & -\sum_{i=1}^{n-1} c_i z_i^2 - \sum_{i=1}^{n-1} k_i z_i z_{n-1} + (\theta - \hat{\theta})^T \Gamma^{-1} (\dot{\theta} - \tau_{n-1}) \\ & + \left(\sum_{i=2}^{n-1} z_i \frac{\partial \hat{h}^{(i-1)}}{\partial \theta} + \sum_{i=3}^{n-1} z_i \frac{\partial \alpha_{i-1}}{\partial \theta} \right) (\dot{\theta} - \tau_{n-1}) + \sigma (\dot{\sigma} + z_{n-1}) \end{aligned}$$

Choosing the dynamic control law

$$\begin{aligned} \dot{v}_1 &= v_2 \\ \dot{v}_2 &= v_3 \\ &\vdots \\ \dot{v}_{n-\rho} &= v_{n-\rho+1} \\ \dot{v}_{n-\rho+1} &= \begin{cases} -v_{n-\rho+1} & |v_{n-\rho+1}| > 1 \\ -\frac{1}{2} \left((\alpha_m + \alpha_M) \text{sign}(\sigma) \right. \\ \quad \left. + (\alpha_M - \alpha_m) \text{sign}(\Delta\sigma) \right) & |v_{n-\rho+1}| \leq 1 \end{cases} \end{aligned} \quad (18)$$

with $v_1 = u$, we can attain $\sigma \equiv 0$ in a finite time. Taking the parameter estimate update law $\dot{\theta} = \tau_{n-1}$, this yields

$$\begin{aligned} \dot{V}_n &= -\sum_{i=1}^{n-1} c_i z_i^2 - \sum_{i=1}^{n-1} k_i z_i z_{n-1} \\ &= -[z_1 \dots z_{n-1}]^T Q [z_1 \dots z_{n-1}] \\ &\leq 0 \end{aligned}$$

where $Q \in \Re^{(n-1) \times (n-1)}$

$$Q = \begin{bmatrix} c_1 & 0 & \dots & 0 \\ 0 & c_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ k_1 & k_2 & \dots & k_{n-1} + c_{n-1} \end{bmatrix} > 0 \quad (19)$$

Hence the error variables z_i will tend to zero, and in particular $h(x) - y_r(t) \rightarrow 0$.

5 Parametric Semi-Strict Feedback Systems

We next consider the parametric semi-strict feedback form (PSSF) [5]-[7],[19] with disturbances

$$\begin{aligned}\dot{x}_i &= x_{i+1} + \varphi_i^T(x_1, x_2, \dots, x_i)\theta + \eta_i(x, w, t), \quad 1 \leq i \leq n-1 \\ \dot{x}_n &= f(x) + g(x)u + \varphi_n^T(x)\theta + \eta_n(x, w, t) \\ y &= x_1\end{aligned}\quad (20)$$

where $x = [x_1, x_2, \dots, x_n]^T$ is the state, y the scalar output, u the scalar control and $\varphi_i(x_1, \dots, x_i) \in \mathbb{R}^p$, $i = 1, \dots, n$, are known functions which are assumed to be sufficiently smooth. $\theta \in \mathbb{R}^p$ is the vector of constant unknown parameters, $\eta_i(x, w, t)$, $i = 1, \dots, n$, are unknown nonlinear scalar functions including all the disturbances, and w is an uncertain time-varying parameter.

Assumption The functions $\eta_i(x, w, t)$, $i = 1, \dots, n$ are bounded by known positive functions $h_i(x_1, \dots, x_i) \in \mathbb{R}^p$, i.e.

$$|\eta_i(x, w, t)| \leq h_i(x_1, \dots, x_i), \quad i = 1, \dots, n. \quad (21)$$

The output y should track a specified bounded reference signal $y_r(t)$ with bounded derivatives up to n -th order. Assume that $\tilde{\theta} = \theta - \hat{\theta}$ where $\hat{\theta}(t)$ is an estimate of the unknown parameter θ .

5.1 Backstepping Algorithm

The PSSF algorithm is summarized as follows [6]:

PSSF Algorithm

Coordinate transformation

$$\begin{aligned}z_1 &= x_1 - y_r \\ z_k &= x_k - \alpha_{k-1} - y_r^{(k-1)}\end{aligned}\quad (22)$$

with

$$\begin{aligned}\omega_k &= \varphi_k(x_1, \dots, x_k) - \sum_{i=1}^{k-1} \frac{\partial \alpha_{k-1}}{\partial x_i} \varphi_i(x_1, \dots, x_i) \\ \zeta_k &= \frac{n}{4\epsilon} e^{at} \left(h_k^2 + \sum_{i=1}^{k-1} \left(\frac{\partial \alpha_{k-1}}{\partial x_i} \right)^2 h_i^2 \right)\end{aligned}\quad (23)$$

$$\begin{aligned}\xi_k &= \eta_k - \sum_{i=1}^{k-1} \frac{\partial \alpha_{k-1}}{\partial x_i} \eta_i \\ \tau_k &= \Gamma \sum_{i=1}^k \omega_i z_i\end{aligned}\quad (24)$$

$$\begin{aligned}\alpha_k = & -z_{k-1} - c_k z_k - \omega_k^T \hat{\theta} + \sum_{i=1}^{k-1} \frac{\partial \alpha_{k-1}}{\partial x_i} x_{i+1} + \frac{\partial \alpha_{k-1}}{\partial t} \\ & - \zeta_k z_k + \frac{\partial \alpha_{k-1}}{\partial \hat{\theta}} \tau_k + \left(\sum_{i=1}^{k-2} z_{i+1} \frac{\partial \alpha_i}{\partial \hat{\theta}} \right) \Gamma \omega_k \quad (c_k > 0)\end{aligned}$$

Parameter update law

$$\dot{\hat{\theta}} = \tau_n = \Gamma \sum_{i=1}^n \omega_i z_i \quad (25)$$

Control law

$$\begin{aligned}u = \frac{1}{g(x)} \Big[& -z_{n-1} - c_n z_n - f(x) - \omega_n^T \hat{\theta} + \sum_{i=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_i} x_{i+1} \\ & + \frac{\partial \alpha_{n-1}}{\partial \hat{\theta}} \tau_n + \frac{\partial \alpha_{n-1}}{\partial t} - \left(\sum_{i=1}^{n-2} z_{i+1} \frac{\partial \alpha_i}{\partial \hat{\theta}} \right) \Gamma w_n + y_r^{(n)} - \zeta_n z_n \Big] \quad (26)\end{aligned}$$

with $c_n > 0$.

5.2 Sliding Backstepping Control

The parameter update law is again (11) and the adaptive sliding mode output tracking controller is found by

$$\begin{aligned}u = \frac{1}{g(x)} \Big[& -z_{n-1} - f(x) - \omega_n^T \hat{\theta} + \frac{\partial \alpha_{n-1}}{\partial \hat{\theta}} \tau_n + \sum_{i=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_i} x_{i+1} \\ & + y_r^{(n)} + \frac{\partial \alpha_{n-1}}{\partial t} - k_1 \left(-c_1 z_1 + z_2 - \frac{n}{4\epsilon} h_1^2 z_1 e^{at} \right) \\ & - \sum_{i=2}^{n-1} k_i \left(-z_{i-1} - c_i z_i + z_{i+1} - \zeta_i z_i - \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}} (\tau_n - \tau_i) \right. \\ & + \left. \left(\sum_{l=1}^{i-2} z_{l+1} \frac{\partial \alpha_l}{\partial \hat{\theta}} \right) \Gamma w_i \right) + \left(\sum_{i=1}^{n-2} z_{i+1} \frac{\partial \alpha_i}{\partial \hat{\theta}} \right) \Gamma \left(\omega_n + \sum_{i=1}^{n-1} k_i \omega_i \right) \\ & \left. - W \sigma - \left(K + \sum_{i=1}^n k_i \nu_i \right) \operatorname{sgn}(\sigma) \right], \quad (27)\end{aligned}$$

where $k_n = 1$, $K > 0$ and $W \geq 0$ are arbitrary real numbers, and

$$\nu_i = h_i + \sum_{j=1}^{i-1} \left| \frac{\partial \alpha_{k-1}}{\partial x_i} \right| h_j, \quad 1 \leq i \leq n \quad (28)$$

6 Illustrative Example

Consider the second order system in PSSF form

$$\begin{aligned}\dot{x}_1 &= x_2 + x_1\theta + Ax_1^2 \cos(Bx_1x_2) \\ \dot{x}_2 &= u\end{aligned}\quad (29)$$

where A and B are considered unknown but it is known that $|A| \leq 2$ and $|B| \leq 3$. We have

$$\begin{aligned}h_1 &= 2x_1^2 & \omega_1 &= x_1 \\ z_1 &= x_1 - y_r & z_2 &= x_2 + x_1\hat{\theta} + c_1z_1 + \frac{2}{\epsilon}x_1^4z_1e^{at} - \dot{y}_r \\ \alpha_1 &= -x_1\hat{\theta} - c_1z_1 - \frac{2}{\epsilon}x_1^4z_1e^{at} & \omega_2 &= -\frac{\partial\alpha_1}{\partial x_1}x_1 \\ \tau_2 &= \Gamma(\omega_1z_1 + \omega_2z_2) & \zeta_2 &= \frac{2}{\epsilon}e^{at}\left(x_1\frac{\partial\alpha_1}{\partial x_1}\right)^2\end{aligned}$$

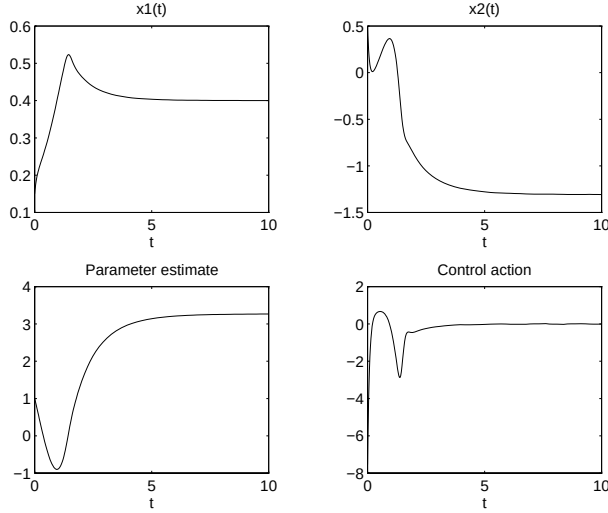


Fig. 2. Regulator responses with nonlinear control (30) for PSSF system

Then the control law (26) becomes

$$u = -z_1 - c_2z_2 - \omega_2^T\hat{\theta} + \frac{\partial\alpha_1}{\partial x_1}x_2 + \frac{\partial\alpha_1}{\partial\hat{\theta}}\tau_2 + \frac{\partial\alpha_1}{\partial t} + y_r^{(2)} - \zeta_2z_2 \quad (30)$$

Simulation results showing desirable transient responses are presented in Fig. 2 with $y_r = 0.4$, $a = 0.1$, $\epsilon = 5$, $\Gamma = 0.5$, $c_1 = 12$ and $c_2 = 0.1$. Alternatively, we can design a sliding mode controller for the system. Assume that the sliding surface is $\sigma = k_1z_1 + z_2 = 0$ with $k_1 > 0$. The adaptive

sliding mode control law (27) is

$$u = (c_1 k_1 - 1) z_1 - k_1 z_2 - \omega_2^T \hat{\theta} + \frac{\partial \alpha_1}{\partial x_1} x_2 + \frac{\partial \alpha_1}{\partial \hat{\theta}} \tau_2 + \frac{\partial \alpha_1}{\partial t} + y_r^{(2)} + \frac{1}{2\epsilon} h_1^2 z_1 e^{at} - W\sigma - \left(K + k_1 + \left| \frac{\partial \alpha_1}{\partial x_1} \right| \right) h_1 \text{sgn}(\sigma) \quad (31)$$

where $\tau_2 = \Gamma(z_1 \omega_1 + \sigma(\omega_2 + k_1 \omega_1))$. Simulation results showing desirable transient responses are shown in Fig. 3 with $y_r = 0.05 \sin(2\pi t)$, $k_1 = 1$, $K = 5$, $W = 0$, $a = 0.2$, $\epsilon = 1$, $\Gamma = 0.05$, $A = 2$, $B = 3$ and $c_1 = c_2 = 2$.

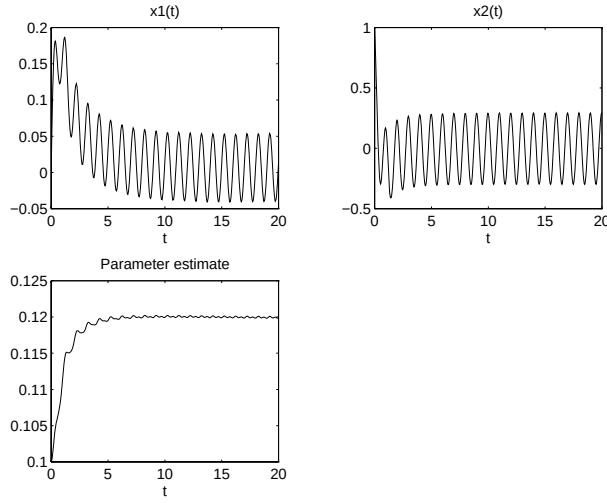


Fig. 3. Tracking responses with sliding control (31) for PSSF system

7 Disturbed DAB

In this section we extend the DAB to affine nonlinear systems with unmodelled or external disturbances. We use a dynamical input-output linearization and assume that there is a well-defined relative degree ρ . For systems where the output is a linearizing function, $\rho = n$, and the new algorithm collapses to the approach of Section 5.

7.1 System

Consider

$$\begin{aligned} \dot{x} &= f_0 + \phi\theta + (g_0 + \psi\theta)u + \eta \\ y &= h(x) \end{aligned} \quad (32)$$

where $x \in \mathbb{R}^n$ is the state; $u, y \in \mathbb{R}$ the input and output respectively; and $\theta = [\theta_1, \dots, \theta_p]^T$ is a vector of unknown parameters. f_0, g_0 and the columns of the matrices $\Phi, \Psi \in \mathbb{R}^{n \times p}$ are smooth vector fields in a neighbourhood R_0 of the origin $x = 0$ with $f_0(0) = 0, g_0(0) \neq 0$; and h is a smooth scalar function also defined in R_0 . $\eta(x, w, t) \in \mathbb{R}^n$ are unknown nonlinear scalar functions including all the disturbances and unmodelled dynamics. w is an uncertain time-varying parameter. It is assumed the relative degree (32) with respect to u is $1 \leq \rho \leq n$, and with respect to θ is one.

Assumption 3 *The functions $\eta(x, w, t)$ are bounded by known positive functions $q(x)$*

$$|\eta_i(x, w, t)| \leq q_i(x) \quad 1 \leq i \leq n \quad (33)$$

Like the unperturbed version, the method is applicable to systems which are observable (1) and minimum-phase (2).

7.2 DAB-EXTENDED Algorithm

For systems of the form (32), the general problem is to track adaptively a bounded desired reference signal $y_r(t)$, with smooth and bounded derivatives up to n -th order, in the presence of unknown parameters θ and disturbances η .

Step 1

Define the output tracking error

$$z_1 = y - y_r = \hat{h}^{(0)} - y_r \quad (34)$$

with $\hat{h}^{(0)} = h$. Then

$$\begin{aligned} \dot{z}_1 &= \frac{\partial h}{\partial x} [f + \phi\theta + (g + \psi\theta)u + \eta] - \dot{y}_r \\ &= \frac{\partial h}{\partial x} [f + \phi\hat{\theta} + (g + \psi\hat{\theta})u] + \frac{\partial h}{\partial x} (\phi + \psi u) (\theta - \hat{\theta}) + \frac{\partial h}{\partial x} \eta - \dot{y}_r \\ &= \hat{h}^{(1)} + \omega_1 \tilde{\theta} + \frac{\partial h}{\partial x} \eta - \dot{y}_r \end{aligned}$$

where $\hat{\theta}(t)$ is an estimate of the unknown parameters θ and

$$\begin{aligned} \hat{h}^{(1)}(x, u, \hat{\theta}) &= \frac{\partial h}{\partial x} [f + \phi\hat{\theta} + (g + \psi\hat{\theta})u] \\ \omega_1 &= \frac{\partial h}{\partial x} (\phi + \psi u) \\ \tilde{\theta} &= \theta - \hat{\theta} \end{aligned}$$

Consider the Lyapunov function

$$V_1 = \frac{1}{2} z_1^2 + \frac{1}{2} \tilde{\theta}^T \Gamma^{-1} \tilde{\theta}$$

where Γ is a symmetric positive definite matrix. The derivative of V_1 is

$$\begin{aligned}\dot{V}_1 &= z_1 \dot{z}_1 + \tilde{\theta}^T \Gamma^{-1} (-\dot{\hat{\theta}}) \\ &= z_1 \left[\hat{h}^{(1)} + \frac{\partial h}{\partial x} \eta - \dot{y}_r \right] + \tilde{\theta}^T \Gamma^{-1} (\Gamma \omega_1^T z_1 - \dot{\hat{\theta}})\end{aligned}$$

Set the first tuning function $\tau_1 = \Gamma \omega_1^T z_1$. If we can set $-\alpha_1 = \hat{h}^{(1)} - \dot{y}_r$, where

$$\alpha_1(x, t) = c_1 z_1 + \left| \frac{\partial h}{\partial x} \right| q \operatorname{sgn}(z_1), \quad q = [q_1, \dots, q_n]^T$$

then we will have $\dot{V}_1 \leq 0$. However, as this result is not valid for all time, we define the second error variable

$$z_2 = \hat{h}^{(1)} + \alpha_1 - \dot{y}_r$$

giving the closed loop derivatives

$$\begin{aligned}\dot{z}_1 &= -c_1 z_1 + z_2 + \omega_1 \tilde{\theta} + \frac{\partial h}{\partial x} \eta - \left| \frac{\partial h}{\partial x} \right| q \operatorname{sgn}(z_1) \\ \dot{V}_1 &= -c_1 z_1^2 + z_1 z_2 + \left(\frac{\partial h}{\partial x} \eta - \left| \frac{\partial h}{\partial x} \right| q \operatorname{sgn}(z_1) \right) z_1 + \tilde{\theta}^T \Gamma^{-1} (\tau_1 - \dot{\hat{\theta}})\end{aligned}$$

Proceeding iteratively, we obtain the k -th step.

Step k , $1 \leq k \leq n-1$

We have

$$z_k = \hat{h}^{(k-1)} + \alpha_{k-1} - y_r^{(k-1)} \quad (35)$$

where

$$\omega_k = \left(\frac{\partial \hat{h}^{(k-1)}}{\partial x} + \frac{\partial \alpha_{k-1}}{\partial x} \right) (\phi + \psi v_1) \quad (36)$$

$$\tau_k = \Gamma \sum_{j=1}^k \omega_j^T z_j \quad (37)$$

$$\begin{aligned}\hat{h}^{(k)} &= \frac{\partial \hat{h}^{(k-1)}}{\partial x} \left[f + \phi \hat{\theta} + (g + \psi \hat{\theta}) v_1 \right] + \frac{\partial \hat{h}^{(k-1)}}{\partial \hat{\theta}} \tau_k + \frac{\partial \hat{h}^{(k-1)}}{\partial t} \\ &\quad + \sum_{j=1}^{k-\rho} \frac{\partial \hat{h}^{(k-1)}}{\partial v_j} v_{j+1} + \sum_{j=2}^{k-1} z_j \frac{\partial \hat{h}^{(j-1)}}{\partial \hat{\theta}} \Gamma \omega_k^T\end{aligned} \quad (38)$$

$$\begin{aligned}\alpha_k &= z_{k-1} + c_k z_k + \frac{\partial \alpha_{k-1}}{\partial x} \left[f + \phi \hat{\theta} + (g + \psi \hat{\theta}) v_1 \right] + \frac{\partial \alpha_{k-1}}{\partial \hat{\theta}} \tau_k \\ &\quad + \frac{\partial \alpha_{k-1}}{\partial t} + \sum_{i=1}^{k-\rho} \frac{\partial \alpha_{k-1}}{\partial v_j} v_{j+1} + \sum_{j=3}^{k-1} z_j \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}} \Gamma \omega_k^T \\ &\quad + \left| \frac{\partial \hat{h}^{(k-1)}}{\partial x} + \frac{\partial \alpha_{k-1}}{\partial x} \right| q \operatorname{sgn}(z_k)\end{aligned} \quad (39)$$

The time derivative is

$$\begin{aligned}
\dot{z}_k &= \frac{\partial \hat{h}^{(k-1)}}{\partial x} [f + \phi \theta + (g + \psi \theta) v_1 + \eta] + \frac{\partial \hat{h}^{(k-1)}}{\partial \hat{\theta}} \dot{\hat{\theta}} + \frac{\partial \hat{h}^{(k-1)}}{\partial t} \\
&\quad + \frac{\partial \alpha_{k-1}}{\partial x} [f + \phi \theta + (g + \psi \theta) v_1 + \eta] + \frac{\partial \alpha_{k-1}}{\partial \hat{\theta}} \dot{\hat{\theta}} + \frac{\partial \alpha_{k-1}}{\partial t} \\
&\quad + \sum_{j=1}^{k-\rho} \frac{\partial \hat{h}^{(k-1)}}{\partial v_j} v_{j+1} + \sum_{j=1}^{k-\rho} \frac{\partial \alpha_{k-1}}{\partial v_j} v_{j+1} - y_r^{(k)} \\
&= \frac{\partial \hat{h}^{(k-1)}}{\partial x} [f + \phi \hat{\theta} + (g + \psi \hat{\theta}) v_1] + \frac{\partial \hat{h}^{(k-1)}}{\partial t} + \frac{\partial \alpha_{k-1}}{\partial t} - y_r^{(k)} \\
&\quad + \frac{\partial \alpha_{k-1}}{\partial x} [f + \phi \hat{\theta} + (g + \psi \hat{\theta}) v_1] + \sum_{j=1}^{k-\rho} \frac{\partial \hat{h}^{(k-1)}}{\partial v_j} v_{j+1} \\
&\quad + \sum_{j=1}^{k-\rho} \frac{\partial \alpha_{k-1}}{\partial v_j} v_{j+1} + \left(\frac{\partial \hat{h}^{(k-1)}}{\partial x} + \frac{\partial \alpha_{k-1}}{\partial x} \right) (\phi + \psi v_1) \tilde{\theta} \\
&\quad + \frac{\partial \hat{h}^{(k-1)}}{\partial \hat{\theta}} \dot{\hat{\theta}} + \frac{\partial \alpha_{k-1}}{\partial \hat{\theta}} \dot{\hat{\theta}} + \left(\frac{\partial \hat{h}^{(k-1)}}{\partial x} + \frac{\partial \alpha_{k-1}}{\partial x} \right) \eta \\
&= \hat{h}^{(k)} + \frac{\partial \alpha_{k-1}}{\partial x} [f + \phi \hat{\theta} + (g + \psi \hat{\theta}) v_1] + \frac{\partial \alpha_{k-1}}{\partial \hat{\theta}} \dot{\hat{\theta}} + \frac{\partial \alpha_{k-1}}{\partial t} \\
&\quad + \sum_{j=1}^{k-\rho} \frac{\partial \alpha_{k-1}}{\partial v_j} v_{j+1} - y_r^{(k)} + \frac{\partial \hat{h}^{(k-1)}}{\partial \hat{\theta}} (\dot{\hat{\theta}} - \tau_k) + \omega_k \tilde{\theta} \\
&\quad + \left(\frac{\partial \hat{h}^{(k-1)}}{\partial x} + \frac{\partial \alpha_{k-1}}{\partial x} \right) \eta - \sum_{j=2}^{k-1} z_j \frac{\partial \hat{h}^{(j-1)}}{\partial \hat{\theta}} \Gamma \omega_k
\end{aligned}$$

where $v_1 = u$, $v_2 = \dot{u}$, ..., $v_j = u^{(j-1)}$.

Augmenting the Lyapunov function

$$\begin{aligned}
V_k &= V_{k-1} + \frac{1}{2} z_k^2 \\
&= \frac{1}{2} \sum_{j=1}^k z_j^2 + \frac{1}{2} \tilde{\theta}^T \Gamma^{-1} \tilde{\theta} \\
\Rightarrow \dot{V}_k &= \dot{V}_{k-1} + z_k \dot{z}_k \\
&= - \sum_{j=1}^{k-1} c_j z_j^2 + \left(\sum_{j=2}^{k-1} z_j \frac{\partial \hat{h}^{(j-1)}}{\partial \hat{\theta}} + \sum_{j=3}^{k-1} z_j \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}} \right) (\dot{\hat{\theta}} - \tau_{k-1}) \\
&\quad + \sum_{j=1}^{k-1} z_j \left[\left(\frac{\partial \hat{h}^{(j-1)}}{\partial x} + \frac{\partial \alpha_{j-1}}{\partial x} \right) \eta - \left| \frac{\partial \hat{h}^{(j-1)}}{\partial x} + \frac{\partial \alpha_{j-1}}{\partial x} \right| q \operatorname{sgn}(z_j) \right]
\end{aligned}$$

$$\begin{aligned}
& + z_k \left(z_{k-1} + \hat{h}^{(k)} + \frac{\partial \alpha_{k-1}}{\partial x} \left[f + \phi \hat{\theta} + (g + \psi \hat{\theta}) v_1 \right] \right. \\
& + \sum_{j=1}^{k-\rho} \frac{\partial \alpha_{k-1}}{\partial v_j} v_{j+1} - y_r^{(k)} + \frac{\partial \hat{h}^{(k-1)}}{\partial \hat{\theta}} (\dot{\hat{\theta}} - \tau_k) + \omega_k \tilde{\theta} \\
& + \frac{\partial \alpha_{k-1}}{\partial \hat{\theta}} \dot{\hat{\theta}} + \frac{\partial \alpha_{k-1}}{\partial t} - \sum_{j=2}^{k-1} z_j \frac{\partial \hat{h}^{(j-1)}}{\partial \theta} \Gamma \omega_k^T \\
& \left. + \left(\frac{\partial \hat{h}^{(k-1)}}{\partial x} + \frac{\partial \alpha_{k-1}}{\partial x} \right) \eta \right) + \tilde{\theta}^T \Gamma^{-1} (\tau_{k-1} - \dot{\hat{\theta}})
\end{aligned}$$

If we can set $-\alpha_k = \hat{h}^{(k)} - y_r^{(k)}$, we will have $\dot{V}_k \leq 0$. However, as this result is not valid for all time, we define the new error variable

$$z_{k+1} = \hat{h}^{(k)} + \alpha_k - y_r^{(k)} \quad (40)$$

This gives

$$\begin{aligned}
\dot{z}_k &= -z_{k-1} - c_k z_k + z_{k+1} + \left(\frac{\partial \hat{h}^{(k-1)}}{\partial \hat{\theta}} + \frac{\partial \alpha_{k-1}}{\partial \hat{\theta}} \right) (\dot{\hat{\theta}} - \tau_k) \\
& + \omega_k \tilde{\theta} - \left(\sum_{j=2}^{k-1} z_j \frac{\partial \hat{h}^{(j-1)}}{\partial \hat{\theta}} + \sum_{j=3}^{k-1} z_j \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}} \right) \Gamma \omega_k^T \\
& + \left(\frac{\partial \hat{h}^{(k-1)}}{\partial x} + \frac{\partial \alpha_{k-1}}{\partial x} \right) \eta - \left| \frac{\partial \hat{h}^{(k-1)}}{\partial x} + \frac{\partial \alpha_{k-1}}{\partial x} \right| q \operatorname{sgn}(z_k) \\
\dot{V}_k &= - \sum_{j=1}^{k-1} c_j z_j^2 + \left(\sum_{j=2}^{k-1} z_j \frac{\partial \hat{h}^{(j-1)}}{\partial \hat{\theta}} + \sum_{j=3}^{k-1} z_j \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}} \right) (\dot{\hat{\theta}} - \tau_{k-1}) \\
& + \sum_{j=1}^{k-1} z_j \left[\left(\frac{\partial \hat{h}^{(j-1)}}{\partial x} + \frac{\partial \alpha_{j-1}}{\partial x} \right) \eta - \left| \frac{\partial \hat{h}^{(j-1)}}{\partial x} + \frac{\partial \alpha_{j-1}}{\partial x} \right| q \operatorname{sgn}(z_j) \right] \\
& + \tilde{\theta}^T \Gamma^{-1} (\tau_{k-1} - \dot{\hat{\theta}}) + z_k (-c_k z_k + z_{k+1} + \omega_k \tilde{\theta} \\
& + \left(\frac{\partial \hat{h}^{(k-1)}}{\partial x} + \frac{\partial \alpha_{k-1}}{\partial x} \right) \eta - \left| \frac{\partial \hat{h}^{(k-1)}}{\partial x} + \frac{\partial \alpha_{k-1}}{\partial x} \right| q \operatorname{sgn}(z_k) \\
& - \left(\sum_{j=2}^{k-1} z_j \frac{\partial \hat{h}^{(j-1)}}{\partial \hat{\theta}} + \sum_{j=3}^{k-1} z_j \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}} \right) \Gamma \omega_k^T \\
& + \left(\frac{\partial \hat{h}^{(k)}}{\partial \hat{\theta}} + \frac{\partial \alpha_k}{\partial \hat{\theta}} \right) (\dot{\hat{\theta}} - \tau_k)
\end{aligned}$$

$$\begin{aligned}
&= -\sum_{j=1}^k c_j z_j^2 + \left(\sum_{j=2}^k z_j \frac{\partial \hat{h}^{(j-1)}}{\partial \hat{\theta}} + \sum_{j=3}^k z_j \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}} \right) (\dot{\hat{\theta}} - \tau_k) \\
&\quad + \sum_{j=1}^k z_j \left[\left(\frac{\partial \hat{h}^{(j-1)}}{\partial x} + \frac{\partial \alpha_{j-1}}{\partial x} \right) \eta - \left| \frac{\partial \hat{h}^{(j-1)}}{\partial x} + \frac{\partial \alpha_{j-1}}{\partial x} \right| q \operatorname{sgn}(z_j) \right] \\
&\quad + z_k z_{k+1} + \tilde{\theta}^T \Gamma^{-1} (\tau_k - \dot{\hat{\theta}})
\end{aligned}$$

Step n

At this step, the actual update law $\dot{\hat{\theta}} = \tau_n$ and the dynamical controller are obtained.

We have $z_n = \hat{h}^{(n-1)} + \alpha_{n-1} - y_r^{(n-1)}$, so

$$\begin{aligned}
\dot{z}_n &= \frac{\partial \hat{h}^{(n-1)}}{\partial x} [f + \phi \theta + (g + \psi \theta) v_1 + \eta] + \frac{\partial \hat{h}^{(n-1)}}{\partial \hat{\theta}} \dot{\hat{\theta}} + \frac{\partial \hat{h}^{(n-1)}}{\partial t} \\
&\quad + \frac{\partial \alpha_{n-1}}{\partial x} [f + \phi \theta + (g + \psi \theta) v_1 + \eta] + \frac{\partial \alpha_{n-1}}{\partial \hat{\theta}} \dot{\hat{\theta}} + \frac{\partial \alpha_{n-1}}{\partial t} \\
&\quad + \sum_{j=1}^{n-\rho} \frac{\partial \hat{h}^{(n-1)}}{\partial v_j} v_{j+1} + \sum_{j=1}^{n-\rho} \frac{\partial \alpha_{n-1}}{\partial v_j} v_{j+1} - y_r^{(n)} \\
&= \left(\frac{\partial \hat{h}^{(n-1)}}{\partial x} + \frac{\partial \alpha_{n-1}}{\partial x} \right) [f + \phi \hat{\theta} + (g + \psi \hat{\theta}) v_1] - y_r^{(n)} + \omega_n \tilde{\theta} \\
&\quad + \sum_{j=1}^{n-\rho} \left(\frac{\partial \hat{h}^{(n-1)}}{\partial v_j} + \frac{\partial \alpha_{n-1}}{\partial v_j} \right) v_{j+1} + \frac{\partial \hat{h}^{(n-1)}}{\partial t} + \frac{\partial \alpha_{n-1}}{\partial t} \\
&\quad + \left(\frac{\partial \hat{h}^{(n-1)}}{\partial \hat{\theta}} + \frac{\partial \alpha_{n-1}}{\partial \hat{\theta}} \right) \dot{\hat{\theta}} + \left(\frac{\partial \hat{h}^{(n-1)}}{\partial x} + \frac{\partial \alpha_{n-1}}{\partial x} \right) \eta
\end{aligned}$$

with

$$\omega_n = \left(\frac{\partial \hat{h}^{(n-1)}}{\partial x} + \frac{\partial \alpha_{n-1}}{\partial x} \right) (\phi + \psi v_1)$$

Augmenting the Lyapunov function

$$\begin{aligned}
V_n &= V_{n-1} + \frac{1}{2} z_n^2 \\
&= \frac{1}{2} \sum_{j=1}^n z_j^2 + \frac{1}{2} \tilde{\theta}^T \Gamma^{-1} \tilde{\theta} \\
\Rightarrow \dot{V}_n &= \dot{V}_{n-1} + z_n \dot{z}_n \\
&= -\sum_{j=1}^{n-1} c_j z_j^2 + \left(\sum_{j=2}^{n-1} z_j \frac{\partial \hat{h}^{(j-1)}}{\partial \hat{\theta}} + \sum_{j=3}^{n-1} z_j \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}} \right) (\dot{\hat{\theta}} - \tau_{n-1})
\end{aligned}$$

$$\begin{aligned}
& + \sum_{j=1}^{n-1} z_j \left[\left(\frac{\partial \hat{h}^{(j-1)}}{\partial x} + \frac{\partial \alpha_{j-1}}{\partial x} \right) \eta - \left| \frac{\partial \hat{h}^{(j-1)}}{\partial x} + \frac{\partial \alpha_{j-1}}{\partial x} \right| q \operatorname{sgn}(z_j) \right] \\
& + z_n \left(z_{n-1} + \left(\frac{\partial \hat{h}^{(n-1)}}{\partial x} + \frac{\partial \alpha_{n-1}}{\partial x} \right) \left[f + \phi \hat{\theta} + (g + \psi \hat{\theta}) v_1 \right] \right. \\
& + \sum_{j=1}^{n-\rho} \left(\frac{\partial \hat{h}^{(n-1)}}{\partial v_j} + \frac{\partial \alpha_{n-1}}{\partial v_j} \right) v_{j+1} + \frac{\partial \hat{h}^{(n-1)}}{\partial t} + \frac{\partial \alpha_{n-1}}{\partial t} \\
& - y_r^{(n)} + \left(\frac{\partial \hat{h}^{(n-1)}}{\partial \hat{\theta}} + \frac{\partial \alpha_{n-1}}{\partial \hat{\theta}} \right) \dot{\hat{\theta}} + \omega_n \tilde{\theta} \\
& \left. + \left(\frac{\partial \hat{h}^{(n-1)}}{\partial x} + \frac{\partial \alpha_{n-1}}{\partial x} \right) \eta \right) + \tilde{\theta}^T \Gamma^{-1} (\tau_{n-1} - \dot{\hat{\theta}})
\end{aligned}$$

We would like to have

$$\begin{aligned}
& \left(\frac{\partial \hat{h}^{(n-1)}}{\partial x} + \frac{\partial \alpha_{n-1}}{\partial x} \right) \left[f + \phi \hat{\theta} + (g + \psi \hat{\theta}) v_1 \right] + \frac{\partial \hat{h}^{(n-1)}}{\partial t} + \frac{\partial \alpha_{n-1}}{\partial t} \\
& - y_r^{(n)} + z_{n-1} + \sum_{j=1}^{n-\rho-1} \left(\frac{\partial \hat{h}^{(n-1)}}{\partial v_j} + \frac{\partial \alpha_{n-1}}{\partial v_j} \right) v_{j+1} = -c_n z_n \\
& - \left| \frac{\partial \hat{h}^{(n-1)}}{\partial x} + \frac{\partial \alpha_{n-1}}{\partial x} \right| q \operatorname{sgn}(z_n) - \left(\frac{\partial \hat{h}^{(n-1)}}{\partial \theta} + \frac{\partial \alpha_{n-1}}{\partial \theta} \right) \tau_n \\
& - \left(\sum_{j=2}^{n-1} z_j \frac{\partial \hat{h}^{(j-1)}}{\partial \hat{\theta}} + \sum_{j=3}^{n-1} z_j \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}} \right) \Gamma \omega_n^T
\end{aligned}$$

Thus our control law $u = v_1$ can be found from

$$\begin{aligned}
& \dot{v}_1 = v_2 \\
& \dot{v}_2 = v_3 \\
& \vdots \\
& \dot{v}_{n-\rho} = \frac{-1}{\frac{\partial \hat{h}^{(n-1)}}{\partial v_{n-\rho}} + \frac{\partial \alpha_{n-1}}{\partial v_{n-\rho}}} \left[\left(\frac{\partial \hat{h}^{(n-1)}}{\partial x} + \frac{\partial \alpha_{n-1}}{\partial x} \right) \left[f + \phi \hat{\theta} + (g + \psi \hat{\theta}) v_1 \right] \right. \\
& \quad + \frac{\partial \hat{h}^{(n-1)}}{\partial t} + \frac{\partial \alpha_{n-1}}{\partial t} + \left(\sum_{j=2}^{n-1} z_j \frac{\partial \hat{h}^{(j-1)}}{\partial \hat{\theta}} + \sum_{j=3}^{n-1} z_j \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}} \right) \Gamma \omega_n^T \\
& \quad \left. + \left(\frac{\partial \hat{h}^{(n-1)}}{\partial \hat{\theta}} + \frac{\partial \alpha_{n-1}}{\partial \hat{\theta}} \right) \tau_n + z_{n-1} + c_n z_n - y_r^{(n)} \right]
\end{aligned} \tag{41}$$

$$\begin{aligned}
& + \sum_{j=1}^{n-\rho-2} \left(\frac{\partial \hat{h}^{(n-1)}}{\partial v_j} + \frac{\partial \alpha_{n-1}}{\partial v_j} \right) v_{j+1} \\
& + \left| \frac{\partial \hat{h}^{(n-1)}}{\partial x} + \frac{\partial \alpha_{n-1}}{\partial x} \right| q \operatorname{sgn}(z_n) \Big]
\end{aligned}$$

and choosing the parameter estimate update law to be

$$\dot{\hat{\theta}} = \tau_n = \Gamma \sum_{j=1}^n \omega_j z_j$$

we get

$$\begin{aligned}
\dot{V}_n &= - \sum_{i=1}^n c_i z_i^2 + \tilde{\theta}^T \Gamma^{-1} (\tau_n - \dot{\hat{\theta}}) \\
&+ \sum_{j=1}^n z_j \left[\left(\frac{\partial \hat{h}^{(j-1)}}{\partial x} + \frac{\partial \alpha_{j-1}}{\partial x} \right) \eta - \left| \frac{\partial \hat{h}^{(j-1)}}{\partial x} + \frac{\partial \alpha_{j-1}}{\partial x} \right| q \operatorname{sgn}(z_j) \right] \\
&+ \left(\sum_{j=2}^n z_j \frac{\partial \hat{h}^{(j-1)}}{\partial \hat{\theta}} + \sum_{j=3}^n z_j \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}} \right) (\dot{\hat{\theta}} - \tau_n) \\
&\leq 0
\end{aligned}$$

8 DAB-EXTENDED-SMC

Robustness can be added to the disturbed DAB algorithm by having sliding mode control at the final stage. This disturbed DAB-SMC algorithm generates the error variables in the same way as the disturbed DAB algorithm. At the n -th step, use the sliding surface (9) and the Lyapunov function

$$\begin{aligned}
V_n &= V_{n-1} + \frac{1}{2} \sigma^2 \\
&= \frac{1}{2} \sum_{i=1}^{n-1} z_i^2 + \frac{1}{2} \sigma^2 + \frac{1}{2} \tilde{\theta}^T \Gamma^{-1} \tilde{\theta} \\
\Rightarrow \dot{V}_n &= \dot{V}_{n-1} + \sigma \dot{\sigma} \\
&= - \sum_{i=1}^{n-1} c_i z_i^2 + \left(\sum_{j=2}^{n-1} z_j \frac{\partial \hat{h}^{(j-1)}}{\partial \hat{\theta}} + \sum_{j=3}^{n-1} z_j \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}} \right) (\dot{\hat{\theta}} - \tau_{n-1}) \\
&+ \sum_{j=1}^{n-1} z_j \left[\left(\frac{\partial \hat{h}^{(j-1)}}{\partial x} + \frac{\partial \alpha_{j-1}}{\partial x} \right) \eta - \left| \frac{\partial \hat{h}^{(j-1)}}{\partial x} + \frac{\partial \alpha_{j-1}}{\partial x} \right| q \operatorname{sgn}(z_j) \right] \\
&+ z_{n-1} z_n + \sigma \left[\left(\frac{\partial \hat{h}^{(n-1)}}{\partial x} + \frac{\partial \alpha_{n-1}}{\partial x} \right) [f + \phi \hat{\theta} + (g + \psi \hat{\theta}) v_1] \right]
\end{aligned}$$

$$\begin{aligned}
& + \frac{\partial \hat{h}^{(n-1)}}{\partial t} + \frac{\partial \alpha_{n-1}}{\partial t} + \sum_{j=1}^{n-\rho-1} \left(\frac{\partial \hat{h}^{(n-1)}}{\partial v_j} + \frac{\partial \alpha_{n-1}}{\partial v_j} \right) v_{j+1} \\
& + \left(\frac{\partial \hat{h}^{(n-1)}}{\partial \hat{\theta}} + \frac{\partial \alpha_{n-1}}{\partial \hat{\theta}} \right) \dot{\hat{\theta}} - y_r^{(n)} + \omega_n \tilde{\theta} \\
& + \left(\frac{\partial \hat{h}^{(n-1)}}{\partial x} + \frac{\partial \alpha_{n-1}}{\partial x} \right) \eta + \sum_{i=1}^{n-1} k_i (-z_{i-1} - c_i z_i \\
& + z_{i+1} + \omega_i \tilde{\theta} + \left(\frac{\partial \hat{h}^{(i-1)}}{\partial \hat{\theta}} + \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}} \right) (\dot{\hat{\theta}} - \tau_i) \\
& + \left(\frac{\partial \hat{h}^{(i-1)}}{\partial x} + \frac{\partial \alpha_{i-1}}{\partial x} \right) \eta - \left| \frac{\partial \hat{h}^{(i-1)}}{\partial x} + \frac{\partial \alpha_{i-1}}{\partial x} \right| q \operatorname{sgn}(z_i) \\
& - \left(\sum_{j=2}^{i-1} z_j \frac{\partial \hat{h}^{(j-1)}}{\partial \hat{\theta}} + \sum_{j=3}^{i-1} z_j \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}} \right) \Gamma \omega_i^T \Big] \\
& + \tilde{\theta}^T \Gamma^{-1} (\tau_{n-1} - \dot{\hat{\theta}})
\end{aligned}$$

Setting $\dot{\hat{\theta}} = \tau_n = \tau_{n-1} + \Gamma \left(\omega_n^T + \sum_{i=1}^{n-1} k_i \omega_i^T \right) \sigma$, and since $z_n = \sigma - \sum_{i=1}^{n-1} k_i z_i$ we can rearrange to give

$$\begin{aligned}
\dot{V}_n = & - \sum_{i=1}^{n-1} c_i z_i^2 - z_{n-1} \sum_{i=1}^{n-1} k_i z_i \\
& + \sum_{j=1}^{n-1} z_j \left[\left(\frac{\partial \hat{h}^{(j-1)}}{\partial x} + \frac{\partial \alpha_{j-1}}{\partial x} \right) \eta - \left| \frac{\partial \hat{h}^{(j-1)}}{\partial x} + \frac{\partial \alpha_{j-1}}{\partial x} \right| q \operatorname{sgn}(z_j) \right] \\
& + \sigma \left[z_{n-1} + \left(\frac{\partial \hat{h}^{(n-1)}}{\partial x} + \frac{\partial \alpha_{n-1}}{\partial x} \right) [f + \phi \hat{\theta} + (g + \psi \hat{\theta}) v_1] \right. \\
& + \frac{\partial \hat{h}^{(n-1)}}{\partial t} + \frac{\partial \alpha_{n-1}}{\partial t} + \sum_{j=1}^{n-\rho-1} \left(\frac{\partial \hat{h}^{(n-1)}}{\partial v_j} + \frac{\partial \alpha_{n-1}}{\partial v_j} \right) v_{j+1} \\
& + \left(\frac{\partial \hat{h}^{(n-1)}}{\partial \hat{\theta}} + \frac{\partial \alpha_{n-1}}{\partial \hat{\theta}} \right) \tau_n - y_r^{(n)} + \left(\frac{\partial \hat{h}^{(n-1)}}{\partial x} + \frac{\partial \alpha_{n-1}}{\partial x} \right) \eta \\
& + \left(\sum_{j=2}^{n-1} z_j \frac{\partial \hat{h}^{(j-1)}}{\partial \hat{\theta}} + \sum_{j=3}^{n-1} z_j \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}} \right) \Gamma \left(\omega_n^T + \sum_{i=1}^{n-1} k_i \omega_i^T \right) \\
& \left. + \sum_{i=1}^{n-1} k_i \left(-z_{i-1} - c_i z_i + z_{i+1} + \left(\frac{\partial \hat{h}^{(i-1)}}{\partial \hat{\theta}} + \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}} \right) (\tau_n - \tau_i) \right) \right]
\end{aligned}$$

$$\begin{aligned}
& + \left(\frac{\partial \hat{h}^{(i-1)}}{\partial x} + \frac{\partial \alpha_{i-1}}{\partial x} \right) \eta - \left| \frac{\partial \hat{h}^{(i-1)}}{\partial x} + \frac{\partial \alpha_{i-1}}{\partial x} \right| q \operatorname{sgn}(z_i) \\
& - \left(\sum_{j=2}^{i-1} z_j \frac{\partial \hat{h}^{(j-1)}}{\partial \theta} + \sum_{j=3}^{i-1} z_j \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}} \right) \Gamma \omega_i^T \Bigg]
\end{aligned}$$

The dynamic controller can be found by solving

$$\begin{aligned}
z_{n-1} & + \left(\frac{\partial \hat{h}^{(n-1)}}{\partial x} + \frac{\partial \alpha_{n-1}}{\partial x} \right) \left[f + \phi \hat{\theta} + (g + \psi \hat{\theta}) v_1 \right] - y_r^{(n)} \\
& + \left(\sum_{j=2}^{n-1} z_j \frac{\partial \hat{h}^{(j-1)}}{\partial \hat{\theta}} + \sum_{j=3}^{n-1} z_j \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}} \right) \Gamma \left(\omega_n^T + \sum_{i=1}^{n-1} k_i \omega_i^T \right) \\
& + \left(\frac{\partial \hat{h}^{(n-1)}}{\partial \hat{\theta}} + \frac{\partial \alpha_{n-1}}{\partial \hat{\theta}} \right) \tau_n + \sum_{j=1}^{n-\rho-1} \left(\frac{\partial \hat{h}^{(n-1)}}{\partial v_j} + \frac{\partial \alpha_{n-1}}{\partial v_j} \right) v_{j+1} \\
& + \sum_{i=1}^{n-1} k_i \left(-z_{i-1} - c_i z_i + z_{i+1} + \left(\frac{\partial \hat{h}^{(i-1)}}{\partial \hat{\theta}} + \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}} \right) (\tau_n - \tau_i) \right. \\
& \quad \left. - \left(\sum_{j=2}^{i-1} z_j \frac{\partial \hat{h}^{(j-1)}}{\partial \theta} + \sum_{j=3}^{i-1} z_j \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}} \right) \Gamma \omega_i^T \right) + \frac{\partial \hat{h}^{(n-1)}}{\partial t} + \frac{\partial \alpha_{n-1}}{\partial t} \\
& = -\lambda \sigma - \beta \operatorname{sgn} \sigma - \left| \frac{\partial \hat{h}^{(n-1)}}{\partial x} + \frac{\partial \alpha_{n-1}}{\partial x} \right| q \operatorname{sgn} \sigma \\
& \quad - 2 \sum_{i=1}^{n-1} k_i \left| \frac{\partial \hat{h}^{(i-1)}}{\partial x} + \frac{\partial \alpha_{i-1}}{\partial x} \right| q \operatorname{sgn} \sigma
\end{aligned} \tag{42}$$

This gives

$$\begin{aligned}
\dot{V}_n & = - \sum_{i=1}^{n-1} c_i z_i^2 - z_{n-1} \sum_{i=1}^{n-1} k_i z_i \\
& + \sum_{j=1}^{n-1} z_j \left[\left(\frac{\partial \hat{h}^{(j-1)}}{\partial x} + \frac{\partial \alpha_{j-1}}{\partial x} \right) \eta - \left| \frac{\partial \hat{h}^{(j-1)}}{\partial x} + \frac{\partial \alpha_{j-1}}{\partial x} \right| q \operatorname{sgn}(z_j) \right] \\
& + \sigma \left[\left(\frac{\partial \hat{h}^{(n-1)}}{\partial x} + \frac{\partial \alpha_{n-1}}{\partial x} \right) \eta - \left| \frac{\partial \hat{h}^{(n-1)}}{\partial x} + \frac{\partial \alpha_{n-1}}{\partial x} \right| q \operatorname{sgn} \sigma \right. \\
& \quad \left. + \sum_{i=1}^{n-1} k_i \left(\left(\frac{\partial \hat{h}^{(i-1)}}{\partial x} + \frac{\partial \alpha_{i-1}}{\partial x} \right) \eta - \left| \frac{\partial \hat{h}^{(i-1)}}{\partial x} + \frac{\partial \alpha_{i-1}}{\partial x} \right| q \operatorname{sgn}(z_i) \right) \right. \\
& \quad \left. - 2 \sum_{i=1}^{n-1} k_i \left| \frac{\partial \hat{h}^{(i-1)}}{\partial x} + \frac{\partial \alpha_{i-1}}{\partial x} \right| q \operatorname{sgn} \sigma - \lambda \sigma + \beta \operatorname{sgn} \sigma \right]
\end{aligned}$$

$$\begin{aligned} &\leq -[z_1 \dots z_{n-1}]^T Q [z_1 \dots z_{n-1}] - \lambda \sigma^2 - \beta |\sigma| \\ &\leq 0 \end{aligned}$$

with Q from (19). This guarantees asymptotic stability for suitably chosen design parameters.

9 Example: Robot with Flexible Joint

Sliding mode backstepping adaptive control offers the potential for control of robotic manipulators in the presence of uncertain flexibilities, changing dynamics due to unknown and varying payloads, and nonlinear interaction without explicit parameter identification. In this section we consider a single link robot arm to illustrate the procedure of PSSF design (22)-(26).

The system of a single link robot arm with a revolute elastic joint rotating in a vertical plane is

$$\begin{aligned} \dot{v}_1 &= v_2 \\ \dot{v}_2 &= \frac{-F_l}{J_l} v_2 - \frac{glM}{J_l} \sin(v_1) + \frac{-k}{J_l} (v_1 - v_3) + \eta_1 \\ \dot{v}_3 &= v_4 \\ \dot{v}_4 &= -\frac{F_m}{J_m} v_4 + \frac{k}{J_m} (v_1 - v_3) + \frac{1}{J_m} u + \eta_3 \end{aligned}$$

in which v_1 , v_2 , v_3 , and v_4 are the link displacement, the velocity of the link, the rotor displacement and the velocity of the rotor, respectively. J_l is the link inertia, J_m the motor rotor inertia, k the elastic constant, M the link mass, g the gravity constant, l the centre of mass. The positive constant parameters F_l and F_m are viscous friction coefficients. The bounded functions

$$\eta_1 = \eta_1(t, v_1, v_2, v_3, v_4, w)$$

and

$$\eta_2 = \eta_2(t, v_1, v_2, v_3, v_4, w)$$

are the perturbation in the system including the motor disturbance. The control u is the torque delivered by the motor. It is assumed that the constants M , F_l and F_m are unknown, and w is an uncertain time-varying parameter.

The transformation

$$\begin{aligned} x_1 &= v_1 \\ x_2 &= x_3 \\ x_3 &= \frac{-k}{J_l} (v_1 - v_3) \\ x_4 &= \frac{-k}{J_l} (v_2 - v_4) \end{aligned}$$

converts the system to

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 + \varphi_2^T \theta + \eta_2(t, x_1, x_2, x_3, x_4, w) \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= -\left(\frac{k}{J_l} + \frac{k}{J_m}\right)x_3 + \varphi_4^T \theta + \frac{k}{J_m J_l}u + \eta_4(t, x_1, x_2, x_3, x_4, w)\end{aligned}$$

with

$$\begin{aligned}\varphi_2^T &= \begin{bmatrix} -\frac{gl}{J_l} \sin(x_1) & -\frac{1}{J_l} x_2 & 0 \end{bmatrix} \\ \varphi_4^T &= \begin{bmatrix} \frac{glk}{J_l^2} \sin(x_1) & \frac{k}{J_l^2} x_2 & -\frac{1}{J_m} \left(\frac{k}{J_l} x_2 + x_4\right) \end{bmatrix} \\ \theta &= [M \quad F_l \quad F_m]^T\end{aligned}$$

Suppose $|\eta_2| \leq h_2$ and $|\eta_4| \leq h_4$ where h_2 and h_4 are known constant real numbers. It is desired for the output $y = x_1$ to track the reference y_r . Define the sliding surface as

$$\sigma = k_1 z_1 + k_2 z_2 + k_3 z_3 + z_4 = 0$$

The error variables are

$$\begin{aligned}z_1 &= x_1 - y_r \\ z_2 &= x_2 - \alpha_1(x_1, t) \\ z_3 &= x_3 - \alpha_2(x_1, x_2, \hat{\theta}, t) \\ z_4 &= x_4 - \alpha_3(x_1, x_2, x_3, \hat{\theta}, t)\end{aligned}$$

with the stabilizing functions

$$\begin{aligned}\alpha_1 &= -c_1 z_1 \\ \alpha_2 &= -z_1 - c_2 z_2 - \varphi_2^T \hat{\theta} - h_2 \text{sgn}(z_2) - c_1 x_2 + \frac{\partial \alpha_1}{\partial t} \\ \alpha_3 &= -z_2 - c_3 z_3 + \varphi_2^T \frac{\partial \alpha_2}{\partial x_2} \hat{\theta} + \frac{\partial \alpha_2}{\partial x_1} x_2 + \frac{\partial \alpha_2}{\partial x_2} x_3 \\ &\quad + \frac{\partial \alpha_2}{\partial \hat{\theta}} \tau_3 - h_2 \left| \frac{\partial \alpha_2}{\partial x_2} \right| \text{sgn}(z_3) + \frac{\partial \alpha_2}{\partial t}\end{aligned}$$

The associated tuning functions are

$$\begin{aligned}\tau_2 &= \Gamma \omega_2 z_2 = \Gamma \varphi_2^T z_2 \\ \tau_3 &= \Gamma(\omega_2 z_2 + \omega_3 z_3) = \Gamma \left(z_2 - \frac{\partial \alpha_2}{\partial x_2} z_3 \right) \varphi_2^T \\ \tau_4 &= \tau_3 + \Gamma \sigma (\omega_4 + k_3 \omega_3 + k_2 \omega_2) \\ &= \Gamma \left[\left(z_2 - \frac{\partial \alpha_2}{\partial x_2} z_3 \right) \varphi_2^T + \sigma \left(\varphi_4 + \left(-\frac{\partial \alpha_3}{\partial x_2} + k_2 - k_3 \frac{\partial \alpha_2}{\partial x_2} \right) \varphi_2^T \right) \right]\end{aligned}$$

where Γ is an adaptation gain matrix. The update parameter law is $\dot{\hat{\theta}} = \tau_4$ and the control is

$$\begin{aligned}
u = \frac{J_l J_m}{k} & \left[- (c_1 k_1 + k_2) z_1 + (k_1 - c_2 k_2 - k_3) z_2 + (k_2 - c_3 K_3 - 1) z_3 \right. \\
& + k_3 z_4 + \left(\frac{k}{J_l} + \frac{k}{J_m} \right) x_3 - \left(\varphi_4 - \frac{\partial \alpha_3}{\partial x_2} \varphi_2 \right)^T \hat{\theta} - \sum_{i=1}^{i=3} \frac{\partial \alpha_3}{\partial x_i} x_{i+1} \\
& + \frac{\partial \alpha_3}{\partial \hat{\theta}} \tau_4 + \frac{\partial \alpha_3}{\partial t} - W \sigma + y_r^{(4)} \\
& - \left(K + h_4 + 2k_2 h_2 + 2k_3 \left| \frac{\partial \alpha_2}{\partial x_2} \right| h_2 + \left| \frac{\partial \alpha_3}{\partial x_2} \right| h_2 \right) \text{sgn}(\sigma) \\
& \left. - k_3 \frac{\partial \alpha_2}{\partial \hat{\theta}} (\tau_4 - \tau_3) + \frac{\partial \alpha_2}{\partial \hat{\theta}} \Gamma \left(\varphi_4 + \left(-\frac{\partial \alpha_3}{\partial x_2} + k_2 - k_3 \frac{\partial \alpha_2}{\partial x_2} \right) \varphi_2^T \right) z_3 \right]
\end{aligned}$$

The simulation results are shown in Fig. 4 with the values $J_l = 5 \text{ Nm}^2$, $J_m = 7 \text{ Nm}^2$, $l = 1 \text{ m}$, $k = 400 \text{ Nm/rad}$. We assume that $F_m = 0$, $F_l = 0$, $h_2 = 0.5$, $h_4 = 0.1$, $\eta_2 = 0.5 \sin(x_1^2)$, $\eta_4 = 0.1 \cos(3x_1 x_3)$ and $y_r = 3$ as the desired point. The design parameters were selected to be $c_1 = 6$, $c_2 = 8$, $c_3 = 5$, $k_1 = 1$, $k_2 = 2$, $k_3 = 1$, $K = 5$, $W = 100$, $\Gamma = 0.002$. The nominal unknown parameter M is taken to be 2 kg.

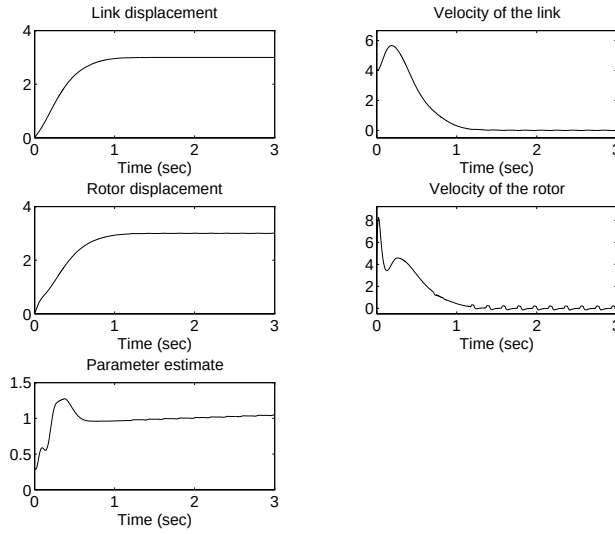


Fig. 4. Responses of Flexible Robot

10 Conclusions

Sliding mode control is a robust control method design and adaptive backstepping is an adaptive control design method. In this chapter the control design has benefited from both design approaches. Backstepping control and sliding backstepping control were developed for a class of nonlinear systems. The plant may have unmodelled or external disturbances. The discontinuous control may contain a gain parameter for the designer to select the velocity of the convergence of the state trajectories to the sliding hyperplane. The previous work of Rios-Bolívar and Zinober [13]-[14] and Koshkouei and Zinober [7], has been extended to include Second-Order SMC and to remove the sufficient existence condition to guarantee that the state trajectories converge to the given sliding surface.

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Sliding Mode Compensation, Estimation and Optimization Methods in Automotive Control

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Abstract. This chapter provides a broad overview of a number of recent automotive applications in a tutorial fashion where several analytical design tools of the sliding mode control theory were primarily used. The design methods are first discussed from a theoretical point of view in three main categories: online functional optimization, disturbance/state estimation and friction compensation. The first automotive control example reported in this chapter is a traction control design which comprises the presented optimization and estimation methods as well as several singular perturbation arguments. A position tracking control problem of a throttle system which has inherent coulomb friction and stiff position feedback is then discussed. A previous sliding mode position tracking control of a pneumatic throttle actuator for an internal combustion engine is also summarized.

1 Introduction

Generally speaking, automotive control problems are highly nonlinear and subject to high amount of disturbances and uncertainties. In most cases, the system to be controlled may operate at diverse operating regimes and include significant nonlinear couplings which make the abundant tools of the linear control system literature not well-suited for a wide range robust operation. On the other hand, sliding mode control theory ([21], [23]) has been investigated in detail over the last three decades and it currently offers numerous systematic design methods applicable to industrial control problems. The use of sliding mode control ideas in automotive control applications has also been reported (see, for instance, [2], [3], [6], [11], [17], [24], [26]).

This chapter presents a couple recent automotive applications which blend several sliding mode control design methods. The theory behind the optimization, estimation and friction compensation tools used in these applications is first discussed in Section 2. The optimization method originates from the on-line unimodular functional optimization method of [23], [1]. The results of a comprehensive investigation of the use of the equivalent control idea for state/disturbance estimation purposes are next summarized from [23], [4],

[7], [8]. The friction compensation method is based on a recent development where the handling of non-smooth nonlinearities operating on manifolds in the state/control space is examined in a broader context via sliding motions [13], [14]. The optimization and estimation methods are then used in Section 3 on a traction control problem where the acceleration characteristics of a vehicle are to be optimized in an engine control framework via dynamic spark advance. The friction compensation method is exemplified on a position tracking control problem of a throttle system with inherent coulomb friction and stiff position feedback in Section 4. Finally, Section 5 summarizes a previous sliding mode position tracking control design for a pneumatic throttle actuator of an internal combustion engine.

2 Sliding Mode Design Methods

2.1 On-line Functional Optimization

To a certain extent, several automotive control objectives can be formulated as an optimization problem. For example, ABS/traction control can be designed to robustly operate around the minimum/maximum point of the tire force-relative slip curve, engine should deliver the desired torque with the least possible fuel consumption, EGR input needs to be determined so as to minimize the emission formation and so on and so forth. The use of sliding modes for on-line optimization of an analytically unknown unimodular functional has been reported in [23]. The basic idea is to make the optimization variable (the signal which is desired to be optimized) follow an increasing/decreasing time function via sliding mode motions. The main difficulty with such a setup is that the unknown gradient term multiplies the control at the differential equation of the optimization variable so that the system itself possesses a variable structure behavior. This idea has been extended in [1] with the introduction of the notion of periodic switching function and applied to ABS/traction control problems in [2], [11] and [24]. Next, we briefly discuss the basics of this optimization method.

Consider a unimodular functional $y = f(x)$ which has a unique extremum at the point (x^*, y^*) . The mathematical expression of $f(x)$ is unknown. For definiteness, the extremum is selected as the maximum which turns the optimization objective into a maximization one. x is assumed to be the output of an integrator which takes u as its input. The control objective is to keep x at the vicinity of the unknown optimal x^* by modulating x by u using the on-line values of y . The performance output (optimization variable), y , is forced to track an increasing time function irrespective of the unknown gradient information via sliding modes. Pick any increasing function $g(t)$ and try to keep $f(x) - g(t)$ at a constant by a proper u . If so, $f(x)$ increases at the same rate with $g(t)$ independent of whether $x < x^*$ or $x > x^*$. To this end, let

$$s = f(x) - g(t) \tag{1}$$

so that

$$\dot{s} = (\partial f / \partial x) u - \dot{g}(t) \quad (2)$$

With the control law of

$$u = M \operatorname{sgn} \sin(2\pi s / \alpha) \quad (3)$$

as in [1] with α being a small positive constant, a sliding motion occurs for $M|\partial f / \partial x| > |\dot{g}(t)|$ and x is steered towards x^* while y tracking $g(t)$. The region, defined by $|\partial f / \partial x| < |\dot{g}(t)| / M$, quantifies the region in which x will be confined with this control. The idea can be extended to more general dynamics by adding the derivatives of the performance variable as well as those of $g(t)$ to the sliding manifold expression so as to compensate the relative degree deficit. In [12], this optimization idea has further been developed for on-line operating point and set point optimization purposes by ending up with a two-time scale sliding mode optimization design. The resulting method allows the optimization of the closed loop operation of a system by exploiting the extra degree of freedom in the available control authority possibly in a different time scale.

2.2 Disturbance Estimation and Compensation

One way of enhancing the robustness of a control system is to estimate the discrepancies between the model used for the control derivation purposes and the actual system by a perturbation estimator and to incorporate this information into the control law in a proper way. To this end, we next present two ways of disturbance estimation. The first one is based on the equivalent control methodology ([23]) and it is in the continuous-time domain. A discrete-time sliding mode disturbance estimation method is also discussed.

First, consider a SISO nonlinear system

$$\dot{x} = f(x) + g(x)u + \delta(t, x, u) \quad (4)$$

where $x \in \mathbf{R}$ is the state, $u \in \mathbf{R}$ is the control, $f(x)$, $g(x)$ are smooth, known functions and $\delta(t, x, u)$ is the disturbance function which lumps all the disturbances and the uncertainties of the system. It is assumed that

$$|\delta(t, x, u)| \leq \rho(t) \quad (5)$$

where $\rho(t)$ is a known bounding function. The objective is to estimate $\delta(t, x, u)$ from x .

The disturbance estimator is given by

$$\dot{\hat{x}} = f(x) + g(x)u + (\rho(t) + \eta) \operatorname{sgn}(x - \hat{x}) \quad (6)$$

which basically repeats what is known about the system with an additional discontinuous injection. The error dynamics follow from the subtraction of Eq. (6) from Eq. (4) as follows:

$$\dot{e}_x = \delta(t, x, u) - (\rho(t) + \eta) \operatorname{sgn} e_x \quad (7)$$

where $e_x = x - \hat{x}$. Ideally $e_x = 0 \forall t \geq t_0$ with $\eta > 0$, $x(t_0) = \hat{x}(t_0)$ so that

$$[(\rho(t) + \eta) \operatorname{sgn} e_x]_{eq} = \delta(t, x, u) \quad (8)$$

The operator $[\bullet]_{eq}$ outputs the equivalent value of its discontinuous argument which is defined as the continuous injection which would satisfy the invariance conditions of the sliding motion ($e_x = 0$, $\dot{e}_x = 0$) that this discontinuous input induces. The equivalent value operator, $[\bullet]_{eq}$, can be approximately realized by an high bandwidth low-pass filter according to the equivalent control methodology ([23]); i.e,

$$\begin{aligned} \tau \dot{v} + v &= (\rho(t) + \eta) \operatorname{sgn} e_x \\ v &= \delta(t, x, u) + \mathcal{O}(\tau, \epsilon/\tau) \end{aligned} \quad (9)$$

where $|e_x| \leq \epsilon \forall t \geq t_0$ with ϵ being an arbitrarily small positive number.

Assume that the disturbance is also differentiable; i.e,

$$\dot{\delta}(t, x, u) = \Delta(t) \quad (10)$$

$$|\Delta(t)| < \bar{\rho}(t) \quad (11)$$

where $\bar{\rho}(t)$ is a known bound. The derivative of the disturbance function can be obtained by

$$\begin{aligned} \dot{\hat{\delta}}(t) &= K \operatorname{sgn} (\delta - \hat{\delta}) \\ \dot{\hat{\delta}}(t) &= K \operatorname{sgn} ([(\rho(t) + \eta) \operatorname{sgn} e_x]_{eq} - \hat{\delta}) \end{aligned}$$

where $K = \bar{\rho}(t) + \kappa$, $\kappa > 0$. Therefore, in sliding mode,

$$[K \operatorname{sgn} ([(\rho(t) + \eta) \operatorname{sgn} e_x]_{eq} - \hat{\delta})]_{eq} = \Delta(t) \quad (12)$$

The design is recursive. Equivalent control operators perform information transfer between two consecutive steps and the design logic can be repeated to estimate higher order derivatives of the disturbance function as long as it is continuously differentiable to a certain order. However, the overall design requires the implementation of the sequential equivalent value operators. The approximability of the equivalent control by low-pass filtering was proven in [23]. The relation between the estimation accuracy and the filter time constants in the implementation of the sequential equivalent value operators by low-pass filters was examined in [8]. In that paper, an ultimate boundedness analysis was carried out for the estimation errors and a theoretical rule of thumb was proposed for the selection of the filter time constants.

Suppose that a baseline control law, $u_n(t, x)$, has already been specified such that it would achieve the control objective for the nominal system. The control can then be complemented with the estimated disturbance as follows:

$$u(t, x) = u_n(t, x) - g^{-1}(t, x)v \quad (13)$$

where v is obtained from Eq. (9). With this new control, the closed loop dynamics are only affected by the residual estimation error which is naturally easier to deal with a less conservative control action. Generally, if the disturbance is matched with respect to the control, a direct cancellation term can be added to the nominal control law to preserve the robustness whereas for mismatched disturbances the nominal control needs to be devised so as to allow freedom for the use of the disturbance estimates. A recent study where these ideas were elaborated in detail can be found in [10].

Most of the today's control algorithms are implemented in discrete-time. However, the discrete-time implementation of a continuous sliding mode control law may cause the well-known chattering problem if no chattering reduction method is employed. As an alternative estimator design where the sampling issues are taken into account at the first place, the continuous-time system of Eq. (14)

$$\dot{x} = u(t) + \delta(t) \quad (14)$$

is first discretized so as to obtain

$$x_{k+1} = x_k + Tu_k + T\bar{\delta}_k \quad (15)$$

where $x_k = x(kT)$, $u_k = u(t)$ for $kT \leq t < (k+1)T$, T is the sampling time,

$$\bar{\delta}_k = \frac{1}{T} \int_{kT}^{(k+1)T} \delta(t) dt \quad (16)$$

and it is assumed that the control is applied through a zero-order-hold. Note that, $\bar{\delta}_k$ cannot be computed unless the future values of the external disturbance function $\delta(t)$ are known. However, if $\delta(t)$ is smooth $\bar{\delta}_k$ can be predicted by $\bar{\delta}_{k-1}$ which can be computed from

$$\bar{\delta}_k = \frac{1}{T} [x_k - x_{k-1}] - u_{k-1} \quad (17)$$

with an $\mathcal{O}(T)$ accuracy according to

$$\begin{aligned} \bar{\delta}_k - \bar{\delta}_{k-1} &= \frac{1}{T} \int_{kT}^{(k+1)T} \delta(t) dt - \frac{1}{T} \int_{(k-1)T}^{kT} \delta(t) dt \\ |\bar{\delta}_k - \bar{\delta}_{k-1}| &= 2T\delta_{\max} = \mathcal{O}(T) \end{aligned} \quad (18)$$

where $|d\delta/dt| < \delta_{\max}$.

Suppose that the control objective is to regulate x to zero. Incorporating the estimated disturbance to the control law

$$u_k = -\frac{1}{T}x_k - \bar{\delta}_{k-1} \quad (19)$$

one gets

$$x_{k+1} = T[\bar{\delta}_k - \bar{\delta}_{k-1}] = \mathcal{O}(T^2) \quad (20)$$

so that at each sampling instant x is actually forced to an $\mathcal{O}(T^2)$ vicinity of zero. This is an increase in performance compared to the direct discretization of a discontinuous sliding mode control law which would achieve only an $\mathcal{O}(T)$ accuracy. A detailed study of this idea can be found in [18], [19] and [20] where it has also been shown that this control leads to an $\mathcal{O}(T^2)$ accuracy in sliding motion also during the inter-sampling behavior for sampled-data systems with control being applied through a zero-order hold.

The basic idea of the discrete-time estimation method presented so far is to calculate the one step previous value of the disturbance from the available quantities and to use it as an estimate for its current value. This results in an $\mathcal{O}(T)$ estimation accuracy provided that the first order derivative of the disturbance is bounded. If the external disturbance is continuously differentiable to a certain order, using more previous values of the disturbance could actually lead to a better estimation accuracy. This idea has been studied in [5] leading to a controller structure where the ideal discrete-time equivalent control law of [22] were complemented with a discrete-time filtering action to increase the robustness of the system as well as the accuracy of the sliding motion at the sampling instances. However, note that the intersampling time behavior will still be $\mathcal{O}(T^2)$ if the control is to be implemented through a zero-order-hold as explained in [19]. Therefore, the new estimator also needs to be complemented with an higher-order-hold mechanism in the control channel to reduce the deviations of x from zero between the two sampling instances as well.

2.3 State Observation

In many industrial applications, the on-line estimation of several signals by an observer rather than using a sensor may lead to more sophisticated and cost effective control systems. There are several state observer design methods reported in the sliding mode control literature. The discussion of all these methods are beyond the scope of this study. Instead, in this chapter, we are specifically interested in a sliding mode observer design method which uses the equivalent control idea as in the continuous time disturbance estimation method of Section 2.2. Next this method is presented similar to its original which was reported in [23]:

Consider a linear system

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx\end{aligned}\tag{21}$$

where $x \in R^n$, $u \in R^p$, $y \in R^m$, the pair (A, C) observable, C has full rank. This system can be transformed into

$$\begin{aligned}\dot{y} &= A_{11}y + A_{12}x_1 + B_1u \\ \dot{x}_1 &= A_{21}y + A_{22}x_1 + B_2u\end{aligned}\tag{22}$$

where A_{11} , A_{12} , A_{21} , A_{22} , B_1 , B_2 are constant matrices of appropriate dimensions. The observer equation for the first part of Eq. (22) is selected as follows:

$$\dot{\hat{y}} = A_{11}\hat{y} + B_1u + L_1 \text{sgn}(y - \hat{y}) \quad (23)$$

Error dynamics are given by

$$\dot{e}_y = A_{11}e_y + A_{12}x_1 - L_1 \text{sgn } e_y \quad (24)$$

where $e_y = y - \hat{y}$. A sliding motion occurs on $e_y = 0$ in finite time with a suitable L_1 and in sliding mode

$$[L_1 \text{sgn } e_y]_{eq} = A_{12}x_1 \quad (25)$$

Therefore, an information on x_1 is indirectly available through a low-pass filter.

At the second step, consider

$$\begin{aligned} \dot{x}_1 &= A_{22}x_1 + A_{21}y + B_2u \\ y_1 &= L_1^{-1}A_{12}x_1 \end{aligned}$$

This reduced order system can also be transformed into

$$\begin{aligned} \dot{y}_1 &= A_{31}y_1 + A_{32}x_2 + A_{33}y + B_3u \\ \dot{x}_2 &= A_{41}y_1 + A_{42}x_2 + A_{43}y + B_4u \end{aligned}$$

Let the second observer equation be

$$\dot{\hat{y}}_1 = A_{31}\hat{y}_1 + A_{33}\hat{y} + B_3u + L_2 \text{sgn}(y_1 - \hat{y}_1) \quad (26)$$

Replacing y_1 with its equivalent in Eq. (25) for implementation, ideally a sliding motion can be guaranteed on $y_1 - \hat{y}_1 = 0$ in finite time as well. This design routine can be repeated so as to result in a full finite time converging observer. The details and the formulatization of this method can be found in [4], [7], [8] among others. As in the disturbance estimation design, the effects of the repeated use of low-pass filtering on the overall estimation accuracy were quantified in [8] in terms of a single variable which parameterizes all the filter time constants. In [4], [7], a discrete-time equivalent control based sliding mode observer design method was also proposed. Next, this method is summarized.

Consider a discrete-time linear system

$$\begin{aligned} x_{k+1} &= \Phi x_k + \Gamma u_k \\ y_k &= C x_k \end{aligned} \quad (27)$$

where $x \in R^n$, $u \in R^p$, $y \in R^m$, the pair (Φ, C) observable and C has full rank.

The discrete time equivalent control definition of [22] is used for a dual discrete-time design. Transform the original system of Eq. (27) into

$$\begin{aligned} y_{k+1} &= \Phi_{11}y_k + \Phi_{12}x_{1,k} + \Gamma_1u_k \\ x_{1,k+1} &= \Phi_{21}y_k + \Phi_{22}x_{1,k} + \Gamma_2u_k \end{aligned} \quad (28)$$

and let the corresponding discrete time sliding mode observer be

$$\begin{aligned} \hat{y}_{k+1} &= \Phi_{11}\hat{y}_k + \Phi_{12}\hat{x}_{1,k} + \Gamma_1u_k - v_k \\ \hat{x}_{1,k+1} &= \Phi_{21}\hat{y}_k + \Phi_{22}\hat{x}_{1,k} + \Gamma_2u_k + Lv_k \end{aligned} \quad (29)$$

Error dynamics are as follows:

$$\begin{aligned} e_{y,k+1} &= \Phi_{11}e_{y,k} + \Phi_{12}e_{x_1,k} + v_k \\ e_{x_1,k+1} &= \Phi_{21}e_{y,k} + \Phi_{22}e_{x_1,k} - Lv_k \end{aligned} \quad (30)$$

where $e_{y,k} = y_k - \hat{y}_k$ and $e_{x_1,k} = x_{1,k} - \hat{x}_{1,k}$. The equivalent value of v_k can be calculated by solving $e_{y,k+1} = 0$ for v_k ([22]) as follows:

$$v_{k,eq} = -\Phi_{11}e_{y,k} - \Phi_{12}e_{x_1,k} \quad (31)$$

A sliding motion occurs on $e_y = 0$ in finite step if $v_k = v_{k,eq}$. In sliding mode

$$e_{x_1,k+1} = (\Phi_{22} + L\Phi_{12})e_{x_1,k} \quad (32)$$

To implement the observer, we define an auxiliary system

$$z_{k+1} = (\Phi_{22} + L\Phi_{12})z_k + (\Phi_{21} + L\Phi_{11})e_{y,k-1} - Le_{y,k} \quad (33)$$

and replace v_k by

$$\begin{aligned} \hat{v}_{k,eq} &= -(\Phi_{11} - \Phi_{12}L)e_{y,k} - \Phi_{12}(\Phi_{21} + L\Phi_{11})e_{y,k-1} \\ &\quad - \Phi_{12}(\Phi_{22} + L\Phi_{12})z_k \end{aligned} \quad (34)$$

By placing the eigenvalues of $(\Phi_{22} + L\Phi_{12})$ at the origin, $\hat{v}_{k,eq} \rightarrow v_{k,eq}$ and $e_y \rightarrow 0$ in finite step. Note that the z -dynamics replace the low-pass filtering of the continuous-time equivalent control based observer design.

2.4 Friction Compensation

This section summarizes the theory behind the friction compensation method of [13], [14]. To this end, we first discuss the system induced manifolds concept and the generalized stiction phenomenon.

Consider the following class of systems

$$\dot{x} = f(x, t) + \mu \cdot \text{sgn}(s(x, t)) + h(x, t)u \quad (35)$$

with right hand-side discontinuities on the k surfaces,

$$s(x, t) = [s_1(x, t), s_2(x, t), \dots, s_k(x, t)]' = 0$$

where $\mu = [\mu_1 \cdots \mu_k] \in \mathbf{R}^{n \times k}$, $h(x, t) \in \mathbf{R}^{n \times p}$, $f(x, t) \in \mathbf{R}^n$ and the entries of f and h are smoothly differentiable functions (in $\mathbf{C}^n$), $u \in \mathbf{R}^p$ is the control input and $s : \mathbf{R}^n \rightarrow \mathbf{R}^k$. The signum operator is defined to operate on every entry of its argument.

The system as given in Eq. (35) appears in the form of an n th-order system with an input that has $k + p$ components k of which have already been specified in the form of sliding mode control with μ and $\sigma = \{x \in \mathbf{R}^n : s(x, t) = 0\}$ being the gain and the manifold, respectively. Note that, this is not exactly the case as these manifolds have not been designed and the associated gains have not been selected by the designer. Instead, they have been induced by the system itself. However, this sort of analogy allows us to analyze the system using the mathematical tools of sliding mode theory.

Consider one of the candidate stiction manifolds, namely s_j for $j = 1, \dots, k$. Under the assumption of the existence of sliding mode, i.e. when

$$\dot{s}_j \cdot s_j < 0; \quad j = 1, \dots, k \quad (36)$$

the system starts to slide on the manifold described by,

$$\sigma_j = \{[x_1 \cdots x_{n-1} \ x_n] \in \mathbf{R}^n \mid s_j(x, t) = 0\} \quad (37)$$

Note that the condition of Eq. (5) defines an open region $\mathcal{A}_j$ in the state space. This region can be found by analyzing the derivative of s_j for $j = 1, \dots, k$

$$\dot{s}_j = \frac{d}{dt} s_j(x_1, \dots, x_{n-1}, x_n) = q_j(\dot{x}_1, \dots, \dot{x}_{n-1}, \dot{x}_n) \quad (38)$$

where $q_j \in \mathbf{R}$ when confined to the trajectories described by Eq. (35), i.e., when the equality in Eq. (35) is used to replace the derivatives of the states in Eq. (38), becomes a function of the states $x_1, \dots, x_n$, the control input $u(t)$ and the combination of the discontinuities given on the right hand side of the system description.

$$\begin{aligned} \dot{s}_j = & \tilde{q}_j(f_1(x), \dots, f_n(x), \text{sgn}(s_1(x)), \dots, \text{sgn}(s_k(x)), \\ & u_1, \dots, u_p, x) + g_j(x) \text{sgn}(s_j(x)) \end{aligned} \quad (39)$$

where $g_j : \mathbf{R}^n \rightarrow \mathbf{R}$ is the gain multiplying the j th discontinuous component, and $\tilde{q}_j : \mathbf{R}^n \rightarrow \mathbf{R}$. Then, $(\dot{s}_j \cdot s_j)$ becomes negative if,

$$-g_j(x, t) > |\tilde{q}_j(\cdot)| \quad (40)$$

$\forall x \in \mathbf{R}^n$. The open region $\mathcal{A}_j$ is then described by,

$$\mathcal{A}_j = \{[x_1 \cdots x_{n-1} \ x_n] \in \mathbf{R}^n \mid -g_j(\cdot) > |\tilde{q}_j(\cdot)|\} \quad (41)$$

Recall that this analysis is prior to the controller design. When the condition of Eq. (40) is satisfied for some j , the system trajectories of Eq. (35) will get stuck at $s_j = 0$ which is not a designed manifold. This phenomenon is induced due to the inherent right hand side discontinuities existing in the

original system. Hence, an open stiction region can now be described in the state space as follows,

$$\mathcal{R}_s = \bigcup_{j=1}^k (\sigma_j \cap \mathcal{A}_j) \quad (42)$$

Note that, $\mathcal{A}_j$ could differ from an empty set to the entire state space, but in general describes an open region which is a subset of $x \in \mathbf{R}^n$. It is also affected by the magnitude of the control input being generated.

So far, the definition of “generalized stiction” has been given. Next, a sliding mode controller design approach which guarantees the avoidance of generalized stiction $\mathbf{R}_s \setminus (\mathbf{R}_s \cap \mathbf{R}_c) \subset \mathbf{R}^n$ in tracking problem for a class of systems where $\mathbf{R}_c$ is the controlled manifold.

Consider now the following class of SISO systems in their companion forms with right hand-side discontinuities on the p surfaces,

$$\begin{aligned} x^{(n)} &= f(x) + \mu \cdot \text{sgn}(s(x)) + h(x)v \\ \dot{v} &= u, \quad y = x \end{aligned} \quad (43)$$

where $\mu = [\mu_1 \ \cdots \ \mu_p] \in \mathbf{R}^{1 \times p}$, $h(\cdot)$, $f(\cdot) : \mathbf{R}^n \rightarrow \mathbf{R}$ are smoothly differentiable functions and moreover $h(\cdot) \neq 0$ for any $x = [x, \dot{x}, \dots, x^{(n-1)}]^T$ (controllability condition over the entire state space), $u, v \in \mathbf{R}$, u is the control input. It is allowed that there are uncertainties in $f(\cdot)$ and/or μ , and only some nominal values $\bar{f}(\cdot)$ and $\bar{\mu}$ are known with bounded errors $\Delta f(\cdot) = |f(\cdot) - \bar{f}(\cdot)|$ and $\Delta\mu = |\mu - \bar{\mu}|$. The control objective is to generate a control input u such that the output $y = x$ tracks the reference signal x_r . Define the tracking error as $e = x - x_r$.

The controller creates multi-layer quasi-sliding manifolds so as to compensate for the uncertainties. To this end, let

$$v_d = \frac{1}{h(\cdot)} \left\{ \left[x_r^{(n)} - \kappa_1 e^{(n-1)} - \cdots - \kappa_n e \right] - \bar{f}(\cdot) - \bar{\mu} \cdot \text{sgn}_k(s(\cdot)) + w \right\} \quad (44)$$

where w is a fictitious input, $\text{sgn}_k(x) = (2/\pi) \arctan(kx)$ is a smooth approximation for the signum function with $\Phi_k(x) = \text{sign}(x) - \text{sgn}_k(x)$ denoting the approximation error.

Select $\sigma_1 = e^{(n-1)} + c_1 e^{(n-2)} + \cdots + c_{n-2} \dot{e} + c_{n-1} e$ such that $\sigma_1 = 0$ will exhibit stable dynamics with negative real poles. Then pick,

$$w = (\kappa_1 - c_1) e^{(n-1)} + \cdots + (\kappa_{n-1} - c_{n-1}) \dot{e} + \kappa_n e - \beta \cdot \text{sgn}_k(\sigma_1) \quad (45)$$

which will ensure $(\dot{\sigma}_1 \cdot \sigma_1) < 0$ for all $|\sigma_1| < \gamma$, provided that k is picked large enough and

$$\beta > \Delta f(\cdot) + |\mu \Phi_k(s(\cdot))| + |\Delta\mu \cdot \text{sgn}_k(s(\cdot))| + \varepsilon \quad (46)$$

Defining the sliding manifold by $\sigma_2 = v - v_d$ and picking the corresponding control input $\dot{v}$ as follows:

$$u = \dot{v} = -\alpha \cdot \text{sgn}(\sigma_2), \quad \text{s.t.} \quad \alpha > (|\dot{v}_d| + \varepsilon) \quad (47)$$

then $\dot{\sigma}_2 \cdot \sigma_2 < 0$ is ensured. In sliding mode, $x \rightarrow x_d$ at the desired rate. A smooth approximation has been used for the signum functions of the first layer to assure that $\dot{v}_d$ is also bounded around $s = 0$ (which may be the case if the signal to be tracked lies on the hyper-surface described by the inherent discontinuity or requires crossing the mentioned hyper-plane). Note that the described control law will guarantee that the system trajectories will be directed towards the subspace described by $|\sigma_1| \leq \gamma$ on the n -dimensional state space. The magnitude of γ can be manipulated by the designer, but cannot be explicitly made zero.

3 Sliding Mode Traction Control

The acceleration characteristics of a vehicle can be improved in an engine control setup where the dynamic spark advance is used to dynamically modulate the engine torque [6]. The primary reason for wheel spin due to sudden changes in the engine air input is closely related to the tire force characteristics of the wheel. The tire force/relative slip curve has ideally only one extrema for each of the acceleration and the braking regions. A sufficiently large throttle input might cause the relative slip to move into the positive feedback region where the tire force is decreasing with increasing slip. Since the availability of a relative slip measurement and an accurate analytic expression of the tire force/relative slip curve are quite unrealistic in the current setup the sliding mode optimization method of [1] were utilized in [11], [24] to robustly operate around the peak driving tire force without any *a priori* information on the tire force/relative slip curve.

In this section, the previous results on this topic are summarized from [11], [24]. The first control logic was to devise an optimal law for the spark angle input in the form of an engine torque multiplier so as to keep the relative slip around its optimal which would produce the maximum traction force. Then, the same idea is used to optimize the performance of a baseline dynamic output feedback spark advance controller (DOFSAC) ([6]) with no additional sensor inputs.

3.1 The Model

The plant model includes a static engine torque map, a first order transmission model, a nonlinear longitudinal tire force model and the vehicle is considered as a point mass with an aerodynamic drag force. The intake manifold dynamics are neglected for simplicity and no driver model is utilized. The model equations can be written as follows:

$$\begin{aligned}\dot{V} &= a_1 V^2 + b_1 F_d(t, \sigma) \\ \dot{w} &= a_2 w + a_3 \Psi + b_2 F_d(t, \sigma) \\ \dot{\Psi} &= a_4 n + a_5 w \\ \dot{n} &= a_6 n + a_7 \Psi + b_3 T_e(n, \Theta) u\end{aligned}\tag{48}$$

where V is the longitudinal speed, w is the wheel speed, Ψ is a transmission variable, n is the engine shaft speed, $\sigma = w/V - 1$ is the slip, $a_1 = -(A_p/J_v)$, $b_1 = (r_e/J_v)$, $a_2 = -(B_w/J_w)$, $a_3 = (K_T K_g/J_w)$, $b_2 = -(r_e/J_w)$, $a_4 = 1$, $a_5 = -K_g$, $a_6 = -(B_e/J_e)$, $a_7 = -(K_T/J_e)$, $b_3 = (1/J_e)$ with relevant physical parameters and the effect of spark retard is modeled as a variable engine torque multiplier denoted by u . The value of u are then translated into spark timing information using an approximate static map.

The engine-wheel coupling through transmission results in a two-time scale behavior. For controller design purposes, this characteristic was utilized to further reduce the order of the actual model based on the singular perturbation theory [15]. The slow system dynamics are summarized as follows:

$$\begin{aligned}\dot{V}_s &= a_1 V_s^2 + b_1 F_d(t, \sigma_s) \\ \dot{w}_s &= \bar{a}_2 w_s + \bar{b}_2 F_d(t, \sigma_s) + \bar{a}_3 T_e(w_s, \Theta) u_s\end{aligned}\quad (49)$$

where V_s , w_s , σ_s and u_s are the slow components of the variables V , w , σ and u , respectively, $\bar{a}_2 = -(B_w + B_e K_g^2)/(J_w + J_e K_g^2)$, $\bar{b}_2 = -r_e/(J_w + J_e K_g^2)$, $\bar{a}_3 = K_g/(J_w + J_e K_g^2)$.

3.2 Sliding Mode Dynamic Spark Advance Controller

The engine RPM and the throttle input are assumed to be available measurements whereas an analytical expression for the tire force/relative slip curve as well as the optimal slip are unknown. The original control problem of robust operation around the optimal slip is formulated as an optimization problem of an analytically unknown criterion using the optimization method summarized in Section 2.1. The control design is carried out on the slow system and the tire force is obtained by the estimator of Section 2.2.

The sliding surface is selected as follows: $s = e + \int_0^t \Lambda(e(\tau)) d\tau$ where $e = F_d - F_d^r(t)$, $F_d^r(t)$ is a user-specified explicit time function and $\Lambda(e)$ is to be chosen. If s can be kept constant, the constrained motion satisfies

$$\frac{de}{dt} + \Lambda(e) = 0 \quad (50)$$

and the tire force behaves as desired with proper selections of $F_d^r(t)$ and $\Lambda(e)$. The error variable is governed by

$$\frac{de}{dt} = \frac{1}{V} \frac{\partial F_d}{\partial \sigma} [A(w, V, F_d) + B(w, \Theta) u] + \frac{\partial F_d}{\partial t} - \frac{dF_d^r}{dt} \quad (51)$$

where

$$A(w, V, F_d) = \bar{a}_2 w + \bar{b}_2 F_d - a_1 w V - b_1 F_d \frac{w}{V} \quad (52)$$

$$B(w, \Theta) = \bar{a}_3 T_e(w, \Theta) \quad (53)$$

Let $A(\bullet) = \bar{A} + \Delta A$, $0 < B_{min} < B(\bullet) < B_{max}$ where $\bar{A}$ represents the nominal part of A whereas the unknown term ΔA is bounded according to

$|\Delta A| \leq \delta_A$ with δ_A known and $\bar{B} = \sqrt{B_{min} B_{max}}$ for which $\beta^{-1} \leq (\bar{B}/B) \leq \beta$ where $\beta = (B_{max}/B_{min})^{1/2}$.

The control law is selected as $u = -\bar{B}^{-1}[\bar{A} + \gamma \Phi(s)]$ where $\gamma = \beta\delta_A + (\beta - 1)|\hat{A}|_\infty + M$ with an $M > 0$ and $\Phi(s) = \text{sgn} \sin(2\pi s/\alpha)$ is the periodic switching function [1]. This selection guarantees that s is kept at $k\alpha$ for some k which depends on the system and the initial conditions, if the following sliding mode existence condition is satisfied:

$$\frac{1}{V} \left| \frac{\partial F_d}{\partial \sigma} \right| M \beta^{-1} > \left| \frac{\partial F_d}{\partial t} - \frac{dF_d^r}{dt} + \Lambda(e) \right| \quad (54)$$

If F_d^r is chosen as a constant, $\Lambda(e) = \lambda e$ with a $\lambda > 0$ and also assume that explicit time dependence of the tire force is negligible, the sliding mode existence condition becomes

$$\frac{1}{V} \left| \frac{\partial F_d}{\partial \sigma} \right| M \beta^{-1} > \lambda |e| \quad (55)$$

In sliding mode, if F_d^r can be reached the tire force converges exponentially towards it with a rate dependent upon λ . On the other hand, if $F_d^r > F_{d,max}$ the tire force behaves as before until it enters the region where the gradient is too small such that the sliding mode existence condition of Eq. (55) can no longer be guaranteed. After that, the system becomes uncontrollable and the tire force behaves arbitrarily. However, the controller creates a region of attraction around the maximum point whose width can be controlled by M . Consider the region $|\partial F_d / \partial \sigma| \leq \Delta$. For any controller parameter $M > 0$, there exists a Δ given by $M > M_\delta = V\lambda|F_d - F_d^r|_{max}/\Delta$ such that the tire force is guaranteed to be kept in this region.

3.3 Optimal Sliding Mode DOFSAC

The original DOFSAC ([6]) controls the engine torque output via dynamic spark advance based on filtered engine RPM measurement. Engine RPM is filtered with a band pass filter for practical differentiation purposes and then it is compared to a constant threshold value. If the filter output is greater than the threshold, the spark timing is retarded proportional to the error. This decouples the high energy terms of the engine from the wheel so that the likelihood of wheel spin reduces. However, the threshold value needs to be selected and originally it is tuned in advance for different conditions through simulation studies and experimental tests.

This section summarizes an optimal sliding mode DOFSAC design where basically an additional loop is devised for the threshold so as to maximize the tire force (Fig. 1). To this end, DOFSAC is first modeled by $u = 1 - K(\dot{n} - \rho) = \bar{\rho} - K\dot{n}$ where ρ denotes the threshold and K is selected such that the error $K(\dot{n} - \rho)$ is properly mapped to a value in the admissible control domain to produce a control within its limits. Using the singular perturbation theory,

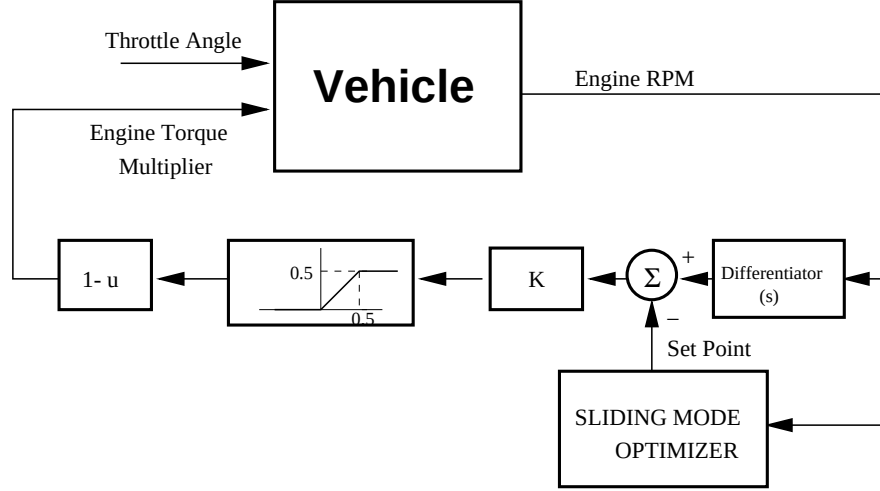


Fig. 1. Optimized DOFSAC (from [11], [24])

the order of the complete system with the control in the loop can also be reduced as in Section 3.2 as follows:

$$\begin{aligned}\dot{V}_s &= a_1 V_s^2 + b_1 F_d(t, \sigma_s) \\ \dot{w}_s &= \hat{a}_2 w_s + \hat{b}_2 F_d(t, \sigma_s) + \hat{a}_3 T_e(w_s, \Theta) \bar{\rho}_s\end{aligned}\quad (56)$$

with state dependent $\hat{a}_2$, $\hat{b}_2$ and $\hat{a}_3$. Repeating the design of Section 3.2, the set point is obtained from $\bar{\rho} = -\hat{B}^{-1}[\hat{A} + \gamma \Phi(s)]$ where $s = e + \int_0^t \Lambda(e) d\tau$, $\Lambda(e)$ is to be chosen and $\gamma = \beta \delta_A + (\beta - 1)|\hat{A}|_\infty + M$. With a sufficiently large M , this set point selection forms a positively invariant region around the optimal slip defined by $|\partial F_d / \partial \sigma| \leq \Delta$. In sliding mode, $\dot{e} + \Lambda(e) = 0$ and the tire force converges to this region as desired with proper F_d^r and $\Lambda(e)$. The size of this region can also be controlled by M . Once $\bar{\rho}$ has been determined ρ can be computed using $\rho = K^{-1}(\bar{\rho} - 1)$. Further details of the presented traction control designs as well as the simulation results can be found in [11].

4 Friction Compensation for the Position Control of a Throttle System

The system involves a plant which is driven by an actuator with faster dynamics. The plant has inherent coulomb-viscous friction and stiff position feedback which are the two sources of stiction in the state space.

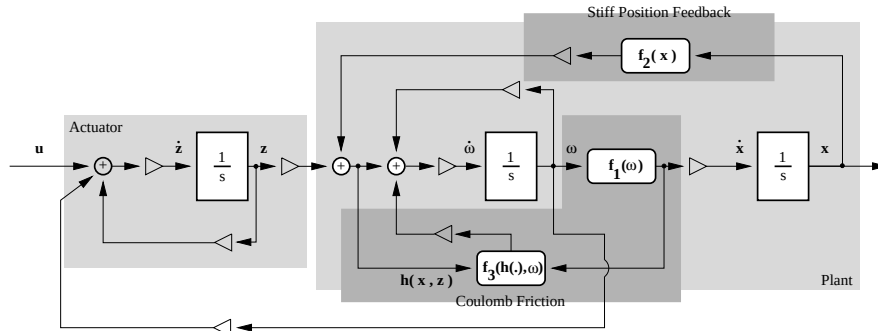


Fig. 2. The system to be controlled (from [14])

4.1 The Plant Model

Consider the plant depicted in Fig. 2. The state space representation is given by

$$\begin{aligned}\dot{x} &= (1/K_g)f_1(\omega) \\ \dot{\omega} &= (1/J)(h(x, z) - (C/K_g)\omega - (1/K_g)f_3(h(x, z), \omega)) \\ \dot{z} &= (1/L)(-Rz - K_t\omega + u)\end{aligned}\tag{57}$$

where

$$f_1(\omega) = \text{deadzone}(\omega, \pm\delta, 1) \quad (58)$$

$$h(x, z) = K_t z - (1/K_g) (f_2(x) + K (180x/\pi - \theta_o)) \quad (59)$$

$$f_2(x)! = \gamma^{\text{sat}}((180x/\pi - \theta_o)\alpha) \quad (60)$$

$$f_3(h(\cdot), \omega) = \beta_{\text{sat}}(f_1(\omega)/\delta) + \beta_{\text{sat}}(K_g h(\cdot) f_4(f_1(\omega))) \quad (61)$$

$$f_4(f_1(\omega)) = 1 - (\text{relaywdzn}(f_1(\omega), \pm\delta, 1))^2 \quad (62)$$

where x is the position, ω is the angular velocity, and z is the auxiliary state variable that describes the dynamics of the first order actuator. Due to the physical limits of the system the position variable $x(t)$ should be between 7° and 85° . The referred nonlinearities, the desired tracking signals and the parametric values can be found in [14].

4.2 An Approximate Model for Control Design

Although friction has been modeled in details so as to include the Stribeck effects as well as stick-slip behavior, it is concluded that a simpler model suffices to describe the motion of the system with good precision while easing the controller design phase. Consider,

$$\begin{aligned}\dot{x} &= a_{12}\omega \\ \dot{\omega} &= a_{21}(x - x_o) + a_{22}\omega - \kappa \operatorname{sgn}(x - x_o) - \mu \operatorname{sgn}(\omega) + a_{23}z \\ \dot{z} &= a_{32}\omega + a_{33}z + bu\end{aligned}\tag{63}$$

where $a_{12} = (1/K_g)$, $a_{21} = -(K/K_g J)$, $a_{22} = -(C/K_g J)$, $a_{23} = (K_t/J)$, $a_{32} = -(K_t/L)$, $a_{33} = (R/L)$, $b = (1/L)$, $x_o = (\pi\theta_o/180)$, $\kappa = (\gamma/K_g J)$ and $\mu = (\beta/K_g J)$. The unforced system (when $u = 0$) converges the stable equilibrium point given by $(x, \omega, z)_{eq} = (x_o^*, 0, 0)$ where $x_o^* = \{x \in \mathbf{R} : |x - x_o| \leq \zeta\}$ in the sliding mode sense starting from any initial conditions due to the existence of the discontinuous terms on the right hand side of the state space representation in Eq. (63). Based on the numerical data, it has been observed that the coupling on the third equation is weak so that the z term in the second equation can be replaced with $z = -(a_{32}/a_{33})\omega$ according to the singular perturbation theory.

4.3 Controller Design

Let $e_x = x - x_r$ where x_r is the reference to be followed. From (63), one obtains,

$$\ddot{x} = a_{12}(a_{21}(x - x_o) + (a_{22}/a_{12})\dot{x} + \dots \dots - \kappa \operatorname{sgn}(x - x_o) - \mu \operatorname{sgn}(\dot{x}/a_{12}) + a_{23}z) \quad (64)$$

Following the design method of Section 2.4, the control law is governed by

$$\bar{z}_{fl} = z_{fl} + (w/a_{12}a_{23}) \quad (65)$$

$$z_{fl} = \frac{1}{a_{12}a_{23}} [\ddot{x}_r + \xi_1 \dot{x}_r + \xi_2 x_r + a_{12}a_{21}x_o + \dots - (a_{12}a_{21} + \xi_2)x - (a_{22} + \xi_1)\dot{x} + \dots + a_{12}\kappa \operatorname{sgn}(x - x_o) + a_{12}\mu \operatorname{sgn}(\dot{x}/a_{12})] \quad (66)$$

$$\begin{aligned} s_w &= \dot{e}_x + C_1 e_x \\ w &= (\xi_1 - C_1)\dot{e}_x + \xi_2 e_x - \tilde{M} \operatorname{sgn}_k(s_w) \\ s_z &= z - \bar{z}_{fl} \\ u &= -M \operatorname{sgn}(s_z) \end{aligned} \quad (67)$$

where $C_1 > 0$, $\tilde{M} > |\Phi(\cdot)|$ with $\Phi(\cdot)$ being the lumped uncertainty originating from the bounded uncertainties in the plant parameters, M is sufficiently large positive number so as to induce a sliding motion on $s_z = 0$. The speed information required for the control implementation is obtained by an equivalent control based observer whose design idea has been presented in Section 2.3. The details of the overall control design summarized above and the simulation results can be found in [14].

5 Position Control of a Throttle System

This section presents a previous throttle angle position controller developed at the Ohio State University as a part of an Intelligent Vehicles and Highway Systems study. The existing pneumatic throttle actuator of a 1992 Honda Accord station-wagon was controlled for vehicle speed control purposes.

5.1 Throttle Actuator Model

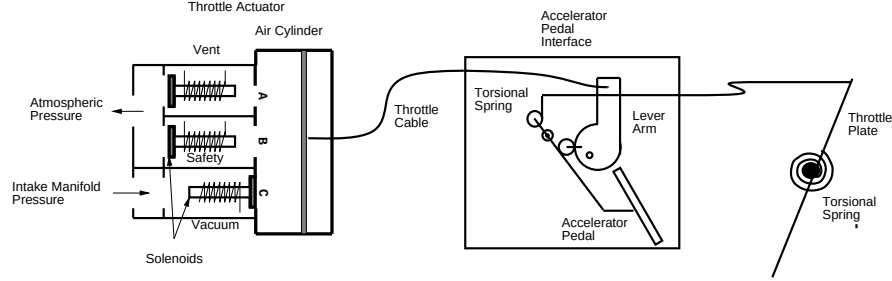


Fig. 3. The pneumatic throttle system (from [16])

The system includes the throttle actuator, throttle cable, and the throttle plate connection (Fig. 3). The throttle actuator is a pneumatic cylinder which creates a force proportional to the ratio of the cylinder's internal air pressure to the external (atmospheric) air pressure. The internal air pressure is controlled using two valves which allow either the engine's intake manifold pressure or the atmospheric pressure be applied to the input of the air cylinder. One cable connects the pneumatic cylinder to the accelerator pedal while a second cable connects the accelerator pedal to the throttle plate. The actuator's internal pressure is controlled using three solenoid actuated valves which control the air flow in and out of the pneumatic cylinder. The throttle angle is controlled by opening or closing the vent and vacuum valves until the internal air pressure that is needed to move the throttle angle to the desired position is achieved.

For control law derivation purposes, two separate second order linear models were experimentally determined, one of which is valid when the vent valve is open and the other when in full vacuum mode. If both the vent and vacuum valves are closed the system is assumed to behave according to the unforced vacuum model.

5.2 Sliding Mode Control of the Throttle Angle

Due to the nature of the control input to the throttle actuator system being either full vacuum or full vent, a sliding mode design was adopted. The sliding surface was defined as $s = \dot{e} + ke$ with $e = \theta - \theta_{des}$, θ is the throttle angle in degrees and θ_{des} was the desired throttle angle. u takes two values, +1 and -1, which represent full vacuum or full vent, respectively, depending on s .

The region in which the sliding mode existence condition can be guaranteed to hold with the available control authority was determined by considering the worst-case scenarios. This region is clearly affected by the value of

k . For implementation, k was selected to have a reasonable decay in sliding mode by giving up the global sliding mode existence although it was possible to select a k which would provide the control objective globally. In order to implement the control, $\dot{\theta}$, which was not directly available, was estimated by a linear Kalman filter. Since there were two possible linear systems, the filter parameters were determined individually and switched according to the input. The further details of the design as well as the experimental results were reported in [16].

6 Concluding Remarks

The usage of sliding mode estimation, optimization and compensation methods in automotive control problems have been demonstrated on there different examples: a traction control design for anti-spin acceleration, a tracking control design for a throttle system subject to stiction nonlinearities and a position tracking control design for a pneumatic throttle system of an internal combustion engine. The theoretical background and the relevant literature on the sliding mode design methods used have also been reported for an easy reference.

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On Quasi-optimal Variable Structure Control Approaches

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Abstract. In this paper, variable structure control (VSC) approaches are integrated with nonlinear optimal control approaches. VSC design consists of two phases: sliding mode design and switching control design. For the first phase, we propose a nonlinear sliding mode design methodology incorporating optimality for a class of nonlinear MIMO systems. Two issues are addressed in detail: 1) how to construct a sliding mode for cascaded nonlinear dynamics where the linear-type sliding mode design is not applicable; 2) how to achieve a nonlinear sliding mode with optimality. As for the second phase, a kind of nonlinear suboptimal control according to the system nominal part is integrated with VSC mechanism with the predesigned nonlinear sliding mode. By integration, we achieve a quasi-optimal controller in which the suboptimal control part and VSC part are made to function in a complementary manner. In particular when the system nominal part is predominant, the nonlinear optimal control part will govern the system response as well as drive the system to approach the equilibrium in an optimal fashion. On the contrary, when the system perturbation becomes the main factor, the VSC will take over the control task to warrant the desired control precision by its robustness property.

1 Introduction

Variable structure control (VSC) has been widely recognized as a powerful control strategy for its ability of making a control system very robust. Numerous theoretical studies as well as application research have been reported [1–7].

A typical VSC design consists of two phases, the first phase is the sliding mode design. In [2] a linear-type optimal sliding mode σ is designed for the linear cascaded systems

$$\begin{cases} \dot{\mathbf{x}}_1 = A_{11}(t)\mathbf{x}_1 + A_{12}(t)\mathbf{x}_2 \\ \dot{\mathbf{x}}_2 = A_{21}(t)\mathbf{x}_1 + A_{22}(t)\mathbf{x}_2 + B(t)(\mathbf{u} + \mathbf{d}) \end{cases}$$

where $\mathbf{d}$ is the matched system uncertainty. Note that the selected sliding mode must stabilize the sliding manifold described by

$$\begin{aligned} \sigma &= C\mathbf{x}_1 + \mathbf{x}_2 = 0 \\ \dot{\mathbf{x}}_1 &= A_{11}(t)\mathbf{x}_1 + A_{12}(t)\mathbf{x}_2, \end{aligned}$$

which leads to well known pole-placement problem for the $\mathbf{x}_1$ -subdynamics

$$\dot{\mathbf{x}}_1 = (A_{11} - A_{12}C)\mathbf{x}_1.$$

A LQR approach was suggested in [2] to conduct a systematic design for the MIMO $\mathbf{x}_1$ -subdynamics and meanwhile achieve the optimality for the sliding manifold.

A limitation of the linear-type sliding mode is that they can hardly handle nonlinear $\mathbf{x}_1$ -subdynamics. Pole-placement or LQR approaches, though are very effective in linear systems, may not be valid in the nonlinear world. Linear approximation of the nonlinear $\mathbf{x}_1$ -subdynamics is only valid locally around the equilibrium where it is linearized. Consider a nonlinear $\mathbf{x}_1$ -subdynamics still with the linear sliding mode

$$\begin{aligned}\boldsymbol{\sigma} &= C\mathbf{x}_1 + \mathbf{x}_2 = 0 \\ \dot{\mathbf{x}}_1 &= f(\mathbf{x}_1, \mathbf{x}_2, t).\end{aligned}$$

In general there may not exist such a matrix C that can stabilize the nonlinear sliding manifold $\dot{\mathbf{x}}_1 = f(\mathbf{x}_1, -C\mathbf{x}_1, t)$. We have to take the system nonlinearities f into account in the sliding mode design, and in the sequel come up with a nonlinear sliding mode. Analogous to [2], another important issue in nonlinear sliding mode design is, whether optimality can be pursued.

In this paper, we propose a systematic sliding mode design methodology for a class of nonlinear systems, which is able to take into account the nonlinearities in $\mathbf{x}_1$ -subdynamics and incorporate optimality. Optimal control not only provides a systematic design approach for MIMO systems but also provides extra degrees of freedom to specify the system performance with selected cost functions. In this work we consider three cases. A nonlinear optimal sliding mode can be constructed if the HJB (Hamilton-Jacobian-Bellman) equation is solvable. In particular if the $\mathbf{x}_1$ -subdynamics is nonlinear only in inputs ($\mathbf{x}_2$), a nonlinear optimal sliding mode can be easily designed by solving an algebraic *Ricatti* equation. If however the HJB is not solvable but there exists a static stabilizing control law for nonlinear $\mathbf{x}_1$ dynamics, a nonlinear sliding mode with inverse optimality can be constructed accordingly.

The second phase is the switching control design. In VSC design, the system nominal part is usually canceled out. Many real physical systems are more or less transparent to us with only a small portion of uncertainties, and the nominal part dominates system response. When the system nominal part is dominant, robustness is no longer the only concern of control design and other performance requirements should be taken into consideration. An interesting question is, while retaining the system stability of VSC, can we introduce another performance index, such as minimizing input energy, achieving faster tracking convergence, etc.? The answer is yes because we can apply optimal control design to the system nominal part, and apply VSC only to norm-bounded uncertainties. Consider a special nonlinear $\mathbf{x}_2$ dynamics (the general one will be presented in subsequent sections)

$$\dot{\mathbf{x}}_2 = \mathbf{f} + \mathbf{u} + \mathbf{d}.$$

where $\mathbf{f}$ is the nominal part. An optimal control $\mathbf{u}_{op}$ can be designed according to $\mathbf{f}$, the VSC part $\mathbf{u}_{vsc}$ according to $\mathbf{d}$, and both work concurrently as $\mathbf{u} = \mathbf{u}_{op} + \mathbf{u}_{vsc}$. It is obvious that, in the region dominated by the system nominal part, i.e. $\|\mathbf{f}\| \gg \|\mathbf{d}\|$, the system behavior is mainly governed by optimal control. On the contrary, in the region where perturbation becomes dominant, i.e. $\|\mathbf{d}\| > \|\mathbf{f}\|$, VSC will naturally take over the main control task. A problem encountered in $\mathbf{u}_{op}$ design is the difficulty in solving the HJB equation. To facilitate optimal design, a suboptimal nonlinear control design is considered [10]. The integration of the nonlinear optimal sliding mode, suboptimal control and VSC results in a new control approach – quasi-optimal VSC, which offers the appreciated robustness property, optimality in control performance, and systematic design for quite general classes of nonlinear dynamic systems.

The organization of this chapter is as follows. In Sec. 2, the typical VSC in the presence of system uncertainties is presented. In Sec. 3, nonlinear time varying sliding modes with optimality are constructed based on nonlinear optimal control, the algebraic *Ricatti* equation and inverse optimal control respectively. In Sec. 4, the suboptimal controller based on CLF is illustrated. In Sec. 5, the VSC and the suboptimal control are integrated. In Sec. 6, illustrative examples are presented to demonstrate the validity of the new idea and effectiveness of the proposed approach.

Nomenclature

In this Chapter the following nomenclature will be adopted.

- V := Lyapunov function;
- V^c := control Lyapunov function (CLF);
- $V_1(\mathbf{x}_1)$:= the first part of V^c , a positive definite function of $\mathbf{x}_1$;
- V^* := optimal value function
- σ := switching surface of VSC;
- σ_1 := the first part of σ , a function of $\mathbf{x}_1$ and t ;
- Q, R := weighting matrices in the performance index of the nonlinear optimal sliding mode;
- $l(\mathbf{x}_1)$:= penalty of system states $\mathbf{x}_1$ of the nonlinear optimal sliding mode;
- $q(\mathbf{x})$:= penalty of system states $\mathbf{x}_1$ and $\mathbf{x}_2$ of the suboptimal control part of quasi-optimal VSC;
- $\mathbf{u}$:= control input;
- $\mathbf{u}_{vsc}$:= VSC (switching) part of $\mathbf{u}$;
- $\mathbf{u}_c$:= compensation part of VSC law;

$\mathbf{u}_{op}$:= suboptimal control law or the suboptimal control part of the quasi-optimal VSC law;
 $\mathbf{u}^*$:= optimal control law;
 $\mathbf{v}$:= virtual control input to the $\mathbf{x}_1$ -subsystem;
 $\mathbf{v}^*$:= virtual optimal control input to the $\mathbf{x}_1$ -subsystem

2 Typical VSC System Construction

Consider the following nonlinear uncertain MIMO system

$$\begin{cases} \dot{\mathbf{x}}_1 = \mathbf{f}_1(\mathbf{x}_1, t) + G_1(\mathbf{x}_1, t)\boldsymbol{\varphi}(\mathbf{x}_1, \mathbf{x}_2, t) \\ \dot{\mathbf{x}}_2 = \mathbf{f}_2(\mathbf{x}_1, \mathbf{x}_2, t) + B_2(\mathbf{x}_1, \mathbf{x}_2, t)\{[I + \Delta B_2(\mathbf{x}_1, \mathbf{x}_2, t)]\mathbf{u} + \mathbf{d}(\mathbf{x}_1, \mathbf{x}_2, t)\} \end{cases} \quad (1)$$

where $\mathbf{x}_1 \in \mathcal{R}^{n_1}$ and $\mathbf{x}_2 \in \mathcal{R}^{n_2}$ are the physically measurable state vectors. $\mathbf{u} \in \mathcal{R}^{n_2}$ is the control input. $\boldsymbol{\varphi} \in \mathcal{R}^{n_2}$. $\frac{\partial \boldsymbol{\varphi}}{\partial \mathbf{x}_1}$ and $\frac{\partial \boldsymbol{\varphi}}{\partial \mathbf{x}_2}$ are bounded and $\frac{\partial \boldsymbol{\varphi}}{\partial \mathbf{x}_2} \neq 0$ in $\mathcal{D} \subset \mathcal{R}^{n_1} \cap \mathcal{R}^{n_2} \cap [0, \infty)$. $\mathbf{f}_1$, $\mathbf{f}_2$, $\boldsymbol{\varphi}$, G_1 and B_2 are the known nominal part of the system with appropriate dimensions. G_1 is continuously differentiable in all arguments. $\mathbf{d}$ and ΔB_2 represent system additive and multiplicative input uncertainties respectively. I denotes an identity matrix with appropriate dimensions.

Assumption 1: $\forall (\mathbf{x}_1, \mathbf{x}_2, t) \in \mathcal{D}$,

$$\begin{aligned} \|\Delta B_2\| &= \sqrt{\lambda_{\max}(\Delta B^T \Delta B)} \leq \varepsilon_b \quad 0 < \varepsilon_b < 1 \\ \|\mathbf{d}\| &= \sqrt{d_1^2 + d_2^2 + \cdots + d_{n_2}^2} \leq \rho_m(\mathbf{x}_1, \mathbf{x}_2, t) \end{aligned}$$

where ρ_m is a known positive upper bounding function.

Assumption 2: $\forall (\mathbf{x}_1, \mathbf{x}_2, t) \in \mathcal{D}$, $\frac{\partial \boldsymbol{\varphi}}{\partial \mathbf{x}_2} B_2$ is full rank.

One of the most effective ways to control this kind of systems is VSC which ensures the stability in the presence of system perturbations $\mathbf{d}$ and ΔB_2 .

To construct a VSC, first choose a nonlinear switching surface

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_1(\mathbf{x}_1) + \boldsymbol{\sigma}_2(\mathbf{x}_2) = \boldsymbol{\sigma}_1(\mathbf{x}_1) + \mathbf{x}_2. \quad (2)$$

Then choose a positive definite function $V = \frac{1}{2} \boldsymbol{\sigma}^T \boldsymbol{\sigma}$ and differentiate it, we have

$$\begin{aligned} \dot{V} &= \boldsymbol{\sigma}^T \dot{\boldsymbol{\sigma}} \\ &= \boldsymbol{\sigma}^T \left\{ \begin{bmatrix} \frac{\partial \boldsymbol{\sigma}_1}{\partial \mathbf{x}_1} & I \end{bmatrix} \begin{bmatrix} \mathbf{f}_1 + G_1 \boldsymbol{\varphi} \\ \mathbf{f}_2 \end{bmatrix} + B_2[(I + \Delta B_2)\mathbf{u} + \mathbf{d}] \right\} \\ &= \boldsymbol{\sigma}^T B_2 \{ B_2^{-1} \begin{bmatrix} \frac{\partial \boldsymbol{\sigma}_1}{\partial \mathbf{x}_1} & I \end{bmatrix} \begin{bmatrix} \mathbf{f}_1 + G_1 \boldsymbol{\varphi} \\ \mathbf{f}_2 \end{bmatrix} + (I + \Delta B_2)\mathbf{u} + \mathbf{d} \} \\ &= \boldsymbol{\alpha}^T [-\mathbf{u}_c + (I + \Delta B_2)\mathbf{u} + \mathbf{d}], \end{aligned} \quad (3)$$

where

$$\begin{aligned}\boldsymbol{\alpha}^T &\triangleq \boldsymbol{\sigma}^T B_2 \\ \mathbf{u}_c &\triangleq -B_2^{-1} \left[\frac{\partial \boldsymbol{\sigma}_1}{\partial \mathbf{x}_1} \quad I \right] \begin{bmatrix} \mathbf{f}_1 + G_1 \boldsymbol{\varphi} \\ \mathbf{f}_2 \end{bmatrix}.\end{aligned}\quad (4)$$

Letting $\mathbf{u} = \mathbf{u}_c + \mathbf{u}_{vsc}$ and substituting it into (3), yields

$$\begin{aligned}\dot{V} &= \boldsymbol{\alpha}^T [\Delta B_2 \mathbf{u}_c + \mathbf{d} + (I + \Delta B_2) \mathbf{u}_{vsc}] \\ &= \boldsymbol{\alpha}^T (\Delta B_2 \mathbf{u}_c + \mathbf{d}) + \boldsymbol{\alpha}^T \mathbf{u}_{vsc} + \boldsymbol{\alpha}^T \Delta B_2 \mathbf{u}_{vsc} \\ &\leq \|\Delta B_2 \mathbf{u}_c + \mathbf{d}\| \cdot \|\boldsymbol{\alpha}\| + \boldsymbol{\alpha}^T \mathbf{u}_{vsc} + \boldsymbol{\alpha}^T \Delta B_2 \mathbf{u}_{vsc}.\end{aligned}\quad (5)$$

Define

$$\rho_v(\mathbf{x}, t) = \varepsilon_b \|\mathbf{u}_c\| + \rho_m(\mathbf{x}, t) \geq \|\Delta B_2 \mathbf{u}_c + \mathbf{d}\|$$

and choose

$$\mathbf{u}_{vsc} = -\frac{[\rho_v(\mathbf{x}, t) + \delta] \boldsymbol{\alpha}}{(1 - \varepsilon_b) \|\boldsymbol{\alpha}\|},$$

where $\delta > 0$ is a constant. By substituting $\mathbf{u}_{vsc}$ into (5), we have

$$\begin{aligned}\dot{V} &\leq \rho_v(\mathbf{x}, t) \|\boldsymbol{\alpha}\| - \frac{\boldsymbol{\alpha}^T [\rho_v(\mathbf{x}, t) + \delta] \boldsymbol{\alpha}}{(1 - \varepsilon_b) \|\boldsymbol{\alpha}\|} + \varepsilon_b \frac{\|\boldsymbol{\alpha}^T\| [\rho_v(\mathbf{x}, t) + \delta] \|\boldsymbol{\alpha}\|}{(1 - \varepsilon_b) \|\boldsymbol{\alpha}\|} \\ &= \rho_v(\mathbf{x}, t) \|\boldsymbol{\alpha}\| - \frac{\rho_v(\mathbf{x}, t) + \delta}{(1 - \varepsilon_b)} \|\boldsymbol{\alpha}\| + \varepsilon_b \frac{\rho_v(\mathbf{x}, t) + \delta}{(1 - \varepsilon_b)} \|\boldsymbol{\alpha}\| \\ &= -\delta \|\boldsymbol{\alpha}\|.\end{aligned}$$

Thus the VSC law

$$\begin{aligned}\mathbf{u} &= \mathbf{u}_c + \mathbf{u}_{vsc} \\ &= -B_2^{-1} \left[\frac{\partial \boldsymbol{\sigma}_1}{\partial \mathbf{x}_1} \quad I \right] \begin{bmatrix} \mathbf{f}_1 + G_1 \boldsymbol{\varphi} \\ \mathbf{f}_2 \end{bmatrix} - \frac{[\rho_v(\mathbf{x}, t) + \delta] \boldsymbol{\alpha}}{(1 - \varepsilon_b) \|\boldsymbol{\alpha}\|}\end{aligned}\quad (6)$$

leads to

$$\dot{V} \leq -\delta \|\boldsymbol{\alpha}\| < 0. \quad (7)$$

The negative definiteness of $\dot{V}$ implies a finite reaching time to the switching surface $\boldsymbol{\sigma} = 0$.

When the system is in the sliding mode, the stability of the sliding manifold is jointly determined by $\boldsymbol{\sigma}_1$ and the $\mathbf{x}_1$ -subdynamics. However, it is in general a difficult task to choose $\boldsymbol{\sigma}_1$ which stabilizes the sliding manifold for the $\mathbf{x}_1$ -subdynamics given in (1). Besides, the system nominal part is simply canceled out by $\mathbf{u}_c$ in the above VSC design. It would be highly preferred if this part of the system knowledge can be better made use of in VSC design.

3 Sliding Mode Design with Optimality

In this section, we propose a systematic nonlinear sliding mode design strategy with optimality. The underlying idea is to convert the sliding mode design into an optimal control design for the $\mathbf{x}_1$ -subsystem. In this way not only can we design a nonlinear sliding mode systematically, but also acquire the desired optimality.

3.1 Nonlinear Optimal Sliding Mode Design

Look at system (1), we can treat $\varphi(\mathbf{x}_1, \mathbf{x}_2, t) \triangleq \mathbf{v}$ as a virtual control input to the $\mathbf{x}_1$ -subsystem, though the actual input is $\mathbf{x}_2$. Now consider the following optimal control task

$$\begin{aligned} \dot{\mathbf{x}}_1 &= \mathbf{f}_1(\mathbf{x}_1) + G_1(\mathbf{x}_1)\mathbf{v} \\ \inf_{\mathbf{v}(\cdot)} \int_0^\infty [l(\mathbf{x}_1) + \mathbf{v}^T R \mathbf{v}] dt \end{aligned} \quad (8)$$

where $R = R^T > 0$ and $l(\mathbf{x}_1) > 0$.

The HJB equation, a standard method to solve optimal problems [9], of this system is

$$V_{\mathbf{x}_1}^* \mathbf{f}_1 - \frac{1}{4} V_{\mathbf{x}_1}^* G_1 R^{-1} G_1^T V_{\mathbf{x}_1}^{*T} + l(\mathbf{x}_1) = 0 \quad (9)$$

where $V_{\mathbf{x}_1}^* = [\frac{\partial V^*}{\partial \mathbf{x}_1}]^T$ and V^* is commonly referred to as the value function and can be thought of as the minimum cost to go from the current state $\mathbf{x}_1(t)$, i.e.,

$$V^*(\mathbf{x}_1(t)) = \inf_{\mathbf{v}(\cdot)} \int_t^\infty [l(\mathbf{x}_1(\tau)) + \mathbf{v}^T(\tau) R \mathbf{v}(\tau)] d\tau.$$

Suppose that there exists a $\mathcal{C}^1$ positive semidefinite function $V^*(\mathbf{x}_1)$ which satisfies the HJB equation (9), then the optimal control law is given by

$$\mathbf{v}^*(\mathbf{x}_1) = -\frac{1}{2} R^{-1} G_1^T V_{\mathbf{x}_1}^{*T}(\mathbf{x}_1).$$

Now define

$$\begin{aligned} \sigma_1(\mathbf{x}_1) &= \frac{1}{2} R^{-1} G_1^T V_{\mathbf{x}_1}^{*T}(\mathbf{x}_1) \\ &= -\mathbf{v}^*(\mathbf{x}_1). \end{aligned} \quad (10)$$

Theorem 1. *The following nonlinear sliding mode*

$$\sigma(\mathbf{x}_1, \mathbf{x}_2, t) = \sigma_1(\mathbf{x}_1, t) + \sigma_2(\mathbf{x}_1, \mathbf{x}_2, t) = \sigma_1(\mathbf{x}_1) + \varphi(\mathbf{x}_1, \mathbf{x}_2, t) = 0 \quad (11)$$

where $\sigma \in \mathcal{R}^{n_2}$, is an optimal sliding mode for system (1) with σ_1 defined in (10).

Proof. Note that the $\mathbf{x}_1$ -subdynamics is as follows

$$\dot{\mathbf{x}}_1 = \mathbf{f}_1(\mathbf{x}_1) + G_1(\mathbf{x}_1)\boldsymbol{\varphi}(\mathbf{x}_1, \mathbf{x}_2, t).$$

When the system is in the sliding mode defined in (11), we have

$$\begin{aligned}\boldsymbol{\varphi}(\mathbf{x}_1, \mathbf{x}_2, t) &= -\boldsymbol{\sigma}_1(\mathbf{x}_1) = -\frac{1}{2}R^{-1}G_1^T V_{\mathbf{x}_1}^{*T}(\mathbf{x}_1) \\ &= \mathbf{v}^*(\mathbf{x}_1)\end{aligned}$$

This shows that the nonlinear sliding mode (11) does lead to a nonlinear optimal control law for the optimal control task (8). As a result, the optimality of the nonlinear sliding mode ensures the asymptotic stability of the nonlinear sliding manifold

$$\begin{aligned}\dot{\mathbf{x}}_1 &= \mathbf{f}_1(\mathbf{x}_1) + G_1(\mathbf{x}_1)\boldsymbol{\varphi}(\mathbf{x}_1, \mathbf{x}_2, t) \\ &= \mathbf{f}_1(\mathbf{x}_1) - \frac{1}{2}G_1(\mathbf{x}_1)R^{-1}G_1^T V_{\mathbf{x}_1}^{*T}(\mathbf{x}_1).\end{aligned}$$

Remark 1. In calculating the nonlinear sliding mode we fully use the knowledge of the system nonlinearities. It is not necessary to solve (11) to get the explicit expression for $\mathbf{x}_2$. Thus $\boldsymbol{\varphi}$ can be non-affine in $\mathbf{x}_2$.

3.2 Nonlinear Optimal Sliding Mode Design with Input Nonlinearity

In the above optimal sliding mode design, in order to solve the optimal control task (8) we need to find the solution of the HJB partial differential equation which may not be feasible for many nonlinear systems. However, if $\mathbf{f}_1$ and G_1 in (1) are linear or linearizable, then the $\mathbf{x}_1$ -subdynamics in (1) can be rewritten as

$$\dot{\mathbf{x}}_1 = \mathbf{f}_1 + G_1\boldsymbol{\varphi}(\mathbf{x}_1, \mathbf{x}_2, t) = A_{11}\mathbf{x}_1 + A_{12}\boldsymbol{\varphi}(\mathbf{x}_1, \mathbf{x}_2, t), \quad (12)$$

where (A_{11}, A_{12}) is controllable. Note that the input to the $\mathbf{x}_1$ -subdynamics may have nonlinear factors. In general $\boldsymbol{\varphi}(\mathbf{x}_1, \mathbf{x}_2, t)$ can be affine or non-affine in $\mathbf{x}_2$, such as $x_2 e^{-x_2}$ in scalar case. Regardless of the presence of nonlinearities, we can construct a nonlinear optimal sliding mode for system (12) through solving the algebraic *Ricatti* equation analogous to the linear cases.

Consider the optimal control task below,

$$\begin{aligned}\dot{\mathbf{x}}_1 &= A_{11}\mathbf{x}_1 + A_{12}\mathbf{v} \\ \inf_{\mathbf{v}(\cdot)} \int_0^\infty [\mathbf{x}_1^T Q \mathbf{x}_1 + \mathbf{v}^T R \mathbf{v}] dt\end{aligned} \quad (13)$$

where $Q = Q^T > 0$ and $R = R^T > 0$. The optimal control law of this linear system is given by

$$\mathbf{v}^*(\mathbf{x}_1) = -R^{-1}A_{12}^T P \mathbf{x}_1,$$

where P is the solution of the algebraic *Ricatti* equation

$$PA_{11} + A_{11}^T P - PA_{12}R^{-1}A_{12}^T P + Q = 0$$

and the optimal value function is

$$V^* = \mathbf{x}_1^T P \mathbf{x}_1.$$

Analogous to the sliding mode construction (11) in the previous subsection

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_1(\mathbf{x}_1) + \boldsymbol{\sigma}_2(\mathbf{x}_1, \mathbf{x}_2) = C\mathbf{x}_1 + \boldsymbol{\varphi}(\mathbf{x}_1, \mathbf{x}_2, t) = 0 \quad (14)$$

with

$$C = R^{-1}A_{12}^T P.$$

This nonlinear sliding mode $\boldsymbol{\varphi}(\mathbf{x}_1, \mathbf{x}_2, t) = -R^{-1}A_{12}^T P$ yields the optimal control law for $\mathbf{x}_1$ -subdynamics (12). As a consequence the resulting nonlinear sliding manifold is asymptotically stable.

Remark 2. If $\boldsymbol{\varphi}(\mathbf{x}_1, \mathbf{x}_2, t)$ is simply a linear vector $\mathbf{x}_2$, and R in the cost function (13) is chosen to be a high pass filter, we can easily reach a frequency shaped optimal sliding mode [2].

3.3 Sliding Mode Design with Inverse Optimality

If the $\mathbf{x}_1$ -subdynamics in (1) cannot be simplified into the linear case (12) or the HJB equation (9) is not solvable, we may look for an alternative way – constructing a nonlinear sliding mode with inverse optimality. The underlying idea is as follows. First design a static control law with respect to a Lyapunov function. Then an optimal control law associated with a specific cost function can be constructed accordingly, which lead to a solution to a specific HJB equation. In the sequel the optimal control law can be used to construct the nonlinear optimal sliding mode.

Rewrite the optimal control task (8),

$$\begin{aligned} \dot{\mathbf{x}}_1 &= \mathbf{f}_1(\mathbf{x}_1) + G_1(\mathbf{x}_1)\mathbf{v} \\ \inf_{\mathbf{v}(\cdot)} \int_0^\infty [l(\mathbf{x}_1) + \mathbf{v}^T R \mathbf{v}] dt. \end{aligned} \quad (15)$$

Assume there exists a static state feedback control law

$$\mathbf{v} = -R^{-1}G_1(\mathbf{x}_1)^T V_{\mathbf{x}_1}^T(\mathbf{x}_1) \quad (16)$$

stabilizes the $\mathbf{x}_1$ -subdynamics with respect to a positive definite and radially unbounded Lyapunov function $V(\mathbf{x}_1)$, i.e.

$$\dot{V} = V_{\mathbf{x}_1} \dot{\mathbf{x}}_1 = V_{\mathbf{x}_1} \mathbf{f}_1 + V_{\mathbf{x}_1} G_1 \mathbf{v} < 0.$$

Then the control law

$$\mathbf{v}^* = 2\mathbf{v} = -2R^{-1}G_1(\mathbf{x}_1)^T V_{\mathbf{x}_1}^T(\mathbf{x}_1)$$

is optimal with respect to the performance index (15) where

$$\begin{aligned} l &= -4(V_{\mathbf{x}_1}\mathbf{f}_1 + V_{\mathbf{x}_1}G_1\mathbf{v}) \\ &= -4(V_{\mathbf{x}_1}\mathbf{f}_1 - V_{\mathbf{x}_1}G_1R^{-1}G_1^T V_{\mathbf{x}_1}^T). \end{aligned} \quad (17)$$

Note that from $\dot{V}$, $\forall \mathbf{x}_1 \neq 0$,

$$V_{\mathbf{x}_1}\mathbf{f}_1 + V_{\mathbf{x}_1}G_1\mathbf{v} < 0,$$

thus $l(\mathbf{x}_1)$ is positive definite for all $\mathbf{x} \neq 0$ and can be chosen as part of the cost function. Now let us show that $\mathbf{v}^*$ and $l(\mathbf{x}_1)$ satisfy the HJB equation (9) with the optimal value function

$$V^* = 4V. \quad (18)$$

This can easily be verified below

$$\begin{aligned} &V_{\mathbf{x}_1}^*\mathbf{f}_1 - \frac{1}{4}V_{\mathbf{x}_1}^*G_1R^{-1}G_1^T V_{\mathbf{x}_1}^{*T} + l(\mathbf{x}_1) \\ &= 4V_{\mathbf{x}_1}\mathbf{f}_1 + \frac{1}{4} \cdot 4V_{\mathbf{x}_1}G_1R^{-1}G_1^T \cdot 4V_{\mathbf{x}_1}^T - 4(V_{\mathbf{x}_1}\mathbf{f}_1 - V_{\mathbf{x}_1}G_1R^{-1}G_1^T V_{\mathbf{x}_1}^T) \\ &= 4V_{\mathbf{x}_1}\mathbf{f}_1 + 4V_{\mathbf{x}_1}G_1R^{-1}G_1^T V_{\mathbf{x}_1}^T - 4V_{\mathbf{x}_1}\mathbf{f}_1 - 4V_{\mathbf{x}_1}G_1R^{-1}G_1^T V_{\mathbf{x}_1}^T \\ &= 0. \end{aligned}$$

Using the above inverse optimal approach we can now define the nonlinear optimal sliding mode in a similar way as in (11)

$$\boldsymbol{\sigma}(\mathbf{x}_1, \mathbf{x}_2) = \boldsymbol{\sigma}_1(\mathbf{x}_1) + \boldsymbol{\sigma}_2(\mathbf{x}_1, \mathbf{x}_2) = \boldsymbol{\sigma}_1(\mathbf{x}_1) + \boldsymbol{\varphi}(\mathbf{x}_1, \mathbf{x}_2, t) = 0$$

with

$$\boldsymbol{\sigma}_1 = -2\mathbf{v} = 2R^{-1}G_1(\mathbf{x}_1)^T V_{\mathbf{x}_1}^T(\mathbf{x}_1)$$

which ensures asymptotic stability and optimality of the nonlinear sliding manifold.

Remark 3. The inverse optimal approach illustrated in this section renders the task of solving the HJB partial differential equation (a tougher one) into the task of looking for a stabilizing control law (16) in conjunction with a Lyapunov function $V(\mathbf{x}_1)$ (a relatively easy one), and still provides certain degrees of freedom in the optimal controller design, such as the selection of the weighting matrix R .

3.4 Robustness Analysis in the Presence of Sector Uncertainties

In sliding mode design, whether for linear or nonlinear cases, it is necessary to know the $\mathbf{x}_1$ -subdynamics *a priori*. If there exists uncertainties in the $\mathbf{x}_1$ -subdynamics, robustness should be taken into consideration in the sliding mode design. In the linear case, the optimal sliding mode using LQR design achieves the infinity gain margin and 60 degrees phase margin. Similarly, the nonlinear optimal sliding mode, based on nonlinear optimal control design [9], achieves a sector margin $(\frac{1}{2}, \infty)$.

Consider the following uncertain MIMO system

$$\begin{cases} \dot{\mathbf{x}}_1 = \mathbf{f}_1(\mathbf{x}_1, t) + G_1(\mathbf{x}_1, t)\boldsymbol{\varphi}(\mathbf{x}_2, t) \\ \dot{\mathbf{x}}_2 = \mathbf{f}_2(\mathbf{x}_1, \mathbf{x}_2, t) \\ \quad + B_2(\mathbf{x}_1, \mathbf{x}_2, t)\{[I + \Delta B_2(\mathbf{x}_1, \mathbf{x}_2, t)]\mathbf{u} + \mathbf{d}\}(\mathbf{x}_1, \mathbf{x}_2, t)\}, \end{cases} \quad (19)$$

with sector-type uncertainty in $\boldsymbol{\varphi}(\mathbf{x}_2, t)$, i.e. $\boldsymbol{\varphi}(\mathbf{x}_2, t) = [\varphi_1, \varphi_2, \dots, \varphi_{n_2}]^T$ and $\varphi_i, i = 1, 2, \dots, n_2$ are unknown but belong to a sector $(\frac{1}{2}, \infty)$

$$\frac{1}{2}x^2 < x\varphi_i(x, t) < \infty.$$

By virtue of the sector margin property of nonlinear optimal control, we can construct a stable sliding mode for system (19), as shown by the following *theorem*.

Theorem 2. *Construct the following sliding mode*

$$\begin{aligned} \boldsymbol{\sigma} &= \boldsymbol{\sigma}_1(\mathbf{x}_1) + \boldsymbol{\sigma}_2(\mathbf{x}_2) = \boldsymbol{\sigma}_1(\mathbf{x}_1) + \mathbf{x}_2 = 0, \\ \boldsymbol{\sigma}_1(\mathbf{x}_1) &= \frac{1}{2}R^{-1}G_1^T V_{\mathbf{x}_1}^*(\mathbf{x}_1) \end{aligned} \quad (20)$$

where $R = \text{diag}(r_1, \dots, r_{n_2}) > 0$ and V^* is the solution of the HJB equation (9). This sliding mode results in an asymptotically stable sliding manifold

$$\dot{\mathbf{x}}_1 = \mathbf{f}_1(\mathbf{x}_1, t) + G_1(\mathbf{x}_1, t)\boldsymbol{\varphi}(-\boldsymbol{\sigma}_1) \quad (21)$$

where $\boldsymbol{\varphi}(\cdot)$ is the sector-type uncertainty.

Proof. Choose a positive definite Lyapunov function

$$V(\mathbf{x}_1) = V^*(\mathbf{x}_1),$$

differentiate it and use the HJB equation (9), we have

$$\begin{aligned} \dot{V} &= V_{\mathbf{x}_1}^*[\mathbf{f}_1 + G_1\boldsymbol{\varphi}(\mathbf{x}_2)] \\ &= -l(\mathbf{x}_1) + \frac{1}{4}V_{\mathbf{x}_1}^* G_1 R^{-1} G_1^T V_{\mathbf{x}_1}^{*T} + V_{\mathbf{x}_1}^* G_1 \boldsymbol{\varphi}(\mathbf{x}_2). \end{aligned}$$

When the system is in the sliding mode $\boldsymbol{\sigma} = 0$

$$\mathbf{x}_2 = -\boldsymbol{\sigma}_1 = \mathbf{v}^*(\mathbf{x}_1) = -\frac{1}{2}R^{-1}G_1^T V_{\mathbf{x}_1}^{*T}(\mathbf{x}_1).$$

Note the sector-type uncertainty satisfies $\varphi(-\mathbf{x}_2) = -\varphi(\mathbf{x}_2)$. Thus

$$\begin{aligned}\dot{V} &= -l(\mathbf{x}_1) + \frac{1}{2}V_{\mathbf{x}_1}^* G_1 \boldsymbol{\sigma}_1 + V_{\mathbf{x}_1}^* G_1 \varphi(-\boldsymbol{\sigma}_1) \\ &= -l(\mathbf{x}_1) - V_{\mathbf{x}_1}^* G_1 [\varphi(\boldsymbol{\sigma}_1) - \frac{1}{2}\boldsymbol{\sigma}_1] \\ &= -l(\mathbf{x}_1) - 2\boldsymbol{\sigma}_1^T R [\varphi(\boldsymbol{\sigma}_1) - \frac{1}{2}\boldsymbol{\sigma}_1] \\ &< -2\boldsymbol{\sigma}_1^T R [\varphi(\boldsymbol{\sigma}_1) - \frac{1}{2}\boldsymbol{\sigma}_1].\end{aligned}$$

Note that $R = \text{diag}(r_1, r_2, \dots, r_{n_2}) > 0$ for $\mathbf{x}_1 \neq \mathbf{0}$ and $\varphi_i(\cdot)$ belongs to a sector $(\frac{1}{2}, \infty)$. Therefore,

$$\dot{V} < -2 \sum_{i=1}^{n_1} \{r_i \sigma_{1,i} [\varphi_i(\sigma_{1,i}) - \frac{1}{2}\sigma_{1,i}]\} < 0.$$

Remark 4. The sliding mode (20) constructed in *theorem 2* can guarantee the asymptotical stability of the $\mathbf{x}_1$ -subdynamics, but may not be optimal.

4 Construction of Nonlinear Suboptimal Control Based on Nominal System and CLF

It has been shown that the system nominal part is canceled out in the typical VSC law (6) by $\mathbf{u}_c$. Instead of cancelation, now let us explore the possibility of constructing an optimal controller for the system nominal part.

The nominal part of system (1) is

$$\dot{\mathbf{x}} = \mathbf{f} + B\mathbf{u} \quad (22)$$

where $\mathbf{x} = [\mathbf{x}_1^T, \mathbf{x}_2^T]^T$, $\mathbf{f} = [\mathbf{f}_1 + G_1 \boldsymbol{\varphi}^T, \mathbf{f}_2^T]^T$, $B = [0, B_2^T]^T$. Consider the performance index

$$\inf_{\mathbf{u}(\cdot)} \int_0^\infty [q(\mathbf{x}) + \mathbf{u}^T \mathbf{u}] dt. \quad (23)$$

Usually it is very hard or impossible to find an optimal controller associated with (22) and (23). Thus, we consider a suboptimal control method based on Sontag's formula [10]. The Sontag's formula for the above problem is

$$\mathbf{u}_{op} = \begin{cases} -\frac{V_{\mathbf{x}}^c \mathbf{f} + \sqrt{[V_{\mathbf{x}}^c \mathbf{f}]^2 + q(\mathbf{x})[V_{\mathbf{x}}^c B B^T V_{\mathbf{x}}^{cT}]} V_{\mathbf{x}}^c B B^T V_{\mathbf{x}}^{cT}}{V_{\mathbf{x}}^c B B^T V_{\mathbf{x}}^{cT}} & V_{\mathbf{x}}^c B \neq 0 \\ 0 & V_{\mathbf{x}}^c B = 0 \end{cases} \quad (24)$$

where V^c is a Control Lyapunov Function for the nominal part of system (1).

Definition 1. According to [9], a smooth, positive definite and radially unbounded function V^c is called a Control Lyapunov Function (CLF) for system (22) if for all $\mathbf{x} \neq \mathbf{0}$,

$$V_{\mathbf{x}}^c B = \mathbf{0} \implies V_{\mathbf{x}}^c \mathbf{f} < 0 \quad (25)$$

Note that if V^c is chosen as a CLF, it guarantees the stability of the controller (24), since

$$\begin{aligned}\dot{V}^c &= V_x^c \mathbf{f} + V_x^c B \mathbf{u} \\ &= V_x^c \mathbf{f} - V_x^c B \frac{V_x^c \mathbf{f} + \sqrt{[V_x^c \mathbf{f}]^2 + q(\mathbf{x})[V_x^c B B^T V_x^{cT}]}}{V_x^c B B^T V_x^{cT}} B^T V_x^{cT} \\ &= \begin{cases} -\sqrt{[V_x^c \mathbf{f}]^2 + q(\mathbf{x})[V_x^c B B^T V_x^{cT}]} < 0 & V_x^c B \neq 0 \\ V_x^c \mathbf{f} < 0 & V_x^c B = 0 \end{cases}\end{aligned}$$

Also, by virtue of CLF, we can prove (see Appendix)

$$\lim_{V_x^c B \rightarrow 0} \mathbf{u}_{op} = \sqrt{q(\mathbf{x})},$$

which means $\mathbf{u}_{op}$ is bounded when $V_x^c B$ goes to zero.

In most VSC design, a positive definite function $V = \frac{1}{2} \boldsymbol{\sigma}^T \boldsymbol{\sigma}$ is chosen to facilitate the derivation of the switching control law. In particular in our case the nonlinear optimal sliding mode is $\boldsymbol{\sigma} = \boldsymbol{\sigma}_1(\mathbf{x}_1) + \boldsymbol{\varphi}(\mathbf{x}_1, \mathbf{x}_2, t)$. Because

$$\begin{aligned}V_x B &= \left[\frac{\partial(\frac{1}{2} \boldsymbol{\sigma}^T \boldsymbol{\sigma})}{\partial \mathbf{x}} \right]^T B \\ &= \boldsymbol{\sigma}^T \left[\frac{\partial \boldsymbol{\sigma}}{\partial \mathbf{x}_1} + \frac{\partial \boldsymbol{\varphi}}{\partial \mathbf{x}_1} \quad \frac{\partial \boldsymbol{\varphi}}{\partial \mathbf{x}_2} \right] \begin{bmatrix} \mathbf{0} \\ B_2 \end{bmatrix} \\ &= \boldsymbol{\sigma}^T \frac{\partial \boldsymbol{\varphi}}{\partial \mathbf{x}_2} B_2, \\ V_x \mathbf{f} &= \boldsymbol{\sigma}^T \left[\frac{\partial \boldsymbol{\sigma}_1}{\partial \mathbf{x}_1} + \frac{\partial \boldsymbol{\varphi}}{\partial \mathbf{x}_1} \quad \frac{\partial \boldsymbol{\varphi}}{\partial \mathbf{x}_2} \right] \begin{bmatrix} \mathbf{f}_1 + G_1 \boldsymbol{\varphi} \\ \mathbf{f}_2 \end{bmatrix}.\end{aligned}$$

and $\frac{\partial \boldsymbol{\varphi}}{\partial \mathbf{x}_2} B_2$ is full rank, we have

$$V_x B = 0 \implies \boldsymbol{\sigma} = 0 \implies V_x \mathbf{f} = 0.$$

Thus, V is not a CLF here.

In order to apply the Sontag's formula, it is necessary to construct a CLF. The following *Theorem* provides such a CLF which incorporates the $\boldsymbol{\sigma}^T \boldsymbol{\sigma}$ as its subset.

Theorem 3. *The positive definite function*

$$V^c = V_1(\mathbf{x}_1) + \frac{1}{2} \boldsymbol{\sigma}^T \boldsymbol{\sigma} \quad (26)$$

is a control Lyapunov function for system (22), where $\boldsymbol{\sigma}$ is defined in (11) and

$$V_1 = V^*,$$

V^* is the $\mathcal{C}^1$ positive semidefinite solution of the HJB equation (9).

Proof. It is easy to obtain

$$\begin{aligned}
V_x^c B &= V_{1x}^T B + \left[\frac{\partial(\frac{1}{2}\sigma^T \sigma)}{\partial x} \right]^T B \\
&= 0 + \sigma^T \left[\frac{\partial \sigma}{\partial x_1} + \frac{\partial \varphi}{\partial x_1} \quad \frac{\partial \varphi}{\partial x_2} \right] \begin{bmatrix} \mathbf{0} \\ B_2 \end{bmatrix} \\
&= \sigma^T \frac{\partial \varphi}{\partial x_2} B_2.
\end{aligned} \tag{27}$$

On the other hand,

$$\begin{aligned}
V_x^c \mathbf{f} &= \left[\frac{\partial V_1}{\partial x} \right]^T \begin{bmatrix} \mathbf{f}_1 + G_1 \varphi \\ \mathbf{f}_2 \end{bmatrix} + \left[\frac{\partial(\frac{1}{2}\sigma^T \sigma)}{\partial x} \right]^T \begin{bmatrix} \mathbf{f}_1 + G_1 \varphi \\ \mathbf{f}_2 \end{bmatrix} \\
&= \left[\frac{\partial V_1}{\partial x_1} \right]^T [\mathbf{f}_1 + G_1 \varphi] + \sigma^T \left[\frac{\partial \sigma_1}{\partial x_1} + \frac{\partial \varphi}{\partial x_1} \quad \frac{\partial \varphi}{\partial x_2} \right] \begin{bmatrix} \mathbf{f}_1 + G_1 \varphi \\ \mathbf{f}_2 \end{bmatrix}
\end{aligned}$$

Since $\frac{\partial \varphi}{\partial x_2} B_2$ is full rank, from (11) and (27) we have

$$V_x^c B = 0 \implies \sigma = 0 \implies \varphi = -\sigma_1(x_1) = -\frac{1}{2} R^{-1} G_1^T V_{x_1}^{*T}.$$

Using the relationship (9),

$$\begin{aligned}
V_x^c \mathbf{f} &= \left[\frac{\partial V_1}{\partial x_1} \right]^T [\mathbf{f}_1 + G_1 \varphi] \\
&= -l(x_1) + \frac{1}{4} \left[\frac{\partial V_1}{\partial x_1} \right]^T G_1 R^{-1} G_1^T \left[\frac{\partial V_1}{\partial x_1} \right] + \left[\frac{\partial V_1}{\partial x_1} \right]^T G_1 \varphi \\
&= -l(x_1) + \frac{1}{4} \left[\frac{\partial V_1}{\partial x_1} \right]^T G_1 R^{-1} G_1^T \left[\frac{\partial V_1}{\partial x_1} \right] - \frac{1}{2} \left[\frac{\partial V_1}{\partial x_1} \right]^T G_1 R^{-1} G_1^T \left[\frac{\partial V_1}{\partial x_1} \right] \\
&= -l(x_1) - \frac{1}{4} \left[\frac{\partial V_1}{\partial x_1} \right]^T G_1 R^{-1} G_1^T \left[\frac{\partial V_1}{\partial x_1} \right]. \\
&< 0
\end{aligned}$$

Hence V^C is a CLF.

Remark 5. To apply the suboptimal controller (24), the system model should be known *a priori*. It will lose the effectiveness when the system perturbation becomes dominant. An effective way to overcome this limitation is to integrate optimal control approaches with VSC to gain the robustness.

5 Quasi-optimal Variable Structure Control

Now we are in a position to give the main result as summarized in the *Theorem* below. A new quasi-optimal VSC scheme is constructed through incorporating the nonlinear optimal sliding mode, suboptimal control with the proposed CLF, and a switching control,

Theorem 4. *The integrated quasi-optimal VSC law*

$$\begin{aligned} \mathbf{u} &= \mathbf{u}_{op} + \mathbf{u}_{vsc} \\ &= -\frac{V_x^c \mathbf{f} + \sqrt{[V_x^c \mathbf{f}]^2 + q(\mathbf{x})[V_x^c B B^T V_x^{cT}]}{V_x^c B B^T V_x^{cT}} B^T V_x^{cT} \\ &\quad - \frac{[\rho(\mathbf{x}, t) + \delta] \boldsymbol{\gamma}}{(1 - \varepsilon_b) \|\boldsymbol{\gamma}\|} \end{aligned} \quad (28)$$

with

$$\begin{aligned} \boldsymbol{\gamma}^T &= \boldsymbol{\sigma}^T \frac{\partial \varphi}{\partial \mathbf{x}_2} B_2 \\ \rho(\mathbf{x}, t) &= \varepsilon_b \|\mathbf{u}_{op}\| + \rho_m(\mathbf{x}, t). \end{aligned} \quad (29)$$

achieves asymptotic stability for system (1), where V^c is a CLF defined in (26).

Proof. Choose the CLF given in (26) and differentiate it,

$$\begin{aligned} \dot{V}^c &= \left[\frac{\partial V_1}{\partial \mathbf{x}} \right]^T \mathbf{f} + \left[\frac{\partial V_1}{\partial \mathbf{x}} \right]^T \begin{bmatrix} \mathbf{0} \\ B_2[(I + \Delta B_2)\mathbf{u} + \mathbf{d}] \end{bmatrix} \\ &\quad + \boldsymbol{\sigma}^T \left[\frac{\partial \boldsymbol{\sigma}}{\partial \mathbf{x}_1} + \frac{\partial \varphi}{\partial \mathbf{x}_1} \quad \frac{\partial \varphi}{\partial \mathbf{x}_2} \right] \left\{ \mathbf{f} + \begin{bmatrix} 0 \\ B_2(I + \Delta B_2) \end{bmatrix} \mathbf{u} + \begin{bmatrix} \mathbf{0} \\ B_2 \mathbf{d} \end{bmatrix} \right\} \\ &= \left[\frac{\partial V_1}{\partial \mathbf{x}} \right]^T \mathbf{f} + \boldsymbol{\sigma}^T \left[\frac{\partial \boldsymbol{\sigma}}{\partial \mathbf{x}_1} + \frac{\partial \varphi}{\partial \mathbf{x}_1} \quad \frac{\partial \varphi}{\partial \mathbf{x}_2} \right] \left\{ \mathbf{f} + \begin{bmatrix} \mathbf{0} \\ B_2 \mathbf{d} \end{bmatrix} + \begin{bmatrix} 0 \\ B_2 \Delta B_2 \end{bmatrix} \mathbf{u}_{op} \right. \\ &\quad \left. + \begin{bmatrix} 0 \\ B_2(I + \Delta B_2) \end{bmatrix} \mathbf{u}_{vsc} \right\} + \boldsymbol{\sigma}^T \left[\frac{\partial \boldsymbol{\sigma}}{\partial \mathbf{x}_1} + \frac{\partial \varphi}{\partial \mathbf{x}_1} \quad \frac{\partial \varphi}{\partial \mathbf{x}_2} \right] B \mathbf{u}_{op}. \end{aligned}$$

For

$$\mathbf{u}_{op} = -\frac{V_x^c \mathbf{f} + \sqrt{[V_x^c \mathbf{f}]^2 + q(\mathbf{x})[V_x^c B B^T V_x^{cT}]}{V_x^c B B^T V_x^{cT}} B^T V_x^{cT}$$

and

$$\begin{aligned} V_x^c B &= \left[\frac{\partial V_1}{\partial \mathbf{x}} \right]^T \begin{bmatrix} 0 \\ B_2 \end{bmatrix} + \boldsymbol{\sigma}^T \left[\frac{\partial \boldsymbol{\sigma}}{\partial \mathbf{x}_1} + \frac{\partial \varphi}{\partial \mathbf{x}_1} \quad \frac{\partial \varphi}{\partial \mathbf{x}_2} \right] B \\ &= \boldsymbol{\sigma}^T \left[\frac{\partial \boldsymbol{\sigma}}{\partial \mathbf{x}_1} + \frac{\partial \varphi}{\partial \mathbf{x}_1} \quad \frac{\partial \varphi}{\partial \mathbf{x}_2} \right] B, \end{aligned}$$

we have

$$\begin{aligned} \dot{V}^c &= \left[\frac{\partial V^c}{\partial \mathbf{x}} \right]^T \mathbf{f} + \boldsymbol{\sigma}^T \frac{\partial \varphi}{\partial \mathbf{x}_2} B_2 [\mathbf{d} + \Delta B_2 \mathbf{u}_{op} + (I + \Delta B_2) \mathbf{u}_{vsc}] \\ &\quad - \boldsymbol{\sigma}^T \left[\frac{\partial \boldsymbol{\sigma}}{\partial \mathbf{x}_1} + \frac{\partial \varphi}{\partial \mathbf{x}_2} \quad \frac{\partial \varphi}{\partial \mathbf{x}_2} \right] B \\ &\quad - \frac{V_x^c \mathbf{f} + \sqrt{[V_x^c \mathbf{f}]^2 + q(\mathbf{x})[V_x^c B B^T V_x^{cT}]}{\boldsymbol{\sigma}^T \left[\frac{\partial \boldsymbol{\sigma}}{\partial \mathbf{x}_1} + \frac{\partial \varphi}{\partial \mathbf{x}_1} \quad \frac{\partial \varphi}{\partial \mathbf{x}_2} \right] B B^T V_x^{cT}} B^T V_x^{cT} \end{aligned}$$

$$\begin{aligned}
&= \sigma^T \frac{\partial \varphi}{\partial \mathbf{x}_2} B_2 [\mathbf{d} + \Delta B_2 \mathbf{u}_{op} + (I + \Delta B_2) \mathbf{u}_{vsc}] - \sqrt{\xi} \\
&= \gamma^T (\mathbf{d} + \Delta B_2 \mathbf{u}_{op}) + \gamma^T \mathbf{u}_{vsc} + \gamma^T \Delta B_2 \mathbf{u}_{vsc} - \sqrt{\xi} \\
&\leq \|\Delta B_2 \mathbf{u}_{op} + \mathbf{d}\| \cdot \|\gamma\| + \gamma^T \mathbf{u}_{vsc} + \gamma^T \Delta B_2 \mathbf{u}_{vsc} - \sqrt{\xi}, \tag{30}
\end{aligned}$$

where $\xi = [V_{\mathbf{x}}^c \mathbf{f}]^2 + q(\mathbf{x}) [V_{\mathbf{x}}^c B B^T V_{\mathbf{x}}^{cT}]$.

Note that $\rho(\mathbf{x}, t) = \varepsilon_b \|\mathbf{u}_{op}\| + \rho_m(\mathbf{x}, t) \geq \|\Delta B_2 \mathbf{u}_{op} + \mathbf{d}\|$. Similar to Section 2, choose

$$\mathbf{u}_{vsc} = -\frac{[\rho(\mathbf{x}, t) + \delta] \gamma}{(1 - \varepsilon_b) \|\gamma\|},$$

where $\delta \geq 0$ is a constant and ρ is defined in (29). Substituting $\mathbf{u}_{vsc}$ into (30), we have

$$\begin{aligned}
\dot{V}^c &\leq \rho(\mathbf{x}, t) \|\gamma\| - \frac{\gamma^T [\rho(\mathbf{x}, t) + \delta] \gamma}{(1 - \varepsilon_b) \|\gamma\|} + \varepsilon_b \frac{\|\gamma^T [\rho(\mathbf{x}, t) + \delta] \gamma\|}{(1 - \varepsilon_b) \|\gamma\|} - \sqrt{\xi} \\
&= -\delta \|\gamma\| - \sqrt{\xi} < 0. \tag{31}
\end{aligned}$$

Remark 6. The proposed controller (28) is synthesized by $\mathbf{u}_{op}$ and $\mathbf{u}_{vsc}$ which are made to function in a complementary manner to fulfill the control objective: while in the region dominated by the nominal part, the suboptimal control $\mathbf{u}_{op}$ will govern the system response and drive the system state approaching the equilibrium in a suboptimal fashion; while in the disturbance dominant region, the VSC part $\mathbf{u}_{vsc}$ will take over the control task to warrant the robustness.

Remark 7. From *Theorem 4*, the advantage of the quasi-optimal VSC is immediately obvious. While the control law (28) in general guarantees the stability and the reaching condition (31), we now have one extra degree of freedom in tuning the controller, in the sequel tuning control performance. In particular, we can choose an appropriate quantity $q(\mathbf{x})$ in (23) to change the optimal control part. A large $q(\mathbf{x})$ will lead to time-optimal control as the system state becomes the main concern. On the contrary, by reduction of $q(\mathbf{x})$, the control input becomes the main concern (more penalized), and one can expect less control effort and smoother responses.

Remark 8. : The switching gains of the typical VSC (6) and the quasi-optimal VSC (28) are $[\varepsilon_b \|\mathbf{u}_c\| + \rho_m(\mathbf{x}, t)] / (1 - \varepsilon_b)$ and $[\varepsilon_b \|\mathbf{u}_{op}\| + \rho_m(\mathbf{x}, t)] / (1 - \varepsilon_b)$ respectively. Note that $\mathbf{u}_{op}$ is bounded with $\sqrt{q(\mathbf{x})}$ and will approach zero when the system state approaches equilibrium, whereas $\mathbf{u}_c$ may still be large if $\mathbf{f}_2(\mathbf{x}_1, \mathbf{x}_2, t)$ is nonvanishing around the equilibrium. Therefore the quasi-optimal VSC may have much lower switching gain than that of the typical VSC near the equilibrium.

6 Illustrative Examples

We present two examples in this section. The first shows how the proposed quasi-optimal VSC method can be successfully applied to a highly nonlinear system. The second example clearly shows that the nonlinear optimal sliding mode cannot be approximated by a locally linearized one.

6.1 Attitude Control of A Rigid Spacecraft

Consider the following attitude motion model of a rigid spacecraft [8],

$$\begin{cases} \dot{\boldsymbol{\rho}} = H(\boldsymbol{\rho})\boldsymbol{\omega} \\ \dot{\boldsymbol{\omega}} = J^{-1}S(\boldsymbol{\omega})J\boldsymbol{\omega} + J^{-1}(\mathbf{u} + \mathbf{d}_o) \end{cases} \quad (32)$$

where $\boldsymbol{\rho} \in \mathcal{R}^3$ is the Cayley-Rodrigues parametric vector describing the body orientation, $\boldsymbol{\omega} \in \mathcal{R}^3$ is the angular velocity vector in a body-fixed frame, $\mathbf{u} \in \mathcal{R}^3$ is the acting control torque, $S(\boldsymbol{\omega})$ is a 3×3 skew-symmetric matrix, that is

$$S(\boldsymbol{\omega}) = \begin{bmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{bmatrix},$$

the matrix-valued function $H: \mathcal{R}^3 \rightarrow \mathcal{R}^{3 \times 3}$ denotes the kinematics Jacobian matrix for the Cayley-Rodrigues parameters, given by

$$H(\boldsymbol{\rho}) = \frac{1}{2}(I - S(\boldsymbol{\rho}) + \boldsymbol{\rho}\boldsymbol{\rho}^T),$$

where $J = \Delta J^{-1}J_0$, $J_0 > 0$ is the nominal part of the inertia matrix, here it is assumed $J_0 = \text{diag}(10, 15, 20) \text{ kgm}$, and

$$\begin{aligned} \Delta J &= \text{diag}(\Delta j_1, \Delta j_2, \Delta j_3), \\ \mathbf{d}_o &= [d_{o1}, d_{o2}, d_{o3}]^T, \end{aligned}$$

where perturbations Δj_i and $d_{oi} (i = 1, 2, 3)$ are as follows

$$\begin{aligned} \Delta j_i &= 1 + 0.1 \sin(it), \\ d_{oi} &= \sin(10t). \end{aligned}$$

The rigid spacecraft model (32) can be converted to our standard format as follows,

$$\begin{cases} \dot{\mathbf{x}}_1 = G_1 \mathbf{x}_2 \\ \dot{\mathbf{x}}_2 = \mathbf{f}_2 + B_2\{[I + \Delta B_2]\mathbf{u} + \mathbf{d}\} \end{cases} \quad (33)$$

where

$$\begin{aligned}
\mathbf{x}_1 &= \boldsymbol{\rho}, \\
\mathbf{x}_2 &= \boldsymbol{\omega} \\
G_1 &= H(\mathbf{x}_1), \\
\mathbf{f}_2 &= J_0^{-1} S(\mathbf{x}_2) J_0 \mathbf{x}_2 \\
B_2 &= J_0^{-1}, \\
\Delta B_2 &= \begin{bmatrix} 0.1 \sin(t) & 0 & 0 \\ 0 & 0.1 \sin(2t) & 0 \\ 0 & 0 & 0.1 \sin(3t) \end{bmatrix}, \\
\mathbf{d} &= [\Delta J S(\mathbf{x}_2) \Delta J^{-1} - S(\mathbf{x}_2)] J_0 \mathbf{x}_2 + \Delta J \mathbf{d}_0
\end{aligned}$$

To design the nonlinear optimal sliding mode, we need to solve the HJB equation for the $\mathbf{x}_1$ -subdynamics. Unfortunately, it is not a feasible task. Thus we will construct an inverse optimal sliding mode by applying the method illustrated in Section 3.3.

Rewrite the $\mathbf{x}_1$ -subdynamics of the model (33) as

$$\dot{\mathbf{x}}_1 = G_1(\mathbf{x}_1) \mathbf{v}. \quad (34)$$

Finding a stabilizing feedback control law which has the form as defined in (16) is a crucial step in the inverse optimal sliding mode construction. Such a control law is chosen to be

$$\mathbf{v} = -R^{-1} G_1(\mathbf{x}_1)^T V_{\mathbf{x}_1}^T \quad (35)$$

where $R = R^T > 0$ and $V = \frac{1}{2} \mathbf{x}_1^T \mathbf{x}_1$ is a positive definite and radially unbounded Lyapunov function. Note that

$$\begin{aligned}
\dot{V} &= \left[\frac{\partial V}{\partial \mathbf{x}_1} \right]^T \dot{\mathbf{x}}_1 \\
&= \mathbf{x}_1^T \dot{\mathbf{x}}_1 \\
&= \mathbf{x}_1^T G_1 (-R^{-1} G_1^T V_{\mathbf{x}_1}^T) \\
&= -\mathbf{x}_1^T G_1 R^{-1} G_1^T \mathbf{x}_1 \\
&< 0,
\end{aligned} \quad (36)$$

for all $\mathbf{x}_1 \neq 0$. Thus the control law (35) stabilizes the system (34).

According to Section 3.3, the optimal control law

$$\mathbf{v}^* = 2\mathbf{v} = -2R^{-1} G_1^T \mathbf{x}_1$$

minimizes the cost

$$J = \int_0^\infty [l(\mathbf{x}_1) + \mathbf{v}^T R \mathbf{v}] dt \quad (37)$$

where

$$l(\mathbf{x}_1) = 4\mathbf{x}_1^T G_1 R^{-1} G_1^T \mathbf{x}_1$$

and the optimal value function is

$$V^* = 2\mathbf{x}_1^T \mathbf{x}_1.$$

Thus the corresponding inverse optimal sliding mode is

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_1 + \mathbf{x}_2 = 2R^{-1}G_1^T \mathbf{x}_1 + \mathbf{x}_2 \quad (38)$$

Now let us construct the suboptimal control in terms of the system nominal part. According to *Theorem 4* in Section 5, the CLF for the nominal part of model (32) is

$$\begin{aligned} V^c &= V_1 + \frac{1}{2} \boldsymbol{\sigma}^T \boldsymbol{\sigma} \\ &= V^* + \frac{1}{2} (\boldsymbol{\sigma}_1 + \mathbf{x}_2)^T (\boldsymbol{\sigma}_1 + \mathbf{x}_2) \\ &= 2\mathbf{x}_1^T \mathbf{x}_1 + \frac{1}{2} (2R^{-1}G_1^T \mathbf{x}_1 + \mathbf{x}_2)^T (2R^{-1}G_1^T \mathbf{x}_1 + \mathbf{x}_2). \end{aligned}$$

Thus, the suboptimal control part of the proposed controller is

$$\mathbf{u}_{op} = - \frac{V_x^c \mathbf{f} + \sqrt{[V_x^c \mathbf{f}]^2 + q(\mathbf{x})[V_x^c B B^T V_x^{cT}]} V_x^c B B^T V_x^{cT}}{V_x^c B B^T V_x^{cT}} B^T V_x^{cT} \quad (39)$$

where

$$\begin{aligned} \mathbf{x} &= [\mathbf{x}_1^T, \mathbf{x}_2^T]^T \\ \mathbf{f} &= [G_1^T, \mathbf{f}_2^T]^T \\ B &= [\mathbf{0}, B_2^T]^T \end{aligned} \quad (40)$$

and $q(\mathbf{x})$ is a positive definite function.

It is straightforward to determine the switching control part $\mathbf{u}_{vsc}$ of the proposed controller

$$\mathbf{u}_{vsc} = - \frac{(\rho + \delta)\boldsymbol{\gamma}}{(1 - \varepsilon_b)\|\boldsymbol{\gamma}\|} \quad (41)$$

where $\delta > 0$ is a constant,

$$\begin{aligned} \varepsilon_b &= 0.1 \\ \boldsymbol{\gamma}^T &= \boldsymbol{\sigma}^T B_2 = (2R^{-1}G_1^T \mathbf{x}_1 + \mathbf{x}_2)^T B_2 \\ \rho &= \varepsilon_b \|\mathbf{u}_{op}\| + \rho_m \\ \rho_m &= \frac{10\sqrt{29}}{11} [\max(\mathbf{x}_{21}, \mathbf{x}_{22}) + \max(\mathbf{x}_{21}, \mathbf{x}_{23}) + \max(\mathbf{x}_{22}, \mathbf{x}_{23})] \|\mathbf{x}_2\| \\ &\quad + 3.3. \end{aligned}$$

The quasi-optimal VSC for the rigid spacecraft attitude motion model (32) is

$$\mathbf{u} = \mathbf{u}_{op} + \mathbf{u}_{vsc} \quad (42)$$

with $\mathbf{u}_{op}$ defined in (39) and $\mathbf{u}_{vsc}$ defined in (41).

Figure 1 shows the $\mathbf{x}_{11} - \mathbf{x}_{21}$ phase planes of the suboptimal controller (39) and the proposed controller (42) respectively. The closeness of the two curves indicates that the quasi-optimal VSC does retain the suboptimal control performance where the system nominal part is predominant.

Figure 2 shows the system responses near the equilibrium under the suboptimal controller (39) and the proposed controller (42). We observe that the system state, when near the equilibrium, cannot converge anymore with the suboptimal controller alone. On the contrary, the system state keeps convergent with the proposed controller, owing to the robustness property of VSC.

Figure 3 and Fig. 4 show the possibility of adjusting system responses by changing two weightings $q(\mathbf{x})$ in the suboptimal control part and R in the inverse optimal sliding mode (38), respectively. In Fig. 3, $q(\mathbf{x})$ in the proposed controller (42) is chosen to be $a(\mathbf{x}_1^T \mathbf{x}_1 + \mathbf{x}_2^T \mathbf{x}_2)$. The solid line, dashed line and dashdotted line in Fig. 3 show the values of the switching quantity σ_1 (the first element of $\boldsymbol{\sigma}$) with $a = 10000, 100, 1$, respectively. We observe that a larger a , i.e. a large $q(\mathbf{x})$, will expedite the system response and drive the system state to reach the sliding mode faster. Figure 76, Fig. 77 and Fig. 6 show the corresponding control profiles. Next let us check R , which according to (37) should have effect on the convergence of the states $\mathbf{x}_1$ when the system is in the sliding mode. For simplicity we only show the responses of the first state $\mathbf{x}_1$. In Fig. 4, the solid line is obtained with $R = \text{diag}(0.1, 1, 1)$ and the dashed line is obtained with $R = \text{diag}(0.5, 1, 1)$. We can observe that a smaller R will expedite the system response. These results clearly illustrate that the new quasi-optimal VSC possesses both robust and optimal properties. Figure 8 shows the control profiles. Note that the control gain is very large, this is because we use a very large $q(x)$, i.e. $10^6(\mathbf{x}_1^T \mathbf{x}_1 + \mathbf{x}_2^T \mathbf{x}_2)$, in order to force the system dynamics to reach the inverse optimal sliding mode in a short time.

6.2 Inverse Optimal Sliding Mode vs. Linear Sliding Mode

Here we show that a nonlinear sliding mode may be indispensable when the system is nonlinear in nature. Consider the following $\mathbf{x}_1$ -subdynamics

$$\dot{\mathbf{x}}_1 = \mathbf{f}_1(\mathbf{x}_1) + G_1(\mathbf{x}_1)\mathbf{x}_2, \quad (43)$$

where $\mathbf{x}_1 = [x_{11}, x_{12}]^T$, $\mathbf{x}_2 = [x_{21}, x_{22}]^T$ and

$$\mathbf{f}_1 = \begin{bmatrix} -x_{11} \sin^2(x_{11}) - x_{11} \\ -x_{12} \sin^2(x_{12}) - x_{12} \end{bmatrix}$$

$$G_1 = \begin{bmatrix} 1 + \cos^2 x_{11} - x_{11}^2 & 0 \\ 0 & 1 + \cos^2 x_{12} - x_{12}^2 \end{bmatrix}.$$

A globally stable inverse optimal sliding manifold

$$\boldsymbol{\sigma} = 2R^{-1}G_1^T \mathbf{x}_1 + \mathbf{x}_2. \quad (44)$$

for system (43) can be achieved by applying the method given in Sec. 3.3.

For comparison purposes, a linear optimal sliding mode is also designed based on the linearized model at $\mathbf{x}_1 = 0$. The linearized model is obtained by removing those higher order infinitesimal terms

$$\dot{\mathbf{x}}_1 = A_{11}\mathbf{x}_1 + A_{12}\mathbf{x}_2, \quad (45)$$

where $A_{11} = -I$ and $A_{12} = 2I$. Consider the optimal control task below

$$\begin{aligned} \dot{\mathbf{x}}_1 &= A_{11}\mathbf{x}_1 + A_{12}\mathbf{v} \\ \inf_{\mathbf{v}(\cdot)} \int_0^\infty [\mathbf{x}_1^T Q \mathbf{x}_1 + \mathbf{v}^T R \mathbf{v}] dt. \end{aligned}$$

where $Q = Q^T > 0$ and $R = R^T > 0$. The algebraic *Ricatti* equation for this optimal control problem is

$$\begin{aligned} PA_{11} + A_{11}^T P - PA_{12}R^{-1}A_{12}^T P + Q &= 0 \\ \text{or } -2P - 4PR^{-1}P + Q &= 0, \end{aligned} \quad (46)$$

and the optimal control law is given by

$$\mathbf{v}^*(\mathbf{x}_1) = -2R^{-1}P\mathbf{x}_1,$$

where P is the solution of the algebraic *Ricatti* equation (46). Then the optimal sliding mode defined in (14) is as follows

$$\boldsymbol{\sigma} = 2R^{-1}P\mathbf{x}_1 + \mathbf{x}_2. \quad (47)$$

Figure 5 shows the responses of x_{11} when the system is in the linear sliding mode (47) with different initial values $\mathbf{x}_1(0) = [0.5, 0.4]^T$ and $\mathbf{x}(0) = [5, 4]$, $Q = R = I$. Fig. 6 shows that the system with larger initial values $\mathbf{x}(0) = [5, 4]$ is unstable! The linear-type sliding mode applied to the nonlinear $\mathbf{x}_1$ -subdynamics results in an unstable sliding manifold. Figure 7 shows the responses of x_{11} when the system is in the nonlinear sliding mode with the smaller initial values $\mathbf{x}_1(0) = [0.5, 0.4]^T$. Fig. 8 shows the system responses with the larger initial values $\mathbf{x}(0) = [5, 4]$. Solid lines and dashdotted lines are obtained with $R = I$ and $R = \text{diag}(0.01, 1, 1)$ respectively. We observe that the nonlinear sliding mode with inverse optimality ensures the global stability of the sliding manifold, and meanwhile retains the optimal control properties, such as the possibility of manipulating responses through tuning the weightings R .

7 Conclusion

In this paper, we proposed a new control strategy - quasi-optimal VSC, by integrating VSC with optimal sliding mode as well as nonlinear suboptimal control based on CLF. The nonlinear optimal sliding mode design provides a

systematic way to guarantee a stable nonlinear sliding manifold, consequently making VSC approaches applicable to more general classes of nonlinear systems. In the new control method, the suboptimal control and VSC are made to function in a complementary manner. It performs as the optimal control when the system is dominated by the nominal part, and as the typical VSC when the system perturbations become dominant. Both nonlinear optimal sliding mode design and suboptimal controller design offer extra degrees of freedom in the weight selection so as to meet the different performance requirements.

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Appendix

Define

$$\begin{aligned}\theta &= V_{\mathbf{x}} \mathbf{f}, \\ \mathbf{w}^T &= V_{\mathbf{x}} \mathbf{B}.\end{aligned}$$

Since $V(x)$ is a Control Lyapunov Function, whenever $\mathbf{x} \neq 0$, we have

$$\mathbf{w} = 0 \Rightarrow \theta < 0.$$

Assume that at a point $\mathbf{x}_0$, $V_x^c B|_{\mathbf{x}=\mathbf{x}_0} = 0$ and $V_x^c \mathbf{f}|_{\mathbf{x}=\mathbf{x}_0} < 0$. Since both $\mathbf{f}$ and B are continuous with respect to all arguments, and V^c is continuously differentiable, $\forall \epsilon > 0$, $\exists \delta > 0$, such that when $\|\mathbf{x} - \mathbf{x}_0\| < \delta$, there are $\|V_x^c B\| < \epsilon$ and $V^c \mathbf{f} < 0$.

Thus, when $\|\mathbf{x} - \mathbf{x}_0\| < \delta$, we have

$$\begin{aligned} \|\mathbf{u}_{op}\| &= \left\| -\frac{V_x^c \mathbf{f} + \sqrt{[V_x^c \mathbf{f}]^2 + q(\mathbf{x})[V_x^c B B^T V_x^{cT}]} B^T V_x^{cT}}{V_x^c B B^T V_x^{cT}} \right\| \\ &= \left| \frac{\theta + \sqrt{\theta^2 + q(\mathbf{x}) \mathbf{w}^T \mathbf{w}}}{\mathbf{w}^T \mathbf{w}} \right| \cdot \|\mathbf{w}\| \\ &\leq \frac{-|\theta| + |\theta| + \sqrt{q(\mathbf{x})} \|\mathbf{w}\|}{\|\mathbf{w}\|^2} \|\mathbf{w}\| \\ &= \sqrt{q(\mathbf{x})}. \end{aligned}$$

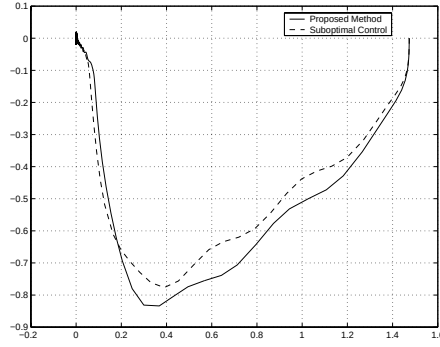


Fig. 1. $\rho_1 - \omega_1$ phase plane under suboptimal control and quasi-optimal VSC

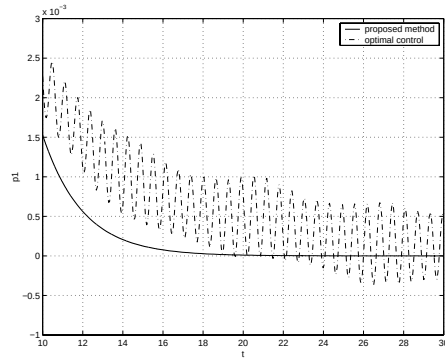


Fig. 2. ρ_1 under suboptimal control and quasi-optimal VSC (near equilibrium)

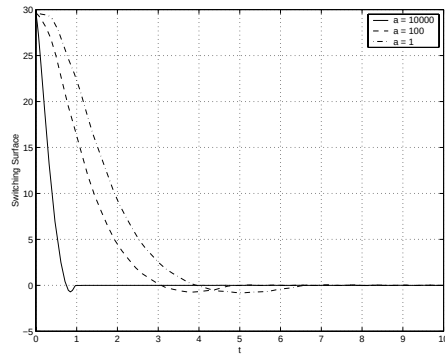


Fig. 3. Switching surface σ_1 under quasi-optimal VSC with different $q(\mathbf{x})$

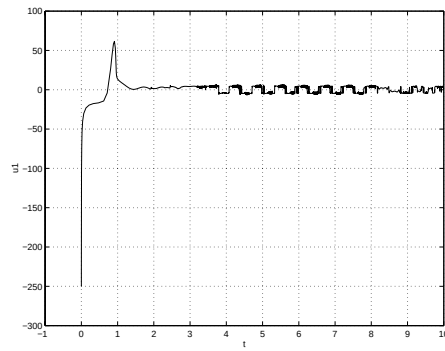


Fig. 4. Control profile of quasi-optimal VSC with $q(\mathbf{x}) = 10000(\mathbf{x}_1^T \mathbf{x}_1 + \mathbf{x}_2^T \mathbf{x}_2)$

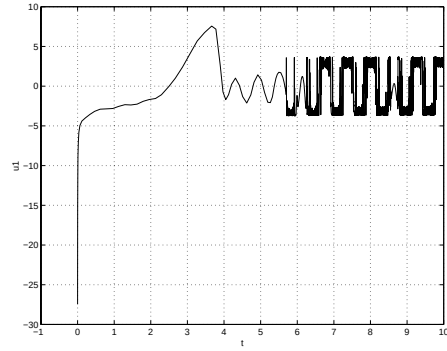


Fig. 5. Control profile of quasi-optimal VSC with $q(\mathbf{x}) = 100(\mathbf{x}_1^T \mathbf{x}_1 + \mathbf{x}_2^T \mathbf{x}_2)$

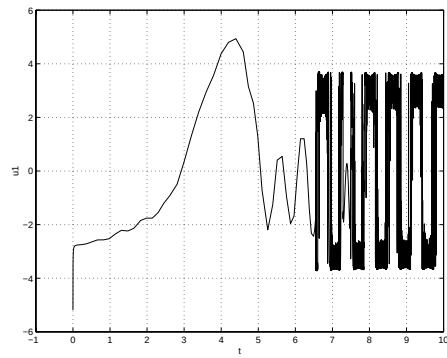


Fig. 6. Control profile of quasi-optimal VSC with $q(\mathbf{x}) = 1(\mathbf{x}_1^T \mathbf{x}_1 + \mathbf{x}_2^T \mathbf{x}_2)$

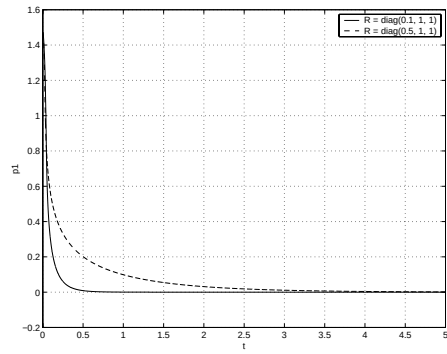


Fig. 7. System responses in inverse optimal sliding mode with different R

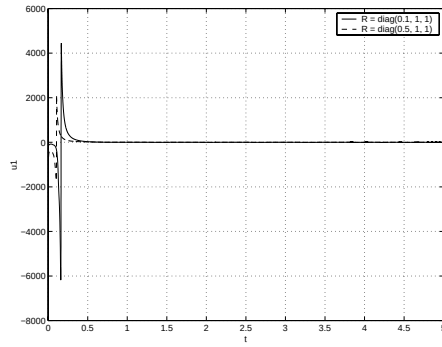


Fig. 8. Control Profiles of quasi-optimal VSC with different R

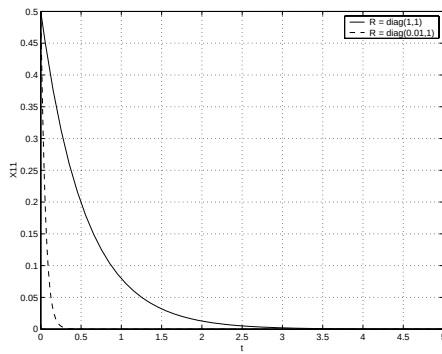


Fig. 9. System responses with a local optimal sliding mode, $x(0) = [0.5, 0.4]$

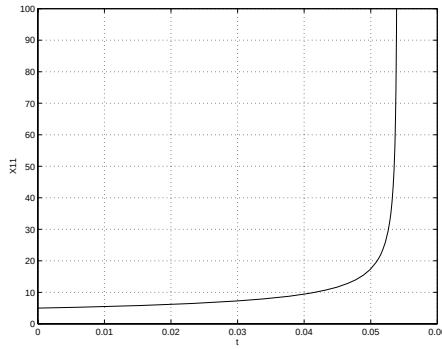


Fig. 10. System responses with a local optimal sliding mode, $x(0) = [5, 4]$

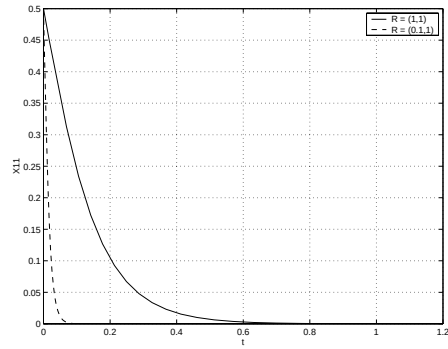


Fig. 11. System responses with inverse optimal sliding mode, $x(0) = [0.5, 0.4]$

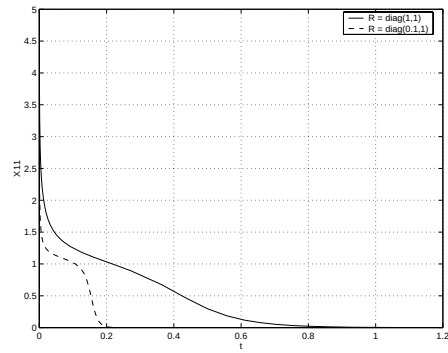


Fig. 12. System responses with inverse optimal sliding mode, $x(0) = [5, 4]$

Robust Control of Infinite-Dimensional Systems via Sliding Modes

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Abstract. Infinite-dimensional control systems, driven by a discontinuous feedback, are under study. Discontinuous control algorithms are developed. The algorithms ensure desired dynamic properties as well as robustness of the closed-loop system against matched disturbances. The theory presented is illustrated by applications to heat processes and mechanical distributed oscillators.

1 Introduction

Many important plants, such as time-delay systems, flexible manipulators and structures as well as heat transfer processes, combustion, and fluid mechanical systems, are governed by functional and partial differential equations or, more general, equations in a Hilbert space. As these systems are often described with a significant degree of uncertainty, robust controller design for these systems presents a challenging problem.

Sliding mode control of finite-dimensional systems is known to guarantee a certain degree of robustness with respect to unmodal dynamics. Since the sliding mode equation is control-independent, the approach based on the deliberate introduction of sliding motions into the control system splits the control problem into two independent problems of lower dimensions. First, a discontinuity manifold with the prescribed dynamic properties of the sliding motion is designed and then a discontinuous control, which ensures the sliding motion on this manifold, is synthesized. Apart from decoupling the original control problem, the sliding mode approach makes the closed-loop system insensitive to matched disturbances. Due to these advantages and simplicity of implementation, sliding mode controllers are widely used in various applications. An overview of finite-dimensional sliding mode control theory and applications can be found in [20].

The first few papers [1,12,16] on the application of sliding mode control algorithms to DPS corroborated their utility for infinite-dimensional systems as well and motivated further theoretical investigations [17,23], which were confined, however, to semilinear parabolic systems with a finite horizon. In order to describe the sliding modes in these systems, the sliding mode equation was shown in [17] to be well-posed via relating the discontinuous control law to the continuous one. The conditions for the infinite-dimensional sliding mode

to exist were obtained in [23] through a finite-dimensional Faedo-Galerkin approximation of the original discontinuous control system.

Subsequently, a set of sliding mode control algorithms has been proposed for distributed parameter plants governed by uncertain partial differential equations (cf. [21,22] and the references quoted therein). All algorithms followed the conventional finite-dimensional approach, which implies that each component of a control action undergoes discontinuities on its own surface and as a result a sliding mode is enforced in their intersection.

This approach, however, becomes invalid in the general infinite-dimensional setting because neither control input nor sliding manifold is representable in a component-wise form. Thus, infinite-dimensional discontinuous control methods developed in the present chapter make a step beyond the finite-dimensional treatment.

The chapter is outlined as follows. In Section 2 we demonstrate some attractive features of discontinuous control systems in a Hilbert space and motivate the subsequent theoretical development. We present here an infinite-dimensional system driven by discontinuous control signals along the discontinuity manifold on an infinite time interval. The discontinuous control law results from the Lyapunov min-max approach, the origins of which may be found in [7,8]. An extension of this approach to infinite-dimensional systems can be found in [15,18]. Based on the extension, the control is synthesized to guarantee that the time-derivative of a Lyapunov function, selected for a nominal, exponentially stable system, is negative on the trajectories of the system with perturbations caused by uncertainties of a plant operator and environment conditions. The approach gives rise to the control action, referred to as a *unit control*, the norm of which is equal to 1 everywhere with the exception of the discontinuity manifold. The closed-loop system enforced by the unit control is shown to be exponentially stable and robust with respect to matched disturbances.

Section 3 presents the decomposition-based synthesis of a discontinuous control law, which imposes the desired dynamic properties as well as robustness with respect to matched disturbances on the closed-loop system. If the undisturbed motion of the system contains two components, one of them stable and another one belonging to a finite-dimensional subspace, the control synthesis, as is shown in Section 4, is split into two independent synthesis procedures. The first procedure uses the standard finite-dimensional setting while the second one is carried out within the infinite-dimensional subspace of the exponentially stable internal dynamics. The latter procedure is developed in Section 2 of the present chapter.

As an illustration of the capabilities of the above procedure, a scalar unit controller for a minimum phase system of a finite relative degree is constructed in Section 5.

2 Unit Feedback Control Synthesis

In this section we present a synthesis of discontinuous unit feedback control signals and discuss some attractive capabilities of discontinuous controllers in addressing both finite and infinite-dimensional systems.

2.1 Synthesis in Eucleadean State Space

We start our investigation with an affine system

$$\dot{x} = f(x, t) + B(x, t)u + h(x, t) \quad (1)$$

with finite-dimensional state and control vectors $x \in R^n$, $u \in R^m$ and state-dependent n -vectors $f(x, t)$ and $h(x, t)$, and matrix $B(x, t) \in R^{n \times m}$. The vector $h(x, t)$ represents the system uncertainty and its influence on the control process should be rejected.

The equation

$$\dot{x} = f(x, t) \quad (2)$$

represents an open loop nominal system which is assumed to be asymptotically stable with some apriori known Lyapunov function:

$$V(x) > 0, \quad W_o = \frac{dV}{dt} \big|_{h=0, u=0} = \{grad(V)\}^T f < 0. \quad (3)$$

The perturbation vector $h(x, t)$ is assumed to satisfy the matching condition [4]

$$h(x, t) \in span(B).$$

In other words, it is assumed that there exists vector $\lambda(x, t) \in R^m$ such that

$$h(x, t) = B(x, t)\lambda(x, t). \quad (4)$$

Here $\lambda(x, t)$ may be an unknown vector with an apriori known upper scalar estimate $\lambda_o(x, t)$, i.e.,

$$\|\lambda(x, t)\| < \lambda_o(x, t). \quad (5)$$

The time derivative of $V(x)$ on the trajectories of the perturbed system (1),(4) is of the form

$$W = \frac{dV}{dt} = W_o + \{grad(V)\}^T B(u + \lambda) < 0. \quad (6)$$

For the control

$$u = -\rho(x, t) \frac{B^T grad(V)}{\|B^T grad(V)\|}, \quad (7)$$

depending on the upper estimate of the unknown disturbance, with a scalar function

$$\rho(x, t) > \lambda_o(x, t)$$

and

$$\| \text{grad}(V) \|^2 = [\{\text{grad}(V)\}^T B][B^T \text{grad}(V)]$$

the time derivative of the Lyapunov function $V(x)$

$$W = W_o - \rho(x, t) \| \{\text{grad}(V)\} \| + \text{grad}(V)^T B \lambda(x, t) <$$

$$W_o - \| B^T \{\text{grad}(V)\} \| [\rho(x, t) - \lambda_o(x, t)] < 0$$

is negative. This means that the perturbed system with control (7) is asymptotically stable, too.

Two important features should be underlined for the system with control (7):

1. The control signal is a discontinuous function of the system state and it undergoes discontinuities in the $(n - m)$ -th dimensional manifold

$$s(x) = B^T \text{grad}(V) = 0. \quad (8)$$

2. The disturbance $h(x, t)$ is rejected due to enforcing the sliding mode in the manifold $s(x) = 0$.

Note that the equivalent value u_{eq} of the control signal (7) is equal to $-\lambda(x, t)$ along the discontinuity manifold $s(x) = B^T \text{grad}(V) = 0$, which is not, generally speaking, the case for the control law (7) beyond the manifold, $s(x) = B^T \text{grad}(V) \neq 0$. Meanwhile, beyond this manifold the norm

$$\left\| \frac{B^T \text{grad}(V)}{\| B^T \text{grad}(V) \|} \right\|$$

of the control signal (7) with the gain $\rho(x, t) = 1$ is equal to 1 for any value of the state vector. This explains the term unit control for the control signal (7).

It is of interest to note that in contrast to the conventional sliding mode control signals which undergo discontinuities whenever a component of the sliding manifold changes sign, the unit control action is a continuous state function until the manifold $s(x) = 0$ is reached. Due to this difference the unit control method turns out to be an appropriate tool to design a discontinuous infinite-dimensional system with control inputs which are not (or even can not be) represented in a component-wise form.

2.2 Synthesis in Hilbert State Space

In order to illustrate the fact that discontinuous infinite-dimensional systems can also be driven along discontinuity manifolds, we consider a dynamical system

$$\dot{x} = e(x), \quad x(0) = x^0 \in H \quad (9)$$

in a real Hilbert space H enforced by the unit control action $e(x) = -x/\|x\|$ which undergoes discontinuities in the trivial manifold $x = 0$. Since the norm $\|x\| = \sqrt{(x, x)}$ in the Hilbert space is defined via its inner product $(\cdot, \cdot)$, then

$$\frac{1}{2} \frac{d\|x\|^2}{dt} = (x(t), \dot{x}(t)) = -\|x(t)\|,$$

and therefore $\|x(t)\| = (\|x^0\| - t)$ for $t \leq \|x^0\|$. Hence, in the infinite-dimensional system (9) starting from the time moment $t = \|x^0\|$, there appears a sliding mode in the discontinuity manifold $x = 0$. Clearly, the sliding mode is unambiguously set by the manifold equation of $x = 0$ regardless of uniformly bounded additive dynamic nonidealities $h(x, t)$ such that $\|h(x, t)\|_H < 1$ for all $t \geq 0$, $x \in H$, which are rejected by the unit control. In this case the sign of the time derivative of the Lyapunov functional along the trajectories of the perturbed system $\dot{x} = e + h$ remains negative. However, in general, neither the unit control belongs to the state space nor the discontinuity manifold is trivial, so that their synthesis presents a formidable problem.

According to the unit feedback approach, developed in the sequel, a linear discontinuity manifold $cx = 0$ in the control space U , which differs from the state space H , may be constructed in compliance with some performance criterion, particularly, according to the Lyapunov min-max approach, whereas a sliding mode in the manifold is enforced by the corresponding unit control $M(x, t)e(cx) = -M(x, t)cx/\|cx\|_U$, possibly with a non-unit gain $M(x, t) \neq 1$. This design idea is now illustrated for an uncertain dynamic system governed by a differential equation

$$\dot{x} = Ax + f(x, t) + bu(x, t), \quad x(0) = x^0 \in \mathcal{D}(A) \quad (10)$$

where the state $x(t)$ and control signal $u(x, t)$ are abstract functions with values in Hilbert spaces H and U , respectively; A is the infinitesimal generator of an exponentially stable semigroup $T_A(t)$ on H ; $b \in L(U, H)$. The operator function $f(x, t)$ with values in H represents the system uncertainties, whose influence on the control process should be rejected. Throughout the present chapter, this function is assumed to be continuously differentiable in all arguments and satisfy the matching condition

$$f(x, t) = bh(x, t) \quad (11)$$

where the uncertain function $h(x, t)$ has an a priori known upper scalar estimate $N(x) \in C^1$, i.e.

$$\|h(x, t)\|_U < N(x) \text{ for all } x \in H, t \geq 0. \quad (12)$$

In order to apply the afore-mentioned Lyapunov approach to the infinite-dimensional system (10), let us note that the positive definite solution $W_A = \int_0^\infty T_A^*(t)T_A(t)dt$ to the Lyapunov equation $W_AA + A^*W_A = -I$ with the identity operator I assigns the quadratic Lyapunov functional

$$V(x) = (W_A x, x)$$

for the nominal system $\dot{x} = Ax$. Then, taking into account (11) and differentiating the Lyapunov functional with respect to t along the trajectories of the perturbed system (10), we obtain

$$\begin{aligned} dV/dt &= (W_A \dot{x}(t), x(t)) + (W_A x(t), \dot{x}(t)) = -(x, x) + \\ &2(W_A x(t), b(u + h)) = -(x, x) + 2(b^* W_A x(t), u + h). \end{aligned} \quad (13)$$

A straightforward application of the Lyapunov min-max approach, which requires minimization of the right-hand side of (13) under the control constraint $\|u(\cdot)\|_U \leq M = \text{const}$, results in the unit control

$$u(x) = -Me(b^* W_A x) = -M \frac{b^* W_A x}{\|b^* W_A x\|_U}. \quad (14)$$

Given the state-dependent gain $M = N(x)$, the time derivative of the Lyapunov functional along the trajectories of the perturbed system (10) driven by the unit control (14) is forced to be negative, namely

$$dV/dt \leq -(x, x) \leq -\frac{1}{\|W_A\|}(W_A x, x) = -\frac{1}{\|W_A\|}V(x) \quad (15)$$

for all $x \in H$ (including the discontinuity manifold!), regardless of admissible plant perturbations $f(x, t)$. This guarantees the exponential stability of the closed-loop system. Thus, the unit control (14) with the gain $M = N(x)$ rejects any admissible perturbation $f(x, t)$ and imposes the desired dynamic and robustness properties on the uncertain system (10). Along with the relative simplicity of the implementation of unit control signals (cf. that of [5,10]), these properties make the use of unit controllers in the infinite-dimensional case attractive.

3 Decomposition of Synthesis Procedure

3.1 Decomposition in Euclidean State Space

According to the finite-dimensional treatment, proposed in [3,19,20] for the affine system (1), the unit control synthesis procedure consists of two steps.

First, a sliding mode is selected to have the prescribed dynamic properties of the motion in the sliding mode by a proper choice of the discontinuity manifold $s = 0$. And second, a discontinuous control is constructed to guarantee existence of the sliding motion along this manifold. Once the manifold $s(x) = 0$ has been selected in compliance with some performance criterion, the control input is designed in the form (7):

$$u = -\rho(x, t) \frac{G^T s(x)}{\|G^T s(x)\|}, \quad (16)$$

with $G = \{\frac{\partial s}{\partial x}\}B$, G is assumed to be nonsingular. The equation of a motion projection of the system (1) on the subspace s is of the form

$$\dot{s} = \{\frac{\partial s}{\partial x}\}(f + h) + Gu. \quad (17)$$

The conditions for the trajectories to converge to the manifold $s(x) = 0$ and the sliding mode to exist in this manifold may be derived based on the Lyapunov function

$$V = \frac{1}{2} s^T s > 0 \quad (18)$$

with the time derivative

$$\begin{aligned} \dot{V} &= s^T \{\frac{\partial s}{\partial x}\}(f + h) - \rho(x, t) \|G^T s(x)\| < \\ &\|G^T s(x)\| \cdot [\|G^{-1}\{\frac{\partial s}{\partial x}\}(f + h)\| - \rho(x, t)]. \end{aligned} \quad (19)$$

For $\rho(x, t) > \|G^{-1}\{\frac{\partial s}{\partial x}\}(f + h)\|$ the value of $\dot{V}$ is negative and therefore the state will reach the manifold $s(x) = 0$ in a finite time interval for any initial conditions and then the sliding mode with the desired dynamics will occur. The boundedness of the interval preceding the sliding motion follows from the inequality resulting from (18),(19):

$$\dot{V} < -\gamma V^{1/2}, \quad \gamma = \text{const} > 0$$

with the solution

$$V(t) < (-\frac{\gamma}{2}t + \sqrt{V_o})^2, \quad V_o = V(0).$$

Since the solution vanishes after some $t_s < \frac{2}{\gamma}\sqrt{V_o}$, the vector s vanishes so that the sliding mode starts after a finite time interval.

In order to describe the sliding mode one should substitute the equivalent control value

$$u_{eq} = -G^{-1}\{\frac{\partial s}{\partial x}\}(f + h),$$

i.e. the continuous solution of the equation $\dot{s} = 0$ with respect to u , into (1) for u . Due to the equivalent control method [20] the resulting equation

$$\dot{x} = f + h - G^{-1} \left\{ \frac{\partial s}{\partial x} \right\} (f + h), \quad (20)$$

referred to as a sliding mode equation, governs the system motion along the sliding manifold $s = 0$. Since the sliding mode equation is control and matched disturbance-independent (indeed, equation (20) subject to (4) takes the form $\dot{x} = f - G^{-1} \left\{ \frac{\partial s}{\partial x} \right\} f$), this approach leads to decomposition of the original design problem into two independent problems and permits construction of a control system which is insensitive to matched disturbances.

3.2 Decomposition in Hilbert State Space

It is clear, the afore-given procedure can also be used in the infinite dimensional system (10) with an infinitesimal operator A which generates a strongly continuous semigroup $T_A(t)$ rather than an exponentially stable semigroup, while the unit feedback disturbance rejection used in Section 2 is no longer in force. The control problem is then split into the selection of a proper Hilbert space S and the linear discontinuity manifold

$$cx = 0, \quad c \in L(H, S) \quad (21)$$

with the desired zero dynamics

$$\dot{x}_1 = (A_{11} - GA_{21})x_1, \quad (22)$$

and design of a unit feedback controller, which ensures the motion of the system along this manifold. In order to derive the sliding mode equation (22) one should represent the Hilbert space

$$H = H_1 \oplus H_2.$$

via the kernel

$$H_1 = \ker c = \{x_1 \in H : cx_1 = 0\} \subseteq H$$

of the operator c and its complementary $H_2 \subseteq H$ (see, e.g., [11]), and then rewrite equation (10) in terms of variables $x_1(t) \in H_1$ and $x_2(t) \in H_2$:

$$\begin{aligned} \dot{x}_1 &= A_{11}x_1 + A_{12}x_2 + P_1f(x_1, x_2, t) + P_1bu(x_1, x_2, t), \quad t \geq 0, \\ x_1(0) &= x_1^0, \end{aligned} \quad (23)$$

$$\begin{aligned} \dot{x}_2 &= A_{21}x_1 + A_{22}x_2 + P_2f(x_1, x_2, t) + P_2bu(x_1, x_2, t), \quad t \geq 0, \\ x_2(0) &= x_2^0. \end{aligned} \quad (24)$$

Here $x_1(t) \oplus x_2(t) = x(t)$, $x_1^0 \oplus x_2^0 = x^0$, P_i is the projector on the subspace H_i , $A_{ij} = P_i A_j$ is the operator from H_j to H_i , $A_j = A|_{H_j}$ is the operator

restriction on H_j , $i, j = 1, 2$. Clearly, the discontinuity manifold (21), written through the new coordinates, takes the form $x_2 = 0$.

Assuming that the operator P_2b from U to H_2 is boundedly invertible (i.e., the operator $(P_2b)^{-1}$ from H_2 to U is bounded) the sliding mode equation

$$\dot{x}_1 = \tilde{A}x_1 + [P_1 - GP_2]f(x_1, 0, t) \quad (25)$$

in the discontinuity manifold $x_2 = 0$ is derived according to the equivalent control method (the extension of the method to infinite-dimensional systems is justified in [15]) by substituting the continuous solution

$$u_{eq}(x, t) = -(P_2b)^{-1}[A_{21}x_1 + P_2f(x_1, 0, t)] \quad (26)$$

of the equation $\dot{x}_2 = 0$ into (23) for $u(x_1, x_2, t)$. Since the external disturbance satisfies the matching condition (11), we obtain that $[P_1 - GP_2]f(x, t) = [P_1 - P_1b(P_2b)^{-1}P_2]bh(x, t) = 0$ and hence the sliding mode equation (25) takes the disturbance-independent form (22).

Example 1: To exemplify the decomposition idea in the infinite-dimensional setting let us consider coupled thermal fields governed by the following partial differential equation

$$\begin{aligned} \frac{\partial Q}{\partial t} &= \frac{\partial^2 Q}{\partial x^2} + DQ + Fu(x, t), \\ 0 &< x < 1, t > 0, \\ \partial Q(0, t)/\partial x &= \partial Q(1, t)/\partial x = 0, t \geq 0, \\ Q(x, 0) &= Q_0(x), 0 \leq x \leq 1 \end{aligned} \quad (27)$$

where $Q(x, t) \in R^n$, $u(x, t) \in R^m$ for all $x \in R^1, t \geq 0$; constant matrices D and F of appropriate dimensions are assumed to be controllable. From the physical viewpoint, the problem consists of heating n similar plants by virtue of m distributed sources; matrix D characterizes heat exchange with the environment and between the plants.

Let the control signal drive system (27) to the manifold

$$S(Q) = cQ = c_1Q_1 + c_2Q_2 = 0, \quad (28)$$

where

$$Q_1 \in R^{n-m}, Q_2 \in R^m, \det c_2 \neq 0, \det (cF) \neq 0.$$

Then the state equation can be represented in terms of Q_1 and S as follows

$$\frac{\partial Q_1}{\partial t} = \frac{\partial^2 Q_1}{\partial x^2} + D_{11}Q_1 + D_{12}S + F_1u(x, t), \quad (29)$$

$$\frac{\partial S}{\partial t} = \frac{\partial^2 S}{\partial x^2} + D_{21}Q_1 + D_{22}S + cFu(x, t). \quad (30)$$

Due to the equivalent control method, the system motion on manifold (28) is governed by the equation

$$\frac{\partial Q_1}{\partial t} = \frac{\partial^2 Q_1}{\partial x^2} + RQ_1, \quad R = D_{11} - F_1(cF)^{-1} \quad (31)$$

that is obtained from substitution of the equivalent control value

$$u_{eq} = -(cF)^{-1}D_{21}Q_1,$$

resulting from (29), (30) with $S(x, t)$ identically equal to zero. Applicability of the equivalent control method to the parabolic partial differential equation (27) is validated by Theorem 1 of [15].

Following the aforegiven design procedure, in the first step one needs to choose a discontinuity manifold (28) to ensure the prescribed properties of the motion in the sliding mode. It is well-known that for the finite-dimensional system

$$\dot{Q} = DQ + Fu \quad (32)$$

the equation of the sliding mode along the manifold $cQ = 0$ takes the form $\dot{Q}_1 = RQ_1$ by virtue of the equivalent control technique. A matrix c for the controllable system may be chosen such that $\det(cF) \neq 0$ and the eigenvalues of the matrix R take up the desired values with negative real parts $\operatorname{Re}\lambda\{R\} < 0$. To specify the matrix c it suffices to represent system (32) in the canonical form where the choice of the matrix becomes straightforward (see [20] for details). Based on this fact, the required rates of L_2 -convergence

$$\lim_{t \rightarrow \infty} \|Q_1(\cdot, t)\|_{L_2(0,1)} = 0 \quad (33)$$

of the state of the distributed parameter system (29) may be imposed as well. In order to obtain the desired rates of convergence (33), let us introduce the Lyapunov functional

$$V(t) = \int_0^1 Q_1^T(x, t)WQ_1(x, t)dx \quad (34)$$

where $W = \int_0^\infty \exp\{R^T t\} \exp\{Rt\} dt$ is the positive definite solution of the Lyapunov equation $R^T W + WR = -I$, I is the identity matrix of an appropriate dimension, and find the time derivative of the functional along the trajectories of (29):

$$\begin{aligned} \dot{V}(t) = & -2 \int_0^1 \left[\frac{\partial Q_1}{\partial x} \right]^T(x, t)W \left[\frac{\partial Q_1}{\partial x} \right] dx - \\ & \int_0^1 Q_1^T(x, t)Q_1(x, t)dx. \end{aligned} \quad (35)$$

Denoting by λ_{max} the maximal eigenvalue of the matrix W and bearing in mind that $Q_1^T W Q_1 \leq \lambda_{max} Q_1^T Q_1$, we arrive at

$$\dot{V}(t) \leq -\lambda_{max}^{-1} V(t). \quad (36)$$

By choosing eigenvalues of R with negative real parts sufficiently large in magnitude, the value of λ_{max} may be made as small as desired. Thus the appropriate choice of matrix c that assigns the desired allocation of eigenvalues

of R , ensures the Lyapunov functional convergence $V(t) \rightarrow 0$ as $t \rightarrow \infty$, as well as (33) at desired rates.

In the second step, a discontinuous control is designed to drive the system state to the manifold $S = 0$. We demonstrate that the unit control

$$u(Q) = -\frac{M(Q)S_1(Q)}{\|S_1(Q(\cdot, t))\|_{L_2(0,1)}} \quad (37)$$

where

$$\begin{aligned} S_1(Q) &= (cF)^{-1}S(Q), \quad M(Q) = M_0\|Q(\cdot, t)\|_{L_2(0,1)}, \\ M_0 &= \|(cF)^{-1}cD\| + M_1, \quad M_1 > 0 \end{aligned} \quad (38)$$

guarantees that in the closed-loop system (27), (37) starting from a finite time moment there appears a sliding motion on the manifold $S(Q) = 0$ or equivalently, $S_1(Q) = 0$. Indeed, differentiating the Lyapunov functional

$$V_1(t) = \int_0^1 S_1^T(Q(x, t))S_1(Q(x, t))dx$$

along the trajectories (27), employing integration by parts, applying boundary conditions and utilizing the control law (37), (38) yield

$$\begin{aligned} \frac{1}{2}\dot{V}_1(t) &= \int_0^1 S_1^T \dot{S}_1 dx = \\ &= \int_0^1 S_1^T \left[\frac{\partial^2 S_1}{\partial x^2} + (cF)^{-1}cDQ + u \right] dx = \\ &= - \int_0^1 \left[\frac{\partial S_1}{\partial x} \right]^T \frac{\partial S_1}{\partial x} dx - \int_0^1 S_1^T \left[\frac{M(Q)S_1(Q)}{\|S_1(Q(\cdot, t))\|_{L_2(0,1)}} - \right. \\ &\quad \left. (cF)^{-1}cDQ \right] dx \leq -M_1\sqrt{V_1(t)}. \end{aligned} \quad (39)$$

The solution to the latter inequality has been shown to vanish after the finite time moment $T = V_1(0)/M_1$. Therefore, starting from T , the unit control signal (37) enforces the system motion in the sliding mode along the manifold (28) and $T \rightarrow 0$ as $M_1 \rightarrow \infty$. Thus the unit control approach leads to a decoupling system design and ensures the desired rate of transient decay.

Further on we shall give general conditions which allow us to reduce the infinite-dimensional control problem and use the well-known synthesis procedures for finite-dimensional systems. Such a situation is shown to appear if the undisturbed motion of (10) under $u(x, t) \equiv f(x, t) \equiv 0$ contains two partial components: one of them is stable and doesn't require to be corrected, and another one belongs to a finite-dimensional subspace.

4 Disturbance Rejection in Exponentially Stabilizable System

The aim of this section is to demonstrate how the uncertainties (11), (12) in the exponentially stabilizable system (10) with a finite-dimensional unstable part can be rejected by means of a unit stabilizing controller.

Throughout this section we assume that the pair $\{A, b\}$ is exponentially stabilizable and the spectrum $\sigma(A) = \sigma_1(A) + \sigma_2(A)$ of the infinitesimal operator A consists of two parts: one of them, $\sigma_1(A) = \{\lambda \in \sigma(A) : \operatorname{Re} \lambda \geq 0\}$, is finite-dimensional and another one, $\sigma_2(A) = \{\lambda \in \sigma(A) : \operatorname{Re} \lambda < 0\}$, is in the open left half-plane.

Let P_1 and P_2 be projectors corresponding to the spectral sets $\sigma_1(A), \sigma_2(A)$, respectively, and $H_j = P_j H$, $j = 1, 2$. Then it is well-known (see, e.g., [9, Section 1.5]) that

1. $H = H_1 \oplus H_2$, H_j are invariant with respect to A , i.e., $AH_j \subset H_j$, $j = 1, 2$;
2. the operator $A_1 = A|_{H_1}$ is finite-dimensional, i.e., $H_1 = \mathbf{R}^n$;
3. the operator $A_2 = A|_{H_2}$ generates an exponentially stable semigroup $T_{A_2}(t)$ with some negative growth bound $-\beta$, i.e.,

$$\|T_{A_2}(t)\| \leq \omega e^{-\beta t}, \omega > 0. \quad (40)$$

If the operator A is compact resolvent then its spectrum $\sigma(A) = \{\lambda_i\}_{i=1}^{\infty}$ would be discrete, and for any $\beta > 0$ there would exist a number l such that $\sigma_2(A) = \{\lambda_i\}_{i=l}^{\infty} < -\beta$, and hence the growth bound of the semigroup $T_{A_2}(t)$ could be arbitrarily prescribed.

The above properties of the operator A admit representation of system (23), (24) in the form

$$\begin{aligned} \dot{x}_1 &= A_1 x_1 + P_1 f(x_1, x_2, t) + P_1 b u(x_1, x_2, t), \quad t \geq 0, \\ x_1(0) &= x_1^0, \end{aligned} \quad (41)$$

$$\begin{aligned} \dot{x}_2 &= A_2 x_2 + P_2 f(x_1, x_2, t) + P_2 b u(x_1, x_2, t), \quad t \geq 0, \\ x_2(0) &= x_2^0. \end{aligned} \quad (42)$$

It should be pointed out that by virtue of $P_1 b \in L(U, \mathbf{R}^n)$, the subspace $U_2 = \ker P_1 b = \{u \in U : P_1 b u = 0\}$ has the finite co-dimension l [11]. Hence, there exists a finite-dimensional subspace $U_1 = \mathbf{R}^m$ such that $U = U_1 \oplus \ker P_1 b$, and due to (11) the finite-dimensional subsystem (41) takes the form

$$\dot{x}_1 = A_1 x_1 + B_1 [h_1(x, t) + u_1(x, t)], \quad (43)$$

where the partition $h(x, t) = h_1(x, t) + h_2(x, t)$,

$$u(x, t) = u_1(x, t) + u_2(x, t) \quad (44)$$

of the exogenous signals $h_1(x, t) \in U_1$, $h_2(x, t) \in U_2$, $u_1(x, t) \in U_1$, $u_2(x, t) \in U_2$ is used; $B_1 = P_1 b|_{U_1}$, and the matrix pair $\{A_1, B_1\}$ turns out to be

controllable, because the pair $\{A, b\}$, otherwise, would not be exponentially stabilizable.

The solution to the afore-mentioned rejection problem is based on the deliberate introduction of sliding modes into the closed-loop system. Following the design procedure for controllable finite-dimensional systems proposed in [20, Chapter 10] we select such a discontinuity manifold $Cx_1 = 0$, $C \in R^{m \times l}$, $\det CB_1 \neq 0$ that ensures the exponential stability

$$\|x_1(t)\| \leq \omega \|x_1(T)\| e^{-\alpha t}, \quad t \geq T \quad (45)$$

of the sliding mode which arises, starting from some time moment $T > 0$, in the finite-dimensional system (43) under the control law

$$u_1(x) = -[N(x) + L\|x_1\|] \frac{Cx_1}{\|Cx_1\|} \quad (46)$$

where α, ω, L are positive constants and α may be as large as desired. Formally, in order to specify the matrix C and constant L in an appropriate manner, one should represent system (43) in the canonical form where the choice of C and L is straightforward and particularly given in Section 5. The sliding motion in (43) is then governed by the disturbance-independent equation

$$\dot{x}_1 = [A_1 - B_1(CB_1)^{-1}CA_1]x_1, \quad (47)$$

obtained through the equivalent control method by substituting the continuous solution

$$u_{1eq}(x, t) = -(CB_1)^{-1}CA_1x_1 - h_1(x, t) \quad (48)$$

of the equation $C\dot{x}_1 = 0$ into (43) for u_1 . Equation (42) is respectively rewritten as follows:

$$\begin{aligned} \dot{x}_2 = & A_2x_2 - B_{21}(CB_1)^{-1}CA_1x_1 + \\ & + B_2[u_2(x, t) + h_2(x, t)], \quad t \geq T \end{aligned} \quad (49)$$

where $B_{21} = P_2b|_{U_1}$, $B_2 = P_2b|_{U_2}$. Due to (40), (45), the unforced system (47), (49) under the zero exogenous inputs $u = f = 0$ is exponentially stable, and it remains to employ the results of Section 3.1 to reject the external disturbance $h_2(x, t)$. Setting $W_{A_2} = \int_0^\infty T_{A_2}^*(t)T_{A_2}(t)dt$, we design the second component

$$u_2(x, t) = -N(x) \frac{B_2^*W_{A_2}x_2}{\|B_2^*W_{A_2}x_2\|} \quad (50)$$

of the control $u(x, t)$ in the unit form (40) that imposes the desired robustness property on the closed-loop system.

If the operator A is compact resolvent then, as mentioned earlier, the growth bound $-\beta$ of the semigroup $T_{A_2}(t)$ and consequently, the value of

$\|W_0\|^{-1}$ could be specified arbitrarily large in magnitude. Combining this with (15), (45) would guarantee the desired decay rate of the closed-loop system (41), (42), (46), (50) whenever the pair $\{A, b\}$ is approximately controllable. Summarizing, the following theorem has been proven.

Theorem 1. *Let the unstable part of the unforced dynamics of (10) under $u = f = 0$ be finite-dimensional and let the pair $\{A, B\}$ be exponentially stabilizable. Then the uncertain infinite-dimensional system (10) is exponentially stabilizable by the discontinuous unit controller (44), (46), (50) which imposes the robustness property with respect to admissible perturbations (11), (12) on the closed loop system. Furthermore, if A has compact resolvent and the pair $\{A, b\}$ is approximately controllable, then the decay rate of the closed-loop system may be specified to be as large as desired.*

Example 2: To support the above result by an example let us consider a distributed parameter system described by the parabolic partial differential equation

$$\begin{aligned} \partial Q(y, t) / \partial t &= \partial^2 Q(y, t) / \partial y^2 + b(y)[u(Q, t) + h(Q, t)] \quad t > 0, \\ Q(y, 0) &= Q_0(y), \quad 0 \leq y \leq 1 \end{aligned} \quad (51)$$

with Dirichlet boundary conditions

$$Q(0, t) = Q(1, t) = 0, \quad t \geq 0. \quad (52)$$

The boundary-value problem (52) describes the propagation of heat in a homogeneous one-dimensional rod with fixed temperature at both ends. Here $Q(y, t)$ is the value of the temperature field of the plant at time moment t at point y along the rod, $Q_0(y)$ is a scalar twice continuously differentiable initial distribution which satisfies the boundary conditions (52); $u(Q, t)$ is a scalar control function; $h(Q, t)$ is a scalar unknown disturbance to be rejected, an upper estimate $N(Q) \in C^1$ of which is known a priori; $b(y)$ is a scalar quadratically integrable function, all Fourier coefficients of which are nonzero. Clearly,

$$b(y) = \begin{cases} \frac{1}{d-c} & \text{if } y \in [c, d] \\ 0 & \text{otherwise,} \end{cases}$$

which corresponds to a spatially uniform temperature input over an interval $[c, d] \subset [0, 1]$, satisfies the above assumptions if $\sin \pi ic - \sin \pi id \neq 0$ for all $i = 1, 2, \dots$

It is required to design a feedback control law which imposes the desired decay rate $-\alpha$ as well as robustness with respect to matched disturbances on the closed-loop system.

If along with the operator b of the multiplication by the function $b(y) \in L_2(0, 1)$ we introduce the operator $\mathcal{A} = -\partial^2 / \partial y^2$ of double differentiation with the dense domain

$$\mathcal{D}(\mathcal{A}) = \{\xi(y) \in L_2(0, 1) : \partial^2 \xi(y) / \partial y^2 \in L_2(0, 1), \xi(0) = \xi(1) = 0\},$$

then the boundary-value problem (51), (52) can be rewritten as the differential equation (10) in the Hilbert space $L_2(0, 1)$. The operator $\mathcal{A}$ generates a strongly continuous semigroup and is compact resolvent [2]. Furthermore, the pair $\{\mathcal{A}, b\}$ is approximately controllable by virtue of the assumption on the function $b(y)$. Hence Theorem 1 is applicable to systems (51), (52).

In order to design a unit control-based solution to the stabilization problem stated above, let us select such a number $n > 1$ that $\pi^2(n+1)^2 \geq \alpha$ and decouple the spectrum $\{-(\pi i)^2\}_{i=1}^\infty = \{-(\pi i)^2\}_{i=1}^n + \{-(\pi i)^2\}_{i=n+1}^\infty$ of $\mathcal{A}$ into two parts. Then

$$\begin{aligned} H_1 &= \text{span}\{\sin(\pi i y)\}_{i=1}^n, H_2 = \{\sin(\pi i y)\}_{i=n+1}^\infty, U_1 = \mathbf{R}^1, U_2 = \{0\}, \\ A_1 &= \text{diag}\{-(\pi i)^2\} \in \mathbf{R}^{n \times n}, B_1 = (P^1 b(\cdot), \dots, P^n b(\cdot))^T, \\ P^i b &= 2 \int_0^1 b(y) \sin(\pi i y) dy, \quad i = 1, \dots, n, \\ x_1(t) &= (P^1 Q(\cdot, t), \dots, P^n Q(\cdot, t))^T \in H_2. \end{aligned}$$

By virtue of the special choice of the row $C = (C_1, \dots, C_n)$ and constant L in (46) the desired decay rate (45) and robustness with respect to the matched disturbances are imposed on the closed loop-systems (46), (51), (52) in the finite-dimensional subspace H_1 . Since the internal dynamics in H_2 is of the desired decay rate by construction then due to the triviality of the subspace U_2 the resulting control law (44) consists of the first component (46) only. Thus, the unit controller

$$u(Q) = -[N(Q) + L\sqrt{\sum_{i=1}^n (P^i Q)^2}] \text{sign}(\sum_{i=1}^n C_i P^i Q)$$

gives a solution to the stated stabilization problem.

Remark 1. To extend the above solution to the case when a point-wise action $b(y)u(Q) = \delta(y - y_0)u(Q)$, $y_0 \in (0, 1)$ is under consideration, the state equation (10) should be interpreted in another Hilbert space (e.g., a Sobolev space) where multiplication by the Dirac function is a bounded operator (see [10] for details).

5 Disturbance Rejection in Minimum Phase System

The problems considered in this subsection are to make the output

$$z(t) = (s, x(t)), \quad s \in H \tag{53}$$

of the uncertain system (10) converge to zero as fast as desired and to ascertain conditions which ensure exponential stability of the closed-loop system. For the sake of simplicity, the development is confined to the scalar output, however, the extension to systems with arbitrary finite-dimensional output is straightforward.

Throughout this section we assume that system (10), (44) has a finite relative degree

$$r := \min\{i = 1, 2, \dots : b^* A^{*^{i-1}} s \neq 0\}$$

and $s \in \mathcal{D}(A^{*^r})$. It follows then that

$$\begin{aligned} z_i(t) &:= z^{(i-1)}(t) = (A^{*^{i-1}} s, x(t)), \quad i = 1, 2, \dots, r, \\ z^{(r)}(t) &= (A^{*^r} s, x(t)) + (A^{*^{r-1}} s, f(x(t), t) + bu(x(t), t)). \end{aligned} \quad (54)$$

Partitioning the state space as

$$\begin{aligned} H &= H_1 \oplus H_2, \quad H_1 = \text{span}\{A^{*^{i-1}} s\}_{i=1}^r, \\ H_2 &= \{x \in H : (A^{*^{i-1}} s, x) = 0, \quad i = 1, 2, \dots, r\} \end{aligned}$$

and utilizing (46), let us represent the original system (10), subject to the matched disturbance (11), in terms of $x_1(t) = \sum_{i=1}^r z_i(t) A^{*^{i-1}} s \in H_1$ and $x_2(t) \in H_2$ as

$$\begin{aligned} \dot{z}_1(t) &= z_2(t), \dots, \dot{z}_{r-1}(t) = z_r(t), \\ \dot{z}_r(t) &= (A^{*^r} s, x(t)) + (b^* A^{*^{r-1}} s, u_1(x, t) + h_1(x, t), t), \end{aligned} \quad (55)$$

$$\begin{aligned} \dot{x}_2 &= A_{21}x_1 + A_{22}x_2 + B_{21}(u_1(x, t) + h_1(x, t)) + \\ &\quad B_{22}(u_2(x, t) + h_2(x, t)) \end{aligned} \quad (56)$$

where

$$\begin{aligned} U &= U_1 \oplus U_2, \quad U_1 = \text{span}\{b^* A^{*^{r-1}} s\}, \quad U_2 = \{u \in U : (b^* A^{*^{r-1}} s, u) = 0\}, \\ u_i, h_i &\in U_i, \quad B_{2i} = P_2 b|_{U_i}, \quad A_{2i} = P_2 A|_{H_i}, \quad i = 1, 2, \end{aligned}$$

P_2 is the projector on H_2 . It should be noted that the operator A_{21} , defined everywhere in H_1 (A_{21} is a densely defined operator on the finite-dimensional space H_1 and hence $\mathcal{D}(A_{21}) = H_1$), is bounded.

The solution of the above problem presented here is based on the deliberate introduction of sliding modes in the manifold

$$cx = \sum_{i=1}^r c_i (A^{*^{i-1}} s, x) = 0 \quad (57)$$

where parameters

$$\begin{aligned} c_r &= 1, \quad c_{r-1} = -\sum_{i=1}^{r-1} \mu_i, \quad c_{r-2} = \sum_{i < k} \mu_i \mu_k, \dots, \\ c_1 &= (-1)^{r-1} \prod_{i=1}^{r-1} \mu_i \end{aligned}$$

are specified to place the roots of the characteristic polynomial of the equation

$$\sum_{i=1}^r c_i z_i(t) = \sum_{i=1}^r c_i z^{(i-1)}(t) = 0 \quad (58)$$

in the open left-half plane at the desired locations μ_i , $i = 1, \dots, r-1$. We demonstrate that the discontinuous unit control law

$$u_1(x) = -M(x) \frac{b^* A^{*^{r-1}} s}{\|b^* A^{*^{r-1}} s\|} \text{sign}(cx) \quad (59)$$

with

$$M(x) = \gamma + N(x) + \frac{\|A^{*r} s\|}{\|b^* A^{*r-1} s\|} \|x(t)\| + \frac{\sum_{i=1}^{r-1} |c_i| |(A^{*i} s, x)|}{\|b^* A^{*r-1} s\|}, \quad \gamma > 0 \quad (60)$$

drives system (10) to the discontinuity manifold (57) within a finite time interval. Indeed, differentiating the functional $V(t) = \frac{1}{2}(cx(t))^2$ along the trajectories of (55) and utilizing (12), (55), (49), (51), we obtain

$$\begin{aligned} \dot{V}(t) &= cx(t)c\dot{x}(t) = cx(t)\{\sum_{i=1}^{r-1} c_i z_{i+1}(t) + \\ &< A^{*r} s, x(t) > + < b^* A^{*r-1} s, h_1(x(t), t) > - \\ &\|b^* A^{*r-1} s\| M(x) \operatorname{sign}(cx(t))\} \leq -2\gamma\sqrt{V(t)}, \end{aligned}$$

that gives rise to (47) for $t \in [T, \infty)$ where $T = \gamma^{-1}\sqrt{V(0)}$. In order to reproduce this conclusion, one should note that for all $t \geq 0$ arbitrary solutions $V(t)$ to the latter inequality is majored $V(t) \leq V_0(t)$ by the solution to the differential equation $\dot{V}_0(t) = -2\gamma\sqrt{V_0(t)}$, initialized with the same initial condition $V_0(0) = V(0)$. Since $V_0(0) = 0$ for all $t \geq T$, $V(t)$ vanishes after the finite time moment T .

Thus, starting from the time moment $T = \gamma^{-1}\sqrt{V(0)}$, in the finite-dimensional system (55), driven by the unit control signal (49), there appears the sliding mode (47), which results in the desired decay rate $-\alpha = \max_{1 \leq i \leq r} \operatorname{Re} \mu_i$ of the variable $x_1(t) = \sum_{i=1}^r z_i(t) A^{*i-1} s \in H_1$:

$$\|x_1(t)\| \leq \omega \|x_1(T)\| e^{-\alpha t}, \quad t \geq T, \quad \omega = \text{const}. \quad (61)$$

In order to derive the sliding mode equation (47) one needs to substitute the continuous solution

$$u_{1eq}(x, t) = -\frac{\sum_{i=1}^r c_i z_{i+1} + (A^{*r} s, x)}{\|b^* A^{*r-1} s\|} - h_1(x, t)$$

of the equation $c\dot{x}(t) = 0$ into (55) for u_1 . By the same substitution equation (56) is rewritten as follows:

$$\begin{aligned} \dot{x}_2 &= \tilde{A}x_2 + A_{21}x_1 - \frac{B_{21}[\sum_{i=1}^r c_i z_{i+1} + (A^{*r} s, x_1)]}{\|b^* A^{*r-1} s\|} + \\ &B_{22}[u_2(x, t) + h_2(x, t)], \quad t \geq T \end{aligned} \quad (62)$$

where $\tilde{A}x_2 = A_{22}x_2 - \frac{B_{21}(A^{*r} s, x_2)}{\|b^* A^{*r-1} s\|}$.

If the operator $\tilde{A}$ generates an exponentially stable semigroup then due to (61) and boundedness of A_{21} , the second control component

$$u_2(x, t) = -N(x) \frac{B_{22}^* \int_0^\infty T_{\tilde{A}}^*(t) T_{\tilde{A}}(t) dt x_2}{\|B_{22}^* \int_0^\infty T_{\tilde{A}}^*(t) T_{\tilde{A}}(t) dt x_2\|}, \quad (63)$$

similar to (50), rejects the external disturbance $h_2(x, t)$ and ensures the exponential stability of the closed-loop system with the same line of reasoning as in Section 4. Apparently, $\tilde{A}$ generates an exponentially stable semigroup iff the input-output system (10), (44) is exponentially minimum phase.

Thus, the following result has been shown.

Theorem 2. *Let $s \in \mathcal{D}(A^{*r})$ and let system (10), (44) be exponentially minimum phase and of the finite relative degree r . Then the uncertain system (10) is exponentially stabilizable by the composition $u(x) = u_1(x) + u_2(x)$ of the unit controllers (49), (63) and the closed-loop system is robust with respect to external disturbances (11), (12).*

We conclude this section with two control problems for heat processes and distributed mechanical oscillators which demonstrate the constructive utilities of the above theorem.

Example 3: Let us modify Example 2 and replace the boundary conditions (52) by the appropriate Neumann boundary conditions

$$\partial Q(0, t)/\partial y = \partial Q(1, t)/\partial y = 0, \quad t \geq 0. \quad (64)$$

These conditions appear to describe the propagation of heat in a one-dimensional rod, insulated at both ends. The Fourier coefficients of the function $b(y)$ are no longer assumed to be nonzero, with the only exception being $\int_0^1 b(y)dy$. The operator $\mathcal{A} = -\partial^2/\partial y^2$ of double differentiation is now defined in $\mathcal{D}(\mathcal{A}) = \{\xi(y) \in L_2(0, 1) : \partial^2 \xi(y)/\partial y^2 \in L_2(0, 1), \partial \xi(0)/\partial y = \partial \xi(1)/\partial y = 0\}$ and the boundary-value problem (51), (64) is still represented as the differential equation (10) in the Hilbert space $L_2(0, 1)$. Since the spectrum $\sigma(\mathcal{A}) = \{-(\pi j)^2\}_{j=0}^\infty$ of $\mathcal{A}$ contains zero eigenvalues the unforced system (51), (64) under $u = h = 0$ is not asymptotically stable.

Specifying the system output (44) as the average temperature

$$z(t) = \int_0^1 Q(y, t)dy \quad (65)$$

of the plant, one can check that the input-output system (51), (64), (65) satisfies all the assumptions of Theorem 4. Indeed, $s = 1$ and hence $s \in \mathcal{D}(\mathcal{A}^l)$ for the self-adjoint operator $\mathcal{A}$ and arbitrary integer l . Furthermore, differentiating (65) with respect to t along the solutions of (51), employing integration by parts and applying the boundary conditions (64) yields

$$\dot{z}(t) = (u + h) \int_0^1 b(y)dy$$

with $\int_0^1 b(y)dy \neq 0$, which proves that system (10), (44) is of the relative degree $r = 1$. Finally, representing the solution

$$Q(y, t) = \int_0^1 G(y, \xi, t)Q_0(\xi)d\xi + \int_0^t \int_0^1 G(y, \xi, t - \tau)b(\xi)d\xi[u(Q, \tau) + h(Q, \tau)]d\tau$$

of the Neumann boundary-value problem (51), (64) via the Green function

$$G(y, \xi, t) = \sum_{j=0}^{\infty} \exp\{-(\pi j)^2 t\} \cos \pi j y \cos \pi j \xi,$$

one can show the exponential stability of the zero dynamics

$$Q_z(y, t) = \sum_{i=1}^{\infty} \int_0^1 \cos(\pi i \xi) Q_0(\xi) d\xi \exp\{-(\pi i)^2 t\} \cos(\pi i y)$$

of (51), (64), (65), written under appropriate initial conditions such that $\int_0^1 Q(y, 0) dy = 0$, and the suitable control signal $u(Q, t) = -h(Q, t)$ produces the system output identically zero.

Thus, Theorem 2 is applicable to systems (51), (64), (65). According to the theorem, the controller

$$u(Q) = -\frac{N(Q)}{\int_0^1 b(y) dy} \operatorname{sign} z(t), \quad (66)$$

imposes a sliding mode along the manifold $z = 0$ so that the closed-loop system is exponentially stable and robust to the matched disturbances.

Remark 2. If the output of the system is replaced by the average temperature over an interval $[c, d] \subset [0, 1]$ such that $\int_c^d b(y) dy \neq 0$, i.e., the output is given as

$$z_1(t) = \frac{1}{d-c} \int_c^d Q(y, t) dy, \quad (67)$$

Theorem 2 is still applicable to the input-output system (51), (64), (67). Indeed, the assumption $s \in \mathcal{D}(A^{*r})$ is now satisfied in the Hilbert space $L_2(c, d)$ rather than in $L_2(0, 1)$ (although such a modification of Theorem 2 requires a separate justification, however, the line of reasoning used in the proof of Theorem 2 applies here as well). Moreover, the exponential stability of the zero dynamics $Q_{z_1}(y, t)$, which is now governed by

$$\begin{aligned} \partial Q_{z_1}(y, t) / \partial t &= \partial^2 Q_{z_1}(y, t) / \partial y^2 + \partial Q_{z_1}(c, t) / \partial y - \\ &- \partial Q_{z_1}(d, t) / \partial y, \quad \partial Q_{z_1}(0, t) / \partial y = \partial Q_{z_1}(1, t) / \partial y = 0, \end{aligned} \quad (68)$$

is straightforwardly shown through the mode representation

$$Q_{z_1}(y, t) = \sum_{j=0}^{\infty} q_j(t) \cos \pi j y$$

where $q_i(t)$, $i = 1, 2, \dots$ satisfy $\dot{q}_i(t) = -(\pi i)^2 q_i(t)$, whereas

$$q_0(t) = -\sum_{i=1}^{\infty} \frac{\sin(\pi i d) - \sin(\pi i c)}{\pi i (d - c)} q_i(t)$$

by virtue of $z_1(t) = 0$. Finally, differentiating (67) yields $\dot{z}_1(t) = Q'(d, t) - Q'(c, t) + (u + h) \int_c^d b(y) dy$, thereby proving that (51), (64), (67) is of the

relative degree $r = 1$. Thus, in accordance with Theorem 2, the control law

$$u(Q) = -\frac{N_0(Q)}{\int_c^d b(y)dy} \text{sign } z_1(t),$$

$$N_0(Q) > N(Q) + |Q'(d, \cdot) - Q'(c, \cdot)|, \quad (69)$$

which ensures a sliding mode along the manifold $z_1 = 0$, also exponentially stabilizes the heat process (51), (64).

Remark 3. It is plausible that the output feedback

$$u(z) = -\frac{M}{\int_0^1 b(y)dy} \text{sign } z(t)$$

with a sufficiently large constant $M > 0$ still drives the system to the discontinuity manifold $z = 0$ and consequently, imposes the desired dynamic properties as well as robustness with respect to the matched disturbances on the closed-loop system. Note that the same target can also be achieved by employing a lumped, e.g., point-wise actuator (see, Remark 1).

Example 4: Let a distributed parameter system be governed by the hyperbolic partial differential equation

$$\begin{aligned} \partial^2 Q / \partial t^2 &= \partial^2 Q / \partial y^2 - 2\partial Q / \partial t + \\ &+ b(y)[u(Q, t) + h(Q, t)], \quad 0 < y < 1, t > 0, \\ Q(y, 0) &= Q_0(y), \quad \partial Q(y, 0) / \partial t = Q_1(y), \quad 0 \leq y \leq 1, \end{aligned} \quad (70)$$

subject to the Neumann boundary conditions (64). The boundary-value problem (64), (70) describes the oscillations of a homogeneous string, insulated at both ends, where the state vector consists of the deflection $Q(y, t)$ of the string and its velocity $\dot{Q}(y, t)$ at time moment $t \geq 0$ and location y along the string. The initial distributions $Q_0(y), Q_1(y)$ are twice continuously differentiable functions which satisfy the boundary conditions (64); b , and u , and h as well as the operator $\mathcal{A}$ and the output z , utilized below, are the same as in Example 3.

If we introduce the operator

$$\tilde{\mathcal{A}} = \begin{bmatrix} 0 & I \\ \mathcal{A} & -2 \end{bmatrix}$$

then the boundary-value problem (64), (70) can be represented as the differential equation (10) in the Hilbert space $L_2(0, 1) \oplus L_2(0, 1)$. The operator $\tilde{\mathcal{A}}$ generates a strongly continuous semigroup (see, e.g., [2]), however, the spectrum $\sigma(\tilde{\mathcal{A}}) = \{-(\pi j)^2\}_{j=0}^\infty$ of $\tilde{\mathcal{A}}$ contains zero eigenvalues and therefore the unforced system (64), (70) under $u = h = 0$ is not asymptotically stable.

Verification of the assumptions of Theorem 2 is similar to that of Example 3, except that the input-output system (64), (65), (70) is of the relative degree $r = 2$, since $\dot{z}(t) = \int_0^1 \dot{Q}(y, t)dy$,

$$\ddot{z}(t) = (u + h) \int_0^1 b(y)dy$$

where $\int_0^1 b(y)dy \neq 0$. Thus, by applying Theorem 2 to (64), (65), (70), the control law

$$u(Q) = -\frac{N_1(Q)}{\int_0^1 b(y)dy} \text{sign} \{ \dot{z}(t) + z(t) \},$$

$$N_1(Q) > N(Q) + |\dot{z}(\cdot)|, \quad (71)$$

which imposes a sliding mode along the manifold $\dot{z} + z = 0$ (thereby yielding $z(t) \rightarrow 0$ as $t \rightarrow \infty$), makes the closed-loop system (65), (64), (70), (71) exponentially stable and robust to the matched disturbances.

Remarks 1-3 remain in force for the stabilization of the distributed oscillator (64), (70) as well.

6 Conclusions

Discontinuous control laws are developed for dynamic systems driven in a Hilbert space. These control laws impose the desired dynamic properties on the closed-loop system while retaining the disturbance rejection and robustness features similar to those possessed by their counterpart in the finite-dimensional case. The control algorithms proposed do not represent simple extensions of the finite-dimensional control laws to the infinite-dimensional case because the general constructions, quite natural for the finite-dimensional systems, become invalid for the infinite-dimensional systems due to the presence of the unbounded operator in the plant equation. The sliding mode control theory developed for infinite-dimensional systems is illustrated by applications to heat processes and distributed oscillators.

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Sliding Modes Applications in Power Electronics and Electrical Drives

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Abstract. Control system design of switching power converters and electrical machines based on the sliding mode approach is presented. The structural similarities among switching converters and electrical machines are used to show that the same structure of the controller could be used for plants under consideration. The controller is designed as a cascade structure with inner current loop designed as a sliding mode system with discontinuous control and outer loop (voltage or mechanical motion) being designed as a discrete-time sliding mode controller.

1 Introduction

The aim of this paper is to present an application of sliding mode control in switching power converters and electrical drives. Our intention is to show that, due to the structural similarities, switching power converters and electrical machines could be analysed in the same framework and that the structure of the control system is the same for both plants. The basis for our approach is the analysis of switching converters and electrical machines as the set of energy storage elements with their interconnections dynamically changed by the operation of the switching matrix [2]. The switching matrix plays the role of a control element determining the power exchange between energy storing elements, introducing change in the structure of the system and, thus making design in the framework of variable structure systems and sliding mode control [1] a natural choice. Engineering methods rather than a historical overview of published results will be presented.

A functional description of switching power converters and electrical machines is presented in section 2. As a result of this analysis mathematical description that treats both converters and machines is devised and a formulation of the converters and electrical drive control, in the framework of VSS is derived. In the third section some results of VSS theory that are used in this paper are reviewed and the control algorithms common in switching converters and electrical machines control are discussed. In the same section design of voltage and power flow control for power converters and design of the motion control of electrical machines is presented. In the fourth section the design of IM observer is discussed in details. The last section presents

experimental example of the neural network realisation of the sliding mode system for induction machine control.

2 Functional description of switching power converters and electrical machines

The role of the power converter is to modulate electrical power flow between power sources (Fig. 1). In general power flow can be bi-directional so both sources may play the role of power generator or power sink (load). The converter should enable that interaction of any input source to any output source of the power systems. It acts as a link having matrix-like structure. Efficient modulation of power flow is realised using switch-like elements having zero voltage drop when conducting and fully blocks current flow when open. Use of switches as structural elements of a converter offers opportunity to consider a converter as switching matrix. Disregarding the wide variety of designs in most switching converters, control of power flow is accomplished by varying the length of time intervals for which one or more energy storage elements are connected to or disconnected from the energy sources. Due to the restriction imposed by Kirchoff's circuit laws the nature of sources at the input and output sides of the switching matrix must be different (voltage or current sources)[2].

For purposes of mathematical modelling the operation of a switch may be described by a two-valued variable $u_{ik}(t)$, ($i = 1, \dots, n; k = 1, \dots, m$), having value 0 when the switch is open and value 1 when the switch is closed with average value $0 \leq \tilde{u}_{ik}(t) \leq 1$. If voltage sources are connected to input side of the switching matrix then restrictions imposed due to Kirchoff's circuit laws allow only one switch connecting one of the n input lines to the k -th ($k = 1, \dots, m$) output line can be closed during any time interval, or mathematically $\sum_{i=1}^n u_{ik}(t) = 1, k = 1, \dots, m$. Allowed connections will be referred as permissible combinations.

An analogous requirement can be derived if the current source is attached at the input side of the switching matrix. The operation of the switching matrix changes the connections among elements of the switching converter

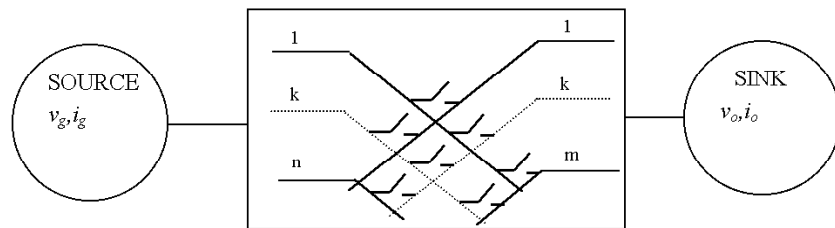


Fig. 1. Converter as a connection between power sources in a switching matrix connecting n -dimensional input and m -dimensional output.

and introduces variation in the dynamical structure of the system. Since the role of switching matrix is to control power flow, the most natural way to model the matrix is by introducing a control vector that will represent the effect of the matrix operation. Let the state of the switches be defined by the vector $\mathbf{s}_{sw}^T = [u_{11} \dots u_{ik} \dots u_{nm}]$ whose elements are two-valued variables $u_{ik}(t)$, ($i = 1, \dots, n; k = 1, \dots, m$) describing the state of switches in each node of the switching matrix [3],[4]. The topological connection of the switches in the switching matrix is defined by the matrix $\mathbf{A}_M$ with elements being from the discrete set $S_3 = \{1, 0, -1\}$. The topological connection of the load with respect to the switching matrix defines the relations of the variables at the output lines of the switching matrix to the load quantities and could be defined by matrix $\mathbf{A}_L$. The operation of the switching matrix can be expressed by vector $\mathbf{u} = \mathbf{A}_M \mathbf{s}_{sw}$. Vector $\mathbf{u}$ has a number of distinctive values equal to the number of permissible switch connection.

2.1 Common converters and their operational properties

In Table 1 topological structure of the most common converters having voltage or current sources on input or output sides are depicted with transformation of variables. The simplest matrix - representing DC-to-DC converters - has only two switches interconnecting two unipolar sources. If unipolar and bipolar sources are to be interconnected then the switching matrix must have at least four switches. If one of the sources to be interconnected is three-phase then the structure has three output or three input lines, depending on the position of the AC source. These structures are representing three phase inverters (DC source at input side) and rectifiers (three phase source at input side).

The role of energy storing elements (L,C) is to balance power flow between source and sink by temporarily storage and release of energy [2]. The dynamics of converters depends on the topological relation of the energy storage elements to the switching matrix. Further analysis will be concentrate on two generic structures - both with inductance energy transfer. In the first - so called buck structure - inductance is connected to the output of the switching matrix, energy flow from the source is pulsating being modulated by the switching matrix. In the other - so called boost structure - inductance is connected to the source and the energy flow from the source is continuous while the switching matrix is modulating discontinuous energy flow to the output side.

Dynamics of DC-to-DC converters. Buck and boost structures of DC-to-DC power converters are shown in Table 2 along with their mathematical models. In the buck structure the dynamical structure of the system remains the same - an LC filter is connected to the variable source. In the case of the boost converter the dynamical structure is changed depending on the

Table 1. Topological structure of the most common switching matrices and their functional characteristics.

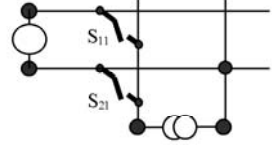
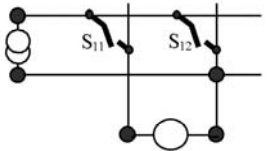
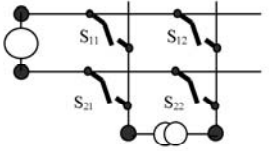
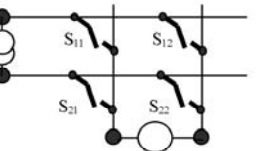
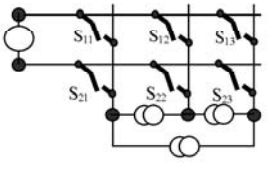
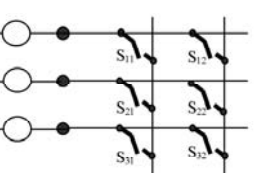
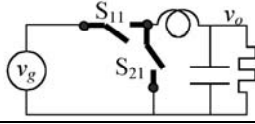
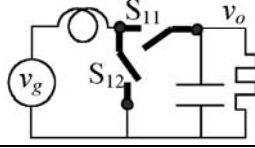
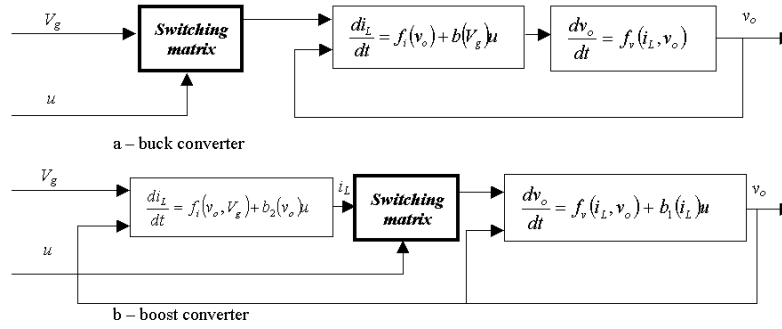
DC-to-DC converters		
Buck structure	Common relations	Boost structure
 $v_s = v_g \mathbf{A}_L \mathbf{u}$ $\mathbf{u} = \mathbf{A}_M \mathbf{s}_{sw}$	$\mathbf{A}_L = [1]$ $\mathbf{A}_M = [1 \ 0]$ $\mathbf{s}_{sw} = [u_{11} \ u_{21}]^T$	 $i_s = i_g \mathbf{A}_L \mathbf{u}$ $\mathbf{u} = \mathbf{A}_M \mathbf{s}_{sw}$
DC-to-AC and AC-to-DC single phase converter		
Buck structure	Common relations	Boost structure
 $v_s = \frac{1}{2} v_g \mathbf{A}_L \mathbf{u}$ $\mathbf{u} = \mathbf{A}_M \mathbf{s}_{sw}$	$\mathbf{A}_L = [1 \ -1]$ $\mathbf{A}_M = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$ $\mathbf{s}_{sw} = [u_{11} \ u_{12} \ u_{21} \ u_{22}]^T$	 $i_s = i_g \mathbf{A}_L \mathbf{u}$ $\mathbf{u} = \mathbf{A}_M \mathbf{s}_{sw}$
Three phase converters		
Inverters	Common relations	Rectifiers
 $v_s = \frac{1}{2} v_g \mathbf{A}_L \mathbf{u}$ $\mathbf{u} = \mathbf{A}_M \mathbf{s}_{sw}$ $\mathbf{A}_L = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}$ $\mathbf{A}_M = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{bmatrix}$	$\mathbf{s}_{sw} = \begin{bmatrix} u_{11} \\ u_{12} \\ u_{13} \\ u_{21} \\ u_{22} \\ u_{23} \end{bmatrix}$ OR $\mathbf{s}_{sw} = \begin{bmatrix} u_{11} \\ u_{12} \\ u_{2'1} \\ u_{22} \\ u_{31} \\ u_{32} \end{bmatrix}$	 $v_s = \mathbf{v}_g^T \mathbf{A}_L \mathbf{u}$ $\mathbf{u} = \mathbf{A}_M \mathbf{s}_{sw}$ $\mathbf{A}_L = \mathbf{E}$ $\mathbf{A}_M = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}$

Table 2. Structures and mathematical models of DC-to-DC converters.

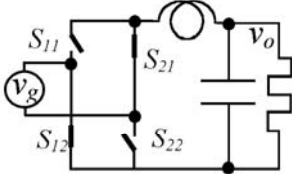
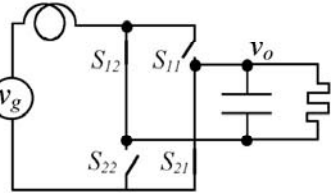
Converter substructure	Converter mathematical model	
Buck structure 	$\frac{dv_o}{dt} = \frac{i_L}{C} - \frac{v_o}{RC}$ $\frac{di_L}{dt} = \frac{V_g}{L}u - \frac{v_o}{L}$ $u = \mathbf{A}_M \mathbf{s}_{sw}$	$\frac{dv_o}{dt} = f_v(i_L, v_o)$ $\frac{di_L}{dt} = f_i(v_o) + b(V_g)u$
Boost structure 	$\frac{dv_o}{dt} = \frac{i_L}{C}(1-u) - \frac{v_o}{RC}$ $\frac{di_L}{dt} = \frac{V_g}{L} - \frac{v_o}{L}(1-u)$ $u = \mathbf{A}_M \mathbf{s}_{sw}$	$\frac{dv_o}{dt} = f_v(i_L, v_o) + b_1(i_L)u$ $\frac{di_L}{dt} = f_i(v_o, V_g) + b_2(v_o)u$

switch position having either two isolated power storage elements (L and C) or a power filter (L,C) connected to the power source. These facts are reflected in the mathematical description of the converters: buck structures being represented in regular form [5] (system is split in the blocks so the first block has the same dimension as control and the second block does not explicitly depend on control input) (see Fig. 2.a). Boost converter have control entering both equations as depicted in Fig. 2.b. These features are common for converters with depicted position of the switching matrix independent on the number of input and output lines of the switching matrix.

**Fig. 2.** Dynamical structure of: a) buck converters and b) boost converters.

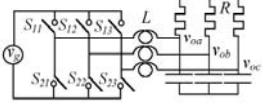
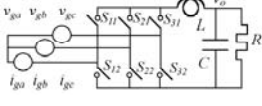
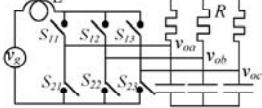
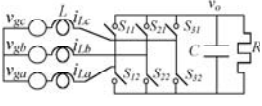
Dynamics of single phase DC-to-AC and AC-to-DC converters. The dynamics of converters with alternative voltage or current sources connected to either input or output side of the switching matrix is the same as for DC-to-DC converters as illustrated in Table 3. The only difference is in the values that control could take being from discrete set $S3 = \{-1, 0, 1\}$ for these converters and from discrete set $S2 = \{0, 1\}$ for DC-to-DC converters. This allows to regard a DC-to-AC converter as a structure in which the load is connected between two DC-to-DC converters.

Table 3. Structures and mathematical models of single-phase converters.

Converter substructure	Converter mathematical model
	$\frac{dv_o}{dt} = \frac{i_L}{C} - \frac{v_o}{RC}$ $\frac{di_L}{dt} = \frac{V_g}{L}u - \frac{v_o}{L}$ $u = \mathbf{A}_M \mathbf{s}_{sw}$
	$\frac{dv_o}{dt} = \frac{i_L}{C}u - \frac{v_o}{RC}$ $\frac{di_L}{dt} = \frac{V_g}{L} - \frac{v_o}{L}u$ $u = \mathbf{A}_M \mathbf{s}_{sw}$

Dynamics of three phase converters. The switching matrix for all three phase converters DC-to-AC (inverters) and AC-to-DC (rectifiers) is the same. Buck and boost structures for both inverters and rectifiers could easily be recognised for three phase converters as clearly shown in Table 4. In our analysis balanced three phase systems is assumed which can be described in different frames of references: stationary three-phase (a, b, c) , orthogonal two-phase (α, β) and synchronous frame of references (d, q) . Mapping between these frames of references is defined by matrix $\mathbf{A}_{abc}^{\alpha\beta}$ for (a, b, c) to (α, β) and for $\mathbf{A}_{\alpha\beta}^{dq}$ to (d, q) . In Table 4 mathematical models are presented in a synchronous frame of references with θ_r as angular position of the selected orthogonal frame of references. Matrix $\mathbf{F}(\theta_r) = \mathbf{A}_{\alpha\beta}^{dq} \mathbf{A}_{abc}^{\alpha\beta}$ is defining the nonlinear transformation between three phase (a, b, c) and synchronous orthogonal (d, q) frames of references. The (d, q) frame of references is determined in such a way that it is synchronous with the three-phase side of a converter (input side for rectifiers and output side for inverters). In the presented models notation is used as follows: $\mathbf{v}_o^T = [v_{od} \ v_{oq}]$ the capacitance

Table 4. Structures and mathematical models of three-phase converters.

Converter substructure and control	Converter dynamics
 $\mathbf{A}_{\alpha\beta}^{dq} = \begin{bmatrix} \cos \theta_r & \sin \theta_r \\ -\sin \theta_r & \cos \theta_r \end{bmatrix}$ $\mathbf{A}_{abc}^{\alpha\beta} = \begin{bmatrix} 1 & -1/2 & -1/2 \\ 0 & \sqrt{3}/2 & -\sqrt{3}/2 \end{bmatrix}$ $\begin{bmatrix} \frac{dv_{od}}{dt} \\ \frac{dv_{oq}}{dt} \end{bmatrix} = \begin{bmatrix} -\frac{v_{od}}{RC} + \omega_r v_{oq} \\ -\frac{v_{oq}}{RC} - \omega_r v_{od} \end{bmatrix} + \frac{1}{C} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} i_{Ld} \\ i_{Lq} \end{bmatrix}$ $\begin{bmatrix} \frac{di_{Ld}}{dt} \\ \frac{di_{Lq}}{dt} \end{bmatrix} = \begin{bmatrix} -\frac{v_{od}}{L} + \omega_r i_{Lq} \\ -\frac{v_{oq}}{L} - \omega_r i_{Ld} \end{bmatrix} + \frac{V_g}{2L} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_d \\ u_q \end{bmatrix}$ $\mathbf{F}(\theta_r) = \mathbf{A}_{\alpha\beta}^{dq} \mathbf{A}_{abc}^{\alpha\beta}$ $\mathbf{u}_{dq} = \mathbf{F}(\theta_r) \mathbf{A}_M \mathbf{s}_{sw}$	
 $\frac{dv_{od}}{dt} = -\frac{v_o}{RC} + \frac{i_L}{C}$ $\begin{bmatrix} \frac{di_L}{dt} \\ i_{gq} \end{bmatrix} = \begin{bmatrix} -\frac{v_o}{L} \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{V_g}{L} & 0 \\ 0 & i_L \end{bmatrix} \begin{bmatrix} u_d \\ u_q \end{bmatrix}$ $\mathbf{F}(\theta_r) = \mathbf{A}_{\alpha\beta}^{dq} \mathbf{A}_{abc}^{\alpha\beta}$ $\mathbf{u}_{dq} = \mathbf{F}(\theta_r) \mathbf{A}_M \mathbf{s}_{sw}$	
 $\mathbf{F}(\theta_r) = \mathbf{A}_{\alpha\beta}^{dq} \mathbf{A}_{abc}^{\alpha\beta}$ $\mathbf{u}_{dq} = \mathbf{F}(\theta_r) \mathbf{A}_M \mathbf{s}_{sw}$ $\begin{bmatrix} \frac{dv_{od}}{dt} \\ \frac{dv_{oq}}{dt} \end{bmatrix} = \begin{bmatrix} -\frac{v_{od}}{RC} + \omega_r v_{oq} \\ -\frac{v_{oq}}{RC} - \omega_r v_{od} \end{bmatrix} + \frac{i_L}{C} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_d \\ u_q \end{bmatrix}$ $\frac{di_L}{dt} = -\frac{v_d}{L} u_d - \frac{v_q}{L} u_q + \frac{V_g}{L}$	
 $\mathbf{F}(\theta_r) = \mathbf{A}_{\alpha\beta}^{dq} \mathbf{A}_{abc}^{\alpha\beta}$ $\mathbf{u}_{dq} = \mathbf{F}(\theta_r) \mathbf{A}_M \mathbf{s}_{sw}$ $\frac{dv_{od}}{dt} = -\frac{v_o}{RC} + \frac{i_{Ld} u_d + i_{Lq} u_q}{2C}$ $\begin{bmatrix} \frac{di_{Ld}}{dt} \\ \frac{di_{Lq}}{dt} \end{bmatrix} = \begin{bmatrix} \omega_r i_{Lq} + \frac{V_g}{L} \\ -\omega_r i_{Ld} \end{bmatrix} - \frac{v_o}{2L} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_d \\ u_q \end{bmatrix}$	

voltage vector, $\mathbf{i}_L^T = [i_{Ld} \ i_{Lq}]$ inductor current vector and $\mathbf{u}^T = [u_d \ u_q]$ is the control vector, V_g is amplitude of input voltage, R, L, C - converter parameters.

The above analysis shows that switching converters can be described as shown in equations (1)-(3) where for particular converter functions $\mathbf{f}_c(v_o, i_L)$, $\mathbf{f}_i(v_o, i_L)$, matrices $\mathbf{B}_c(i_L, V_g)$, $\mathbf{B}_u(i_L, V_g)$ and control $\mathbf{u}$ could be determined from the above tables. The DC-to-DC and DC-to-AC single phase converters are SISO systems while three-phase converters are MIMO systems with the three dimensional control vector. The relation (3) is shown for the purpose of having complete definition of the control input in the (d, q) frame of references.

Buck structure	Boost structure	
$\frac{dv_o}{dt} = \mathbf{f}_c(v_o, i_L)$ $\frac{di_L}{dt} = \mathbf{f}_i(v_o, i_L) + \mathbf{B}_u(i_L, V_g)\mathbf{u}$	$\frac{dv_o}{dt} = \mathbf{f}_c(v_o, i_L) + \mathbf{B}_c(i_L, V_g)\mathbf{u}$ $\frac{di_L}{dt} = \mathbf{f}_i(v_o, i_L) + \mathbf{B}_u(i_L, V_g)\mathbf{u}$	(1)
$\mathbf{u}_{sw} = \mathbf{A}_M \mathbf{s}_{sw} = \mathbf{u}$		(2)
$\mathbf{F}(\theta_r) = \mathbf{A}_{\alpha\beta}^{dq} \mathbf{A}_{abc}^{\alpha\beta}, \quad \mathbf{u}_{dq} = \mathbf{F}(\theta_r) \mathbf{A}_M \mathbf{s}_{sw}$ $\mathbf{A}_{\alpha\beta}^{dq} = \begin{bmatrix} \cos \theta_r & \sin \theta_r \\ -\sin \theta_r & \cos \theta_r \end{bmatrix}, \quad \mathbf{A}_{abc}^{\alpha\beta} = \begin{bmatrix} 1 & -1/2 & -1/2 \\ 0 & \sqrt{3}/2 & -\sqrt{3}/2 \end{bmatrix}$		(3)

Dynamics of electrical machines. To realise necessary power flow a switching converter (or at least a switching matrix) must be inserted between the source and a machine. Since the electrical subsystem of machines are modelled as predominantly inductive, the voltage source shall be used at the source side of the switching matrix. Mathematical models of the most common machines connected to a switching matrix are shown in Table 5. Mechanical motion is the same for all rotating machines and is described as a second order system with electromagnetic torque as the input. The electromagnetic subsystem depends on the magnetic circuitry of the particular machine and can be described by a simple first order equation for DC machines, by a second order system for PM machines and by a fourth order system for induction machines. Descriptions of the electromagnetic subsystem of a machine in Table 5 are presented in (d, q) synchronous orthogonal frame of references with orientation for all of the machines under consideration along the rotor flux vector (so called field orientation). The notation is

Table 5. Structure and mathematical models of common electrical machines

Dynamics of electromagnetic system
DC machine supplied from DC source
$\frac{di_q}{dt} = -\frac{R}{L}i_q - \frac{K_e}{L}\omega + \frac{V_g}{L}u_q, \quad u = \mathbf{A}_M \mathbf{s}_{sw}$
DC machine supplied from three-phase source
$\frac{di_q}{dt} = -\frac{R}{L}i_q - \frac{K_e}{L}\omega + \frac{V_g}{L}u_q, \quad \mathbf{u} = \mathbf{F}(\theta_r) \mathbf{A}_M \mathbf{s}_{sw}$
$i_q(\text{machine}) = i_d(\text{converter}), \quad i_q(\text{converter}) = i_d(\text{machine})(\text{power factor})$
PM three-phase machine supplied from DC source
$\begin{bmatrix} \frac{di_d}{dt} \\ \frac{di_q}{dt} \end{bmatrix} = \begin{bmatrix} -\frac{R}{L}i_d - \omega i_q \\ -\frac{R}{L}i_q - \omega i_d - \frac{K_e}{L}\omega \end{bmatrix} + \begin{bmatrix} \frac{1}{L} & 0 \\ 0 & \frac{1}{L} \end{bmatrix} \begin{bmatrix} u_d \\ u_q \end{bmatrix}, \quad \mathbf{u} = \mathbf{F}(\theta_r) \mathbf{A}_M \mathbf{s}_{sw}$
Induction machine supplied from DC source
$\begin{bmatrix} \frac{di_d}{dt} \\ \frac{di_q}{dt} \end{bmatrix} = \begin{bmatrix} -a_1 i_d + a_3 \omega_r \Psi_q - a_2 \Psi_d \\ -a_1 i_q - a_3 \omega_r \Psi_d + a_2 \Psi_q \end{bmatrix} + \begin{bmatrix} K_{id} & 0 \\ 0 & K_{idq} \end{bmatrix} \begin{bmatrix} u_d \\ u_q \end{bmatrix}; \quad \mathbf{u} = \mathbf{F}(\theta_r) \mathbf{A}_M \mathbf{s}_{sw}$
$\begin{bmatrix} \frac{d\Psi_d}{dt} \\ \frac{d\Psi_q}{dt} \end{bmatrix} = \begin{bmatrix} -\omega \Psi_q - a_4 \Psi_d \\ a_4 \Psi_d \quad \omega \Psi_d \end{bmatrix} + \begin{bmatrix} K_\Psi & 0 \\ 0 & K_\Psi \end{bmatrix} \begin{bmatrix} i_d \\ i_q \end{bmatrix}; \quad K_{id} = K_{iq} = \frac{1}{\sigma_m L_s}$
$K_\Psi = \frac{R_r}{L_r} L_m$
$a_1 = \frac{1}{\sigma_m L_s} \left(R_s + R_r \left(\frac{L_m}{L_r} \right)^2 \right), \quad a_2 = \frac{R_r}{\sigma_m L_s} \left(\frac{L_m}{L_r} \right)^2, \quad a_3 = \frac{L_m}{\sigma_m L_s L_r},$
$a_4 = \frac{R_r}{L_r}, \quad \sigma_m = 1 - \frac{L_m}{L_s L_r}$
Dynamics of mechanical subsystem for all machines
$\begin{bmatrix} \frac{d\theta}{dt} \\ \frac{d\omega}{dt} \end{bmatrix} = \begin{bmatrix} \omega \\ -\frac{T_L(\theta, \omega, t)}{J} \end{bmatrix} + \begin{bmatrix} 0 \\ K_T(i_d) \end{bmatrix} i_q$

as follows: θ, ω angular position and speed of machine, $\Psi^T = [\Psi_d \ \Psi_q]$ is rotor flux vector; $i^T = [i_d \ i_q]$ is stator current vector; $\mathbf{u}^T = [u_d \ u_q]$ is the control vector; V_g is amplitude of input voltage; J is moment of inertia; $T_L(\theta, \omega, t)$ is load torque; $K_T(i_d)$ is torque coefficient which depends on the d -component of stator current; $K_e, K_{iq}, K_{id}, K_\Psi$ are coefficients depending of the machine parameters and flux, R_r, R_s are rotor and stator resistance, L_r, L_s are rotor

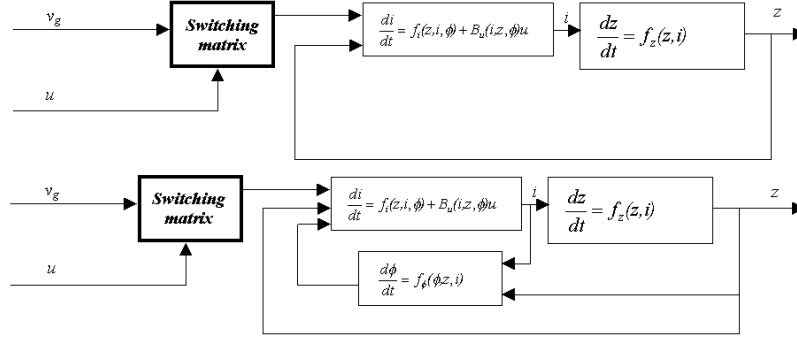


Fig. 3. Dynamical structure of electrical machines: a) DC machines and b) induction three phase machines.

and stator inductance, L_m mutual inductance, σ_m leakage factor. For a DC machine supplied from a three-phase rectifier the selection of the dq frame is related to supply voltage and is reflected in the change of the d and q coordinates in comparison with dc machine supplied from the DC source. A model of a DC machine is given for machines without field winding.

The dynamical structure of a machines is presented in Fig. 3. The difference among DC and AC machines is in the structure of the switching matrix and thus the dimensionality of the control input. For an induction machine, the dynamics of the rotor flux, with currents as input and rotor flux vector as output, should be added to the structure. A mathematical model similar to the one describing buck switching power converters could be used to describe the dynamics of electrical machines [7]

$$\begin{aligned} \frac{d\mathbf{i}_{dq}}{dt} &= \mathbf{f}_i(\mathbf{z}, \mathbf{i}_{dq}, \mathbf{\Psi}) + \mathbf{B}_u(\mathbf{i}_{dq}, \mathbf{z}, \mathbf{\Psi})\mathbf{u}, \\ \frac{d\mathbf{z}}{dt} &= \mathbf{f}_z(\mathbf{z}, \mathbf{i}_{dq}); \quad \frac{d\mathbf{\Psi}}{dt} = \mathbf{f}_\phi(\mathbf{\Psi}, \mathbf{z}, \mathbf{i}_{dq}). \end{aligned} \quad (4)$$

The elements of vector $\mathbf{z}$ are the angular position and the angular velocity of the machine. The third equation describing the change of the rotor flux is present only for the induction machine. The control input for DC machines supplied through DC-to-DC or single-phase converters is scalar. If the DC machine is supplied from a three phase source than the control input is the same as for three phase rectifiers and three phase inverters and it has the form as given in (5):

$$\begin{aligned} \mathbf{u}_{dq} &= \mathbf{F}(\theta_r) \mathbf{A}_M \mathbf{s}_{sw}, \\ \mathbf{F}(\theta_r) &= \begin{bmatrix} \cos \theta_r & \sin \theta_r \\ -\sin \theta_r & \cos \theta_r \end{bmatrix} \begin{bmatrix} 1 & -1/2 & -1/2 \\ 0 & \sqrt{3}/2 & -\sqrt{3}/2 \end{bmatrix}, \end{aligned} \quad (5)$$

where θ_r is the position of the synchronous frame of reference.

3 Control of power converters and electrical machines

The goal of the control system design of switching converters, in most of the practical cases, is reduced to the requirement that the output voltage of the converter tracks its reference while satisfying certain dynamical constraints (overshoot, load rejection etc.). The same is true for electrical machines for which the control goal could be stated as the requirement to have tracking in the torque, or velocity or angular position. In the following sections the unified approach to the control of switching converters and electrical machines based on the introduction of sliding mode in the control system will be shown. First we will briefly discuss some results in sliding mode design applicable to switching converters and electrical machines control.

3.1 Some results in sliding mode control

Variable structure systems are originally defined for dynamic systems described by ordinary differential equations with a discontinuous right hand side. In such a system so-called sliding mode motion can result. This motion is represented by the state trajectories in the sliding mode manifold and high frequency changes in the control. For sliding mode applications the equations of motion and the existence conditions are two basic questions to be discussed. Since models of switching converters and electrical machines are linear with respect to control further analysis will be restricted to the systems defined in the following form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) + \mathbf{B}(\mathbf{x}, t)\mathbf{u}, \quad (6)$$

where $\mathbf{B}(\mathbf{x}, t)$ is an $n \times m$ matrix, $\mathbf{x} \in \mathcal{R}^n$, $\mathbf{u} \in \mathcal{R}^m$. For such a system boundary-layer regularisation [1],[9] enables the substantiation of the so-called equivalent control method. In accordance with this method, in (6) control should be replaced by the equivalent control, which is the solution to $\dot{\boldsymbol{\sigma}} = \mathbf{G}\mathbf{f}(\mathbf{x}, t) + \mathbf{G}\mathbf{B}(\mathbf{x}, t)\mathbf{u}_{eq} = \mathbf{0}$, $\mathbf{G} = \{\partial\boldsymbol{\sigma}/\partial\mathbf{x}\}$, where $\boldsymbol{\sigma} = \mathbf{0}$, $\boldsymbol{\sigma} \in \mathcal{R}^m$ is defining sliding mode manifold while $\sigma_i = 0$ describe the so-called switching surfaces. For $\det \mathbf{GB} \neq 0$ equivalent control is $\mathbf{u}_{eq} = -(\mathbf{GB})^{-1}\mathbf{G}\mathbf{f}$, the sliding mode equation in the manifold $\boldsymbol{\sigma} = \mathbf{0}$ is

$$\dot{\mathbf{x}} = (\mathbf{E} - (\mathbf{GB})^{-1}\mathbf{G})\mathbf{f}, \quad \boldsymbol{\sigma} = \mathbf{0}. \quad (7)$$

From $\boldsymbol{\sigma} = \mathbf{0}$, m components $\mathbf{x}_2 \in \mathcal{R}^m$ of the state vector $\mathbf{x}$ may be found as a function of the rest $(n - m)$ components $\mathbf{x}_1 \in \mathcal{R}^{n-m}$ as $\mathbf{x}_2 = -\boldsymbol{\sigma}_0(\mathbf{x}_1)$, $\boldsymbol{\sigma}_0 \in \mathcal{R}^m$ and the order of the sliding mode equation (7) may be reduced by m :

$$\dot{\mathbf{x}} = \mathbf{f}_1(\mathbf{x}_1, -\boldsymbol{\sigma}_0(\mathbf{x}_1)), \quad \mathbf{f}_1 \in \mathcal{R}^{n-m}. \quad (8)$$

For a system subject to disturbances $\mathbf{h}(\mathbf{x}, t)$ it has been shown [17] that if $\mathbf{h}(\mathbf{x}, t) \in \text{range } \mathbf{B}$ the sliding mode motion is independent of the disturbance $\mathbf{h}(\mathbf{x}, t)$ [7].

To derive the sliding mode existence conditions in analytical form the stability of the projection of the system motion on subspace σ

$$\dot{\sigma} = Gf(x, t) + GB(x, t)u \quad (9)$$

should be analysed. If GB is an identity matrix system (9) is decomposed on m first order systems and selecting the control such that signs of each component σ_i and its derivative are opposite the sliding mode motion will occur in each discontinuity surface. Other procedures and selection of the Lyapunov functions for VSS are discussed in details in [1]. The most interesting fact is that the Lyapunov function affirming the convergence to the sliding mode manifold is finite function of time. It vanishes after the finite time interval testifying that sliding mode motion arises in a finite time instant. Sliding mode equations (8) and existence conditions constitute the basis for the variety of design procedures in VSS. To demonstrate some of the design procedures, let us write system (6), subject to disturbance $Dh = B\lambda$, in the so-called regular form

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2), \\ \dot{x}_2 &= f_2(x_1, x_2) + B_2 u + B_2 \lambda, \end{aligned} \quad (10)$$

where $x_1 \in \mathcal{R}^{n-m}$, $x_2 \in \mathcal{R}^m$, $f_1(x_1, x_2)$, and $f_2(x_1, x_2)$ are vectors of appropriate dimensions and $\text{rank} B_2 = \text{rank} B = m$. Assume that the sliding mode manifold is defined as $\sigma = \sigma_0(x_1) + x_2 = 0$, $\sigma \in \mathcal{R}^m$. Then the equivalent control is expressed as $u_{eq} = -B_2^{-1}(G_1 f_1(x_1, x_2) + f_1(x_1, x_2)) - \lambda$. It depends on disturbance and in most cases its realization is unpractical. Calculating $(u + \lambda)$ from $d\sigma/dt = 0$ and substituting it to second equations in (10) then, when the sliding mode appears in this manifold, the system behaviour is governed by the $(n - m)$ order equation

$$\dot{x}_1 = f_1(x_1, -\sigma_0(x_1)), \quad x_2 = -\sigma_0(x_1). \quad (11)$$

In (11) vector $\sigma_0(x_1)$ could be treated as "virtual control" and should be selected to satisfy the desired system dynamics. For control input selection the projection of the system motion on m -dimensional space σ is found

$$\begin{aligned} \dot{\sigma} &= f(x_1, x_2, \lambda) + B_2 u, \quad f \in \mathcal{R}^m, \\ f(x_1, x_2, \lambda) &= G_1 f_1 + f_2 x_2 + B_2 \lambda. \end{aligned} \quad (12)$$

The discontinuous control $u = -B_2^{-1}M \text{sign}(\sigma)$, $M = \text{const.} > 0$ leads to

$$\dot{\sigma}_i = f_i(x_1, x_2, \lambda) - M u_i, \quad i = 1, \dots, m. \quad (13)$$

There exists large enough $M > 0$ such that the functions σ_i , $i = 1, \dots, m$ and derivatives $\dot{\sigma}_i$ have opposite signs, sliding mode will occur in each of the discontinuity surfaces.

Discrete time sliding mode was introduced for discrete time plants [11],[13],[15]. The most significant difference with the continuous time sliding mode is that

motion in the sliding mode manifold may occur in discrete time systems with continuous right hand side. By applying the sample and hold process with sampling period T , and integrating the solution over interval $t \in [kT, (k+1)T]$ with $\mathbf{u}(t) = \mathbf{u}(kT)$ and $\mathbf{d}(t) = \mathbf{d}(kT)$, the discrete time model of plant (6) may be represented as

$$\mathbf{x}_{k+1} = \mathbf{F}_k \mathbf{x}_k + \bar{\mathbf{B}} \mathbf{u}_k + \mathbf{P} \mathbf{d}_k. \quad (14)$$

The sliding manifold is defined as $\boldsymbol{\sigma}_k = \mathbf{G} \mathbf{x}_k$, $k = 1, 2, \dots$. In [11] the equivalent control $\mathbf{u}_k^{eq} = \mathbf{u}_k^{eq}(kT)$ is defined as the solution of

$$\boldsymbol{\sigma}_{k+1} = \mathbf{G} \mathbf{x}_{k+1} = \mathbf{G} \mathbf{F}_k \mathbf{x}_k + \mathbf{G} \bar{\mathbf{B}} \mathbf{u}_k^{eq} + \mathbf{G} \mathbf{P} \mathbf{d}_k = \mathbf{0}. \quad (15)$$

Provided that $\det \mathbf{G} \bar{\mathbf{B}} \neq 0$ the equivalent control can be expressed as

$$\mathbf{u}_k^{eq} = -(\mathbf{G} \bar{\mathbf{B}})^{-1} \mathbf{G} (\mathbf{F}_k \mathbf{x}_k + \mathbf{P} \mathbf{d}_k). \quad (16)$$

It is important to note that matching conditions now are defined in terms of matrices $\mathbf{G}$, $\bar{\mathbf{B}}$, $\mathbf{P}$. The required magnitude of control (16) may be large and limitation should be applied, so the final form of the control is

$$\mathbf{u}_k^{eq} = \begin{cases} -(\mathbf{G} \bar{\mathbf{B}})^{-1} \mathbf{G} (\mathbf{F}_k \mathbf{x}_k + \mathbf{P} \mathbf{d}_k^*), & \text{if } |\mathbf{u}_k| < U_o \\ -U_o \text{sign}(\boldsymbol{\sigma}_k), & \text{if } |\mathbf{u}_k| \geq U_o \end{cases}, \quad (17)$$

where $\mathbf{d}_k^*$ is the estimated disturbance and U_o is a control input bound. In another approach for system (6) asymptotic stability of the solution $\boldsymbol{\sigma}(\mathbf{x}) = \mathbf{0}$ can be assured if one can find a control input such that the stability criteria are satisfied for the following Lyapunov function $\nu = \boldsymbol{\sigma}^T \boldsymbol{\sigma} / 2$ with the requirement that the time derivative $(d\nu/dt)$ has a certain form, for example $d\nu/dt = -\boldsymbol{\sigma}^T \mathbf{D} \boldsymbol{\sigma}$, $\mathbf{D} > \mathbf{0}$, [16]. Then the control input, with sampling interval T , that satisfy the given requirements is in the form

$$\mathbf{u}_k = \mathbf{u}_{k-1} - (\mathbf{G} \mathbf{B} T)^{-1} ((\mathbf{E} + \mathbf{T} \mathbf{D}) \boldsymbol{\sigma}_k - \boldsymbol{\sigma}_{k-1}). \quad (18)$$

The realisation of control (18) requires information on the sliding functions and the plant gain matrix, which is much easier to obtain than information necessary to implement algorithm (17).

Mathematical models of the switching converters and electrical machines could be presented in regular form (10) with discontinuous control influencing the change of the currents and currents being treated as "virtual control" in the voltage dynamics (for converters) or mechanical motion dynamics (for machines). The structure of the boost converters is more complicated with control entering all the equations of the system. Despite the differences in the dynamical structure the control system design for buck and boost converters and electrical machines may follow the two-step procedure:

- Select control $\mathbf{u}$ such that inductor current (or electromagnetic torque in electrical machines) tracks its reference;

- Select the current reference (virtual control) so that capacitance voltage (or mechanical coordinates) satisfy prescribed dynamical behaviour.

This procedure is not so obvious for the boost structures since control enters both equations. In the framework of sliding mode systems the above procedure for boost converters requires substitution of the equivalent control to the first equation and then taking current reference as "virtual control" input. In the following sections we will show the consequent application of the above procedure in details without presenting unnecessary details for particular converters or machines.

3.2 Control of DC-to-DC converters and DC machines

In this section the control of DC-to-DC converters and DC machines both having scalar control input is discussed. Control of DC machines supplied from three-phase sources will be discussed in the section dealing with control of three-phase converters and AC machines.

Control of DC-to-DC buck converter. Assume the current reference as continuous function $i_L^{ref}(t)$ then, for the tracking error $\sigma = i_L^{ref}(t) - i_L(t)$ and the control selected as $u = (1 - \text{sign}\sigma)/2$ the sliding mode exists if the equivalent control $0 \leq u_{eq} \leq 1$ is calculated as:

$$\begin{aligned} \frac{d\sigma}{dt} &= \frac{d(i_L^{ref} - i_L)}{dt} = \frac{di_L^{ref}}{dt} + \frac{v_o}{L} - \frac{V_g}{L}u_{eq} = 0 \\ \Rightarrow u_{eq} &= \frac{1}{V_g} \left(L \frac{di_L^{ref}}{dt} + v_o \right). \end{aligned} \quad (19)$$

Substituting the equivalent control to the original equations of the system one can obtain:

$$\frac{dv_o}{dt} = \frac{i_L^{ref}}{C} - \frac{v_o}{RC}; \quad i_L(t) = i_L^{ref}(t). \quad (20)$$

From (20) reference current could be selected using well-established design procedures for linear systems. For example, if the first order response $\sigma_v = \rho(v_o^{ref} - v_o) + d(v_o^{ref} - v_o)/dt = 0$ of closed loop system is required, the reference current could be easily determined as

$$i_L^{ref} = \text{sat} \left(\frac{v_o}{R} + \rho C(v_o^{ref} - v_o) + C \frac{dv_o^{ref}}{dt} \right) \Rightarrow i_L^{ref} = \text{sat}(i_L + C\sigma_v) \quad (21)$$

where $\text{sat}(\bullet)$ is the saturation function. Structure of system (21) is shown in Fig. 4. Another structure of the control system may be determined from the required closed loop dynamics by inserting $dv_o/dt = i_L/C - v_o/RC$ into the expression for σ_v which leads to $\sigma_v = \rho(v_o^{ref} - v_o) + dv_o^{ref}/dt + v_o/RC - i_L/C =$

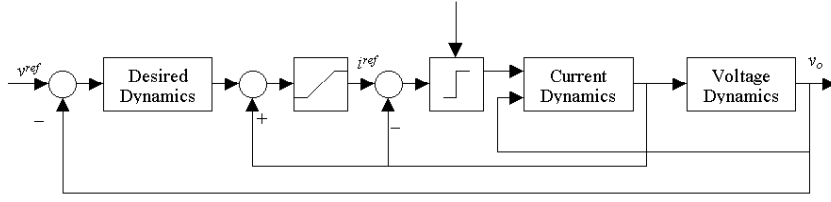


Fig. 4. Structure of the converter control system.

$(i_L^{ref} - i_L)/C$ and selecting the control as $u = (1 - \text{sign}(C\sigma_v))/2$ the sliding mode motion is established in manifold $\sigma_v = 0$ if $0 \leq u_{eq} \leq 1$ and the same result is obtained as in algorithm (21).

The structure in Fig. 4 is suitable for the implementation of different control algorithms in designing the reference current, thus it leaves more room for merging other control techniques with the sliding mode. This will be especially clear in the control of the boost type of power converters.

Control of DC machines. By following the same procedure as for buck converters the DC machine tracking error could be defined as $\sigma = i_q^{ref}(t) - i_q(t)$ with the control $u = (1 - \text{sign}\sigma)/2$ the sliding mode exists if the equivalent control satisfies $-1 \leq u_{eq} = (L/V_g)(di_q^{ref}/dt + R/Li_q + K_e/L\omega) \leq 1$. By substituting u_{eq} to the original system the motion of the Dc machine is reduced to the second order system

$$\frac{d\theta}{dt} = \omega; \quad \frac{d\omega}{dt} = -\frac{T_L(\theta, \omega, t)}{J} + \frac{K_T(i_d)}{J}i_q^{ref}. \quad (22)$$

By requiring the closed loop transient to satisfy $\sigma_\theta = C_\theta(\theta^{ref} - \theta) + C_\omega \frac{d(\theta^{ref} - \theta)}{dt} + \frac{d^2(\theta^{ref} - \theta)}{dt^2} = 0$ determined by the design parameters C_θ and C_ω the reference current becomes

$$\sigma_\theta = C_\theta(\theta^{ref} - \theta) + C_\omega \frac{d(\theta^{ref} - \theta)}{dt} + \frac{d^2(\theta^{ref} - \theta)}{dt^2} = \frac{K_T}{J}(i_q^{ref} - i_q),$$

$$i_q^{ref} = \text{sat} \left(\frac{T_L}{K(i_d)} + C_\theta(\theta^{ref} - \theta) + C_\omega \frac{d(\theta^{ref} - \theta)}{dt} + \frac{d^2\theta^{ref}}{dt^2} \right). \quad (23)$$

Note that for $C_\theta = 0$ the above dynamics reduces to $\sigma_\omega = C_\omega(\omega^{ref} - \omega) + C_\omega(d(\omega^{ref} - \omega)/dt) = 0$ and defines the desired dynamics for velocity control. Implementation of control (23) requires information on the machine load which could be obtained using disturbance observer techniques proposed by Ohnishi in [18]. Manipulating (23) one can determine a much more simple, way of calculating reference current

$$i_q^{ref} = \text{sat} \left(i_q + \frac{J}{K_T(i_d)}\sigma_\theta \right). \quad (24)$$

Structure of the control system is the same as the one depicted in Fig. 4. These results depict earlier shown similarities between buck converters and electrical machines. The structure of the control is the same, with necessary changes in measured variables. In [22], [23] the application of the sliding mode control to DC machine combining the acceleration, the velocity and the position control is proposed. Proposed algorithm ensures robustness against parameters and disturbance changes even in the acceleration stage. Design of the sliding mode control for DC electrical machines based on the reduced order model (dynamics of the electrical current is neglected) is discussed in details in [4] and [26].

Control of boost DC-to-DC converter. Assume the current reference as the continuous function $i_L^{ref}(t)$ then, selecting tracking error $\sigma = i_L^{ref}(t) - i_L(t)$ and control $u = (1 - \text{sign}\sigma)/2$ the sliding mode exists if equivalent control (25) satisfies conditions $0 \leq u_{eq} \leq 1$

$$\begin{aligned} \frac{d\sigma}{dt} &= \frac{d(i_L^{ref} - i_L)}{dt} = \frac{di_L^{ref}}{dt} + \frac{v_o}{L}(1 - u_{eq}) - \frac{V_g}{L} = 0 \\ \Rightarrow u_{eq} &= \frac{1}{v_o} \left(L \frac{di_L^{ref}}{dt} + v_o - V_g \right). \end{aligned} \quad (25)$$

By substituting u_{eq} to the original equations of the boost converter one could determine

$$\frac{C}{2} \frac{d(v_o)^2}{dt} + \frac{v_o^2}{R} = V_g i_L^{ref} - \frac{L}{2} \frac{d(i_L^{ref})^2}{dt} = (V_g - v_L) i_L^{ref}, \quad i_L = i_L^{ref}, \quad (26)$$

where v_L represents the voltage drop on the inductance L . System (26) may be interpreted as the description of power conservation in the circuit $(C/2)d(v_o)^2/dt + v_L i_L^{ref} = V_g i_L^{ref} - (v_o^2/R)$. From the control point of view it could be regarded as a linear first order system with the square of the output voltage v_o^2 as output and reference inductor current as control. With such definition of variables the mathematical description reduces to the first order system nonlinear with respect to the control. For DC-to-DC converters the change of energy stored in inductance could be neglected (average $v_L = 0$) and the system reduces to the first order linear system $(C/2)d(v_o)^2/dt + (v_o^2/R) = V_g i_L^{ref}$ and selection of the reference current may follow the same procedure as for buck converters [3],[17]. Another often applied solution is the feed-forward calculation of the reference current from the reference voltage [26].

3.3 Control of three phase converters and three phase machines

Three phase converters and electrical machines structurally differ from their DC counterparts in the number of energy storage elements and in the structure of the switching matrix. The dynamical structure of the systems remains

the same as for their DC counterparts except that the there-phase are MIMO systems. That allows conduct design of the control in the same two-step procedure as applied for dc systems.

Current control in three phase systems. For modelling and control design purposes the three phase switching matrix has been defined by the three-dimensional control vector $\mathbf{u} = \mathbf{A}_M \mathbf{s}_{sw}$. Vector $\mathbf{s}_{sw}$ has six elements but due to electric circuits constraints for buck inverter and boost rectifier, vector $\mathbf{s}_{sw}$ may have the following values $\mathbf{S}_1 = [1\ 0\ 0\ 1\ 1]$, $\mathbf{S}_2 = [1\ 1\ 0\ 0\ 1]$, $\mathbf{S}_3 = [0\ 1\ 0\ 1\ 0]$, $\mathbf{S}_4 = [0\ 1\ 1\ 0\ 0]$, $\mathbf{S}_5 = [0\ 0\ 1\ 1\ 0]$, $\mathbf{S}_6 = [1\ 0\ 1\ 0\ 1]$, $\mathbf{S}_7 = [1\ 1\ 1\ 0\ 0]$, $\mathbf{S}_8 = [0\ 0\ 0\ 1\ 1]$; for boost inverter and buck rectifier vector $\mathbf{s}_{sw}$ may have the following values $\mathbf{S}_1 = [1\ 0\ 0\ 0\ 1]$, $\mathbf{S}_2 = [0\ 0\ 1\ 0\ 1]$, $\mathbf{S}_3 = [0\ 1\ 1\ 0\ 0]$, $\mathbf{S}_4 = [0\ 1\ 0\ 0\ 1]$, $\mathbf{S}_5 = [0\ 0\ 0\ 1\ 1]$, $\mathbf{S}_6 = [1\ 0\ 0\ 1\ 0]$, $\mathbf{S}_7 = [1\ 1\ 0\ 0\ 0]$, $\mathbf{S}_8 = [0\ 0\ 1\ 1\ 0]$, $\mathbf{S}_9 = [0\ 0\ 0\ 0\ 1]$.

For inverters and rectifiers the number of independent control inputs is three. As depicted in Fig. 5 for three-phase inverters the number of independent variables to be controlled is two: for inverters these are the d and q components of output voltage, for AC machines they are the same as the components of supply voltage. For inverters and machines supplied from DC sources there is no variable to be controlled on the input side. For three-phase rectifiers the output is the DC source and thus only one independent variable (output current or voltage) is to be controlled. On the input side of the rectifier, the magnitude of voltage or current are defined thus only phase shift between the voltage and the current vector could be controlled. This allows introduction of an additional requirement to the control system design. The

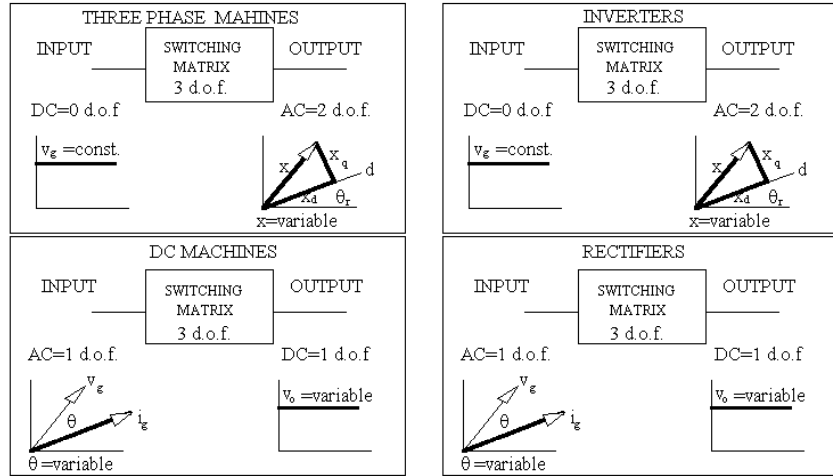


Fig. 5. The assignment of the degrees of freedom in control for three-phase switching matrix.

most natural choice is to relate this additional requirement to the selection of the switching pattern.

Let us now look at design of the switching pattern for the three phase converters in more details. Current control is based on the sliding mode existence in the manifold $\sigma^T = [i_d^{ref}(t) - i_d, i_q^{ref}(t) - i_q]^T = \mathbf{0}$ where vector $\sigma^T = [\sigma_d \ \sigma_q]^T$ with $\sigma_d = i_d^{ref}(t) - i_d$, $\sigma_q = i_q^{ref}(t) - i_q$ and i_d^{ref} , i_q^{ref} are continuous functions to be determined later. Design of the current controller is based on the system description (4) $d\mathbf{i}_{dq}/dt = \mathbf{f}_{dq} + \mathbf{B}_{dq}\mathbf{u}_{dq}$ where matrix $\mathbf{B}_{dq}$ is diagonal. The structure of function $\mathbf{f}_{dq}$ and matrix $\mathbf{B}_{dq}$ could be easily found from mathematical models given in Tables 4 and 5. The time derivative of $\sigma^T = [\sigma_d \ \sigma_q]^T$ is determined as

$$\frac{d\sigma}{dt} = \frac{di_d^{ref}}{dt} - \frac{di_d}{dt} = \frac{di_d^{ref}}{dt} - \mathbf{f}_{dq} - \mathbf{B}_{dq}\mathbf{u}_{dq}, \quad \mathbf{u}_{dq}^T = [u_d \ u_q]. \quad (27)$$

Equivalent control can be calculated as $\mathbf{B}_{dq}^{-1}[di_d^{ref}/dt - \mathbf{f}_{dq}] = \mathbf{u}_{eq}$ and equation (27) is expressed as

$$\frac{d\sigma_{dq}}{dt} = \mathbf{B}_{dq}[\mathbf{u}_{eq} - \mathbf{u}_{dq}(S_i)], \quad i = 1, \dots, 9. \quad (28)$$

Control vectors could take values from the discrete set $\mathbf{S}_8 = \{\mathbf{S}_1, \mathbf{S}_2, \mathbf{S}_3, \mathbf{S}_4, \mathbf{S}_5, \mathbf{S}_6, \mathbf{S}_7, \mathbf{S}_8, \mathbf{S}_9\}$ as depicted in Fig. 6.a. All realizable values of the equivalent control lie inside the hexagon spanned by the elements of the set $\mathbf{S}_8$ [7]. The rate of change of error is proportional to the differences between the vector of equivalent control and the realisable control vectors. For a particular combination of errors all permissible vectors $\mathbf{S}_i$ that satisfy the sliding mode existence conditions could be determined from $(d\sigma_d/dt)\sigma_d < 0$ and $(d\sigma_q/dt)\sigma_q < 0$ or $\text{sign}(\mathbf{u}_{eq} - \mathbf{u}_{dq}(\mathbf{S}_i)) = -\text{sign}(\sigma_{dq})$ as shown in Fig. 6.b. For some combinations of errors there are more than one permissible vector

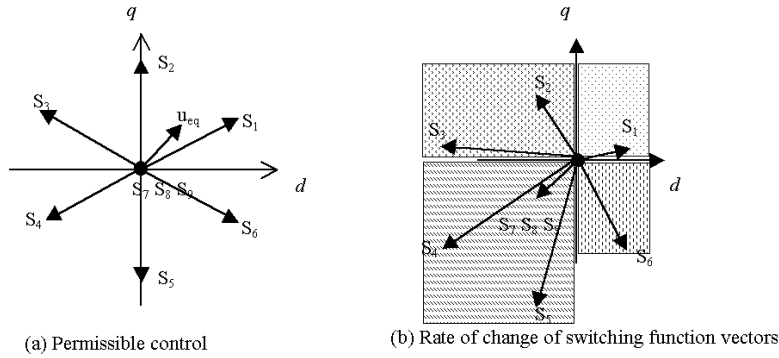


Fig. 6. Control vectors and selection of permissible control for given combination of the signs of control errors.

that leads to an ambiguous selection of the control and consequently existence of more than one solution for the selection of switching pattern .

The same could be concluded from $\text{rank} \mathbf{F}(\theta_r) = \text{rank}(\mathbf{A}_{\alpha\beta}^{dq} \mathbf{A}_{abc}^{\alpha\beta}) = 2$. Ambiguity in selection of the control vector based on selected u_d and u_q allows us to have a number of different PWM algorithms based on satisfying sliding mode conditions in (d, q) frame of references. In early works [7] the following solution was proposed: add an additional requirement $\vartheta(t) = 0$ to the control system specification so that (27) is augmented to have the form

$$\begin{bmatrix} \frac{d\sigma_{dq}}{dt} \\ \frac{d\vartheta}{dt} \end{bmatrix} = \begin{bmatrix} \frac{d\mathbf{i}^{ref}}{dt} - \mathbf{f}_{dq} \\ \mathbf{f}_{\vartheta} \end{bmatrix} - \begin{bmatrix} \mathbf{B}_{udq} \mathbf{F}(\theta_r) \\ \mathbf{b}_{\vartheta}^T \end{bmatrix} \mathbf{u}(\mathbf{S}_i), \quad (29)$$

$$\frac{d\sigma_N}{dt} = \mathbf{f}_N - \mathbf{B}_N \mathbf{u}(\mathbf{S}_i), \quad \mathbf{u}^T(\mathbf{S}_i) = [u_a \ u_b \ u_c]. \quad (30)$$

Vector $\mathbf{b}_{\vartheta}$ should be selected so that $\text{rank} \mathbf{B}_N = 3$. the simplest solution is for $\vartheta(t) = u_a + u_b + u_c$ [7],[26] then matrix $\mathbf{B}_N$ will have full rank. To determine the switching pattern, the simplest way is to use the nonlinear transformation $\sigma_s = \mathbf{B}_N^{-1} \sigma_N$, then the sliding mode conditions are satisfied if the control is selected as

$$\text{sign}(u_j(\mathbf{S}_i)) = -\text{sign}(\sigma_{sj}), \quad -1 \leq u_{eq} \leq 1. \quad (31)$$

This line of reasoning with some variations has been the most popular in designing the sliding mode based switching pattern [4],[7].

Another solution implicitly applied in most of the so-called space vector PWM algorithms is based on the simple idea [5] using transformation

$$\mathbf{u}_{abc} = \text{rank}(\mathbf{A}_{\alpha\beta}^{dq} \mathbf{A}_{abc}^{\alpha\beta})^T \mathbf{u}_{dq}$$

to the (a, b, c) reference frame. Then components u_a, u_b and u_c of $\mathbf{u} = \mathbf{A}_M \mathbf{s}_{sw}$ are selected according to the following rule

$$\mathbf{S}_i = \begin{cases} \text{sign}(u_a(\mathbf{S}_i)) = \text{sign}(u_d \cos \theta_r - u_q \sin \theta_r) \\ \text{sign}(u_b(\mathbf{S}_i)) = \text{sign}(u_d \cos(\theta_r - 2\pi/3) - u_q \sin(\theta_r - 2\pi/3)) \\ \text{sign}(u_c(\mathbf{S}_i)) = \text{sign}(u_d \cos(\theta_r - 4\pi/3) - u_q \sin(\theta_r - 4\pi/3)) \end{cases} \quad (32)$$

$i = (1, 2, \dots, 8, 9)$.

This idea is analysed in details in [7]. Further simplification of this algorithm leads to its implementation using a look-up table [19]. In [24] and [25] the so-called space vector PWM based on the sliding mode approach is discussed. The solution is based on the expression (32) but it is realized using space sectors.

In the application of above algorithms switching is realized using hysteresis which, as follows from (28) directly determine the current ripple to be equal to the half of the hysteresis width. For the given current ripple (constant hysteresis width) the time between two switching for each component is directly proportional to $[u_{eqj} - u_{dqj}]$, $(j = d, q)$, $(i = 1, \dots, 9)$. A new class

of the switching algorithms based on the simple requirement that control should be selected to give the minimum rate of change of control error could be designed for which the same error will be achieved with less switching effort. The algorithm can be formulated in the following form [3]

$$S_i = \begin{cases} \min \| \mathbf{u}_{eq}(t) - \mathbf{u}_j(S_i) \| \\ \text{sign}([u_{eqd} - u_d(S_i)] \bullet \sigma_d(t)) = -1, & i = (1, 2, \dots, 8, 9). \end{cases} \quad (33)$$

$$\text{sign}([u_{eqq} - u_q(S_i)] \bullet \sigma_q(t)) = -1$$

The difference in behaviour for algorithm (32) and (33) is depicted in Fig. 7, where the steady state operation of buck inverter current control is shown. The operating point for both algorithms is the same and the width of hysteresis is also kept the same. The difference in the switching frequency of the voltage is easily detectable.

All of the above algorithms naturally include so-called over-modulation functionality. This can be seen in the diagrams depicted in Fig. 8. For algorithm (33) behaviour of an induction machine ($P=4\text{kVA}$, $p=2$, $U=220\text{ V}$) current control loop is depicted. (α, β) currents and control are shown in Fig. 8.a. for a loaded machine and $w^{ref}=50\text{p [rad/s]}$. Note that switching is regular between the two closest vectors and zero vector. In Fig. 8.b. the current vector is depicted for $w^{ref} = 95\pi \text{ [rad/s]}$. For this condition the sliding mode existence conditions are violated in certain regions of the plane. In Fig. 8.c. the current vector is depicted for $w^{ref} = 100\pi \text{ [rad/s]}$. For this condition the sliding mode existence conditions cannot be satisfied and the system operates under a six-step mode.

Due to the specifics of three-phase balanced systems the number of independent controls for the switching matrix is higher than the dimension of the controlled current vector. This is the basic reason that three-phase PWM, under many different names, is still attractive as a research topic. The solution proposed in [15] shows that the formalization of the minimization of TDF in

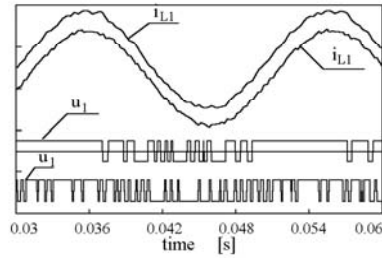


Fig. 7. The steady state operation of buck inverter current control for switching algorithms (32) - lower trace and (33) - upper trace.

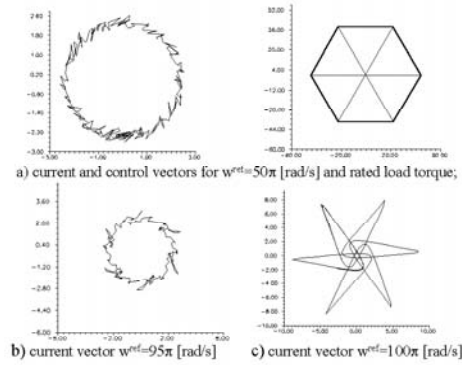


Fig. 8. Change of the current and control vectors in (α, β) frame of references.

the above framework has lead to a solution that is far better in comparison with other so-called space vector approaches [16].

In the sliding mode dynamics of current the control loop is reduced to $\sigma^T = [i^{ref}(t) - i]^T = \mathbf{0}$ or $i_d^{ref}(t) = i_d$ and $i_q^{ref}(t) = i_q$ with equivalent control being determined as $\mathbf{B}_{udq}^{-1} [di^{ref}/dt - \mathbf{f}_{idq}] = \mathbf{u}_{eq}$. In order to complete the design of converters and electrical machines the reference currents shall be determined.

Voltage and mechanical motion control system design. For buck converters and electrical machines with reference currents interpreted as virtual control inputs the description could be easily transformed to the following form

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}_x(\mathbf{x}) + \mathbf{B}_i(\mathbf{x})\mathbf{i}^{ref}, \quad (34)$$

where vector $\mathbf{x}^T = [x_1 \ x_2]$ represent either the vector of output voltage (for converters) or the vector with velocity and position (for electrical machines).

For the buck inverter both components of the reference current could be determined from the specification of the voltage loop, but for buck rectifiers only the d -component of the source current can be determined from the voltage loop specification. The q -component of the source current does not influence the output voltage and thus represent current circulating between supply sources and creating reactive power flow from sources. The same is directly applicable for a DC machine supplied by the three-phase rectifier.

Since all machines have the same structure of mechanical subsystem the results obtained for DC machines may be directly applicable to AC machines thus giving a way of determining one component of the current vector. The other component of the current vector should be determined from the requirement of the magnetic circuits of the machine and is specific for each type of the machine. For PM and induction three-phase machines the d -component of the current defines the rotor flux so it should be selected taking rotor flux behavior into consideration.

The requirements for converters and machines are presented in Table 6. the selection of the components of the switching function vector is given along with the expression for the reference current calculation. The reference current is selected following the discrete time sliding mode control design and for all systems under consideration it is $\mathbf{i}_k^{ref} = \mathbf{i}_{k-1}^{ref} - (\mathbf{GBT})^{-1}((\mathbf{E} + \mathbf{T}\mathbf{D})\sigma_k - \sigma_{k-1})$; $\mathbf{G} = \{\partial\sigma/\partial\mathbf{x}\}$ where T is the sampling interval. The realisation of this control algorithm requires information on the sliding functions and the plant gain matrix.

Consider the three-phase boost rectifier connected as a controllable current source as shown in Fig. 9. Its role is to generate current flow so that the energy source is loaded by active power or the reactive power of source is zero $Q_s = 0$ which gives $i_{sq}^{ref} = 0$. The d -component reference current

Table 6. The selection of desired dynamics and control vector

Type	Function σ	Reference currents
Buck rectifier	$\sigma_d = v_o^{ref} - v_o$ $\sigma_q = i_{qav}^{ref} - i_{qav}$ usually $i_{qav}^{ref} = 0$	$\mathbf{i}_k^{ref} = \mathbf{i}_{k-1}^{ref} - (\mathbf{GBT})^{-1} ((\mathbf{E} + \mathbf{T}\mathbf{D})\boldsymbol{\sigma}_k - \boldsymbol{\sigma}_{k-1});$ $\mathbf{G} = \{\partial\boldsymbol{\sigma}/\partial\mathbf{x}\}$ $\mathbf{i}_{dq}^{ref} = [i_d^{ref} \ i_q^{ref}]^T$ $\boldsymbol{\sigma}_{dq}^{ref} = [\sigma_d \ \sigma_q]^T$
Dc machine	$\sigma_d = C_\theta \Delta\theta + \frac{d\Delta\theta}{dt}; \ \Delta\theta = \theta^{ref} - \theta$ $\sigma_q = i_{qav}^{ref} - i_{qav}$ usually $i_{qav}^{ref} = 0$	
Buck inverter	$\sigma_d = v_{od}^{ref} - v_{od}$ $\sigma_q = v_{oq}^{ref} - v_{oq}$	
PM synchronous machine	$\sigma_d = i_{dav}^{ref} - i_{dav}$ usually $i_{dav}^{ref} = 0$ $\sigma_q = C_\theta \Delta\theta + \frac{d\Delta\theta}{dt}; \ \Delta\theta = \theta^{ref} - \theta$	
Induction machine	$\sigma_d = \sigma_\phi(\phi, \phi^{ref}) = \Delta\phi;$ $\Delta\phi = \phi^{ref} - \phi$ $\sigma_q = C_\theta \Delta\theta + \frac{d\Delta\theta}{dt}; \ \Delta\theta = \theta^{ref} - \theta$	
Boost rectifier	$\sigma_d = \mu ((v_o^{ref})^2 - (v_o)^2)$ $\sigma_q = i_{qav}^{ref} - i_{qav}$ usually $i_{qav}^{ref} = 0$	

shall be determined from the capacitance voltage. It is easy to show that $(C/2)(dv_{od}^2/dt) + v_{od}^2/R = V_g i_d^{ref} - v_L i_d^{ref}$ which is the same as for the DC-to-DC boost converter and all comments regarding the DC-to-DC boost converter control are applicable.

The structure of the power converters and electrical machines system can be presented as in Fig. 10. where the current control loop operates in sliding mode with discontinuous control. The structure is the same as one shown in Fig. 4. with additional details on the outer loop controller. The selected structure is only one of the several possible solutions, other structures may be derived by applying some other design procedures many of which are developed in the framework of motion control systems. By doing so, essential features of the sliding mode are preserved by current loop design.

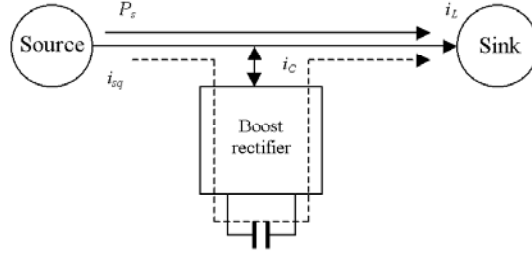


Fig. 9. Power flow in system when boost converter is used as an active power filter.

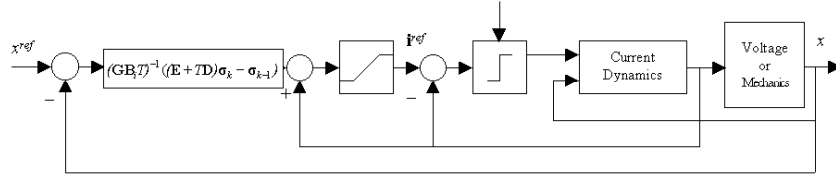


Fig. 10. Structure of the converters and electrical machine control system.

Application of the above algorithms requires information on currents and voltages for converters and mechanical coordinates for electrical machines. Usually measurement of electrical quantities is not considered demanding so realization of the control algorithms in the case of switching converters does not represent any problem. This may not be true for AC electrical machines and especially for the induction machine. For these machines the synchronous frame of references is determined by the rotor flux vector, which is not accessible for measurement and should be derived using observer. Since induction machine is a nonlinear system the observer design may not be so straight forward a task. In the section 4 of this paper we will be discussing the sliding mode based observers of induction machine rotor flux and velocity.

4 Induction machine observer

Design of a IM sensorless drives is still a challenge. The basic problem is speed estimation especially at the low speed range and under light load conditions. In this section the VSS approach to rotor flux and speed estimation of an induction machine will be discussed.

The description of the machine in the (α, β) frame of references is

$$\frac{d\boldsymbol{\Psi}_r}{dt} = -\mathbf{B}_r \boldsymbol{\Psi}_r + \frac{R_r L_m}{L_r} \mathbf{i}_s; \quad \mathbf{B}_r = \begin{bmatrix} \frac{R_r}{L_r} & -\omega \\ \omega & \frac{R_r}{L_r} \end{bmatrix}, \quad (35)$$

$$\frac{d\mathbf{i}_s}{dt} = \frac{1}{\sigma L_s} \left(\frac{L_m}{L_r} \left(-\mathbf{B}_r \boldsymbol{\Psi}_r + \frac{R_r L_m}{L_r} \mathbf{i}_s \right) - R_s \mathbf{i}_s + \mathbf{u}_s \right). \quad (36)$$

Formally it is possible to design a stator current observer based on voltage and current measurements and with a rotor flux vector derivative as the control input:

$$\frac{d\hat{\mathbf{i}}_s}{dt} = \frac{1}{\sigma L_s} \left(-\frac{L_m}{L_r} \mathbf{u}_\psi - R_s \hat{\mathbf{i}}_s + \mathbf{u}_s \right). \quad (37)$$

This selection of the observer control input is different from the usually used current error feedback (well known Gopinath's method), application of sliding mode control based on current feedback [17], or selecting the unknown velocity as the control input [18],[4],[26].

The estimation error is determined as

$$\frac{d\boldsymbol{\varepsilon}_i}{dt} = \frac{d(\mathbf{i}_s - \hat{\mathbf{i}}_s)}{dt} = \frac{1}{\sigma L_s} \left(\frac{L_m}{L_r} \left((-\mathbf{B}_r \boldsymbol{\Psi}_r + \frac{R_r L_m}{L_r} \mathbf{i}_s) + \mathbf{u}_\psi \right) - R_s \boldsymbol{\varepsilon}_i \right) \quad (38)$$

If the sliding mode exists then from $d\boldsymbol{\varepsilon}_i/dt = \mathbf{0}$, and $\boldsymbol{\varepsilon}_i = \mathbf{0}$. Under the assumption that the angular velocity is known, from (38) one can find

$$\hat{\boldsymbol{\Psi}}_r = \mathbf{B}_r^{-1} \left(\mathbf{u}_{\psi eq} - \frac{R_r L_m}{L_r} \mathbf{i}_s \right). \quad (39)$$

In the observer design suitable for the sensorless drive, the observer control input should be a known function of the motor speed so that, after establishing sliding mode in current tracking loop, the speed can be determined as a unique solution. This leads to the following selection of the structure of the stator current observer

$$\frac{d\hat{\mathbf{i}}_s}{dt} = \frac{1}{\sigma L_s} \left(\frac{L_m}{L_r} \left(-\mathbf{u}_\phi + \frac{R_r L_m}{L_r} \hat{\mathbf{i}}_s \right) - R_s \hat{\mathbf{i}}_s + \mathbf{u}_s \right) \quad (40)$$

and the estimation error becomes:

$$\frac{d\boldsymbol{\varepsilon}_i}{dt} = \frac{d(\mathbf{i}_s - \hat{\mathbf{i}}_s)}{dt} = \frac{1}{\sigma L_s} \left(\frac{L_m}{L_r} (-\mathbf{B}_r \boldsymbol{\Psi}_r + \mathbf{u}_\phi) - \left(R_s + \frac{R_r L_m^2}{L_r^2} \right) \boldsymbol{\varepsilon}_i \right) \quad (41)$$

Algorithm (18) could be used to calculate the control $\mathbf{u}_{\phi k} = \mathbf{u}_{\phi k-1} + (\sigma L_s L_r / L_m T)((1 + \rho T)\boldsymbol{\varepsilon}_{ik} - \boldsymbol{\varepsilon}_{ik-1})$ with $\boldsymbol{\varepsilon}_i = [\varepsilon_{i\alpha} \ \varepsilon_{i\beta}]$. From $d\boldsymbol{\varepsilon}_i/dt = \mathbf{0}$ the equivalent control is determined as $\mathbf{u}_{\phi eq} = \mathbf{B}_r \boldsymbol{\Psi}_r$. The rotor flux observer could be selected as having the same structure as (35) with the additional convergence term $\mathbf{f}$ whose structure will be explained later

$$\frac{d\hat{\boldsymbol{\Psi}}_r}{dt} = -\mathbf{u}_\phi + \frac{R_r L_m}{L_r} \hat{\mathbf{i}}_s + \mathbf{f}. \quad (42)$$

Now flux estimation error can be calculated as:

$$\frac{d\boldsymbol{\varepsilon}_\psi}{dt} = \frac{d(\boldsymbol{\Psi}_r - \hat{\boldsymbol{\Psi}}_r)}{dt} = -\mathbf{B}_r \boldsymbol{\Psi}_r + \mathbf{u}_\phi + \frac{R_r L_m}{L_r} \mathbf{i}_s - \mathbf{f}. \quad (43)$$

To ensure convergence $\mathbf{f}$ could be selected in the following form:

$$\mathbf{f} = \mathbf{k}\hat{\mathbf{e}}_\Psi, \quad (44)$$

where

$$\begin{bmatrix} \bar{\varepsilon}_{\Psi\alpha} \\ \bar{\varepsilon}_{\Psi\beta} \end{bmatrix} = \begin{bmatrix} \hat{\Psi}_\alpha & -\hat{\Psi}_\beta \\ \hat{\Psi}_\beta & \hat{\Psi}_\alpha \end{bmatrix} \begin{bmatrix} \mu \\ \eta \end{bmatrix}, \quad \mu = \frac{\Delta\hat{x}_r \hat{x}_r + \Delta\hat{\omega} \omega}{\hat{\omega}^2 + \hat{x}_r^2}, \quad \eta = \frac{\Delta\hat{x}_r \hat{\omega} - \Delta\hat{\omega} \hat{x}_r}{\hat{\omega}^2 + \hat{x}_r^2},$$

and

$$\begin{bmatrix} \hat{\omega} \\ \hat{x}_r \end{bmatrix} = \frac{1}{\|\hat{\Psi}_r\|^2} \begin{bmatrix} \hat{\Psi}_\beta & -\hat{\Psi}_\alpha \\ \hat{\Psi}_\alpha & \hat{\Psi}_\beta \end{bmatrix} \begin{bmatrix} u_{\phi\alpha} \\ u_{\phi\beta} \end{bmatrix}, \quad (45)$$

$$\begin{bmatrix} \Delta\hat{\omega} \\ \Delta\hat{x}_r \end{bmatrix} = \frac{\sigma_m L_s L_r}{L_m T} \begin{bmatrix} \hat{\Psi}_\alpha & -\hat{\Psi}_\beta \\ \hat{\Psi}_\beta & \hat{\Psi}_\alpha \end{bmatrix} \left(\rho \varepsilon_i + \frac{d\varepsilon_i}{dt} \right). \quad (46)$$

If the sliding mode in the current control loop (41) exists and both $\Delta\hat{\omega} = 0$ and $\Delta\hat{x}_r = 0$, then $\mu = 0$ and $\eta = 0$ and consequently $\hat{\mathbf{e}}_\Psi = \mathbf{0}$. The convergence of the observer is easier to analyse by projecting errors in the (d, q) frame of references as given by (47)

$$\begin{bmatrix} e_d \\ e_q \end{bmatrix} = \begin{bmatrix} \hat{\Psi}_\alpha & -\hat{\Psi}_\beta \\ \hat{\Psi}_\beta & \hat{\Psi}_\alpha \end{bmatrix} \begin{bmatrix} \varepsilon_{\Psi\alpha} \\ \varepsilon_{\Psi\beta} \end{bmatrix}. \quad (47)$$

After some algebra one can find

$$\begin{bmatrix} \frac{de_d}{dt} \\ \frac{de_q}{dt} \end{bmatrix} = - \begin{bmatrix} -k & T_e + \hat{\omega} \\ T_e - \hat{\omega} & -k \end{bmatrix} \begin{bmatrix} e_d \\ e_q \end{bmatrix} + k \|\Psi\| \begin{bmatrix} \mu \\ \eta \end{bmatrix}. \quad (48)$$

The design parameter k could be selected from (48) so that the estimated rotor flux tends to its real value. T_e denotes the electromagnetic torque of the machine.

5 Neural network application in sliding mode systems

As an example of the application of the above ideas in this section a neural network controller for the induction machine will be discussed [21]. The procedure below is valid for all converters and electrical machines since all of them could be presented in the form (34). The structure of the system with a neural network controller is depicted in Fig. 11. For system (34) and the sliding mode manifold selected as given in Table 6. The equivalent control could be determined as $\mathbf{i}_{eq}^{ref} = -(\mathbf{GB}_i)^{-1}(\mathbf{G}\mathbf{f}_x)$. In the system of Fig. 11 the neural network is used to determine the unknown part of equivalent control in the system. The control input could be expressed as

$$\mathbf{i}^{ref} = -(\mathbf{GB}_i)^{-1}(\mathbf{GN}(\mathbf{x}, t)) - (\mathbf{GB}_i)^{-1}\mathbf{D}\boldsymbol{\sigma}, \quad (49)$$

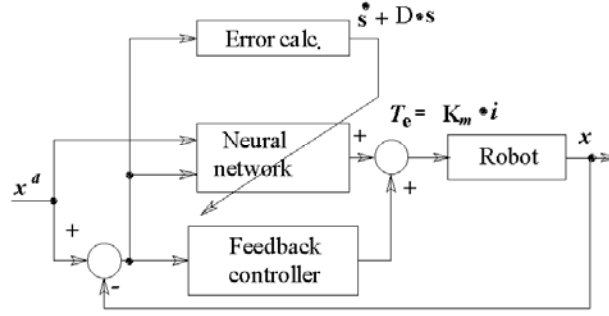


Fig. 11. The structure of the control system.

where $\mathbf{GN}(\mathbf{x}, t)$ is the output of the neural network. This control input gives the time derivative of the Lyapunov function candidate $V = \boldsymbol{\sigma}^T \boldsymbol{\sigma} / 2$ as [20]

$$\frac{dV}{dt} = \boldsymbol{\sigma}^T \frac{d\boldsymbol{\sigma}}{dt} = \boldsymbol{\sigma}^T (\mathbf{G}(\mathbf{f}_x - \mathbf{N}(\mathbf{x}, t))) - \boldsymbol{\sigma}^T \mathbf{D}\boldsymbol{\sigma}. \quad (50)$$

The stability conditions will be satisfied if $\|\boldsymbol{\sigma}^T \mathbf{G}(\mathbf{f}_x - \mathbf{N}(\mathbf{x}, t))\| < \|\boldsymbol{\sigma}^T \mathbf{D}\boldsymbol{\sigma}\|$ holds, in other words if neural network is trained to make zero the error function $\mathbf{Z} = \mathbf{G}(\mathbf{f}(\mathbf{x}, t) - \mathbf{N}(\mathbf{x}, t)) = \mathbf{D}\boldsymbol{\sigma} + \dot{\boldsymbol{\sigma}}$. This means that the neural network approximation error could be calculated from the sliding mode function. An obvious selection of the role of the neural network is to minimise

$$\Gamma = \frac{1}{2} \mathbf{Z}^T \mathbf{Z} = \frac{1}{2} (\mathbf{D}\boldsymbol{\sigma} + \dot{\boldsymbol{\sigma}})^T (\mathbf{D}\boldsymbol{\sigma} + \dot{\boldsymbol{\sigma}}) \quad (51)$$

thus to assure mapping $\mathbf{GN} \rightarrow \mathbf{Gf}$. An important feature of the developed scheme is that the approximation error is available on-line. It is important to notice that the feedback controller assures the stability of the overall system. Assume neural network model

$$y_i^l = g_i^l \left(\sum_{j=1}^k w_{ij}^l a_j^l + b_i^l \right), \quad g_i^l(\text{net}_i^l) = \tanh(\text{net}_i^l) = \frac{2}{1 + e^{-2\text{net}_i^l}} - 1,$$

where a_j^l is the j -th input of the l -th layer, y_i^l is the output of the i -th neuron in the l -th layer, w_{ij}^l is the weight connecting the j -th input and the i -th neuron in the l -th layer and b_i^l is the bias of the i -th neuron in the l -th layer of the neural network. Learning proceeds using a gradient descent where η denotes the learning rate $\Delta w_{ij}^l = -\eta \partial E / \partial w_{ij}^l$. For the output layer, denoted by L , the following equations can be used:

$$\frac{\partial \Gamma}{\partial y_i^{L-1}} = \mathbf{G}(\mathbf{f}_x - \mathbf{N}(\mathbf{x}))(-g_m), \quad (52)$$

$$\Delta w_{ij}^L = \eta g_m (\mathbf{D}\boldsymbol{\sigma} + \dot{\boldsymbol{\sigma}}) g_i^L (a_i^L) y_j^L. \quad (53)$$

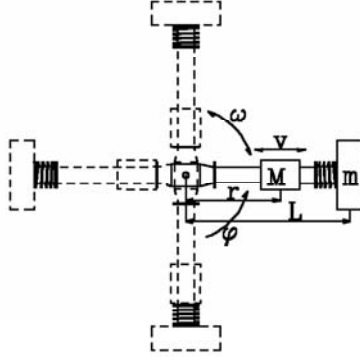


Fig. 12. Experimental system.

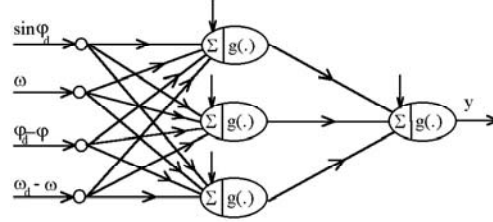


Fig. 13. The structure of the neural network.

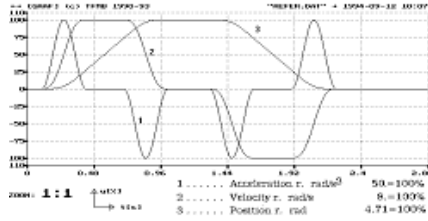


Fig. 14. Desired values of position, velocity and acceleration.

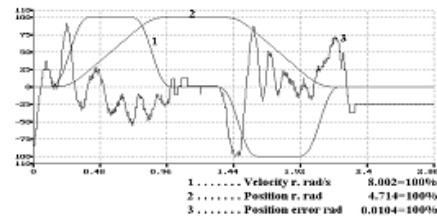


Fig. 15. Experimental results for performing task presented in Fig. 14.

For the hidden layer a back-propagation learning algorithm as is used with $\Delta w_{ij}^l = \eta \delta_j^l y_j^l$.

The mechanism featured in Fig. 12 was used in the experiment [21]. A hammer, mounted on the axis of the induction motor was used and a mass - spring - damper load was added to show effects of coupling. The two layered neural network used as an on-line estimator is featured in Fig. 13. φ_d is the desired position and ω_d is the desired velocity of the system. The system task was to learn to perform the operation, described in Fig. 14. Movement is performed using the so-called $\sin^2$ shaped acceleration. The results are depicted in Fig. 15, and good tracking performance is obtained.

6 Conclusions

A unified approach to control system design for switching converters and electrical machines is discussed. It has been shown that, due to the structural similarities, switching power converters and electrical machines could be analysed in the same framework and the control system structure is the same for both classes of the plant. The switching matrix plays the role of a control element, introducing change in the structure of the system and, thus

making design in the framework of variable structure systems and sliding mode control a natural choice. Engineering methods rather than historical overview of published results is presented. As an example of the possibility of combining neural networks and VSS approaches the induction machine neural network controller is discussed. Due to the appropriate selection of the system's error the learning procedure has been determined so that it does not require any measurements except those needed for the sliding mode function calculation. This error signal has been used for the gradient descent algorithm. Theoretical results have been confirmed by experiments on IM loaded with a nonlinear load.

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On the Development and Application of Sliding Mode Observers

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1 Introduction

This chapter will provide a perspective on the development of sliding mode observers for continuous time systems – primarily those which can be well represented as linear systems subject to bounded nonlinearities/uncertainties. This encompasses a wide range of real engineering systems. Sliding mode observers for systems modelled by nonlinear systems, and whose design methodology explicitly exploits the nonlinear system structure, have recently been considered in [1]; sliding mode observers for discrete time systems is addressed elsewhere in this volume.

Consider initially a nominal linear system

$$\dot{x}(t) = Ax(t) + Bu(t) \quad (1a)$$

$$y(t) = Cx(t) \quad (1b)$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$ and $C \in \mathbb{R}^{p \times n}$. Without loss of generality assume that C has full row rank; i.e. there is no redundancy amongst the measurements. The objective here is to obtain an estimate of the state $x(t)$ by measuring the quantities $y(t)$ and $u(t)$. The design of a linear time invariant dynamical system to estimate $x(t)$ from the inputs and outputs was extensively studied in the 1960's by Luenberger. An algebraic condition on the matrix pair (A, C) – the notion of observability – was proposed as a necessary and sufficient condition for state reconstruction [24]. (For simplicity, throughout this chapter, the observability condition will be assumed to hold, although technically some of the developments only require the weaker restriction of detectability on the pair (A, C) . See for example [3].) One way of viewing the approach of Luenberger is to regard the observer as a model of the system in which the discrepancy between its output, and the output of the system, is feedback through a designer specified gain. Generally speaking, in sliding mode observers, instead of feeding back the output error between the observer and the system linearly, the output error is feedback via a discontinuous switched signal.

2 A Discontinuous Observer

To be specific, consider the dynamical system

$$\dot{z}(t) = Az(t) + Bu(t) + G_n \nu \quad (2)$$

where $G_n \in \mathbb{R}^{n \times p}$ is to be specified, and ν is a discontinuous injection signal depending on the output error $e_y(t) = Cz(t) - y(t)$.

As is common in sliding mode approaches, the analysis is greatly simplified if an appropriate change of coordinates is introduced. In the case of observer design, the coordinate transformation associated with the invertible matrix

$$T_c = \begin{bmatrix} N_c^T \\ C \end{bmatrix} \quad (3)$$

where $N_c \in \mathbb{R}^{n \times (n-p)}$ spans the null-space of C , is appropriate. As a result of this transformation, the triple (A, B, C) has the form

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \quad C = \begin{bmatrix} 0 & I_p \end{bmatrix} \quad (4)$$

where the partitions of A and B are conformable with respect to C . Define the state estimation error $e(t) = z(t) - x(t)$ and suppose (in the new coordinate system) that the output error injection gain

$$G_n = \begin{bmatrix} G_{n,1} \\ I_p \end{bmatrix} \quad (5)$$

where $G_{n,1} \in \mathbb{R}^{(n-p) \times p}$ is a gain matrix to be specified.

Suppose the state estimation error $e(t)$ in the coordinates of (4) is partitioned as (e_1, e_y) then

$$\dot{e}_1(t) = A_{11}e_1(t) + A_{12}e_y(t) + G_{n,1}\nu \quad (6a)$$

$$\dot{e}_y(t) = A_{21}e_1(t) + A_{22}e_y(t) + \nu \quad (6b)$$

As argued in [32], if the components of the discontinuous term

$$\nu_i = -\rho \operatorname{sign} e_{y,i} \quad i = 1 \dots m \quad (7)$$

where $e_{y,i}$ is the i -th component of e_y , then for a large enough scalar ρ an ideal sliding motion is induced in finite time on the surface

$$\mathcal{S} = \{e \in \mathbb{R}^n : e_y = Ce = 0\} \quad (8)$$

During the sliding motion $e_y = \dot{e}_y = 0$ and equation (6b) can be written as

$$0 = A_{21}e_1 + \nu_{eq} \quad (9)$$

where ν_{eq} represents the *equivalent output error injection* term necessary to maintain a sliding motion on $\mathcal{S}$. This is the natural analogue of the so-called equivalent control occurring in the design of sliding mode controllers [32]. Rearranging (9) and substituting back into equation (6a) it follows that the reduced order sliding motion is governed by

$$\dot{e}_1(t) = (A_{11} - G_{n,1}A_{21})e_1(t) \quad (10)$$

In order to be able to sustain a sliding motion, and for (asymptotic) state estimation error decay, the eigenvalues of $(A_{11} - G_{n,1}A_{21})$ must be stable. However if (A, C) is observable then so is (A_{11}, A_{21}) ; see [32] for example. Therefore there exists $G_{n,1}$ for which $(A_{11} - G_{n,1}A_{21})$ is stable.

Consider the example

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad C = [1 \ 1] \quad (11)$$

An appropriate coordinate change to generate the canonical form in (4) is given by $x \mapsto Tx$ where

$$T = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \quad (12)$$

This results in (with abuse of notation)

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad C = [0 \ 1] \quad (13)$$

In the following the gain $G_{n,1} = -1$ which gives $(A_{11} - G_{n,1}A_{21}) = -1$ and the nonlinear gain $\rho = 2$. The sliding mode observer from (2) can then be written as

$$\dot{z}_1(t) = z_2(t) - u(t) + 2\text{sign } e_y(t) \quad (14a)$$

$$\dot{z}_2(t) = -z_1(t) + u(t) - 2\text{sign } e_y(t) \quad (14b)$$

For simplicity assume that $u(t) \equiv 0$. In the simulations which follow the initial condition for the observer (in the transformed coordinates) is $[0 \ 1]^T$ and the initial condition of the system (in the transformed coordinates) is $[1 \ 0]^T$. Figure 1 is a plot of the output estimation error e_y against time. This may also be viewed as the switching function, and as expected, it is forced to zero in finite time.

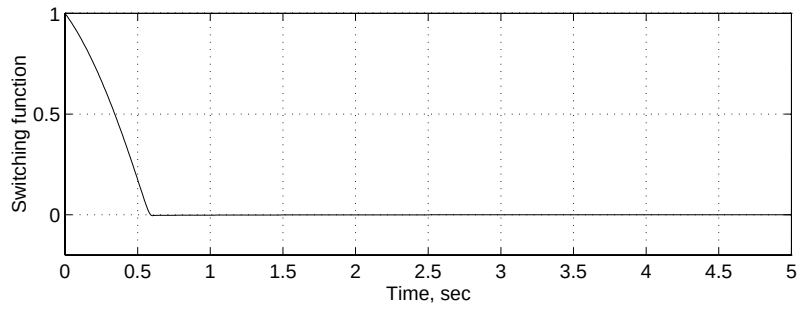


Fig. 1. Switching function

Figure 2 shows a comparison of the states of the system and the states of the plant. The observer states are given by the solid line, and the plant states by a dotted line. It is readily apparent that although $e_y = 0$ in finite time (after approximately 0.5 seconds) the state estimation error $e(t) \rightarrow 0$ exponentially and is different from zero for several seconds.

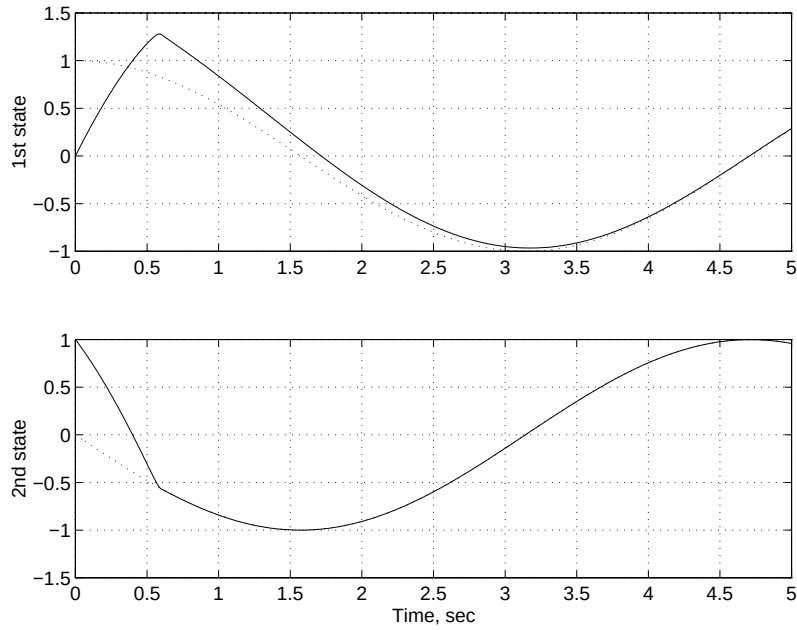


Fig. 2. A comparison of the observer and system states

The description thus far describes the first published¹ work on sliding mode observers by Utkin [32].

An important consequence of inducing a sliding motion is that robust state estimation can be obtained. In the same way that sliding mode controllers exhibit complete rejection of a class of uncertainty, so will the observer just described. Suppose that the real system is governed by

$$\ddot{x}(t) = -\sin x(t) \quad (15)$$

and that the measured output corresponds to

$$y(t) = x(t) + \dot{x}(t) \quad (16)$$

¹ It is more accurate to say the first published work in English on sliding observers is [32]. Earlier published work on this theme appears in Russian; see for example the references in [8].

Physically this may be viewed as a pendulum system where $x(t)$ represents angular displacement from the vertical. The system in (11) may be viewed as a linearization of the system above at the equilibrium point $(0,0)$. Using the observer given in (14a)-(14b), the error system is governed by

$$\dot{e}_1(t) = e_y(t) - \sin x_1(t) - \nu \quad (17a)$$

$$\dot{e}_y(t) = -e_1(t) + \sin x_1(t) + \nu \quad (17b)$$

where $\nu = -2\text{sign } e_y$. During the sliding motion $e_y = \dot{e}_y = 0$ and

$$\nu_{eq} = e_1(t) - \sin x_1(t) \quad (18)$$

Substituting into equation (17a) the sine terms cancel leaving

$$\dot{e}_1(t) = -e_1(t) \quad (19)$$

The sliding motion is therefore independent of the nonlinearity/uncertainty resulting from the linearization, and the observer states track the real states asymptotically. Once again, in the simulations which follow the initial condition for the observer is given by $[0 \ 1]^T$ and the initial condition of the (nonlinear) system is $[1 \ 0]^T$. Figure 3 is a plot of the output estimation error e_y against time. Again the switching function is forced to zero in finite time.

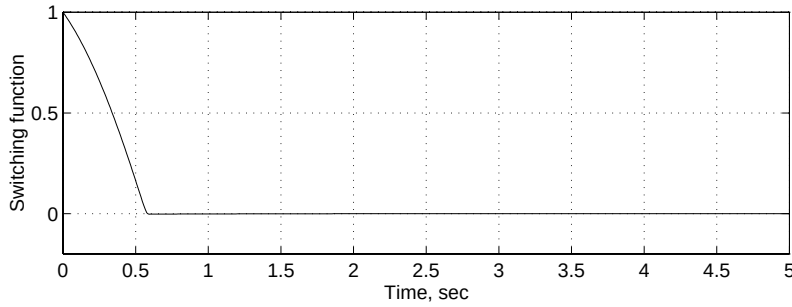


Fig. 3. Switching function

Figure 4 shows a comparison of the states of the system and the states of the plant. Once again perfect asymptotic tracking is obtained despite the mismatch between the system and the linear model about which the observer was designed.

3 Observers with Linear and Discontinuous Injection

The formulation just described generally requires large values of ρ in order to ensure sliding for a broad range of initial state estimation errors – particularly

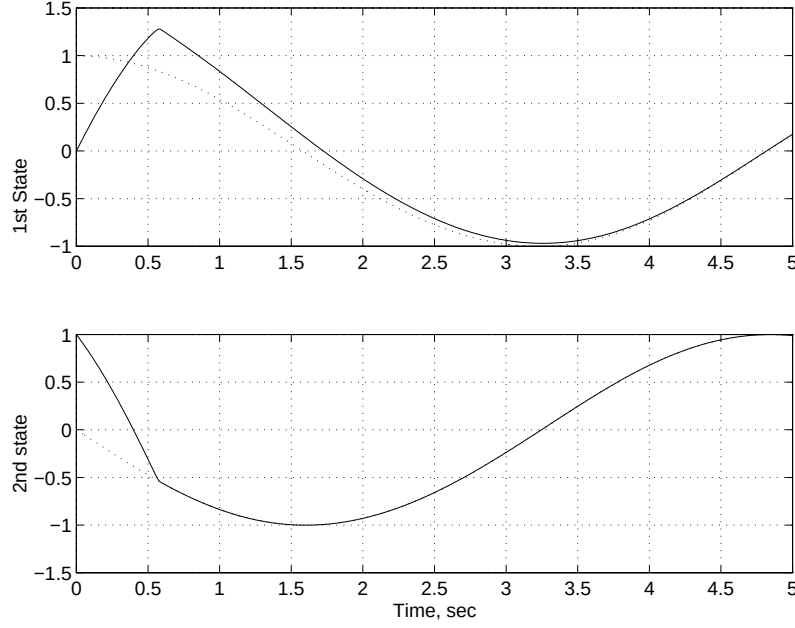


Fig. 4. A comparison of the observer and system states

if the underlying system is unstable. As discussed in [7], from a practical viewpoint, this may cause difficulties. A trade-off is usually necessary between the requirement of a large ρ to ensure a sliding mode occurs and its subsequent reduction to prevent excessive chattering (whilst still ensuring sliding). For this reason it may be preferable to include a linear output error injection term. Rather than the formulation described in (2) consider the dynamical system

$$\dot{z}(t) = Az(t) + Bu(t) - G_l e_y(t) + G_n \nu \quad (20)$$

where $G_l, G_n \in \mathbb{R}^{n \times p}$ and $e_y(t) = Cz(t) - y(t)$. This is effectively the observer structure of Slotine *et al.* [27], where it is argued that the linear gain should be chosen to enhance the size of the so-called sliding patch i.e. the domain in the state estimation error space in which sliding occurs. With a well designed linear gain G_l this observer enjoys the same robustness properties. Indeed in certain situations (which will be discussed in more detail later) global state estimation error convergence properties can be proven. To demonstrate this, consider once more the nonlinear system (15)-(16) and its linearization (11). Again change coordinates according to (12) to obtain the realization in (13). The observer design and analysis will be performed in this coordinate system. Choose as the output error injection feedback gains

$$G_l = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \quad G_n = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad (21)$$

It can readily be shown that $\lambda(A - G_l C) = \{-1, -1\}$. Furthermore the state estimation error system is governed by

$$\dot{e}_1(t) = e_y(t) - \sin x_1(t) - \nu \quad (22a)$$

$$\dot{e}_y(t) = -e_1(t) - 2e_y(t) + \sin x_1(t) + \nu \quad (22b)$$

Consider now the quadratic form

$$V(e_1, e_y) = e_1^2 + 2e_1 e_y + 2e_y^2 = (e_1 + e_y)^2 + e_y^2 \quad (23)$$

as a candidate Lyapunov function for the error system above. Clearly the expression in (23) is positive definite. Furthermore, after some algebra, it can be shown that

$$\begin{aligned} \dot{V} &= -2e_1^2 - 6e_y e_1 - 6e_y^2 + 2e_y \sin x_1 + 2e_y \nu \\ &\leq -2e_1^2 - 6e_y e_1 - 6e_y^2 + 2|e_y| |\sin x_1| - 2\rho|e_y| \\ &\leq -2e_1^2 - 6e_y e_1 - 6e_y^2 - |e_y| \\ &\leq -2(e_1 + \frac{3}{2}e_y)^2 - \frac{3}{2}e_y^2 - |e_y| \leq 0 \end{aligned}$$

and therefore (global) asymptotic stability of the error system has been proven.

4 The Walcott and Żak Observer

The problem of robust state estimation for systems with bounded matched uncertainty was first explored by Walcott & Żak [34]. They sought to design a sliding mode observer for the system

$$\dot{x}(t) = Ax(t) + Bu(t) + Bf(t, y, u) \quad (24a)$$

$$y(t) = Cx(t) \quad (24b)$$

where $f : \mathbb{R}_+ \times \mathbb{R}^p \times \mathbb{R}^m \mapsto \mathbb{R}^m$ represents lumped uncertainty or nonlinearities. The function is assumed to be unknown but bounded so that

$$\|f(t, y, u)\| \leq \rho(t, y, u) \quad (25)$$

where $\rho(\cdot)$ is known. The observer proposed in [34] has the form

$$\dot{z}(t) = Az(t) + Bu(t) - GCe(t) - P^{-1}C^T F^T \nu \quad (26)$$

where the discontinuous scaled unit vector term

$$\nu = \rho(t, y, u) \frac{F C e(t)}{\|F C e(t)\|} \quad (27)$$

and $e(t) = z(t) - x(t)$. The symmetric positive definite matrix $P \in \mathbb{R}^{n \times n}$ and the gain matrix G are assumed to satisfy

$$PA_0 + A_0^T P < 0 \quad (28)$$

where $A_0 := A - GC$, and the structural constraint

$$PB = (FC)^T \quad (29)$$

for some $F \in \mathbb{R}^{m \times p}$. Under these circumstances the quadratic form given by $V(e) = e^T P e$ can be shown to guarantee quadratic stability [34]. Furthermore an ideal sliding motion takes place on

$$\mathcal{S}_F = \{e \in \mathbb{R}^n : F C e = 0\}$$

in finite time.

Remarks:

- If $p > m$ then sliding on $\mathcal{S}_F$ is not the same as sliding on $Ce(t) = 0$ and so the observer structure in (26) is different from the observer in (20).
- A system theoretic interpretation of (28)-(29) by Steinberg & Corless [29] is that the transfer function matrix $G(s) = FC(sI - A_0)^{-1}B$ is strictly positive real.

4.1 Synthesizing the Gains

The problem of synthesizing P, G and F (and incorporating some sort of design element) is non-trivial. The original work in [34] postulated the use of symbolic manipulation tools to solve a sequence of constraints arising from ensuring that the principal minors of both P and the right hand-side of (28) are positive and negative respectively. In the original work it is also not apparent for which class of systems the design problem has a solution. For low order systems the synthesis problem is quite tractable: the observer designed for the pendulum described earlier is in fact a Walcott & Żak observer. The structural requirements (28)-(29) of Walcott & Żak were shown in [14] to be solvable if and only if

- $\text{rank}(CB) = m$
- any invariant zeros of $(A, B, C) \in \mathbb{C}_-$

Details of the constructive design algorithms are described in [14] and follow on from earlier work described in [9]. Under these circumstances a (semi) analytic expression for the solution to the observer design problem is given in terms of a gain matrix $L \in \mathbb{R}^{(n-p) \times (p-m)}$ and a stable matrix $A_{22}^s \in \mathbb{R}^{p \times p}$.

5 A Linear Matrix Inequality Design Method

This section considers a more general problem than that of Walcott & Żak and proposes a numerically based solution methodology which exploits all the degrees of freedom available in the design.

Consider the dynamical system

$$\dot{x}(t) = Ax(t) + Bu(t) + D\xi(t, x, u) \quad (30a)$$

$$y(t) = Cx(t) \quad (30b)$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{p \times n}$ and $D \in \mathbb{R}^{n \times q}$ where $p \geq q$. Assume C and D are full rank and the function $\xi : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^q$ is unknown but bounded so that

$$\|\xi(t, x, u)\| \leq r_1\|u\| + \alpha(t, y) \quad (31)$$

where r_1 is a known scalar and $\alpha : \mathbb{R}_+ \times \mathbb{R}^p \rightarrow \mathbb{R}_+$ is a known function.

An observer of the form (as described earlier in §3)

$$\dot{z}(t) = Az(t) + Bu(t) - G_l e_y(t) + G_n \nu \quad (32)$$

will be considered where $G_l \in \mathbb{R}^{n \times p}$ and $G_n \in \mathbb{R}^{n \times p}$ and $e_y(t) = Cz(t) - y(t)$. The discontinuous vector ν is defined by

$$\nu = \begin{cases} -\rho(t, y, u) \|P_o \mathcal{D}_2\| \frac{e_y}{\|e_y\|} & \text{if } e_y \neq 0 \\ 0 & \text{otherwise} \end{cases} \quad (33)$$

where $P_o \in \mathbb{R}^{p \times p}$ is symmetric positive definite. The matrices P_o and $\mathcal{D}_2$ will be defined formally later. The function $\rho : \mathbb{R}_+ \times \mathbb{R}^p \times \mathbb{R}^m \rightarrow \mathbb{R}_+$ satisfies

$$\rho(t, y, u) \geq r_1\|u\| + \alpha(t, y) + \gamma_0 \quad (34)$$

where γ_0 is a positive scalar. If the state estimation error $e(t) := z(t) - x(t)$, then it is straightforward to show from equations (30a)-(30b) and (32) that

$$\dot{e}(t) = A_0 e(t) + G_n \nu - D\xi(t, x, u) \quad (35)$$

where $A_0 := A - G_l C$. In [14] it is argued that necessary and sufficient conditions for the existence of a stable sliding motion on

$$\mathcal{S} = \{e \in \mathbb{R}^n : e_y = 0\}$$

that is independent of ξ are

1. $\text{rank}(CD) = q$
2. any invariant zeros of (A, D, C) lie in the left half plane

These of course are analogous to those for the Walcott & Żak observer described in §4.1. Again a (semi) analytic expression for the gains G_l and G_n in terms of a matrix $L \in \mathbb{R}^{(n-p) \times (p-q)}$ and a stable matrix $A_{22}^s \in \mathbb{R}^{p \times p}$ can be demonstrated [14]. This parameterization however only represents a specific subclass of possible solutions. Additional degrees of freedom are available which are not exploited because the simplicity of solution is lost. The next section considers the design of the matrices G_l , G_n and P_o so that a sliding motion takes place on $\mathcal{S}$. Assuming that conditions 1 and 2 are satisfied, a new parameterization is given which seeks to exploit all of the design freedom which is available.

5.1 A convex parameterization

A canonical form from [9] will constitute a useful starting point. If conditions 1 and 2 above are satisfied there exists a coordinate transformation T_0 in which the system (A, D, C) can be written as

$$\bar{A} = \left[\begin{array}{c|c} \bar{A}_{11} & \bar{A}_{12} \\ \hline \bar{A}_{211} & \bar{A}_{22} \end{array} \right] \quad \bar{D} = \begin{bmatrix} 0 \\ \bar{D}_2 \end{bmatrix} \quad \bar{C} = [0 \ T] \quad (36)$$

where the sub-matrices $\bar{A}_{11} \in \mathbb{R}^{(n-p) \times (n-p)}$, $\bar{A}_{211} \in \mathbb{R}^{(p-q) \times (p-n)}$ represent a detectable pair; $\bar{D}_2 \in \mathbb{R}^{q \times q}$ is nonsingular; and $T \in \mathbb{R}^{p \times p}$ is orthogonal. Further the unobservable modes of $(\bar{A}_{11}, \bar{A}_{211})$ are the invariant zeros of (A, D, C) [12]. In order to make the partitions in (36) conformable it is convenient to introduce the term

$$D_2 := \begin{bmatrix} 0_{(p-q) \times q} \\ \bar{D}_2 \end{bmatrix} \quad (37)$$

Let $\bar{G}_l$ and $\bar{G}_n$ represent the observer gain matrices in the new coordinate system and define $\bar{A}_0 = \bar{A} - \bar{G}_l \bar{C}$. The gain matrix $\bar{G}_l$ is to be determined but assume

$$\bar{G}_n = \begin{bmatrix} -\bar{L}T^T \\ T^T \end{bmatrix} P_o^{-1} \quad (38)$$

where $\bar{L} \in \mathbb{R}^{(n-p) \times p}$ and $\bar{L} = [L \ 0]$ with $L \in \mathbb{R}^{(n-p) \times (p-q)}$; and the orthogonal matrix T is part of the output distribution matrix $\bar{C}$ from (36).

Proposition 1. *If there exists a positive definite Lyapunov matrix $\bar{P}$, that satisfies $\bar{P}\bar{A}_0 + \bar{A}_0^T \bar{P} < 0$, with the structure*

$$\bar{P} = \begin{bmatrix} \bar{P}_1 & \bar{P}_1 \bar{L} \\ \bar{L}^T \bar{P}_1 & \bar{P}_2 + \bar{L}^T \bar{P}_1 \bar{L} \end{bmatrix} > 0 \quad (39)$$

where $\bar{P}_1 \in \mathbb{R}^{(n-p) \times (n-p)}$ and $\bar{P}_2 \in \mathbb{R}^{p \times p}$, then the error system in equation (35) is quadratically stable.

Proof : Consider the quadratic form given by

$$\mathcal{V}(\bar{e}) = \bar{e}^T \bar{P} \bar{e} \quad (40)$$

as a candidate Lyapunov function where $\bar{e} := T_0 e$. Notice that if $\bar{P}_1, \bar{P}_2 > 0$ then $\bar{P} > 0$ from the Schur expansion. From (35) the derivative along the system trajectory

$$\dot{\mathcal{V}} = \bar{e}^T (\bar{A}_0^T \bar{P} + \bar{P} \bar{A}_0) \bar{e} + 2\bar{e}^T \bar{P} \bar{G}_n \nu - 2\bar{e}^T \bar{P} \bar{D} \xi \quad (41)$$

From the definitions in (36), (38) and (39)

$$\bar{P} \bar{G}_n = \begin{bmatrix} 0 \\ \bar{P}_2 T^T \end{bmatrix} P_o^{-1} = \bar{C}^T \quad (42)$$

if the symmetric positive definite matrix

$$P_o := T\bar{P}_2T^T \quad (43)$$

Using the special structures of $\bar{L}$ and D_2 , $\bar{L}D_2 = 0$ and therefore

$$\bar{P}\bar{D} = \begin{bmatrix} 0 \\ \bar{P}_2D_2 \end{bmatrix} = \bar{C}^T P_o \mathcal{D}_2 \quad (44)$$

if the matrix

$$\mathcal{D}_2 := T D_2 \quad (45)$$

Consequently, (41) becomes

$$\begin{aligned} \dot{\bar{V}} &= \bar{e}^T (\bar{A}_0^T \bar{P} + \bar{P} \bar{A}_0) \bar{e} + 2e_y^T \nu - 2e_y^T P_o \mathcal{D}_2 \xi \\ &\leq \bar{e}^T (\bar{A}_0^T \bar{P} + \bar{P} \bar{A}_0) \bar{e} - 2\rho \|P_o \mathcal{D}_2\| \|e_y\| - 2e_y^T P_o \mathcal{D}_2 \xi \end{aligned}$$

Using the uncertainty bounds for ξ from equations (31) and (34)

$$\begin{aligned} \dot{\bar{V}} &\leq \bar{e}^T (\bar{A}_0^T \bar{P} + \bar{P} \bar{A}_0) \bar{e} - 2\rho \|P_o \mathcal{D}_2\| \|e_y\| + 2\|P_o \mathcal{D}_2\| [r_1 \|u\| + \alpha(y)] \|e_y\| \\ &\leq \bar{e}^T (\bar{A}_0^T \bar{P} + \bar{P} \bar{A}_0) \bar{e} - 2\gamma_0 \|P_o \mathcal{D}_2\| \|e_y\| \end{aligned}$$

Since $(\bar{A}_0^T \bar{P} + \bar{P} \bar{A}_0) < 0$ it follows that $\dot{\bar{V}} < 0$ for all $\bar{e} \neq 0$. $\square$

Corollary 1. *An ideal sliding motion takes place on S in finite time. Furthermore the sliding dynamics are given by the system matrix $\bar{A}_{11} + L\bar{A}_{211}$.*

Proof : Using Proposition 1, a modification to Corollary 6.1 in [12] shows that sliding takes place on S in finite time. Using the concept of equivalent output error injection, the sliding motion is governed by

$$(I - \bar{G}_n(\bar{C}\bar{G}_n)^{-1}\bar{C})\bar{A}_0 = \begin{bmatrix} \bar{A}_{11} + L\bar{A}_{211} & \bar{A}_{12} + \bar{L}\bar{A}_{22} \\ 0 & 0 \end{bmatrix}$$

Hence the sliding motion is governed by $\bar{A}_{11} + L\bar{A}_{211}$ as claimed. $\square$

Remarks: Since $(\bar{A}_{11}, \bar{A}_{211})$ is detectable by construction, there exists a family of matrices $L \in \mathbb{R}^{(n-p) \times (p-q)}$ such that $\bar{A}_{11} + L\bar{A}_{211}$ is stable. If a further linear change of co-ordinates

$$T_L = \begin{bmatrix} I_{n-p} & \bar{L} \\ 0 & T \end{bmatrix} \quad (46)$$

is applied to the triple $(\bar{A}, \bar{D}, \bar{C})$ and its Lyapunov matrix $\bar{P}$, the system matrix, disturbance distribution matrix and the output distribution matrix will be in the form

$$\mathcal{A} = \begin{bmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} \\ \mathcal{A}_{21} & \mathcal{A}_{22} \end{bmatrix} \quad \mathcal{D} = \begin{bmatrix} 0 \\ \mathcal{D}_2 \end{bmatrix} \quad \mathcal{C} = [0 \quad I_p] \quad (47)$$

where $\mathcal{A}_{11} = \bar{A}_{11} + L\bar{A}_{211}$. In the new co-ordinate system, the Lyapunov matrix will be

$$\mathcal{P} = (T_L^{-1})^T \bar{P} (T_L^{-1}) = \begin{bmatrix} \bar{P}_1 & 0 \\ 0 & P_o \end{bmatrix} \quad (48)$$

where P_o is defined in (43). The nonlinear output error injection gain matrix

$$\mathcal{G}_n = \begin{bmatrix} 0 \\ P_o^{-1} \end{bmatrix} \quad (49)$$

As argued in [9], the fact that $\mathcal{P}$ is block diagonal Lyapunov matrix for $\mathcal{A}_0 = \mathcal{A} - \mathcal{G}_l \mathcal{C}$ implies that $\mathcal{A}_{11}$ is stable and hence the sliding motion is stable.

The remainder of this chapter focuses on *design methods* to synthesise the gain $\bar{G}_l$ and the Lyapunov matrix $\bar{P}$ which has the structure given in (39). The problems will be posed in such a way that Linear Matrix Inequalities (LMIs) [2] can be used to numerically synthesise the required matrices.

5.2 Synthesis procedure for the gain matrices

In this section, $\bar{P}$ and $\bar{G}_l$ will be chosen so that the matrix inequality

$$\bar{A}_0^T \bar{P} + \bar{P} \bar{A}_0 < -\bar{P} W \bar{P} - \bar{P} \bar{G}_l V \bar{G}_l^T \bar{P} \quad (50)$$

is satisfied, where the design weighting matrices W and V are assumed to be symmetric positive definite, and $\bar{P}$ has the structure in (39). Substituting for $\bar{A}_0$, the inequality (50) can be written as

$$\bar{A}^T \bar{P} + \bar{P} \bar{A} - (\bar{Y} \bar{C})^T - \bar{Y} \bar{C} + \bar{P} W \bar{P} + \bar{Y} V \bar{Y}^T < 0 \quad (51)$$

where $\bar{Y} := \bar{P} \bar{G}_l$. Using standard matrix manipulations, inequality (51) is identical to

$$\bar{P} \bar{A} + \bar{A}^T \bar{P} + (\bar{Y}^T - V^{-1} \bar{C})^T V (\bar{Y}^T - V^{-1} \bar{C}) - \bar{C}^T V^{-1} \bar{C} + \bar{P} W \bar{P} < 0 \quad (52)$$

Using inequality (52), the necessary and sufficient condition for (51) to hold is that $\bar{P}$ satisfies

$$\bar{P} \bar{A} + \bar{A}^T \bar{P} - \bar{C}^T V^{-1} \bar{C} + \bar{P} W \bar{P} < 0 \quad (53)$$

since choosing

$$\bar{Y}^T = V^{-1} \bar{C} \quad (54)$$

eliminates the third term in (52). The problem considered here is one of minimizing $\text{trace}(\bar{P}^{-1})$ subject to $\bar{P}$ satisfying inequality (53). The observer gain $\bar{G}_l$ can then be directly calculated as $\bar{G}_l = \bar{P}^{-1} \bar{C}^T V^{-1}$ which follows

from equation (54) and the definition of $\bar{Y}$. The matrix inequality in (53) is equivalent to

$$\begin{bmatrix} \bar{P}\bar{A} + \bar{A}^T\bar{P} - \bar{C}^T V^{-1} \bar{C} & \bar{P} \\ \bar{P} & -W^{-1} \end{bmatrix} < 0 \quad (55)$$

by using the Schur complement. If $\bar{X} \in \mathbb{R}^{n \times n}$ is symmetric positive definite then the LMI

$$\begin{bmatrix} -\bar{P} & I \\ I & -\bar{X} \end{bmatrix} < 0 \quad (56)$$

is equivalent to $\bar{X} > \bar{P}^{-1}$. Thus minimizing $\text{trace}(\bar{P}^{-1})$ subject to (53) is equivalent to minimizing $\text{trace}(\bar{X})$ subject to the LMIs (55) and (56). Writing $\bar{P}$ from (39) as

$$\bar{P} = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix} > 0 \quad (57)$$

where $P_{11} \in \mathbb{R}^{(n-p) \times (n-p)}$, $P_{22} \in \mathbb{R}^{p \times p}$ and

$$P_{12} := [P_{121} \ 0] \quad (58)$$

with $P_{121} \in \mathbb{R}^{(n-p) \times (p-q)}$, it follows there is a one-to-one correspondence between the variables $(P_{11}, P_{121}, P_{22})$ and $(\bar{P}_1, L, \bar{P}_2)$ since

$$P_{11} = \bar{P}_1 \quad (59a)$$

$$L = P_{11}^{-1} P_{121} \quad (59b)$$

$$\bar{P}_2 = P_{22} - P_{12}^T P_{11}^{-1} P_{12} \quad (59c)$$

The problem can then be formally defined as

Minimize $\text{trace}(\bar{X})$ with respect to P_{11}, P_{121}, P_{22} and $\bar{X}$ subject to (55), (56) and (57).

This represents a convex optimization problem. Standard LMI software, such as [16], can be employed to synthesise numerically $\bar{P}$ and $\bar{X}$.

Remark: The motivation for the choice of the inequality posed in (50), and the minimisation of $\text{trace}(\bar{P}^{-1})$ subject to (53) and (56), is that in the absence of uncertainty and as $\gamma_0 \rightarrow 0$, the observer behaves with sub-optimal Linear Quadratic Gaussian performance properties. For details see [31].

5.3 Design of the sliding motion system matrix

In the previous section, the dynamics of the sliding motion, although guaranteed to be stable, were designed somewhat implicitly. This section considers the sliding motion design problem and shows how additional LMI constraints

can be augmented with those used previously to tune the sliding mode performance. Using the new co-ordinates obtained after applying the transformation T_L in equation (46), the matrix inequality (53) can be written as

$$\mathcal{P}\mathcal{A} + \mathcal{A}^T\mathcal{P} - \mathcal{C}^TV^{-1}\mathcal{C} + \mathcal{P}\mathcal{W}\mathcal{P} < 0 \quad (60)$$

where $\mathcal{W} := T_L \mathcal{W} T_L^T$. The top left block of (60) can be identified to be

$$\bar{P}_1\mathcal{A}_{11} + \mathcal{A}_{11}^T\bar{P}_1 + \bar{P}_1\mathcal{W}_1\bar{P}_1 < 0 \quad (61)$$

where $\mathcal{W}_1 \in \mathbb{R}^{(n-p) \times (n-p)} > 0$ is the top left sub-block of the matrix $\mathcal{W}$ and $\mathcal{A}_{11} = \bar{A}_{11} + L\bar{A}_{211}$. If the weighting matrix W is partitioned as

$$W = \begin{bmatrix} W_{11} & W_{12} \\ W_{12}^T & W_{22} \end{bmatrix} \quad (62)$$

where $W_{11} \in \mathbb{R}^{(n-p) \times (n-p)}$ and W_{12} is the null space of $\bar{L}^T$ for all L , then inequality (61) can be written as

$$\bar{P}_1\mathcal{A}_{11} + \mathcal{A}_{11}^T\bar{P}_1 + \bar{P}_1W_{11}\bar{P}_1 + \bar{P}_1\bar{L}W_{22}\bar{L}^T\bar{P}_1 < 0 \quad (63)$$

This is identical in structure to inequality (50) and hence W_{11} and W_{22} may be interpreted as playing the roles of performance and noise attenuation matrices in a Linear Quadratic Gaussian sense for the observer problem associated with the pair $(\bar{A}_{11}, \bar{A}_{211})$. Thus the choice of W_{11} and W_{22} can be used to tune the sliding motion. However since

$$\bar{P}_1\mathcal{A}_{11} = P_{11}\bar{A}_{11} + P_{121}\bar{A}_{211} \quad (64)$$

which is affine with respect to the LMI optimization variables P_{11} and P_{121} , additional LMIs can be employed together with (55) and (56) to tune the sliding mode performance. One approach is to use root clustering [17] methods to achieve pole placement of $\mathcal{A}_{11}$ in regions of the complex plane. Typically the poles may be required to lie in

- a conic sector centred at (0,0) with inner angle θ_a
- a disc of radius r_a and centre $(q_a, 0)$
- a vertical strip $a_a < x < b_a$

Chilali and Gahinet [6] prove that the following inequalities describe these regions

$$\begin{bmatrix} (\bar{P}_1\mathcal{A}_{11} + \mathcal{A}_{11}^T\bar{P}_1) \sin \frac{1}{2}\theta_a & -(\bar{P}_1\mathcal{A}_{11} - \mathcal{A}_{11}^T\bar{P}_1) \cos \frac{1}{2}\theta_a \\ (\bar{P}_1\mathcal{A}_{11} - \mathcal{A}_{11}^T\bar{P}_1) \cos \frac{1}{2}\theta_a & (\bar{P}_1\mathcal{A}_{11} + \mathcal{A}_{11}^T\bar{P}_1) \sin \frac{1}{2}\theta_a \end{bmatrix} < 0 \quad (65a)$$

$$\begin{bmatrix} -r_a\bar{P}_1 & \bar{P}_1\mathcal{A}_{11} - q_a\bar{P}_1 \\ \mathcal{A}_{11}^T\bar{P}_1 - q_a\bar{P}_1 & -r_a\bar{P}_1 \end{bmatrix} < 0 \quad (65b)$$

$$\begin{bmatrix} \bar{P}_1\mathcal{A}_{11} + \mathcal{A}_{11}^T\bar{P}_1 - 2b_a\bar{P}_1 & 0 \\ 0 & -(\bar{P}_1\mathcal{A}_{11} + \mathcal{A}_{11}^T\bar{P}_1) + 2a_a\bar{P}_1 \end{bmatrix} < 0 \quad (65c)$$

Writing $\bar{P}_1 \mathcal{A}_{11}$ as in (64), and substituting into (65a) - (65c), results in a well-posed convex problem since the inequalities (65a) - (65c) are affine in P_{11} and P_{121} . A new optimization problem can be stated as

Minimize $trace(\bar{X})$ with respect to P_{11}, P_{121}, P_{22} and $\bar{X}$ subject to the LMIs (56), (55), (57) and (65a) - (65c).

5.4 An example

The design method described in §5 will now be demonstrated by using a 7th order aircraft example taken from [20]. The system matrices are

$$A = \begin{bmatrix} 0 & 0 & 1.0000 & 0 & 0 & 0 & 0 \\ 0 & -0.1540 & -0.0042 & 1.5400 & 0 & -0.7440 & -0.0320 \\ 0 & 0.2490 & -1.0000 & -5.2000 & 0 & 0.3370 & -1.1200 \\ 0.0386 & -0.9960 & -0.0003 & -2.1170 & 0 & 0.0200 & 0 \\ 0 & 0.5000 & 0 & 0 & -4.0000 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -20.0000 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -25.0000 \end{bmatrix}$$

$$D^T = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 20 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 25 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & -0.1540 & -0.0042 & 1.5400 & 0 & -0.7440 & -0.0320 \\ 0 & 0.2490 & -1.0000 & -5.2000 & 0 & 0.3370 & -1.1200 \\ 1.0000 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.0000 & 0 & 0 \end{bmatrix}$$

where the states are respectively bank angle (rad), yaw rate (rad/s), roll rate (rad/s), sideslip angle (rad), washout filter state, rudder deflection (rad) and aileron deflection (rad). The inputs are rudder command(rad) and aileron command(rad). The measured outputs are roll acceleration (rad/s^2), yaw acceleration(rad/s^2), bank angle (rad) and washout filter state. Here D is chosen to be the input distribution matrix.

The eigenvalues of the sliding motion represented by the system matrix $\mathcal{A}_{11}$ were required to lie in the intersection of the following regions:

- a circle of centre (0,0) and radius 5
- a vertical upper bound at $x = -2$
- a conic sector symmetric about the real axis, with inner angle $\theta = 80^\circ$

The weighting matrices associated with the Ricatti-like inequality (53) were chosen respectively as $W = \text{diag}(0.08, 0.08, 0.08, 0.08, 0.08, 0.05, .05)$ and $V = 0.2 I_2$. The following matrices were obtained for the canonical form

described in (36):

$$\bar{A} = \begin{bmatrix} -2.0722 & 5.0994 & 1.6893 & 0 & -0.1801 & 0.5527 & -0.6465 \\ 0.0000 & -0.0000 & -0.0000 & 0 & 0.0000 & -0.4607 & -0.8969 \\ 0.0000 & 0.0000 & 0.0000 & 0 & -0.0000 & 0.9884 & -0.4625 \\ -0.0000 & 0.4962 & 0.0226 & -4.0000 & 0 & 0.0000 & 0.0000 \\ 0.0000 & 0.0122 & -0.9159 & 0 & 0 & 0 & 0.0000 \\ -20.1535 & -3.2909 & -6.9001 & 0 & 0.1583 & -25.7219 & 0.6257 \\ 15.4751 & 5.0661 & -1.3973 & 0 & -0.1370 & 2.3305 & -20.4769 \end{bmatrix}$$

$$\bar{D}^T = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 16.3353 \\ 0 & 0 & 0 & 0 & 0 & 25.8356 & -10.8242 \end{bmatrix}$$

$$\bar{C} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -0.4126 & -0.9109 \\ 0 & 0 & 0 & 0 & 0 & -0.9109 & 0.4126 \\ 0 & 0 & 0 & 0 & 1.0000 & 0 & 0 \\ 0 & 0 & 0 & 1.0000 & 0 & 0 & 0 \end{bmatrix}$$

From this representation

$$\bar{A}_{11} = \begin{bmatrix} -2.0722 & 5.0994 & 1.6893 \\ 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 \end{bmatrix} \quad \bar{A}_{211} = \begin{bmatrix} 0.0000 & 0.4962 & 0.0226 \\ 0.0000 & 0.0122 & -0.9159 \end{bmatrix}$$

The triple $(\bar{A}_{11}, \bar{A}_{211})$ has an unobservable mode and thus the system has an invariant zero at -2.0722.

Following the synthesis procedure in §5.2 and imposing the design constraints described earlier the following matrices were obtained:

$$L = \begin{bmatrix} -3.5281 & -0.1519 \\ -1.6175 & -0.2722 \\ 0.5549 & 0.8274 \end{bmatrix} \quad \text{and} \quad P_o = \begin{bmatrix} 1.8468 & 0.4484 & 0.2924 & 3.3827 \\ 0.4484 & 0.2267 & 0.0900 & 0.4751 \\ 0.2924 & 0.0900 & 2.9843 & -1.3817 \\ 3.3827 & 0.4751 & -1.3817 & 84.7852 \end{bmatrix}$$

The associated gain matrices (in the original coordinates) are

$$G_l = \begin{bmatrix} -0.3118 & -0.1570 & 1.7300 & 0.0415 \\ -0.3436 & 0.2567 & 0.4682 & 0.1131 \\ -0.3279 & -0.0215 & 1.3387 & 0.0662 \\ 0.1763 & -0.1187 & -0.2839 & -0.0560 \\ -0.1715 & 0.1856 & 0.0415 & 0.0655 \\ -7.4899 & 15.5815 & -0.2850 & 0.0872 \\ 6.7480 & -33.0829 & 0.2813 & 0.0864 \end{bmatrix}$$

$$G_n = \begin{bmatrix} 0 & 0 & 1.0000 & 0 \\ 0.0000 & -0.0000 & 0.2327 & 1.5800 \\ -0.0000 & 0.0000 & 0.7611 & 0.5280 \\ 0.0000 & -0.0000 & -0.1458 & -0.7626 \\ 0 & 0 & 0 & 1.0000 \\ -1.3269 & 0.0379 & -0.3518 & -2.0294 \\ -0.3993 & -0.8814 & -0.0568 & 2.8100 \end{bmatrix}$$

The eigenvalues of $\mathcal{A}_0$ are

$$\{-70.3610, -23.6149, -4.0551 - 0.7589 \pm 0.9881i, -1.9097, -1.2967\}$$

and the eigenvalues of $\mathcal{A}_{11}$ are $\{-2.0722, -0.5273, -1.0239\}$

6 Fault Detection and Isolation

It has been seen that numerically tractable algorithms are available for the design of sliding observers. One obvious application for such observers is as estimators in state feedback control schemes [10,22]. Another, perhaps less obvious, use is in fault detection and isolation (FDI) schemes. The fundamental purpose of an FDI scheme is to generate an alarm when an abnormal condition, such as a component malfunction or variation in operating condition, develops in the process being monitored and to identify the source or location. Overviews and surveys of early work are given in [26,15,5]. Many of the approaches proposed in the literature utilise Luenberger observers or Kalman filters designed around mathematical models of the system to generate estimates of the system states. Usually this additional information is used to generate so-called residuals formed from the weighted difference between the measured system outputs and the outputs from the observer. Different approaches are largely distinguished in the way in which the available design freedom in the output error injection gain matrix is utilized and the way in which the residual information is processed. For example the gain matrix may be designed to make the residuals sensitive to certain faults and not others, or else to make the residual vector lie in a specific direction in response to a particular fault. An overview of the current state of the art is detailed in [25] and the references therein. Here however the approach will be one of fault estimation. Fault estimation provides a direct estimate of the size and severity of the fault, which can be important in many applications [4,30]. The sliding mode observers considered previously will be designed for FDI purposes. Their efficacy will be substantiated by considering their application to the ship propulsion benchmark problem [23].

Consider a nominal linear system subject to certain faults described by

$$\dot{x}(t) = Ax(t) + Bu(t) + Df_i(t) \quad (66a)$$

$$y(t) = Cx(t) + f_o(t) \quad (66b)$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{p \times n}$, $D \in \mathbb{R}^{n \times q}$ with $q \leq p < n$ and the matrices C and D are of full rank. The functions $f_i(t)$ and $f_o(t)$ are deemed to represent actuator and sensor faults respectively and are assumed to be unknown but bounded. It is further assumed that the states of the system are unknown and only the signals $u(t)$ and $y(t)$ are available. (Note: in the case when $p = n$ all the states will be known and the approach of Sreedhar [28] may be adopted.) The objective is to synthesise an observer to generate

a state estimate $z(t)$ and output estimate $Cz(t)$ such that a sliding mode is attained in which the output error

$$e_y(t) = Cz(t) - y(t) \quad (67)$$

is forced to zero in finite time. The observer structure (32) in §5 will be considered. It will be shown that, provided a sliding motion can be attained, estimates of $f_i(t)$ and $f_o(t)$ can be computed from approximating the equivalent output error injection signal required to maintain a sliding motion.

6.1 Reconstruction of Actuator Faults

Consider initially the case when $f_o = 0$. Assume the conditions necessary for the existence of a sliding mode observer are met for the triple (A, D, C) . After first establishing the canonical form (36) from §5 and applying the coordinate transformation associated with T_L from (46), the state estimation error system satisfies

$$\dot{e}_1(t) = \mathcal{A}_{11}e_1(t) + \mathcal{A}_{12}e_2(t) - \mathcal{G}_{1,1}e_y(t) \quad (68a)$$

$$\dot{e}_2(t) = \mathcal{A}_{21}e_1(t) + \mathcal{A}_{22}e_2(t) - \mathcal{G}_{2,2}e_y(t) + P_o^{-1}\nu - D_2f_i(t) \quad (68b)$$

where (e_1, e_2) represents a partition of the state estimation error commensurate with the partition in (47) and the matrix $\mathcal{A}_{11}$ (which governs the sliding motion) has *stable* eigenvalues. In this scenario $e_y = e_2$. During the sliding motion, $e_y = 0$ and $\dot{e}_y = 0$ and therefore equation (68b) becomes

$$0 = \mathcal{A}_{21}e_1(t) - D_2f_i(t) + P_o^{-1}\nu_{eq} \quad (69)$$

where ν_{eq} is the equivalent output error injection signal. From (68a), and using the fact that $\mathcal{A}_{11}$ is stable, it follows that $e_1(t) \rightarrow 0$ and therefore

$$P_o^{-1}\nu_{eq} \rightarrow D_2f_i(t) \quad (70)$$

One way to recover the equivalent output error injection signal is by the use of a low pass filter [32,12]. Here an alternative approach will be employed: suppose that the discontinuous component in (33) is replaced by the continuous approximation

$$\nu_\delta = -\rho \|P_o D_2\| \frac{e_y}{\|e_y\| + \delta} \quad (71)$$

where δ is a small positive scalar. It can be shown that the equivalent output error injection signal can be approximated to any degree of accuracy by (71) for a small enough choice of δ . Since $\text{rank}(D_2) = q$ it follows from (70) that

$$f_i(t) \approx -\rho \|P_o D_2\| (D_2^T D_2)^{-1} D_2^T P_o^{-1} \frac{e_y(t)}{\|e_y(t)\| + \delta} \quad (72)$$

The key point is that the signal on the right hand side of the above can be computed on-line and depends only on the output estimation error e_y .

6.2 Reconstruction of Sensor Faults

Now consider the case when $f_i(t) \equiv 0$ and consider the effect of $f_o(t)$. In this situation

$$e_y(t) = e_2(t) - f_o(t) \quad (73)$$

and it follows that

$$\dot{e}_1(t) = \mathcal{A}_{11}e_1(t) + \mathcal{A}_{12}e_2(t) - \mathcal{G}_{l,1}e_y(t) \quad (74a)$$

$$\dot{e}_2(t) = \mathcal{A}_{21}e_1(t) + \mathcal{A}_{22}e_2(t) - \mathcal{G}_{l,2}e_y(t) + P_o^{-1}\nu \quad (74b)$$

or more conveniently

$$\dot{e}_1(t) = \mathcal{A}_{11}e_1(t) + \mathcal{A}_{12}f_o(t) + (\mathcal{A}_{12} - \mathcal{G}_{l,1})e_y \quad (75a)$$

$$\dot{e}_y(t) = \mathcal{A}_{21}e_1(t) + \mathcal{A}_{22}f_o(t) + (\mathcal{A}_{22} - \mathcal{G}_{l,2})e_y - \dot{f}_o(t) + P_o^{-1}\nu \quad (75b)$$

Arguing as before, provided a sliding motion can be attained,

$$0 = \mathcal{A}_{21}e_1 - \dot{f}_o(t) + \mathcal{A}_{22}f_o(t) + P_o^{-1}\nu_{eq} \quad (76)$$

If the fault signal $f_o(t)$ is slowly varying compared with the dynamics of the sliding motion in (75a), $e_1 \approx -\mathcal{A}_{11}^{-1}\mathcal{A}_{12}f_o$ and the derivative $\dot{f}_o$ in (76) can be ignored. Consequently

$$P_o^{-1}\nu_{eq} \approx -(\mathcal{A}_{22} - \mathcal{A}_{21}\mathcal{A}_{11}^{-1}\mathcal{A}_{12})f_o \quad (77)$$

As in the previous section, the continuous approximation to the equivalent output injection ν_{eq} can be calculated from (71) and if $(\mathcal{A}_{22} - \mathcal{A}_{21}\mathcal{A}_{11}^{-1}\mathcal{A}_{12})$ is nonsingular the fault signal can be obtained from equation (77). Note, if $(\mathcal{A}_{22} - \mathcal{A}_{21}\mathcal{A}_{11}^{-1}\mathcal{A}_{12})$ is singular, inference can still be made about certain fault channels depending on the precise nature of the rank deficiency [14].

The remainder of the chapter considers the application of this approach to the ship propulsion system benchmark proposed in [23].

7 An FDI Scheme for the Ship Benchmark

This section considers the use of a sliding mode FDI scheme applied to a ship propulsion system benchmark problem [23]. The benchmark consists of a propulsion system for a low speed marine vehicle. The consequences of faults in such a system have serious safety implications as well as significant losses associated with capital investment. The propulsion system consists of two identical engine/propeller systems which are placed in parallel. A continuous time state-space description of one engine is

$$\dot{x}(t) = Ax(t) + Bu(t) + Ed(t) + F_1f(t) + w_s(t) \quad (78a)$$

$$y(t) = Cx(t) + F_2f(t) + w(t) \quad (78b)$$

where $d(t)$ represents a vector of unknown disturbances, the reference/control demand vector

$$u(t) = \begin{bmatrix} \theta_{ref} \\ Y \end{bmatrix} \begin{array}{l} \text{pitch angle reference} \\ \text{fuel index} \end{array} \quad (79)$$

and $f(t)$ is a vector of faults. The outputs represent

$$y(t) = \begin{bmatrix} \theta \\ n \\ U \end{bmatrix} \begin{array}{l} \text{propeller pitch angle} \\ \text{shaft speed} \\ \text{ship speed} \end{array} \quad (80)$$

and $w(t)$ and $w_s(t)$ are noise vectors. In particular,

$$w_s(t) = [-k_t w_1(t) \ 0 \ 0 \ 0]^T \quad (81)$$

where the constant $k_t = 0.1510$ and $w_1(t)$ is the first component of $w(t)$. The physical meaning of the components of the unknown fault signal $f(t)$ is given below:

$$f(t) = \begin{bmatrix} f_1(t) \\ f_2(t) \\ f_3(t) \end{bmatrix} \begin{array}{l} \text{Pitch angle measurement fault} \\ \text{Hydraulic leakage (incipient) fault} \\ \text{Shaft angular measurement fault} \end{array} \quad (82)$$

In addition, as in [23], a multiplicative fault corresponding to a failure associated with the diesel engine governor will be considered so that

$$u(t) = \begin{bmatrix} \theta_{ref}(t) \\ Y(t) + \Delta k_y Y(t) \end{bmatrix} \quad (83)$$

where Δk_y represents an unknown change in gain.

A linearization of the nonlinear model from [23] is given by the triple

$$\begin{aligned} A &= \begin{bmatrix} -0.1510 & 0 & 0 & 0 \\ -2.2936 & -0.3154 & 0.1964 & 8.5252 \\ 0.1424 & 0.0056 & -0.0149 & 0 \\ 0 & 0 & 0 & -1.8745 \end{bmatrix} & B &= \begin{bmatrix} 0.1500 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1.0000 \end{bmatrix} \\ C &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{aligned} \quad (84a)$$

The fault distribution structure in the framework described earlier can be expressed as

$$f_o(t)^T := [f_1(t) \ f_3(t) \ 0] \quad (85)$$

where $f_1(t)$ and $f_3(t)$ are components of $f(t)$ as in (82). The actuator fault distribution matrix is

$$D := \begin{bmatrix} 1 & 0 \\ 0 & 4.548 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad f_i(t) := \begin{bmatrix} f_2(t) - k_t f_1(t) \\ f_{\Delta k_y Y}(t) \end{bmatrix} \quad (86)$$

Details of this derivation are given in [13].

7.1 Observer design

This system (A, D, C) has a stable invariant zero at -1.8745 and in the canonical form of (36) the triple $(\mathcal{A}, \mathcal{D}, \mathcal{C})$ is given by

$$\mathcal{A} = \begin{bmatrix} -1.8745 & 0 & 0 & 0 \\ 0 & -0.1510 & 0 & 0 \\ 8.5252 & -2.2936 & -0.3154 & 0.1964 \\ 0 & 0.1424 & 0.0056 & -0.0149 \end{bmatrix} \quad \mathcal{D} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 4.548 \\ 0 & 0 \end{bmatrix} \quad (87a)$$

$$\mathcal{C} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (87b)$$

Choosing $W = \text{diag}\{0.3, 6.0, 6.0, 1.0\}$ and $V = \text{diag}\{0.0018, 0.0132, 0.0100\}$ yields observer gain matrices (in the original coordinates) as

$$G_l = \begin{bmatrix} 23.4158 & -0.1584 & 0.0128 \\ -1.1613 & 21.7424 & 0.1045 \\ 0.0712 & 0.0792 & 24.4800 \\ 0.0839 & 1.8379 & -0.0067 \end{bmatrix} \quad G_n = \begin{bmatrix} 0.0421 & -0.0021 & 0.0001 \\ -0.0021 & 0.2784 & 0.0011 \\ 0.0001 & 0.0011 & 0.2448 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$P_o = \begin{bmatrix} 23.7352 & 0.1829 & -0.0132 \\ 0.1829 & 3.5939 & -0.0158 \\ -0.0132 & -0.0158 & 4.0851 \end{bmatrix}$$

7.2 A sliding mode fault detection system

From (70), and writing $\tilde{\nu}_{eq} := P_o^{-1}\nu_{eq}$ for notational convenience, the actuator faults satisfy

$$\begin{bmatrix} \tilde{\nu}_{eq,1} \\ \tilde{\nu}_{eq,2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 4.548 \end{bmatrix} \begin{bmatrix} f_{i,1} \\ f_{i,2} \end{bmatrix} \quad (88)$$

It can readily be verified that

$$-(\mathcal{A}_{22} - \mathcal{A}_{21}\mathcal{A}_{11}^{-1}\mathcal{A}_{12}) = -\mathcal{A}_{22} = \begin{bmatrix} 0.1510 & 0 & 0 \\ 2.2936 & 0.3154 & -0.1964 \\ -0.1424 & -0.0056 & 0.0149 \end{bmatrix} \quad (89)$$

since $\mathcal{A}_{12} = 0$. Note: this means that equation (76) no longer depends on $e_1(t)$. From equation (77)

$$\begin{bmatrix} \tilde{\nu}_{eq,2} \\ \tilde{\nu}_{eq,3} \end{bmatrix} = \begin{bmatrix} 2.2936 & 0.3154 \\ -0.1424 & -0.0056 \end{bmatrix} \begin{bmatrix} f_{o,1} \\ f_{o,2} \end{bmatrix} \quad (90)$$

or equivalently, since these equations furnish a unique solution

$$\begin{bmatrix} f_{o,1} \\ f_{o,2} \end{bmatrix} = \begin{bmatrix} -0.1740 & -9.8235 \\ 4.4366 & 71.4434 \end{bmatrix} \begin{bmatrix} \tilde{\nu}_{eq,2} \\ \tilde{\nu}_{eq,3} \end{bmatrix} \quad (91)$$

Define

$$\begin{bmatrix} f_{est,1} \\ f_{est,2} \end{bmatrix} := \begin{bmatrix} 0 & -0.1740 & -9.8235 \\ 0 & 4.4366 & 71.4434 \end{bmatrix} \tilde{\nu}_{eq} \quad (92)$$

so that the signals $\tilde{\nu}_{eq,1}$, $\tilde{\nu}_{eq,2}$ and $f_{est,1}$ and $f_{est,2}$ can be used to reconstruct different fault scenarios. Note four distinct fault events need to be isolated from only three available equivalent output error injection signals. To be able to distinguish between actuator faults and sensor faults additional information needs to be exploited. One possibility is to consider the output error (or switching function) signal e_y since step changes in sensor faults will cause a break in the sliding motion. This can be seen from equation (73) since at the instant a step change occurs in f_o , e_y momentarily becomes equal to the fault before the nonlinear output error injection term quickly forces e_y back to zero. Thus sensor faults can be detected by monitoring e_y for brief breaks in the sliding motion and the nature of the fault can be deduced from $f_{est,1}$ and $f_{est,2}$. Conversely actuator faults tend not to interrupt the sliding motion and manifest themselves in terms of changes in $\tilde{\nu}_{eq,1}$ and $\tilde{\nu}_{eq,2}$.

The following remarks can be made:

- an actuator fault f_2 , corresponding to an incipient hydraulic/pitch rate fault, affects only $f_{i,1}$ and hence in turn (from equation (88) and (92)) only affects $\tilde{\nu}_{eq,1}$ and not f_{est} .
- a sensor fault f_3 corresponding to a change in shaft angular velocity affects only $f_{o,2}$ and thus from equations (91) and (92) can be detected by a peak in the second component of e_y and replicated from the second component of f_{est} .
- a fault f_1 corresponding to a change in pitch affects both $f_{i,1}$ (from equation (86)) and also $f_{o,1}$ although it can be seen from equations (91) and (92) that replication of the fault is only made through $f_{est,1}$. At the instant at which a fault occurs a peak will be observed in the first component of e_y .
- an actuator fault relating to a change in the controller gain $\Delta k_y Y$ will be replicated by $\tilde{\nu}_{eq,2}$ (from equation (88)) but will also cause a variation in f_{est} as shown in equation (92). As argued earlier, this type of fault will not interrupt the sliding motion and therefore can be distinguished from sensor faults.

In the simulations which follow $\rho = 20$ and $\delta = 0.001$. The fault events have been simulated as shown in Table 1.

In Figure 5 six signals are displayed and used to deduce both the occurrence and the exact nature of any faults. Specifically the signals plotted represent the difference between $\tilde{\nu}_{eq,1}$, $\tilde{\nu}_{eq,2}$, $f_{est,1}$ and $f_{est,2}$ for both engines and the $\|e_y\|$ signals. For clarity, the initial portion of the time histories of

² The incipient hydraulic fault as modelled in Figure 3 [23] is argued to result in a slow drift in pitch angle - hence the symbol $\Delta\theta$ in the notation of [23].

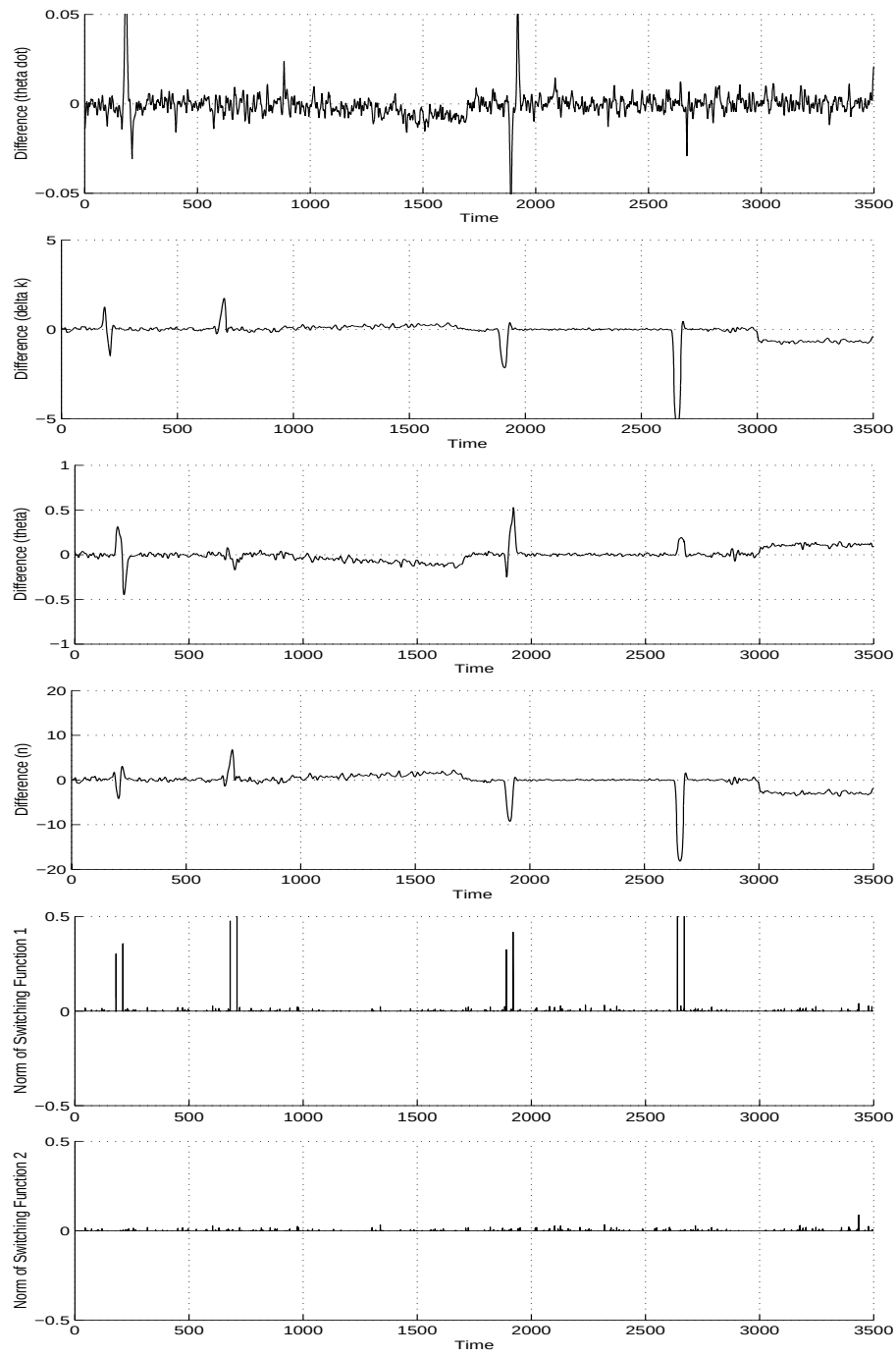


Fig. 5. Switching function and fault reconstruction signals

Event	Notation	Start	End	Type	Size
Pitch angle measurement fault ($\Delta\theta$)	f_1	180	210	step	-1.00
Hydraulic leakage incipient fault ¹ ($\Delta\dot{\theta}$)	f_2	800	1700	ramp	0.01
Pitch angle measurement fault ($\Delta\theta$)	f_1	1890	1920	step	0.64
Shaft angular velocity measurement fault (Δn)	f_3	680	710	step	-4.30
Shaft angular velocity measurement fault (Δn)	f_3	2640	2670	step	-13.70
Gain drop in diesel engine (Δk_y)	Δk_y	3000	3500	step	1.00

Table 1. Fault events for the linear simulation

the reconstruction signals are given in Figure 6. After 180 seconds the peak in the first component of the switching function $e_{y1,1}$ indicates the occurrence of a pitch angle sensor fault. Examining $f_{est,1}$ and $f_{est,2}$ it can be seen that both react (Figure 6). In addition the pitch rate/hydraulic leakage reconstruction signal $\tilde{\nu}_{eq,1}$ has also reacted strongly. It transpires that on the nonlinear simulation a pitch fault affects the shaft angular velocity and vice versa. However, a pitch fault causes the *pitch rate* estimator to react; this does not happen for faults relating to the angular velocity measurement of the shaft speed. Also the ‘step input’ nature of the fault is modified by the fact the corrupted pitch angle measurement signal is fed-back and regulated by the controller. The result is the upward/downward ‘double hit’ response.

A peak in the second component of the switching function $e_{y1,2}$ occurs after 680 seconds, indicating the presence of a shaft speed sensor fault. Examining $f_{est,1}$ and $f_{est,2}$ in Figure 6 it can be seen that both react. Importantly no significant change occurs in the pitch rate estimator $\tilde{\nu}_{eq,1}$ which confirms that there has been a fault associated with the measurement of the angular velocity of the shaft. (Using similar logic, the switching function peaks at 1890 seconds and 2640 seconds in Figure 5 indicate pitch and shaft speed faults respectively.)

No further peaks occur in the switching functions. However in Figure 7 (at approximately 800 seconds), it can be seen that a drift starts to occur in the hydraulic leakage/pitch rate estimate. This suggests an actuator fault - specifically one associated with hydraulic leakage/pitch rate - which can be seen to induce a drift in both the pitch and shaft indicator signals. A recovery is made at approximately 1700 seconds and the fault estimation signals return to near zero.

From Figure 5, after 3000 seconds, a step change occurs in both the f_{est} signals (and in the $\tilde{\nu}_{eq,2}$ signal). This is not heralded by a peak in the switching function signals indicating the cause is an actuator fault. Since the hydraulic leakage/pitch rate estimator does not react, $\tilde{\nu}_{eq,2}$ indicates the presence of a control gain change. An appropriate threshold needs to be set based on the results of a fault free simulation.

It is readily seen that the outputs from the sliding mode based FDI observer have enabled all the faults present in the test sequence to be identified despite the presence of model uncertainty, load changes and noise.

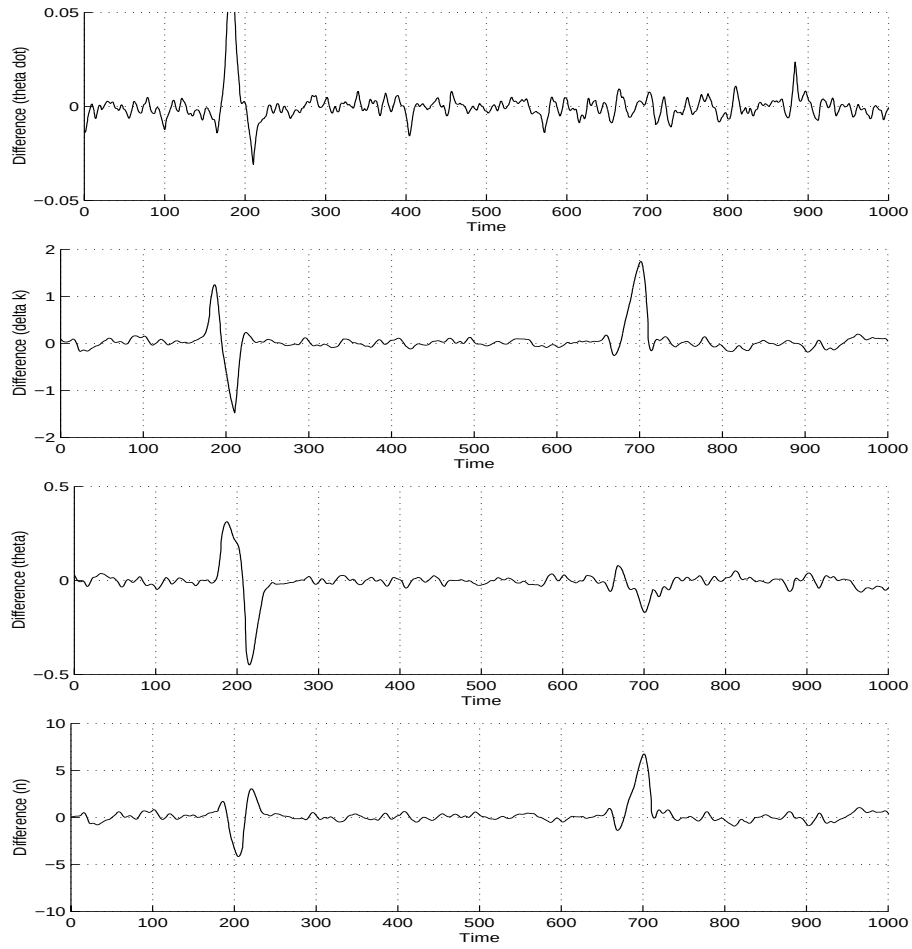


Fig. 6. Fault reconstruction signals (magnified)

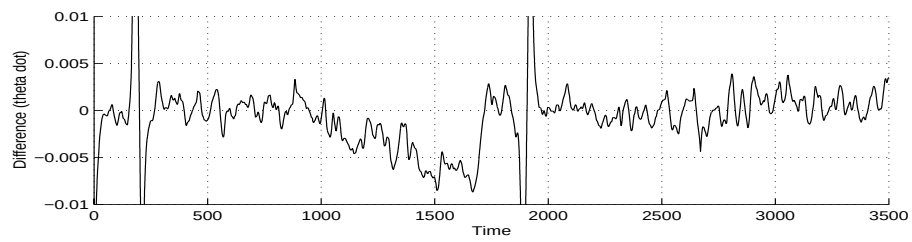


Fig. 7. Fault reconstruction signals (magnified)

The plots presented in Figures 5-7 are associated with the simulations specified in the benchmark [23]. However it is possible to modify the setup to remove the noise which is introduced on the three ‘measured’ signals and produce ‘ideal’ reconstructions. In this way it is possible to explore the effects of noise on the reconstruction signals. In the following simulations the fault scenarios are those described in Table 1. Figures 8 and 9 correspond to Figures 6 and 7; the only difference is the simulations associated with Figures 8 and 9 are noise free. The solid lines represent the ideal reconstruction signals from the noise free simulation, the dotted lines are the data in Figures 6 and 7 for comparison. It can be seen that the reconstruction signal associated with the benchmark setup represent a ‘noisy’ version of the ideal reconstructions.

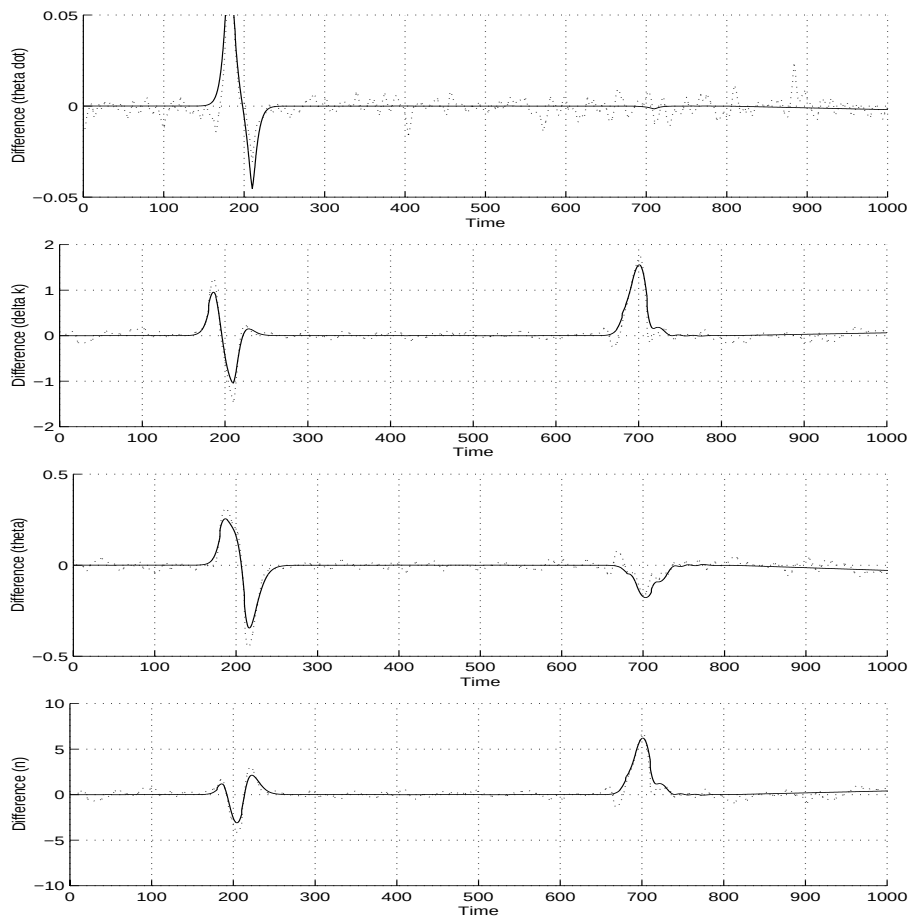


Fig. 8. Comparisons between the benchmark and noise free reconstructions

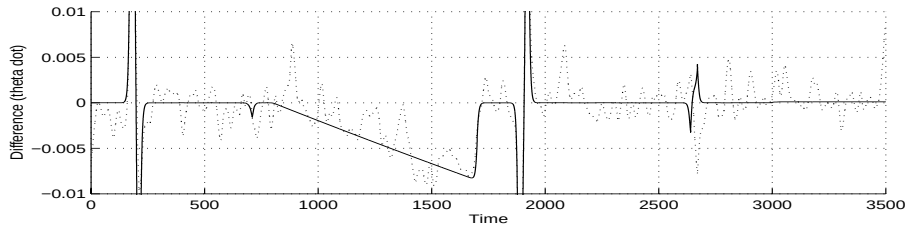


Fig. 9. Comparisons between the benchmark and noise free reconstructions

It should be noted that in the benchmark setup the noise has zero mean and appears to have no low frequency affect on the reconstruction signals.

8 Summary

This chapter has discussed the development of sliding mode observers for a class of uncertain systems. Recent work has been reported which uses Linear matrix Inequalities to formulate the design problem in convex terms so that efficient numerical methods can be employed to synthesise the observer gains. A case study has also been presented which demonstrates how sliding mode observers can be used for fault Detection and Isolation.

9 Notes and References

The pendulum example is effectively the case study considered in [33]. Interestingly in this paper a sliding mode observer was compared to other direct nonlinear observer design methodologies for the pendulum example. It was argued that the sliding mode observer had the best performance and was the least involved in terms of design.

An almost completely analytic solution to the Walcott & Żak observer problem is given in [11] for square systems. In [10] an analysis is given of the closed loop system arising from using a state feedback sliding mode controller implemented using the state estimates generated from a sliding mode observer. For a class of systems with matched uncertainty the closed loop retains the invariance properties of its full state feedback counterpart. An overview of much of this work appears in [12].

In Herrmann *et al.* [22] the minimum phase condition is alleviated to some extent for systems with many outputs. In essence, rather than seeking a closed-loop observer based controller which is robust to matched uncertainty, robustness with respect to a strict subset of matched uncertainty is achieved by effectively generating an uncertainty distribution matrix of lower dimension which is contained within the range space of the input distribution matrix. For details see [22].

The use of sliding mode observers for disturbance estimation is discussed in [19]. In fact this work describes a design framework for a broader class of systems than the one described in this chapter – namely linear time varying systems.

Sliding mode ideas have previously been used in the literature for fault detection. Sreedhar *et al.* [28] consider a model-based sliding mode observer approach although in their design procedure it is assumed that the states of the system are available; a different approach is adopted by Hermans & Zarrop [21] who attempt to design an observer in such a way that in the presence of a fault the sliding motion is destroyed. The approach in [14] seeks to design the sliding mode observer to maintain sliding even in the presence of faults. More detailed simulations and analysis of a (slightly different) sliding mode based FDI scheme is described in [13].

An interesting use of the concept of equivalent output error injection is described in [18]. A recursive design procedure based on successive evaluations of appropriate output error injection signals is shown to provide estimation of all unknown states of observable linear systems in finite time.

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Multivariable Output-Feedback Sliding Mode Control

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Abstract. This chapter addresses the problem of model-reference sliding mode control of uncertain linear multi-input-multi-output systems by output-feedback. After a brief revision of the main approaches and results available in the literature, a new method to solve the problem is presented using unit vectors as switching functions to induce sliding modes. It is named *Unit Vector Model-Reference Adaptive Control* (UV-MRAC).

Some features of the UV-MRAC are: (i) No explicit state observers are employed. (ii) It applies to plants of any uniform relative degree. (iii) Less restrictive a priori knowledge about the *high frequency gain matrix* of the plant is required for its implementation, namely, only a Hurwitz condition is required.

The analysis of the algorithm shows that, in spite of possible bounded input disturbances, the model following error equation is exponentially stable with respect to some small residual set. The *peaking phenomena*, which is a problem in most observer based variable structure controllers, does not occur in the UV-MRAC.

1 Introduction

In real world applications, the model of a plant is often uncertain with respect to the system parameters, nonlinearities and disturbances. In order to guarantee stability and good performance of a control system even in the presence of significant uncertainty, robust and/or adaptive control schemes have been widely investigated in the last two decades.

A large number of works have used the Variable Structure Control (VSC) approach with sliding modes [39,8]. Under ideal conditions, this technique can guarantee invariant system performance once sliding mode is attained. For this reason, some authors consider VSC as an adaptive control method [20,17,6,7]. In this sense, the term *adaptive* is used in the present work. However, adaptation in this case is based on signal synthesis rather than parameter estimation.

A close approach to VSC is the nonlinear robust control design based on the Lyapunov method [16,15] which relies on *unit vector control* (UVC) instead of the usual *sign* switching control of VSC. The common feature of these approaches is that they are directly or indirectly based on sliding modes.

Hence, in what follows, we will use the term *sliding mode control* (SMC) to refer either to one or the other.

The sliding mode is designed to occur on some manifold of the state space expressed by $\sigma(x) = 0$, where $\sigma : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is some smooth function. To convey the main idea in a simple way, consider a linear plant given by

$$\dot{x} = Ax + Bu, \quad y = Cx, \quad (1)$$

$x \in \mathbb{R}^n$, $u, y \in \mathbb{R}^m$, and a plane given by $\sigma(x) = Sx = 0$. A necessary condition for the sliding mode to occur on $Sx = 0$ is $\text{rank}(SB) = m$, i.e., full rank. In other words, we say that the *relative degree* n^* of the transfer function from u to σ must be one. This necessary condition is not a difficulty if the plant states are available. However, when only the plant output y is measurable, the condition applies also to the matrix $K_p = CB$ which must also be full rank, i.e., the plant transfer function must have relative degree one. Unfortunately, in many practical applications, this condition does not hold and the design of output-feedback VSC becomes a difficult problem, particularly when the plant is uncertain. One way to solve the problem is to use state observers.

1.1 State Observers in VSC

Owing to the practical difficulty of measuring all states, output-feedback strategies for VSC and SMC were developed [30,33]. In this respect, a seminal paper is [4] where asymptotic observers were introduced in VSC to address a fundamental problem of SMC, namely, the realization of “ideal” sliding modes in practice. The key idea of [4] (see also [38]) is to design the sliding modes in the observed state ($\hat{x}$). The sliding manifold is given in terms of $\sigma(\hat{x}) = S\hat{x}$, which has time derivative

$$\dot{\sigma} = SA\hat{x} + SBu + SKC(\hat{x} - x). \quad (2)$$

Now, the rank condition for the existence of the sliding mode applies to SB , i.e., S must be chosen so that SB has full rank. Note that σ is the auxiliary error which can be calculated for any S since $\hat{x}$ is totally known.

Then, ideal sliding is possible using asymptotic state observers, independently of the original relative degree of the plant transfer function $G_{yu}(s)$.

It is also worthwhile noting that $G_{yu}(s)$ is not required to be minimum phase. This requirement applies only to $G_{\sigma u}(s)$ since the sliding motion is governed by the transmission zeros of $G_{\sigma u}(s)$ [43].

The main point is that the observer based system allows the existence of a linear transfer function with $n^* = 1$ and stable transmission zeros closing a loop, called the *ideal sliding loop* (ISL), around the SMC block (discontinuous control), independently of the plant relative degree, as shown in Fig.1. The existence of an ISL is necessary for the realization of ideal sliding modes.

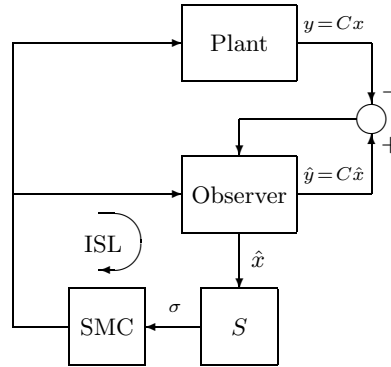


Fig. 1. Sliding mode control using a state observer

1.2 High Gain Observers for Uncertain Systems

Full knowledge of the plant is assumed in [4]. To guarantee robustness to plant uncertainties, sliding observers or high gain observers were used [34,40,10] [11,30,28].

Usually the goal is robust stabilization by output-feedback, e.g., [40,44,30]. A simple illustrative example of SMC stabilization using high gain observers can be found in [30]. In this approach, the *inverse* of the observer gain is characterized by a small parameter ε and as typical result the controller will be guaranteed to work only if there exists a constant $\varepsilon^* > 0$ such that $0 < \varepsilon < \varepsilon^*$. For example, consider the plant

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = u + d(x_1), \quad y = x_1. \quad (3)$$

Then, regarding $d(x_1)$ as a disturbance, a high gain observer could be [30]

$$\dot{\hat{x}}_1 = \hat{x}_2 + \frac{1}{\varepsilon}(y - \hat{x}_1), \quad \dot{\hat{x}}_2 = u + \frac{1}{\varepsilon^2}(y - \hat{x}_1). \quad (4)$$

Besides the problems with noise sensitivity inherent to high-gain systems, the undesirable phenomena of *peaking* in the observer and controller signals can arise [11,30].

1.3 Model-Reference Approach for Sliding Mode Control

A more general problem is to specify the desired closed loop response by a reference model which is to be followed by the plant. This approach was introduced in the VSC literature in [42] where state feedback was utilized and full state model following was required. It was further developed in [45]. Output-feedback VSC for output tracking was considered in [1,3] for single-input-single-output (SISO) systems. In these references, the structure of Model-Reference Adaptive Controllers (MRAC) by output-feedback (as in

[29]) was introduced in the VSC theory. Following this approach, new controllers were proposed in [20,17,19] for SISO plants. An early reference on output-feedback adaptive control of multi-input-multi-output (MIMO) systems is [37]. It should be noted that in the latter, SMC was used as an auxiliary control signal to achieve robustness and disturbance rejection. The MIMO case was considered in [6], [7].

In this chapter, an output-feedback based unit vector MRAC (UV-MRAC) for MIMO plants is presented. While the controller of [35] uses a state space description of the plant and a nonlinear observer, the approach here, likewise [37,6,7], relies on the plant transfer matrix and follows the development of MRAC without explicit state observers [9,36]. The controller presented here is an extension of the scheme developed in [20,17,19,23] for SISO systems. The controller guarantees global exponential stability with respect to some small residual set in the error space which can be made arbitrarily small depending on a design parameter. The required apriori knowledge about the high frequency gain (HFG) matrix is less restrictive than in previous works and no *peaking phenomena* [11] occurs in the closed loop system.

2 Preliminaries

2.1 Norms and notations

The following notation and basic concepts are employed in this chapter.

- The maximum and the minimum singular values of a matrix A are denoted as $\sigma_{max}(A)$ and $\sigma_{min}(A)$, respectively.
- The maximum and minimum eigenvalues of a symmetric matrix P are denoted as $\lambda_{max}(P)$ and $\lambda_{min}(P)$, respectively.
- $\|x\|$ denotes the Euclidean norm of a vector x and $\|A\| = \sigma_{max}(A)$ denotes the corresponding induced norm of a matrix A .
- The $\mathcal{L}_{\infty}$ norm of the signal $x(t) \in \mathbb{R}^n$ is defined as [26]

$$\|x_t\|_{\infty} := \sup_{0 \leq \tau \leq t} \|x(\tau)\| . \quad (5)$$

- Mixed time-domain and Laplace transform domain (operator) representations will be adopted. However, to give a precise meaning for such representations, the following notation and concepts are adopted. The output y of a linear time-invariant system with transfer function $H(s)$ and input u is given by $H(s)u$. Pure convolution operations $h(t) * u(t)$, $h(t)$ being the impulse response from $H(s)$, will be eventually written, for simplicity, as $H(s) * u$. Consider the realization $\dot{x} = Ax + Bu$, $y = Cx + Du$, of $H(s)$. Then,

$$y(t) = H(s)u(t) = h(t) * u(t) + C \exp(At)x(0) , \quad (6)$$

where the exponential term is the homogeneous response of the system ($u(t) \equiv 0$).

- The symbol s denotes either the complex variable of Laplace transforms or the differential operator $\frac{d}{dt}$ in time-domain expressions.
- The norm of an operator $H(s)$ is defined as

$$\|H(s)\| := \|h(t)\|_1 = \int_0^{+\infty} \|h(\tau)\| d\tau. \quad (7)$$

The following inequality holds for $y(t) = h(t) * u(t)$ ([26], p. 80)

$$\|y_t\|_\infty \leq \|h(t)\|_1 \|u_t\|_\infty. \quad (8)$$

- A *stability margin* δ of a Hurwitz polynomial $p(\lambda)$ is defined as

$$0 < \delta \leq \min_i \operatorname{Re}(-\lambda_i), \quad (9)$$

where λ_i are the roots of $p(\lambda)$. Similarly, for a Hurwitz matrix A or a Hurwitz transfer function $G(s)$, a stability margin is defined with λ_i being the eigenvalues of A or the poles of $G(s)$.

2.2 Basic MIMO Systems Concepts

Let $\{A, B, C\}$ be a realization of a strictly proper and nonsingular $m \times m$ rational transfer function matrix $G(s) = C(sI - A)^{-1}B$.

- The *observability index* of the pair $\{C, A\}$ ($A \in \mathbb{R}^{n \times n}$, $C \in \mathbb{R}^{m \times n}$) is the smallest integer ν , ($1 \leq \nu \leq n$), such that

$$\mathcal{O}_\nu = [C^T \ (AC)^T \ \dots \ (A^{\nu-1}C)^T]^T \quad (10)$$

has full rank. The observability index has a nice system interpretation: $(\nu - 1)$ is the largest number of derivatives of y required to determine $x(0)$ [27] (pp. 356–357). In other words, it gives information about the order of the state variable filters required in the structure of a MIMO MRAC design.

- Another important figure in a MIMO MRAC design is the relative degree of the plant, which requires the concept of interactor matrix for its definition. The interactor matrix associated to $G(s)$ is a special lower triangular matrix with the form [41]

$$\xi(s) = H(s) \operatorname{diag} \left\{ s^{n_1^*}, s^{n_2^*}, \dots, s^{n_m^*} \right\}, \quad (11)$$

$$H(s) = \begin{bmatrix} 1 & 0 & \dots & 0 \\ h_{21}(s) & 1 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ h_{m1}(s) & h_{m2}(s) & \dots & 1 \end{bmatrix}, \quad (12)$$

where the polynomials $h_{ij}(s)$ are either zero or divisible by s , such that

$$K_p = \lim_{s \rightarrow \infty} \xi(s)G(s) \quad (13)$$

is finite and nonsingular. Notice that $\xi(s)$ defined in (11) is unique and it is shown in [41] that $\xi(s)$ is invariant under dynamic compensation. The matrix $K_p \in \mathbb{R}^{m \times m}$ in (13) is called *high frequency gain* (HFG) matrix and $(n_1^*, n_2^*, \dots, n_m^*)$ is the *vector relative degree* of $G(s)$. If $n_i^* = n^*$, ($i = 1, \dots, m$), we say that $G(s)$ has uniform vector relative degree n^* .

2.3 Discontinuous Differential Equations

- Since we shall deal with discontinuous systems, it is necessary to define precisely the meaning of a solution for such systems. Here, Filippov's definition is assumed [12]. Note that the control signal u is not necessarily a function of t in the usual sense when sliding modes take place. In order to avoid clutter, we will denote by $u(t)$ the locally integrable functions which are equivalent to u , in the sense of *equivalent control*, along any given Filippov solution $z(t)$ of the closed-loop system. It should be stressed that $z(t)$ is, *by definition*, absolutely continuous. This definition is motivated by the adequate representation of the behavior of physical systems when the actual switching mechanism tends to an ideal switching device which corresponds to the given discontinuous differential equation [13, pp. 94–99] [39]. Then, along any such solution, u can be replaced by $u(t)$ in the right-hand side of the governing differential equations. Although the equivalent control $u(t) = u_{eq}(t)$ is not directly available, for affine systems filtering u with any strictly proper filter $G(s)$ gives $G(s)u = G(s)u(t) = G(s)u_{eq}(t)$.
- The *extended equivalent control* is defined as an equivalent control which applies for a complete system motion, i.e., on and outside the sliding mode surface $\sigma(x(t), t) = 0$. Consider the class of nonlinear systems described by $\dot{x} = f(x, t) + B(x, t)u$, where $f(\cdot)$ and $B(\cdot)$ are smooth vector fields. Let $x(t)$ be a solution of this system, for $t \in [0, T)$. Then, the extended equivalent control is a locally integrable function, defined almost everywhere in the interval $[0, T)$, and is given by

$$u_{eq} = -[GB(x(t), t)]^{-1}[Gf(x(t), t) + \frac{d}{dt}\sigma(x(t), t)], \quad (14)$$

where $G = \frac{\partial}{\partial x}\sigma(x(t), t)$. The above expression is well defined since the solution $x(t)$ is absolutely continuous by definition and, thus, has derivatives almost everywhere.

3 Problem Statement

3.1 Plant Description

We consider an observable and controllable MIMO linear time-invariant plant described by

$$\dot{x}_p = A_p x_p + B_p [u + d(t)], \quad (15)$$

$$y = C_p x_p, \quad (16)$$

where $x_p \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the input, $d \in \mathbb{R}^m$ is an unmeasurable input disturbance, and $y \in \mathbb{R}^m$ is the output. The corresponding input-output model is

$$y = G(s)[u + d(t)], \quad (17)$$

where $G(s) = C_p(sI - A_p)^{-1}B_p$ is an $m \times m$ transfer function matrix. We assume that the parameters of the plant model are uncertain, i.e., only known within certain finite bounds.

3.2 Basic Assumptions

The following assumptions regarding the plant are taken as granted:

- (A1) The transmission zeros of $G(s)$ have negative real parts.
- (A2) $G(s)$ has full rank and is strictly proper.
- (A3) The observability index ν of $G(s)$ is known.
- (A4) The interactor matrix $\xi(s)$ is diagonal and $G(s)$ has known uniform vector relative degree n^* .
- (A5) A matrix S_p is known such that $-K_p S_p$ is Hurwitz.
- (A6) The disturbance $d(t)$ is piecewise continuous and a bound $\bar{d}(t)$ is known such that $\|d(t)\| \leq \bar{d}(t) \leq \bar{d}_{sup} < +\infty, \forall t \geq 0$.

Assumptions (A1) to (A3) are usual in MIMO adaptive control. Some prior knowledge of the interactor is usually assumed in MIMO adaptive control and also VSC literature [36,37,6,7]. However, if uncertain systems are to be considered, some assumptions may not be acceptable in practice. In this sense, assumption (A4), also made in [21], may look too strong. However, it can be argued that a diagonal interactor can be achieved with an appropriate precompensator. Indeed, in most cases (in a generic sense), Lemma 2.6 in [36] guarantees that there exists a precompensator $W_p(s)$ so that $G(s)W_p(s)$ has diagonal interactor matrix (see also [14,32]). Moreover, $W_p(s)$ does not depend on the parameters of $G(s)$. Once the interactor is known to be diagonal and if the relative degree of each element of $G(s)$ (or of $G(s)W_p(s)$) is known, then $\xi(s)$ can be determined without any prior knowledge about the transfer function parameters [41]. In order to achieve uniform vector relative degree,

one can follow the approach of [7] which employs a precompensator to render the relative degree uniform and equal to $n^* = \max_i \{n_i^*\}$.

Assumption (A5) represents a considerable extension of the class of acceptable high frequency gain matrices. In other published works the more restrictive assumption of positive definiteness of $K_p S_p$ (and also symmetry in some approaches) is required [26,36,37,6,7].

3.3 Reference Model

The reference model is defined by

$$y_M = W_M(s)r, \quad r, y_M \in \mathbb{R}^m, \quad (18)$$

where, without loss of generality,

$$W_M(s) = \text{diag} \left\{ \frac{1}{s + \gamma_j} \right\} L^{-1}(s), \quad \gamma_j > 0, \quad (j = 1, \dots, m), \quad (19)$$

$$L(s) = L_1(s)L_2(s) \cdots L_N(s), \quad N = n^* - 1, \quad (20)$$

$$L_i(s) = (s + \alpha_i), \quad \alpha_i > 0, \quad (i = 1, \dots, N), \quad (21)$$

The reference signal $r(t)$ is assumed piecewise continuous and uniformly bounded. $W_M(s)$ has the same uniform vector relative degree n^* as $G(s)$ and its high frequency gain is the identity matrix (i.e., $\lim_{s \rightarrow \infty} \xi(s)W_M(s) = I$). The tracking of more general reference models could be obtained by simply preshaping the reference signal r through a precompensator at the input of the above model.

3.4 Control Objective

The objective is to design a control law u so that the plant output y asymptotically tracks the output y_M of the reference model, within some small residual error. More precisely, the control objective is to achieve asymptotic convergence of the output error

$$e(t) = y(t) - y_M(t) \quad (22)$$

to zero, or to a small residual neighborhood of zero in the error space, as $t \rightarrow +\infty$.

4 Unit Vector Control

The unit vector control law has the form

$$u = -\varrho(x, t) \frac{v(x)}{\|v(x)\|}, \quad (23)$$

where x is the state vector, $v(x)$ is a vector function of the (partial) state of the system (e.g., the output) and $\varrho(x, t) \geq 0, \forall x, \forall t$. We refer to $\varrho(\cdot)$ as the unit vector modulation function, which is designed to induce a sliding mode on the manifold $v(x) = 0$. We will henceforth assume that $u = 0$ if $v(x) = 0$ only to have a complete definition of the control law.

4.1 Basic Lemmas

Some lemmas regarding the application of the unit vector control into the MRAC framework are now introduced. These lemmas generalize their SISO counterparts found in [23] and are instrumental for the controller synthesis and stability analysis.

In what follows, we assume $t \in \mathbb{R}^+$ so that $\forall t$ means $\forall t \geq 0$, except otherwise stated. We use “LI” to denote *locally integrable* in the sense of Lebesgue and omit the term “*almost everywhere*” since its need is believed obvious where necessary. We denote by $\pi(t)$ any exponentially decreasing signal, i.e., $\|\pi(t)\| \leq Re^{-\lambda t}, \forall t$, for some unknown positive scalars R and λ .

Proposition 1. *Consider the MIMO system*

$$\dot{\varepsilon}(t) = A\varepsilon(t) + K[u + d(t) + \pi(t)], \quad (24)$$

where $A, K \in \mathbb{R}^{m \times m}$, $d(t)$ and $\pi(t)$ are LI. Assume that $-K$ is Hurwitz. If

$$u = -\varrho(\varepsilon, t) \frac{\varepsilon}{\|\varepsilon\|}, \quad \varrho(\varepsilon, t) \geq \delta + c_\varepsilon \|\varepsilon(t)\| + (1 + c_d) \|d(t)\|, \quad (25)$$

where ϱ is LI, $c_\varepsilon \geq 0$ and $c_d \geq 0$ are appropriate constants, and $\delta \geq 0$ is an arbitrary constant, then, for the closed loop system (24)–(25), the inequality

$$\|\varepsilon(t)\| \leq (c_1 \|\varepsilon(0)\| + c_2 R) e^{-\lambda_1 t} \quad (26)$$

holds $\forall t$ for some positive constants c_1, c_2 and λ_1 . Therefore, the system is globally exponentially stable when $\pi(t) \equiv 0$. Moreover, if $\delta > 0$, then the sliding mode at the point $\varepsilon = 0$ is reached after some finite time $t_s \geq 0$.

Proof. see Appendix B.1.

Lemma 1. *Consider the MIMO system*

$$\varepsilon(t) = M(s)[u + d(t) + \pi(t)], \quad (27)$$

where $M(s)$ is a minimum phase $(m \times m)$ transfer function matrix with uniform vector relative degree one and high frequency gain matrix K , $-K$ being Hurwitz, and $d(t)$ and $\pi(t)$ are LI. If

$$u = -\varrho(\varepsilon, t) \frac{\varepsilon}{\|\varepsilon\|}, \quad \varrho(\varepsilon, t) \geq \delta + c_\varepsilon \|\varepsilon\| + (1 + c_d) \|d(t)\| + c_f \varepsilon_f, \quad (28)$$

where ϱ is LI , $\delta \geq 0$ is an arbitrary constant, $c_f, \gamma_0, c_\varepsilon, c_d$ are appropriate nonnegative constants, and ε_f is generated by the filter

$$\dot{\varepsilon}_f = -\gamma_0 \varepsilon_f + \gamma_0 \|\varepsilon\|, \quad (29)$$

then, the inequality

$$\|\varepsilon(t)\| \text{ and } \|x_{ea}(t)\| \leq [c_1 \|x_{ea}(0)\| + c_2 R] e^{-\lambda_1 t} \quad (30)$$

holds $\forall t \geq 0$ for some positive constants c_1, c_2, λ_1 , where $x_{ea}^T = [x_e^T \ \varepsilon_f^T]$ is the complete state of the closed loop system and x_e is the state of any stabilizable and detectable realization of (27) (possibly nonminimal). Moreover, if $\delta > 0$, then $\varepsilon(t)$ becomes identically zero after some finite time $t_s \geq 0$.

Proof. see Appendix B.2.

A more specialized version of Lemma 1 is the following corollary.

Corollary 1 (Lemma 1). Suppose, in Lemma 1, that $M(s) = \text{diag}\{1/(s + \alpha_i)\}K$ with $-K$ Hurwitz. Then all the properties of Lemma 1 hold for

$$\varrho(\varepsilon, t) \geq \delta + c_\varepsilon \|\varepsilon\| + (1 + c_d) \|d(t)\|, \quad \forall t. \quad (31)$$

Moreover, if $\alpha_i = \alpha > 0$, $(\forall i)$, then $c_\varepsilon = 0$.

Proof. see Appendix B.3.

Lemma 2. Consider the MIMO system

$$\dot{\varepsilon}(t) = -\alpha \varepsilon(t) + K[u + d(t)], \quad \varepsilon(t) = \hat{\varepsilon}(t) + \pi(t) + \beta(t), \quad (32)$$

which has input-output relationship given by

$$\varepsilon(t) = L^{-1}(s)K[u + d(t)] + \pi(t) + \beta(t), \quad (33)$$

where $L(s) = (s + \alpha)I$, I is the $m \times m$ identity matrix, $\hat{\varepsilon}, \varepsilon, u, d \in \mathbb{R}^m$, $\alpha > 0$, $-K \in \mathbb{R}^{m \times m}$ is Hurwitz, $d(t)$ is LI , and $\pi(t)$ and $\beta(t)$ are absolutely continuous, $(\forall t)$. If $u = -\varrho(t) \frac{\varepsilon}{\|\varepsilon\|}$, where ϱ is LI and $\varrho(t) \geq (1 + \hat{c}_d) \|d(t)\|$, $\forall t$, for some appropriate $\hat{c}_d \geq 0$, then, the signals $\varepsilon(t)$ and $\hat{\varepsilon}(t)$ are bounded by

$$\|\varepsilon(t)\| \text{ and } \|\hat{\varepsilon}(t)\| \leq c_1 \|\hat{\varepsilon}(0)\| e^{-\alpha t} + c_2 \left[R e^{-\min(\alpha, \lambda)t} + \|\beta_t\|_\infty \right], \quad (34)$$

for some positive constants c_1, c_2 .

Proof. see Appendix B.4.

4.2 Unit Vector Versus Variable Structure Control

Traditional VSC systems are based on the sign function ($x \in \mathbb{R}^m$)

$$\text{sgn}(x) := \begin{bmatrix} \text{sgn}(x_1) \\ \vdots \\ \text{sgn}(x_m) \end{bmatrix}, \quad \text{sgn}(x_i) := \begin{cases} 1 & \text{if } x_i > 0, \\ 0 & \text{if } x_i = 0, \\ -1 & \text{if } x_i < 0. \end{cases} \quad (35)$$

To compare the stability properties of VSC and UVC, let us consider the VSC system given by

$$\dot{x} = K \operatorname{sgn}(x), \quad K = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}, \quad x \in \mathbb{R}^2. \quad (36)$$

The following Theorem was proved in [22].

Theorem 1. *Consider the discontinuous system (36). Then $x(t) = 0$ is a globally asymptotically stable solution if and only if one of the following conditions holds:*

- (1) $k_{12}k_{21} = 0$, $k_{11} < 0$ and $k_{22} < 0$.
- (2) $k_{12}k_{21} \neq 0$, $\frac{k_{11}}{|k_{12}|} + \frac{k_{22}}{|k_{21}|} < 0$, as well as $\det(K) > 0$.

Equivalent conditions are also given in [13] (pp. 256–258). Unfortunately, no necessary and sufficient conditions are known for systems of dimension greater than two. In contrast, the following Theorem for UVC systems of arbitrary dimension was proved in [2].

Theorem 2. *Consider the system*

$$\dot{x} = K(x) \frac{x}{\|x\|}, \quad (37)$$

where $x \in \mathbb{R}^m$, $m \geq 1$, $K : \mathbb{R}^m \rightarrow \mathbb{R}^{m \times m}$, and $\det(K(x)) \neq 0$, $\forall x$. The origin of the state-space of system (37), with bounded $K(x)$ and its derivatives, is stable (asymptotically stable, unstable) if and only if the system $\dot{z} = K(z)z$ is stable (asymptotically stable, unstable).

In particular, if $K(x)$ is a constant matrix, we conclude that the origin of the UVC system (37) is globally asymptotically stable if and only if K is Hurwitz. It is straightforward to find matrices K which result in asymptotic stability with both control laws, e.g., $K = -I$. However there exist matrices which result in stable closed-loop systems with one control law only.

Another clear difference between VSC and UVC is that sliding modes in the former may take place in any individual switching surfaces before reaching their intersection $x = 0$, whereas, for UVC, the sliding mode occurs only at the origin.

4.3 Relationship with Adaptive Stabilization

A connection of UVC with *adaptive stabilizers* of [5,24] is now discussed. Consider the plant (15)–(16) with $d(t) \equiv 0$ under the same assumptions of minimum phase and uniform vector relative degree one as in Lemma 1. One can apply the Byrnes-Willems adaptive stabilizer

$$u(t) = -k(t)y(t), \quad \dot{k}(t) = \|y(t)\|^2, \quad k(0) \in \mathbb{R}, \quad (38)$$

where $k : \mathbb{R}^+ \rightarrow \mathbb{R}$ is a scalar adaptive gain. Such control law leads to an asymptotically stable output y if $-K_p$ is Hurwitz [5].

The design of the UVC requires the knowledge of bounds for the plant parameters, which is not needed in adaptive stabilization. An advantage of UVC is that *exponential stability* and even finite time convergence of the plant output can be achieved, whereas the adaptive stabilizers can only achieve *asymptotic* [5] and *exponential convergence* [24].

5 Control Parameterization for Output-Feedback

If the plant is perfectly known, then a control law which achieves matching between the closed-loop transfer matrix and $W_M(s)$ is given by the following parameterization, which appears in the adaptive control literature

$$u^* = \theta^{*T} \omega - W_d(s) * d(t), \quad (39)$$

where the *parameter matrix* θ^* and the *regressor vector* $\omega(t)$ are given by

$$\theta^* = [\theta_1^{*T} \ \theta_2^{*T} \ \theta_3^{*T} \ \theta_4^{*T}]^T, \quad \theta^* \in \mathbb{R}^{2m\nu \times m}, \quad (40)$$

$$\omega = [\omega_1^T \ \omega_2^T \ y^T \ r^T]^T, \quad \omega \in \mathbb{R}^{2m\nu}, \quad (41)$$

$$\omega_1 = \frac{B(s)}{\Lambda(s)} u, \quad \omega_2 = \frac{B(s)}{\Lambda(s)} y, \quad \omega_1, \omega_2 \in \mathbb{R}^{m(\nu-1)}, \quad (42)$$

$$B(s) = [Is^{\nu-2} \ Is^{\nu-3} \ \dots \ Is \ I]^T, \quad (43)$$

$$W_d(s) = I - \theta_1^{*T} \frac{B(s)}{\Lambda(s)}, \quad (44)$$

$\theta_3^*, \theta_4^* \in \mathbb{R}^{m \times m}$ and $\Lambda(s)$ is a monic Hurwitz polynomial of degree $\nu-1$. The matching condition requires that $\theta_4^{*T} = K_p^{-1}$. Let $X = [x_p^T \ \omega_1^T \ \omega_2^T]^T$. The open-loop system composed by the plant (15)–(16) and the filters (42) can be written as

$$\dot{X} = A_o X + B_o u + B_{od} d, \quad (45)$$

$$y = C_o X. \quad (46)$$

Then, the regressor vector is given by

$$\omega = \Omega_1 X + \Omega_2 r, \quad \Omega_1 = \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & I \\ C_p & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \Omega_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ I \end{bmatrix}. \quad (47)$$

Substituting u by $\theta^{*T} \omega$ in (45) and omitting d , we obtain the following nonminimal realization of $W_M(s)$

$$\dot{X}_M = A_c X_M + B_c r, \quad (48)$$

$$y_M = C_o X_M, \quad (49)$$

where $A_c = A_o + B_o \theta^{*T} \Omega_1$, $B_c = B_o \theta^{*T} \Omega_2 = B_o \theta_4^{*T}$. Note that A_c is Hurwitz. The state space error equation is obtained by subtracting (48)–(49) from (45)–(46) resulting ($X_e := X - X_M$)

$$\dot{X}_e = A_c X_e + B_c (\theta_4^{*T})^{-1} [u - \theta^{*T} \omega + W_d(s) * d(t)] , \quad (50)$$

$$e = C_o X_e , \quad (51)$$

or in input-output form

$$e = W_M(s) K_p [u - \theta^{*T} \omega + W_d(s) * d(t)] . \quad (52)$$

From the error equations, it becomes clear that if u is replaced by the ideal matching control law u^* (39), then $e(t) \rightarrow 0$ exponentially as $t \rightarrow +\infty$.

From the control parameterization described above, we now make the following assumption on the class of admissible control laws.

(A7) The control law satisfies the inequality

$$\|u_t\|_\infty \leq K_\omega \|\omega_t\|_\infty + K_{rd} , \quad (53)$$

where K_ω , K_{rd} are positive constants.

This assumption guarantees that no finite time escape occurs in the system signals. Indeed, in this case the system signals will be *regular* and therefore can grow at most exponentially [31].

6 Design and Analysis of the UV-MRAC

This section considers the MRAC problem when the plant states are not fully available and only output-feedback is possible. The solution described here stems from the variable structure model-reference adaptive controller (VS-MRAC) structure developed for SISO plants in [20,17,19] and generalized to the MIMO case in [6,7]. The novelty is the use of unit vector control instead of the sign function of VSC and therefore the controller is referred to as UV-MRAC. Compared to the results of [6,7], the main new features are: (a) global exponential stability properties can be demonstrated, (b) less restrictive assumption on the plant high frequency gain matrix is required, and (c) the controller is shown to be free of peaking.

6.1 The Case of Relative Degree One

For $n^* = 1$, we have $N = n^* - 1 = 0$ and $L(s) = I$. Therefore, from the output error equations (50)–(52), and according to Corollary 1, the proposed unit vector control law is

$$u = u^{nom} - S_p \varrho \frac{e}{\|e\|} , \quad u^{nom} = \theta^{nomT} \omega , \quad (54)$$

where $S_p \in \mathbb{R}^{m \times m}$ is a design matrix which verifies assumption (A5) and θ^{nom} is some nominal value for θ^* .

From Corollary 1, exponential stability is achieved if the modulation signal $\varrho \in \mathbb{R}^+$ satisfies the inequality:

$$\varrho \geq \delta + c_\varepsilon \|e\| + (1 + c_d) \|S_p^{-1} [(\theta^{nomT} - \theta^{*T}) \omega + W_d(s) * d(t)]\|, \quad (55)$$

where $c_\varepsilon \geq 0$, $c_d \geq 0$ are appropriate constants and $\delta \geq 0$ is an arbitrary constant. From Lemma 4, one possible choice for ϱ which satisfies (55) *modulo* exponentially decaying terms is:

$$\varrho = \delta + c_1 \|\omega\| + c_2 \|e\| + c_3 \hat{d}(t), \quad (56)$$

with

$$\hat{d}(t) = \bar{d}(t) + \frac{c_4}{s + \lambda_d} * \bar{d}(t) \quad \left(\geq \left\| \left(I - \theta_1^{*T} \frac{B(s)}{\Lambda(s)} \right) * d(t) \right\| \right), \quad (57)$$

where $c_i \geq 0$ ($i = 1, \dots, 4$) are appropriate constants, λ_d is the stability margin of $\Lambda(s)$ and $\bar{d}(t) \geq \|d(t)\|$. It should be noted that the constants c_1 , c_2 and c_3 can be estimated based on some nominal parameters of the plant. The constant c_4 should be such that the inequality in (57) holds.

Now, in order to state the stability theorem for this case, consider the error system (50)–(51) and the augmented state vector $z^T = [X_e^T, \hat{d}^0]$, where $\hat{d}^0$ is the transient state of the filter (57). Then we can state the following stability result.

Theorem 3. *The UV-MRAC strategy (54), (56)–(57) for plants of uniform vector relative degree is globally exponentially stable, i.e. $\|z(t)\| \leq k e^{-\lambda t} \|z(0)\|$ ($k, \lambda > 0$), $\forall t$. Moreover, if $\delta > 0$ in the modulation function (55), the output error $e(t)$ becomes zero after some finite time.*

Proof. The proof is a simple application of Corollary 1 to the nonminimal realization of (52) given by the error system (50)–(51) and the equations for the transient state of the filter that generates $\hat{d}$. The transient state is incorporated to the π term of Lemma 1. ■

6.2 The Case of Higher Relative Degree

For higher uniform relative degree the unit vector control strategy cannot be applied directly. Similarly to the SISO case, to overcome this difficulty, [19,23], the controller structure is now modified according to Figs. 2 and 3. A key idea for the controller generalization is the introduction of the prediction error [19]

$$\hat{e} = W_M(s)L(s)K^{nom}(U_0 - L^{-1}(s)U_N), \quad (58)$$

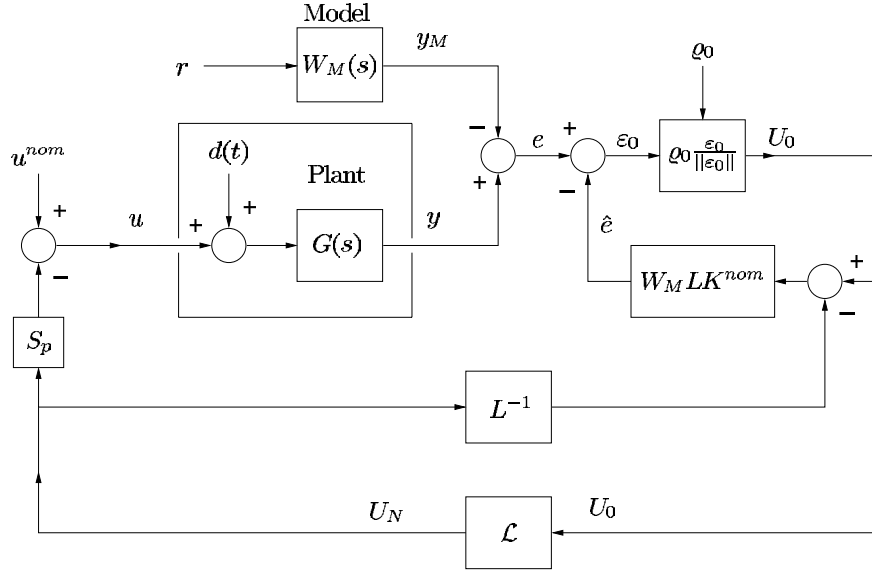


Fig. 2. UV-MRAC for plants of higher uniform relative degree. The state filters and the computation of ϱ_0 are omitted to avoid clutter. The realization of the operator $\mathcal{L}$ is presented in Fig. 3

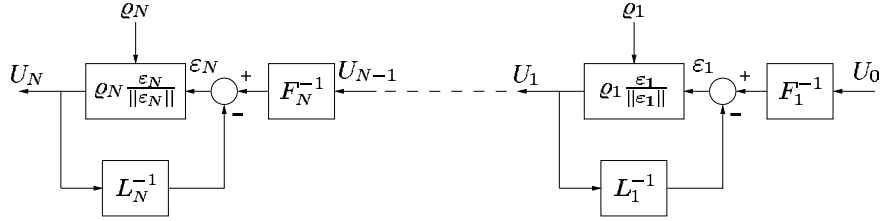


Fig. 3. Unit vector implementation of the operator $\mathcal{L}$

where K^{nom} is a nominal value of $K = K_p S_p$ and the operator $L(s)$ is given by (20). The purpose of $L(s)$ is to make $G(s)L(s)$ and $W_M(s)L(s)$ of uniform vector relative degree one. The operator $L(s)$ is noncausal but can be approximated by the unit vector lead filter $\mathcal{L}$ presented in Fig. 3. The case of relative degree one is recovered by letting $L(s) = \mathcal{L} = I$. Then, $\hat{e} \equiv 0$ and thus internal prediction error loop is eliminated resulting in the scheme of Sect. 6.1.

The averaging filters $F_i^{-1}(\tau_i s)$ in Fig. 3 are low-pass filters with transfer function given by

$$F_i^{-1}(\tau_i s) = \frac{1}{f_{av}(\tau_i s)}, \quad (59)$$

with $f_{av}(\tau_i s)$ being Hurwitz polynomials in $\tau_i s$ such that the filter has unit DC gain ($f_{av}(0) = 1$), e.g., $f_{av}(\tau_i s) = \tau_i s + 1$. If the time constants $\tau_i > 0$ are sufficiently small, the averaging filters give an approximation of the equivalent control signals [39]

$$(U_{i-1})_{eq} \approx F_i^{-1}(\tau_i s) U_{i-1}. \quad (60)$$

In [17], this approximation was used to justify the system stability properties. A complete theoretical justification of the VS-MRAC, taking into account the averaging filter dynamics, was presented later in [19,23].

Simplified Analysis

A simplified analysis is helpful in understanding of the UV-MRAC. As in [17], assume that the modulation functions ϱ_i are such that ideal sliding modes start after some finite time t_{si} in each loop around the unit vector blocks. Then, neglecting the time constant τ of the averaging filters we have $(U_{i-1})_{eq} = F_i^{-1}(\tau_i s) (U_{i-1})$ and thus $(U_N)_{eq} = L(s)(U_0)_{eq}$. Then, the input and output of the block $W_M LK^{nom}$ would be just exponentially decaying signals. However, since sliding takes place in all unit vector loops, $\varepsilon_0 = 0$ and hence, the output error $e(t)$ must decrease to zero exponentially fast. In the case of higher relative degree the error convergence is asymptotic. Finite time transient is no longer possible due to the dynamics of the block $W_M LK^{nom}$.

This result is a guideline for the complete stability analysis in the sense that it represents the limiting situation when the averaged controls tend to the corresponding equivalent controls as the time constants $\tau_i \rightarrow +0$.

The role of the prediction error ($\hat{e}$) loop is not evident from the simplified analysis. However, without the prediction error loop there would be no possibility of having an ideal sliding mode around the first unit vector block corresponding to U_0 [18] due to the small lags introduced by averaging filters. Hence, chattering would be unavoidable. The main purpose of the prediction error loop is to create the necessary ideal sliding loop around the first unit vector block.

Error Equations

We develop the expressions for the auxiliary error signals which are convenient for the controller design and stability analysis.

From (52) and (58), the auxiliary error signal $\varepsilon_0 = e - \hat{e}$ can be rewritten as

$$\begin{aligned} \varepsilon_0 = & W_M(s) K_p [u - \theta^{*T} \omega + W_d(s) * d(t)] - \\ & - W_M(s) L(s) K^{nom} [U_0 - L^{-1}(s) U_N]. \end{aligned} \quad (61)$$

Using $u = \theta^{nomT} \omega - S_p U_N$, $K = K_p S_p$ from assumption (A5) and

$$\begin{aligned} \tilde{U} := & (K^{nom})^{-1} K_p [(\theta^{*T} - \theta^{nomT}) \omega - W_d(s) * d(t)] - \\ & - (I - (K^{nom})^{-1} K) U_N, \end{aligned} \quad (62)$$

the auxiliary error ε_0 in (61) can be rewritten as

$$\varepsilon_0 = W_M(s)L(s)K^{nom} [-U_0 - L^{-1}(s)\bar{U}] , \quad (63)$$

where it is clear that in the nominal case ($\theta^* = \theta^{nom}$ and $K = K^{nom}$) the control signal U_0 has to reject only the effect of the disturbance $d(t)$. The auxiliary errors in the lead filters are given by

$$\varepsilon_i = F_i^{-1}(\tau s)U_{i-1} - L_i^{-1}(s)U_i , \quad (64)$$

where, for the sake of simplicity, it is assumed that all the averaging filters time constants are the same ($\tau_i \equiv \tau$). Applying U_0 (obtained from (63)) in (64), the auxiliary error in the first lead filter is written as

$$\varepsilon_1 = L_1^{-1}(s) \{ -F_1^{-1}(\tau s)L_1(s)L^{-1}(s) [(K^{nom})^{-1}W_M^{-1}(s)\varepsilon_0 + \bar{U}] - U_1 \} .$$

Through the recursive application of this procedure, the auxiliary errors can be rewritten as ($i = 1, \dots, N-1$):

$$\varepsilon_i = L_i^{-1}(s) \left[-U_i - F_{1,i}^{-1}(\tau s)L_{i+1,N}^{-1}(s)\bar{U} \right] - \pi_{ei} - \pi_{0i} , \quad (65)$$

$$\begin{aligned} \varepsilon_N = & -L_N^{-1}(s)(K^{nom})^{-1}K \left[U_N + F_{1,N}^{-1}(\tau s)U_d \right] - \\ & - (I - (K^{nom})^{-1}K) \beta_{uN} - \pi_{eN} - \pi_{0N} , \end{aligned} \quad (66)$$

where $L_{i,j}(s) = \prod_{k=i}^j L_k(s)$ ($L_{i,j}(s) = 1$ if $j < i$), $F_{i,j}(\tau s)$ is defined in similar way and (by convention, $\pi_{e1} \equiv 0$)

$$U_d = S_p^{-1} [(\theta^{*T} - \theta^{nomT}) \omega - W_d(s) * d(t)] , \quad (67)$$

$$\beta_{uN} = (F_{1,N}(\tau s) - I) F_{1,N}^{-1}(\tau s)L_N^{-1}(s)U_N , \quad (68)$$

$$\pi_{ei} = \sum_{j=1}^{i-1} L_{j,i-1}(s)F_{j+1,i}^{-1}(\tau s) \varepsilon_j = L_{i-1}(s)F_i^{-1}(\tau s)[\pi_{e,i-1} + \varepsilon_{i-1}] , \quad (69)$$

$$\pi_{0i} = (W_M(s)F_{1,i}(\tau s)L_{i,N}(s)K^{nom})^{-1} \varepsilon_0 . \quad (70)$$

Bounds for the Auxiliary Errors

The error system to be considered here is composed of (50)–(51), (63), (65), and (66). Let X_ε denote the state vector of (63) and x_{FL}^0 denote the transient state corresponding to the following operators: L^{-1} in (63), $F_{1,i}^{-1}L_{i+1,N}^{-1}$ in (65) and all the remaining operators associated with $\beta_{uN}, \pi_{ei}, \pi_{0i}$ in (68)–(70) and $\hat{d}_i$ in (79), (81). Since all these operators are stable, there exist positive constants K_{FL} and a_{FL} such that

$$\|x_{FL}^0(t)\| \leq K_{FL} e^{-a_{FL}t} \|x_{FL}^0(0)\| .$$

In order to fully account for the initial conditions, the following state vector z is used

$$z^T = [(z^0)^T, \varepsilon_N^T, X_e^T] , \quad (z^0)^T = [X_\varepsilon^T, \varepsilon_1^T, \varepsilon_2^T, \dots, \varepsilon_{N-1}^T, (x_{FL}^0)^T] . \quad (71)$$

In what follows, all K 's and a 's denote positive constants, operator norms $(\|\cdot\|)$ are L_∞ induced norms, and " Π " and " Π^0 " denote any term of the form $K\|z(0)\|e^{-at}$ and $K\|z^0(0)\|e^{-at}$, respectively, where K and a are (generic) positive constants.

Theorem 4. *For $N \geq 1$ consider the auxiliary errors (63), (65) and (66). If $-K^{nom}$ and $-(K^{nom})^{-1}K$ are Hurwitz, and the relay modulation functions satisfy*

$$\begin{aligned}\varrho_0 &\geq (1 + c_{d0})\|L^{-1} * \bar{U}\| + c_{\varepsilon 0}\|\varepsilon_0\|, \\ \varrho_i &\geq (1 + c_{di})\|(F_{1,i}^{-1}L_{i+1,N}^{-1}) * (\bar{U})\|, \quad (i = 1, \dots, N-1), \\ \varrho_N &\geq (1 + c_{dN})\|F_{1,N}^{-1} * U_d\|,\end{aligned}\tag{72}$$

for all $t \geq 0$, with some appropriate constants $c_{\varepsilon 0} \geq 0$ and $c_{di} \geq 0$ for $i = 0, \dots, N$, then the auxiliary errors ε_i , ($i = 0, \dots, N-1$), tend to zero at least exponentially. Moreover,

$$\|\varepsilon_i(t)\|, \|X_\varepsilon(t)\| \leq \Pi^0, \tag{73}$$

$$\|\varepsilon_N(t)\| \leq \tau \|I - (K^{nom})^{-1}K\| K_{eN}C(t) + \Pi, \tag{74}$$

and

$$\|\pi_{ei}(t)\|, \|\pi_{0i}(t)\| \leq \Pi^0; \quad i = 1, \dots, N, \tag{75}$$

$$\|\beta_{uN}(t)\| \leq \tau K_{\beta N}C(t) + \Pi^0, \tag{76}$$

where

$$C(t) = M_\theta \|\omega_t\|_\infty + M_{red}, \tag{77}$$

with some positive constants M_θ and M_{red} .

Proof. see Appendix B.5.

Remark 1. Note that in the above theorem the Hurwitz condition on $-K^{nom}$ and $-(K^{nom})^{-1}K$ could be satisfied choosing $K^{nom} = k^{nom}I$, with $k^{nom} \in \mathbb{R}$, $k^{nom} > 0$. In particular, with $K^{nom} = k^{nom}I$ the HFG condition is simply $-K$ Hurwitz.

Error System Stability

The following stability theorem will be demonstrated for the full error system given by (63), (65) and (66). For $N = 0$ ($n^* = 1$) exponential stability follows from Theorem 3.

Theorem 5. *For $N \geq 1$ assume that $-K^{nom}$ and $-(K^{nom})^{-1}K$ are Hurwitz, and that the modulation functions satisfy (72). Then, for sufficiently small $\tau > 0$, the error system (50), (63), (65) and (66) with state z as defined in (71) is globally exponentially stable with respect to a residual set of order τ , i.e., there exist positive constants a and K_z such that $\forall z(0), \forall t \geq 0$, $\|z(t)\| \leq K_z e^{-at} \|z(0)\| + O(\tau)$.*

Proof. see Appendix B.6.

7 Implementation of the UV-MRAC

In this section we address the problem of obtaining explicit implementable modulation functions for the UV-MRAC. We also present a very simplified version which can be used in practice when only local stability is sufficient.

7.1 Modulation Functions

For the case $n^* = 1$, explicit modulation functions were obtained in the form (56–57). We now develop similar expressions for the case of $n^* > 1$. For the design of ϱ_i , $i = 0, \dots, N$, we apply Lemma 4 to the inequalities (72) obtaining, for ϱ_0 ,

$$\varrho_0 = \delta_0 + c_{\varepsilon 0} \|\varepsilon_0\| + c_{\omega 0} \|L^{-1}(s)\omega\| + c_{U0} \|L^{-1}(s)U_N\| + \hat{d}_0(t), \quad (78)$$

$$\hat{d}_0(t) = \bar{c}_{d0} \bar{d}(t) + \frac{c_{d0}}{s + \lambda_{d0}} * \bar{d}(t), \quad (79)$$

with appropriate constants $c_{\varepsilon 0}$, $c_{\omega 0}$, c_{U0} , $\bar{c}_{d0}$, $c_{d0} > 0$, arbitrary $\delta_0 \geq 0$, and λ_{d0} being the stability margin of the filter $L^{-1}(s)W_d(s)$.

Similarly, for ϱ_i , $i = 1, \dots, N-1$, we obtain

$$\begin{aligned} \varrho_i = & \delta_i + c_{\omega i} \left\| F_{1,i}^{-1}(\tau s) L_{i+1,N}^{-1}(s) \omega \right\| + \\ & + c_{U i} \left\| F_{1,i}^{-1}(\tau s) L_{i+1,N}^{-1}(s) U_N \right\| + \hat{d}_i(t), \end{aligned} \quad (80)$$

$$\hat{d}_i(t) = \bar{c}_{di} \bar{d}(t) + \frac{c_{di}}{s + \lambda_{di}} * \bar{d}(t), \quad (81)$$

with arbitrary $\delta_i \geq 0$ and appropriate constants $c_{\omega i}$, $c_{U i}$, $\bar{c}_{di}$, $c_{di} > 0$, and λ_{di} is a stability margin of the filter $F_{1,i}^{-1}(\tau s) L_{i+1,N}^{-1}(s) W_d(s)$. For ϱ_N we have,

$$\varrho_N = \delta_N + c_{\omega N} \left\| F_{1,N}^{-1}(\tau s) \omega \right\| + \hat{d}_N(t), \quad (82)$$

with arbitrary $\delta_N \geq 0$ and appropriate constants $c_{\omega N}$, $\bar{c}_{dN}$, $c_{dN} > 0$, $\hat{d}_N(t)$ given by (81) and λ_{dN} is a stability margin of the filter $F_{1,N}^{-1}(\tau s) W_d(s)$.

7.2 Simplified Modulation Functions

One can further simplify the modulation ϱ by using Lemma 4 to compute a simple upper bound for $\|\omega\|$ which does not involve the utilization of a large amount of filtering devices that would be needed for the direct computation of ω . To this end notice that, by virtue of Lemma 4, one can write (*modulo exponentially decaying terms*)

$$\|\omega_1\| \leq \frac{c_1}{s + \lambda} \|y\|,$$

where λ is the stability margin of $A(s)$. Similarly, one could find an upper bound for $\|\omega_2\| \leq c_2/(s + \lambda) \|u\|$. However, since u involves discontinuous

terms, this would result in a conservative estimate [23]. For this reason we develop another bound using the following identity which holds for any $\lambda > 0$ and $\tau > 0$:

$$\frac{1}{s + \lambda} = \frac{\tau}{\tau s + 1} + \frac{1 - \tau\lambda}{(\tau s + 1)(s + \lambda)}. \quad (84)$$

Then, it follows that

$$\left\| \frac{1}{s + \lambda} u \right\| \leq \tau \|u_{av}\| + \frac{c_\lambda}{s + \lambda} \|u_{av}\|, \quad u_{av} = \frac{u}{\tau s + 1}, \quad (85)$$

where we see that the latter bound does not involve $\|u_{eq}(t)\|$ but rather the averaged control $u_{av}(t)$. This, in principle, leads to less conservative (smaller) modulation functions.

Then, we have the following upper bound for $\|\omega\|$:

$$\|\omega\| \leq \frac{c_1}{s + \lambda} \|y\| + \tau c_\tau \|u_{av}\| + \frac{c_{\lambda 1}}{s + \lambda} \|u_{av}\| + \|y\| + \|r\|. \quad (86)$$

This bound can be used to simplify the computation of the modulation functions terms such as

$$\|F_{1,i}^{-1}(\tau s) L_{i+1,N}^{-1}(s) \omega\| \leq \frac{c_{\omega i}}{s + \lambda_{Li}} \|\omega\|, \quad (i = 0, \dots, N - 1). \quad (87)$$

The above bound was obtained by application of Corollary 2 (see Appendix A). An important point is that, combined with (86), it does not require the regressor vector to be computed. This certainly eliminates a lot of computational burden.

Relay UV-MRAC

Significant simplification is possible if local stability is sufficient for a particular application. It consists in using constant control amplitudes $\varrho_i \equiv c_i$ ($i = 0, \dots, N$). Then, the filters applied in the computation of the modulation functions are not needed. Tracking with disturbance rejection can be obtained with appropriate amplitudes c_i . The resulting controller is denoted Relay UV-MRAC because in the SISO case the control signals are generated by simple relays $U_i = c_i \text{sgn}(\varepsilon_i)$ [19].

8 Simulation Results

A simulation example is presented in order to evaluate the performance of the UV-MRAC. No nominal parameter matrix is applied ($\theta^{nom} = 0$) to allow the design of the controller to be carried out with minimum knowledge about the plant.

An articulated suspension system is described in Fig. 4. The objective is to make the load position (y_2) and orientation angle (ϕ) track the reference model output through the control of two linear displacement actuators. The control signals are the actuator forces $\bar{u}_1$ and $\bar{u}_2$. The plant is linearized in the neighborhood of $\phi = 0 \text{ rad}$, resulting in the input-output representation

$$y = \text{diag} \left\{ \frac{1}{s^2}, \frac{1}{s^2} \right\} K_p [\bar{u} + d], \quad (88)$$

$$K_p = \begin{bmatrix} l_1/J & -l_2/J \\ 1/m & 1/m \end{bmatrix}, \quad d = K_p^{-1} \begin{bmatrix} 0 \\ g \end{bmatrix}, \quad \bar{u} = u - d^{nom}, \quad (89)$$

where the plant output vector is $y = [\phi \ y_2]^T$ and the control vector is $\bar{u} = [\bar{u}_1 \ \bar{u}_2]^T$. The term $d^{nom} = [50 \ 50]^T N$ is used for gravity compensation, thus allowing the reduction of the amplitude of the controller signal u . We set $\hat{d}_0(t) \equiv 250$ and $\hat{d}_1(t) \equiv 50$ in the modulation functions to account for the residual disturbance term $(d - d^{nom})$.

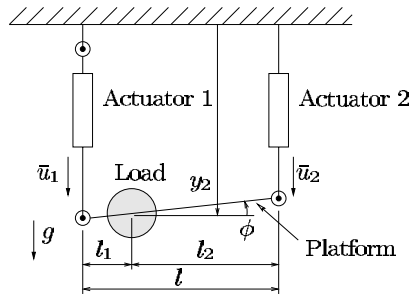


Fig. 4. Diagram of the suspension system

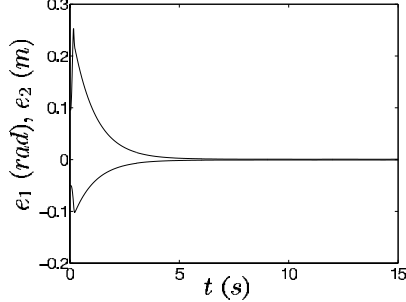


Fig. 5. Output error signals for the suspension system

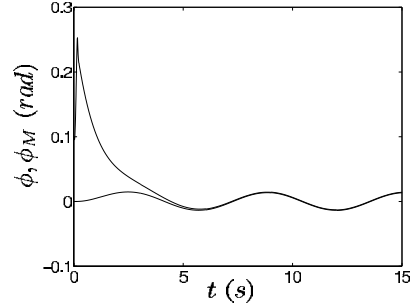


Fig. 6. Load orientation angle and model output (ϕ, ϕ_M)

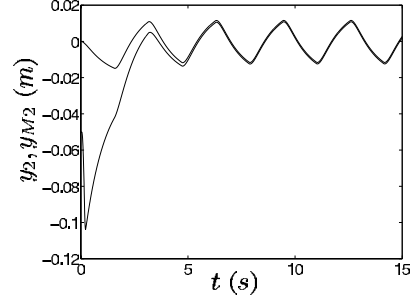


Fig. 7. Load position and model output (y_2, y_{M2})

The parameters of the plant are: load mass $m = 10 \text{ kg}$, load moment of inertia $J = 1 \text{ kg m}^2$, gravity acceleration $g = 9.81 \text{ m/s}^2$, platform length

$l = 4m$ and load position $l_2 = 3m$. Such parameters were not used in the UV-MRAC design. The platform mass, the actuator mass and the friction are neglected.

The reference model is $W_M(s) = [(s+1)(s+5)]^{-1}I$. The reference signals r_1 and r_2 are a sine wave of amplitude 0.1 and frequency 1 *rad/s* and a square wave of amplitude 0.1 and frequency 2 *rad/s*, respectively. The controller parameters are: $c_{\omega 0} = c_{U0} = 1$, $c_{\omega 1} = 0.2$, $\delta_0 = \delta_1 = 0.1$, $L(s) = \Lambda(s) = s+5$, $K^{nom} = 0.1I$ and $S_p = I$.

The convergence of the error signals observed in Fig. 5 is exponentially fast. However, quite large peaks, of amplitudes 0.25 *rad* ($\approx 14^\circ$) in e_1 and 0.1 *m* in e_2 , are present in the beginning of the convergence phase due to unfavorable initial conditions. This justifies the practice of initializing the reference model such that the initial error is small.

The Hurwitz condition required to apply the UV-MRAC is satisfied for $S_p = I$ if and only if $l_1 > -J/m$, which is always true. However, if the control law requires that $K_p S_p$ to be positive definite, such as in [37,7], then the necessary and sufficient conditions are $l_1 > -J/m$ and

$$l_2^2 + \frac{2J}{m}l_2 + \frac{J^2}{m^2} - \frac{4Jl}{m} < 0. \quad (90)$$

In this numerical example the load position should be kept within $l_2 < 1.164$ to allow the application of [37,7], otherwise, an appropriate S_p matrix should be chosen to make $K_p S_p$ positive definite. It should be stressed that the positive definiteness of $K_p S_p$ implies that $(-K_p S_p)$ be Hurwitz, but the converse is not true, thus the UV-MRAC can control plants with HFG belonging to a broader class of matrices.

The averaging filters time constant ($\tau = 0.003$ s) was chosen small enough to keep the closed loop system stable and the output error small.

The constants in the modulation functions ($c_{\omega 0}$, $c_{\omega 1}$ and c_{U0}) are such that the amplitude of the control signal u is as small as possible, but the controller stability and performance are robust to the plant parameter uncertainties.

9 Conclusion

A model-reference sliding mode control strategy for uncertain linear MIMO systems by output-feedback was presented. Although the derivation of the new controller, called UV-MRAC, follows in essence its SISO counterpart, the development of some new results for unit vector control systems was necessary. A notable stability condition, which already appeared in different contexts of Control Theory, was that the plant high frequency gain (HFG) matrix, possibly compensated by some static matrix gain, should be Hurwitz. This condition is more stringent than the simple knowledge of the sign of the HFG in the SISO case. Yet, for the MIMO case it does not seem to be overly

restrictive since it is known to be a necessary and sufficient condition for the existence of sliding modes with unit vectors as switching control laws.

The new controller UV-MRAC utilizes averaging filters with sufficiently small time constant τ . This time constant is similar to the small parameter ε that characterizes the high gain observers (HGO) of output-feedback sliding mode controllers. In the latter, peaking of control signals and observer variables may arise as $\varepsilon \rightarrow 0$. In both types of controllers, tracking errors tend to zero as the small parameters tend to zero. However, in contrast to the HGO case, an important feature of the UV-MRAC is that it possesses global exponential properties uniformly with respect to $\tau \in (0, \tau^*]$ for some small enough τ^* . This implies that the UV-MRAC preserves global stability and is free of peaking as $\tau \rightarrow 0$.

10 Acknowledgment

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Appendix

A Complementary Lemmas

Lemma 3. *Let $r(t)$ be an absolutely continuous scalar function. Suppose $r(t)$ is nonnegative and while $r > 0$ it satisfies $\dot{r} \leq -\delta - \gamma r + Re^{-\lambda t}$, where $\delta, \gamma, \lambda, R$ are nonnegative constants. Then, one can conclude that: (a) $r(t)$ is bounded by*

$$r(t) \leq [r(0) + cR]e^{-\min(\lambda, \gamma)t} \quad (\forall t \geq 0), \quad (91)$$

where c is an appropriate positive constant; (b) if $\delta > 0$ then there exists $t_s < +\infty$ such that $r(t) \equiv 0$ for all $t \geq t_s$.

Proof. According to the Comparison Theorem [12], an upper bound $\bar{r}(t)$ of $r(t)$, is given by the solution of:

$$\dot{\bar{r}} = -\gamma \bar{r} + Re^{-\lambda t}, \quad \bar{r}(0) = r(0).$$

Hence, $r(t) \leq [r(0) + cR]e^{-\min(\gamma, \lambda)t}$, which proves the first part of the Lemma. Now, if $\delta > 0$, the comparison equation is

$$\dot{\bar{r}} = -\delta - \gamma \bar{r} + Re^{-\lambda t}, \quad \bar{r}(0) = r(0), \quad (92)$$

which implies that while $r(t) > 0$ one has $r(t) \leq [r(0) + \delta/\gamma + cR]e^{-\min(\gamma, \lambda)t} - \delta/\gamma$. Defining $t_s = [\min(\lambda, \gamma)]^{-1} \ln\{\gamma(r(0) + \delta/\gamma + cR)/\delta\} < +\infty$, one notes that the right hand side of the latter inequality is negative $\forall t > t_s$. Thus, since $r(t)$ continuous and $r(t) \geq 0$ one concludes that $r(t)$ becomes identically zero for $t \geq t_s$. ■

Lemma 4. *Consider the system*

$$z = W(s)d, \quad z \in \mathbb{R}^m, \quad d \in \mathbb{R}^p, \quad (93)$$

where $W(s)$ is a stable and strictly proper $m \times p$ transfer matrix. Let γ_0 be the stability margin of $W(s)$, i.e., $0 < \gamma_0 < \min_j |\operatorname{Re}(p_j)|$, where p_j are the poles of $W(s)$. Let $\bar{d}(t)$ be an instantaneous upper bound of $d(t)$, i.e., $\|d(t)\| \leq \bar{d}(t) \forall t$. Then, there exists a positive constant c_1 such that the impulse response $w(t)$ satisfies $\|w(t)\| \leq c_1 \gamma_0 e^{-\gamma_0 t}$ and the following inequalities hold

$$\|w(t) * d(t)\| \leq c_1 \gamma_0 e^{-\gamma_0 t} * \bar{d}(t) = c_1 \frac{\gamma_0}{s + \gamma_0} * \bar{d}(t), \quad (94)$$

$$\|z(t) - z^0(t)\| \leq c_1 \|d_f(t) - d_f^0(t)\|, \quad d_f = \left(\frac{\gamma_0}{s + \gamma_0}\right) \bar{d}, \quad (95)$$

$$\|z(t)\| \leq c_1 d_f(t) + \exp, \quad (96)$$

where z^0, d_f^0 and “exp” depend on the initial conditions and decay exponentially to zero with rate γ_0 .

Proof. The proof follows from a direct extension of the scalar case in [25].

Corollary 2 (Lemma 4). *Consider*

$$z = G_F(\tau s) G_L(s) d = G_F(\tau s) \frac{1}{s + \alpha} \bar{G}_L(s) d, \quad (97)$$

where G_F, G_L are rational, stable, strictly proper, $\bar{G}_L$ has positive impulse response, $\alpha > 0$ is the stability margin of $G_L(s)$. If $\tau \in [0, \bar{\tau}]$ and $\bar{\tau}$ is sufficiently small, there exists $k > 0$ such that (95) and (96) hold with

$$d_f(t) = k \frac{1}{s + \alpha} \bar{d}(t). \quad (98)$$

Proof. The proof follows from a direct extension of the scalar case in [23].

B Proof of Lemmas and Theorems

B.1 Proof of Proposition 1

Since $-K$ is Hurwitz, there exists $P = P^T > 0$ and $Q = Q^T > 0$ such that $K^T P + P K = Q$. Thus, consider the quadratic form $V(\varepsilon) = \varepsilon^T P \varepsilon$ which has time derivative bounded by

$$\dot{V} \leq -\varrho \lambda_{\min}(Q) \|\varepsilon\| + \lambda_{\max}(A^T P + P A) \|\varepsilon\|^2 + 2\sigma_{\max}(PK) \|\varepsilon\| (\|d\| + \|\pi\|)$$

Now, choosing ϱ as in (25) with

$$c_\varepsilon \geq \max \left\{ \frac{\lambda_{\max}(A^T P + P A)}{\lambda_{\min}(Q)} + \bar{\delta}, 0 \right\}, \quad c_d \geq 2 \frac{\sigma_{\max}(PK)}{\lambda_{\min}(Q)} - 1, \quad (99)$$

where the constant $\bar{\delta} > 0$ provides some desired stability margin, we obtain

$$\dot{V} \leq -\lambda_{\min}(Q) [\bar{\delta} + \bar{\delta} \|\varepsilon\| - (1 + c_d) R e^{-\lambda t}] \|\varepsilon\|. \quad (100)$$

Now, from the Rayleigh-Ritz inequality $\lambda_{\min}(P) \|\varepsilon\|^2 \leq V(\varepsilon) \leq \lambda_{\max}(P) \|\varepsilon\|^2$, and denoting $c_{Q1} = \lambda_{\min}(Q)/\sqrt{\lambda_{\max}(P)} (> 0)$, $c_{Q2} = \lambda_{\min}(Q)/\lambda_{\max}(P) (> 0)$ and $c_D = (1 + c_d)\lambda_{\min}(Q)/\sqrt{\lambda_{\min}(P)} (> 0)$, inequality (100) can be rewritten as

$$\dot{V} \leq -\delta c_{Q1} \sqrt{V} - \bar{\delta} c_{Q2} V + c_D R e^{-\lambda t} \sqrt{V}. \quad (101)$$

Then, defining $r := \sqrt{V}$, one obtains

$$2\dot{r} \leq -\delta c_{Q1} - \bar{\delta} c_{Q2} r + c_D R e^{-\lambda t}. \quad (102)$$

Thus from Lemma 3 (see Appendix A), we can conclude that $r(t) \leq [r(0) + cR]e^{-\lambda_1 t}$, where $c > 0$ is an appropriate constant and $\lambda_1 = \min(\lambda, \bar{\delta} c_{Q2}/2)$. Applying the Rayleigh-Ritz inequality, we finally obtain inequality (26). If $\delta > 0$ in (25), from Lemma 3, one can further conclude that there exists $t_1 < +\infty$ such that $r(t) \equiv 0, \forall t > t_1$, hence, the sliding mode at $\varepsilon = 0$ starts in some finite time $t_s, 0 \leq t_s \leq t_1$.

B.2 Proof of Lemma 1

Consider a stabilizable and detectable realization of (27)

$$\dot{x} = Ax + B(u + d + \pi), \quad \varepsilon = Cx. \quad (103)$$

The high frequency gain is $K = CB$. System (103) can be transformed to the regular form [39]

$$\dot{x}_1 = A_{11}x_1 + A_{12}\varepsilon, \quad (104)$$

$$\dot{\varepsilon} = A_{21}x_1 + A_{22}\varepsilon + K(u + d + \pi). \quad (105)$$

The state vector of this realization is $x_e^T = [x_1^T \ \varepsilon^T]$. The zero dynamics is given by $\dot{x}_1 = A_{11}x_1$. Since $M(s)$ is minimum phase, A_{11} is Hurwitz. Let γ_0 be the stability margin of A_{11} . Then, according to Lemma 4 applied to (104) written as $x_1 = (sI - A_{11})^{-1}A_{12}\varepsilon$, one has the bound

$$\|x_1(t)\| \leq c_1[|\varepsilon_f(t)| + |\varepsilon_f^0(t)|] + \|x_1^0(t)\|, \quad \varepsilon_f = \frac{\gamma_0}{s + \gamma_0} * \|\varepsilon\|, \quad (106)$$

where ε_f^0 and x_1^0 are exponentially decaying terms due to initial conditions. On the other hand, since $-K$ is Hurwitz, and thus nonsingular, equation (105) can be rewritten as

$$\dot{\varepsilon} = A_{22}\varepsilon + K[u_1 + d + K^{-1}A_{21}(sI - A_{11})^{-1}A_{12} * \varepsilon + \pi_1], \quad (107)$$

where $u_1 = -\varrho_1 \frac{\varepsilon}{\|\varepsilon\|}$ and $\varrho_1 = (1 + c_d)\|d(t)\| + c_\varepsilon\|\varepsilon\| + c_f \frac{\gamma_0}{s+\gamma_0} * \|\varepsilon\|$. The term π_1 takes into account the terms π, x_1^0 and ε_f^0 and is bounded by $\|\pi_1\| \leq [c_1\|x_{ea}(0)\| + R]e^{-\lambda_1 t}$, where $x_{ea} = [x_1^T \ \varepsilon^T \ \varepsilon_f^T]^T$ is the complete state vector of the closed loop system and $\lambda_1 = \min(\gamma_0, \lambda)$. Thus, from Proposition 1, inequality (30) is satisfied for $\|\varepsilon\|$, provided that $c_d, c_\varepsilon, c_f \geq 0$ are appropriately chosen. Since, $\|x_1\|$ and $\|\varepsilon_f\|$ satisfy similar inequality, then $\|x_{ea}(t)\|$ also satisfies (30).

Also from Proposition 1, one can further conclude that ε becomes identically zero after some finite time t_s , provided that $\delta > 0$ in (28).

B.3 Proof of Corollary 1

For simplicity, we will consider a controllable realization. In this case, if there are unobservable states, the element A_{21} of the regular form (104)–(105) is identically zero, i.e., $A_{21} = 0$. Then, the result follows directly from the proof of Lemma 1. In the case of a nonminimal realization which is noncontrollable and/or nonobservable, the proof follows in a similar way, using the Kalman Decomposition.

Now, if $\alpha_i = \alpha, (\forall i)$, then $A_{22} = -\alpha I$ satisfies the Lyapunov equation $PA_{22} + A_{22}^T P = -Q_2, Q_2 > 0$ for any $P = P^T > 0$. Since the constant c_ε , in Lemma 1, comes from inequality (99) in Proposition 1, one can choose $c_\varepsilon = 0$.

B.4 Proof of Lemma 2

The time derivative of the quadratic form $V(t) = \varepsilon^T P \hat{\varepsilon}$ ($P = P^T > 0$) is given by

$$\dot{V} = -2\alpha \hat{\varepsilon}^T P \hat{\varepsilon} + [\hat{\varepsilon}^T P K u + u^T K^T P \hat{\varepsilon}] + 2\hat{\varepsilon}^T P K d.$$

Then, with $u = -\varrho \frac{\varepsilon}{\|\varepsilon\|}$, we have that

$$\dot{V} = -2\alpha V - \varrho \frac{\hat{\varepsilon}^T P K \varepsilon + \varepsilon^T K^T P \hat{\varepsilon}}{\|\varepsilon\|} + 2\hat{\varepsilon}^T P K d.$$

Now, define the auxiliary functions

$$\dot{V}_s = -2\alpha V_s, \quad \bar{V} = V_s + \gamma,$$

where γ is an absolutely continuous function satisfying $\dot{\gamma} \geq -2\alpha\gamma$ and $\gamma \geq c^2\|\beta + \pi\|^2$ for some positive constant c , yet to be defined. Since $\dot{\bar{V}} = -2\alpha V_s + \dot{\gamma} = -2\alpha(\bar{V} - \gamma) + \dot{\gamma}$, we can write the following *comparison* equation

$$\frac{d}{dt}[\bar{V} - V] = -2\alpha[\bar{V} - V] + \dot{\gamma} + 2\alpha\gamma + 2\varrho \frac{\hat{\varepsilon}^T P K \varepsilon}{\|\varepsilon\|} - 2\hat{\varepsilon}^T P K d. \quad (108)$$

Considering γ given by $\gamma(t) = c^2[R e^{-\min(\alpha, \lambda)t} + \|\beta_t\|_\infty]^2$, it is easy to verify that $V_s(0) > 0$ implies that

$$V_s(t) = e^{-2\alpha t} V_s(0) > 0, \quad (109)$$

$$\bar{V}(t) > c^2[R e^{-\min(\alpha, \lambda)t} + \|\beta_t\|_\infty]^2, \quad \forall t. \quad (110)$$

Then, if $\bar{V}(t_0) = V(t_0)$ for any $t_0 \geq 0$, the inequality (110) implies that $V(t_0) > c^2[R e^{-\min(\alpha, \lambda)t_0} + \|\beta_{t_0}\|_\infty]^2$ ($\forall t_0 \geq 0$). From the Rayleigh-Ritz inequality, $\lambda_{\min}(P)\|\hat{\varepsilon}\|^2 \leq V \leq \lambda_{\max}(P)\|\hat{\varepsilon}\|^2$, one concludes that

$$\|\hat{\varepsilon}(t_0)\| \geq \frac{c}{\sqrt{\lambda_{\max}(P)}}[R e^{-\min(\alpha, \lambda)t_0} + \|\beta_{t_0}\|_\infty] \quad (111)$$

and also, for $t = t_0$,

$$\frac{d}{dt}[\bar{V} - V] \geq 2\varrho \frac{\hat{\varepsilon}^T P K \varepsilon}{\|\varepsilon\|} - 2\hat{\varepsilon}^T P K d. \quad (112)$$

Since $K^T P + P K = Q > 0$, we can rewrite (112) as

$$\frac{d}{dt}[\bar{V} - V] \geq \varrho \frac{\varepsilon^T Q \varepsilon - 2\varepsilon^T K^T P(\beta + \pi)}{\|\varepsilon\|} - 2\hat{\varepsilon}^T P K d$$

and consequently, denoting $c_d = 2\sigma_{\max}(PK)/\lambda_{\min}(Q) - 1 (\geq 0)$,

$$\begin{aligned} \frac{d}{dt}[\bar{V} - V] &\geq \lambda_{\min}(Q) \left[\varrho \frac{\|\varepsilon\| - (c_d + 1)\|\beta + \pi\|}{\|\hat{\varepsilon}\|} - (c_d + 1)\|d\| \right] \|\hat{\varepsilon}\| \\ &\geq \lambda_{\min}(Q) \left[\varrho \left(1 - (c_d + 2) \frac{\|\beta + \pi\|}{\|\hat{\varepsilon}\|} \right) - (c_d + 1)\|d\| \right] \|\hat{\varepsilon}\|. \end{aligned}$$

Then, choosing $c \geq (1 + k)(c_d + 2)\sqrt{\lambda_{\max}(P)}$ in (111) for some $k > 0$, one has

$$\frac{\|\beta(t_0) + \pi(t_0)\|}{\|\hat{\varepsilon}(t_0)\|} \leq \frac{1}{(1 + k)(c_d + 2)}. \quad (113)$$

Therefore, we have for $t = t_0$,

$$\frac{d}{dt}[\bar{V} - V] \geq \lambda_{\min}(Q) \left[\varrho \frac{k}{1 + k} - (c_d + 1)\|d\| \right] \|\hat{\varepsilon}\|.$$

Now, if we choose $\varrho(t)$ such that $\varrho(t) \geq (\hat{c}_d + 1)\|d(t)\|$, $\forall t$, where $\hat{c}_d = (1 + k^{-1})(c_d + 1) - 1 > 0$, we have that, for $t = t_0$, $\frac{d}{dt}[\bar{V} - V] \geq 0$. Thus using the *Comparison Theorem* in [12, Theorem 7], we conclude that $\dot{V}(t_0) \leq \dot{\bar{V}}(t_0)$ implies $V(t) \leq \bar{V}(t)$, $\forall t \geq t_0$. Then, from the Rayleigh-Ritz inequality and since $\bar{V}(t) = e^{-2\alpha t} V_s(0) + \gamma(t)$, we obtain inequality (34), with $c_1 = \sqrt{\lambda_{\max}(P)/\lambda_{\min}(P)}$ and, without loss of generality, $t_0 = 0$.

B.5 Proof of Theorem 4

By Corollary 1, ε_0 (63), as well as the state X_ε of (63), converge to zero, at least exponentially, if the signal ϱ_0 satisfies (72). Note that the transient part of L^{-1} is represented by $\pi(t)$ which is thus bounded by Π^0 . Consequently, one concludes $\|\varepsilon_0(t)\| \leq \Pi^0$.

From (70) one can write $\pi_{0i} = H_i(s)\varepsilon_0 = h_i(t) * \varepsilon_0(t) + \pi_{0i}^0(t)$. Since $\|\pi_{0i}^0\| \leq \Pi^0$ and $\|h_i(t) * \varepsilon_0(t)\| \leq \Pi^0$, then $\|\pi_{0i}\| \leq \Pi^0$. Now, for $N > 1$ and $i = 1$, since $\pi_{e1} \equiv 0$, equation (65) results in $\|\varepsilon_1(t)\| \leq \Pi^0$, from Lemma 2 with $\beta(t) \equiv 0$. With similar argument, we recursively conclude from (65), (69) and Lemma 2 that $\|\pi_{ei}(t)\| \leq \Pi^0$ and $\|\varepsilon_i(t)\| \leq \Pi^0$ ($i = 2, \dots, N-1$).

Now consider ε_N (see (66)). Note that, from assumption (A7) (see Sect. 5), M_θ and M_{red} can be chosen so that $\|U_N(t)\|_\infty \leq C(t)$. Since $\pi_{e,N-1}$ and ε_{N-1} are bounded by Π^0 , so is π_{eN} , i.e., $\|\pi_{eN}(t)\| \leq \Pi^0$. From (68) one has

$$\begin{aligned} \|(\beta_{uN} - \beta_{uN}^0)_t\|_\infty &\leq \left\| \underbrace{(F_{1,N}(\tau s) - I) F_{1,N}^{-1}(\tau s) L_N^{-1}(s)}_{O(\tau)} \right\| C(t) \\ &= \tau K_{\beta N} C(t), \end{aligned} \quad (114)$$

where $\beta_{uN}^0(t)$ is bounded by Π^0 . Then, applying Lemma 2 to equation (66), which can be rewritten as $\varepsilon_N = -L_N^{-1}(K^{nom})^{-1}K[U_N + F_{1,N}^{-1}U_d] - (I - (K^{nom})^{-1}K)[\beta_{uN} - \beta_{uN}^0] - [(I - (K^{nom})^{-1}K)\beta_{uN}^0 + \pi_{eN} + \pi_{0N}]$, one readily concludes (74). Since $\|(\beta_{uN} - \beta_{uN}^0)_t\|_\infty \geq \|\beta_{uN} - \beta_{uN}^0\| \geq \|\beta_{uN}\| - \Pi^0$, then (76) follows from (114).

B.6 Proof of Theorem 5

It is convenient to rewrite (66) as

$$\varepsilon_N = L_N^{-1}[-U_N - \bar{U}] + \bar{\beta}_{uN} - \pi_{eN} - \pi_{0N}, \quad (115)$$

where $\bar{\beta}_{uN} = (F_{1,N} - I)F_{1,N}^{-1}L_N^{-1}\bar{U}$ can be bounded by $C(t)$ (see (77)) similarly as β_{uN} in (114), i.e., $\|(\bar{\beta}_{uN} - \bar{\beta}_{uN}^0)_t\|_\infty \leq \tau \bar{K}_{\beta N} C(t)$. Remembering that $u = u^{nom} - S_p U_N$, we note that L_N^{-1} in (115) operates on the same signal U_N as the one in (50). From (50) and (66), the model following error can be rewritten as: $\dot{X}_e = A_c X_e + B_c K^{nom} [\dot{\hat{e}}_N + \alpha_N \hat{e}_N]$, where $\hat{e}_N := \varepsilon_N - (\bar{\beta}_{uN} - \pi_{eN} - \pi_{0N})$. To eliminate the derivative term $\dot{\hat{e}}_N$, a variable transformation $\bar{X}_e := X_e - B_c K^{nom} \hat{e}_N$ is performed yielding

$$\dot{\bar{X}}_e = A_c \bar{X}_e + (A_c + \alpha_N I) B_c K^{nom} \hat{e}_N. \quad (116)$$

The bound (74) (Theorem 4) and the exponential stability of A_c imply that $\bar{X}_e(t)$ is bounded by $\|\bar{X}_e(t)\| \leq \tau \bar{K} C(t) + \Pi$. Moreover, as explained below,

$$\|X_e(t)\| \leq \tau K_e C(t) + \Pi, \quad \text{and} \quad \|e(t)\| \leq \tau K_0 C(t) + \Pi, \quad (117)$$

$$\|\omega_t\|_\infty \leq \tau K_1 C(t) + K_2 \|z(0)\| + K_m, \quad (118)$$

$$C(t) \leq \frac{K'_{red} + K_4 \|z(0)\|}{1 - \tau K_3}. \quad (119)$$

Indeed, inequalities (117) follow from $\bar{X}_e = X_e - B_c K^{nom} \hat{e}_N$. From the relation (47), it follows that $\|\omega\| \leq K_M + K_D \|X_e\|$ and, then from (117) we obtain (118), where $K_2 \|z(0)\|$ comes from the initial value of the term Π appearing in the bound (117). Now, from (77) and (118), $C(t) \leq \tau K_3 C(t) + K_4 \|z(0)\| + K'_{red}$, whereby, after a simple algebraic manipulation one obtains (119), which is valid for $\tau < K_3^{-1}$. Now, as explained below, we can also write

$$\|z^0(t)\| \leq K_z e^{-a_z t} \|z^0(0)\|, \quad (120)$$

$$\|z_e(t)\| \leq \tau K_5 (\|z_e(0)\| + \|z^0(0)\|) + O(\tau) + \Pi. \quad (121)$$

Indeed, the variables $X_\varepsilon, \varepsilon_{i=0,\dots,N-1}, x_{FL}^0$ are bounded by Π^0 in Theorem 4. Therefore, with the partition $z^T = [(z^0)^T, z_e^T]$, where $z_e^T := [\varepsilon_N^T, X_e^T]$ one gets (120), where only the initial condition on z^0 appears. Now, from (74), (117) and (119) follows (121), where $O(\tau)$ is independent of the initial conditions. Noting that the initial time is irrelevant in deriving the above expressions, we can write, for arbitrary $t \geq t_k \geq 0$ ($k = 0, 1, \dots$) and some $T_1 = t_{k+1} - t_k > 0$

$$\|z_e(t)\| \leq \left[\tau K_5 + K_6 e^{-a(t-t_k)} \right] [\|z_e(t_k)\| + \|z^0(t_k)\|] + O(\tau), \quad (122)$$

$$\|z^0(t)\| \leq K_z e^{-a_z(t-t_k)} \|z^0(t_k)\|, \quad (123)$$

$$\|z_e(t_{k+1})\| \leq \lambda (\|z_e(t_k)\| + \|z^0(t_k)\|) + O(\tau), \quad (124)$$

$$\|z^0(t_{k+1})\| \leq \lambda \|z^0(t_k)\|. \quad (125)$$

Equations (124) and (125) are obtained from (122) and (123) as follows: for $\tau < K_5^{-1}$, there exists $T_1 > 0$ such that $\lambda = \max(\tau K_5 + K_6 e^{-aT_1}, K_z e^{-a_z T_1}) < 1$. Then, the simple linear recursive inequalities (124) and (125) hold and easily lead to the conclusion that, for τ small enough, the error system is globally exponentially stable with respect to a residual set of order τ .

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Sliding Modes, Differential Flatness and Integral Reconstructors

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Abstract. The relevance of the differential flatness property in the context of sliding mode controller design is explored for the case of linear and nonlinear SISO and MIMO systems. We also explore, for linear systems, the possibilities of extending the idea of Generalized PI control, based on state reconstructors, to the problem of sliding surface asymptotic synthesis requiring no state measurements, or asymptotic observer design. An experimental example is presented, dealing with the sliding mode control of an electro-mechanical system, where there is no need for mechanical sensors.

1 Introduction

In this chapter we examine the relevance of the differential flatness property in the design of sliding mode controllers for nonlinear Single-Input Single-Output (SISO) and Multiple-Input Multiple-Output (MIMO) systems. It is shown that flatness considerably simplifies the sliding mode controller design process by reducing the problem to that of controlling a linearized system in Brunovsky's canonical form in the case of SISO systems or, in the MIMO case, to that of controlling a suitable set of independent integrator chains placed also in Brunovsky's canonical form.

A new development, regarding the possibilities of sliding mode control synthesis *without* state measurements is envisioned from the perspective of the recently introduced Generalized PI control or, more properly called, “Integral Reconstruction” schemes for state feedback policies (see Fliess *et al* [2]-[4]).

Section 2 introduces in a tutorial manner the concept of differential flatness for nonlinear systems. Some illustrative examples are presented regarding the uncovering of this important, though non-generic, property. Section 3 presents the sliding mode control of flat systems from the perspective of Generalized PI control. Here, we consider the use of *integral reconstructors* in the stabilization and trajectory tracking, without state measurements, of

linearized flat systems. Section 4 contains the results of an actual experimental application of GPI sliding mode control to a linear electro-mechanical system. The conclusions and suggestions for further work are collected at the end of the chapter.

2 Sliding Mode Control of Differentially Flat Systems

2.1 Differential Flatness

Differential flatness was introduced by Prof. M. Fliess and his co-workers in a series of interesting articles nearly 10 years ago (see Fliess *et al* , [2], and the references therein). Important contributions to the subject have also been given by Pomet [9], Rathinham [10], Sluis [13] and Nieuwstadt [8]. The concept of flatness arose from the early work of E. Cartan [1] in connection with studies of underdetermined systems of differential equations.

Roughly speaking, an n -dimensional system, of the form $\dot{x} = f(x, u)$, $y = h(x)$, with multiple outputs denoted by the vector y which is provided with m independent inputs u , entering *regularly* into the system equations, is differentially flat if we can find m artificial outputs F of the form:

$$F = \psi(x, u, \dot{u}, \dots, u^{(\alpha)})$$

such that all variables in the system, i.e., states, original outputs, and all control inputs, can be written in terms of the vector F and its time derivatives, *without* integrating any differential equations. In other words a *differential parameterization* is possible for *all* system variables in terms of F alone, i.e.

$$\begin{aligned} x &= \Phi(F, \dot{F}, \dots, F^{(\beta)}) & y &= \Psi(F, \dot{F}, \dots, F^{(\beta)}) \\ u &= \Theta(F, \dot{F}, \dots, F^{(\beta+1)}) \end{aligned}$$

where β is to be understood as a *vector of integers* containing in each entry the order of derivation of the corresponding component of the vector F , i.e.

$$F^{(\beta)} = (F_1^{(\beta_1)}, \dots, F_m^{(\beta_m)})$$

It is clear that the flatness property trivializes the feedback linearization problem. For instance, for the special case in which the function Θ is locally regular with respect to $F^{(\beta+1)}$, the local input coordinate transformation

$$u = \Theta(F, \dot{F}, \dots, F^{(\beta)}, v) \quad (1)$$

takes the system into a set of decoupled systems in Brunovsky's canonical form:

$$F_i^{(\beta_i+1)} = \vartheta_i, \quad i = 1, \dots, m$$

When the function Θ is *not* regular in $F^{(\beta+1)}$ then, generally speaking, an obstruction exists for static feedback linearization and further derivations

must be taken in *some* of the control input expressions, i.e. in some of the components of the mapping Θ , so as to achieve the required regularity and local invertibility. This procedure, modulo possible singularities, leads to *dynamic* feedback linearization achievable in an *endogenous* manner.

2.2 Sliding mode control based on flatness

It is clear that, in the regular case, a sliding mode controller for the nonlinear system can also be prescribed in terms of the flat outputs by defining a set of m suitable *sliding surface coordinate functions*, denoted by σ_i , $i = 1, \dots, m$, specified on the basis of the state variables characterizing each one of the independent Brunovsky chains of integrators, i.e. we may adopt as sliding surface coordinate functions,

$$\sigma_i = \sum_{j=1}^{\beta_i+1} \alpha_{ji} \left(F_i^{(j-1)} - [F_i^*(t)]^{(j-1)} \right)$$

for suitable Hurwitz coefficients α_{ji} , $j = 1, \dots, \beta_i + 1$, with, $\alpha_{\beta_i+1,i} = 1$, for all i , and where the signals: $F_i^*(t)$, $i = 1, \dots, m$, represent the desired trajectories for the flat outputs.

The *sliding surface* is defined as the non-empty intersection of the *zero level sets* of all the sliding surface coordinate functions.

The time derivative of the i -th sliding surface coordinate function is obtained as

$$\begin{aligned} \dot{\sigma}_i &= \sum_{j=1}^{\beta_i+1} \alpha_{ji} \left(F_i^{(j)} - [F_i^*(t)]^{(j)} \right) \\ &= \sum_{j=1}^{\beta_i} \alpha_{ji} \left(F_i^{(j)} - [F_i^*(t)]^{(j)} \right) + \vartheta_i - [F_i^*(t)]^{(\beta_i+1)} \end{aligned}$$

The discontinuous feedback control actions for the auxiliary control variables, ϑ_i , $i = 1, \dots, m$, is prescribed to be,

$$\vartheta_i = -W_i \text{sign } \sigma_i, \quad i = 1, \dots, m$$

with, $W_i = \delta_i + \eta_i$, $i = 1, 2, \dots, m$ where $\eta_i > 0$ and, δ_i , chosen so as to satisfy,

$$\delta_i > \left| -[F_i^*(t)]^{(\beta_i+1)} + \sum_{j=1}^{\beta_i} \alpha_{ji} \left(F_i^{(j)} - [F_i^*(t)]^{(j)} \right) \right| \quad \forall t, \quad i = 1, 2, \dots, m$$

These choices result in the following inequalities for each sliding surface coordinate function, σ_i ,

$$\frac{1}{2} \frac{d}{dt} (\sigma_i^2) \leq -\eta_i |\sigma_i|, \quad i = 1, 2, \dots, m$$

thus yielding a local sliding motion on the intersection manifold:

$$\mathcal{S} = \bigcap_{i=1}^m \{x : \sigma_i = 0\}$$

The actual feedback control inputs u are obtained by reversing the regular input coordinate transformation (1).

Example 1. Extended two wheeled non-holonomic car

Consider the following model of a non-holonomic two wheeled car

$$\dot{x} = v \cos \theta, \quad \dot{y} = v \sin \theta, \quad \dot{\theta} = u_2, \quad \dot{v} = u_1$$

The system is differentially flat, with flat outputs given by the wheel's common axis midpoint coordinates $(F_1, F_2) = (x, y)$. Indeed, all system variables are differentially parameterizable in terms of F_1 , F_2 and a finite number of their time derivatives

$$\begin{aligned} x &= F_1, \quad y = F_2, \quad \theta = \arctan \left(\frac{\dot{F}_2}{\dot{F}_1} \right), \quad v = \sqrt{\dot{F}_1^2 + \dot{F}_2^2} \\ u_1 &= \frac{\dot{F}_1 \ddot{F}_1 + \dot{F}_2 \ddot{F}_2}{\sqrt{\dot{F}_1^2 + \dot{F}_2^2}}, \quad u_2 = \frac{\dot{F}_1 \ddot{F}_2 - \dot{F}_2 \ddot{F}_1}{\dot{F}_1^2 + \dot{F}_2^2} \end{aligned}$$

The parameterization of the control inputs is locally *regular* with respect to the second order derivatives of the flat outputs $\ddot{F}_1, \ddot{F}_2$.

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \frac{\dot{F}_1}{\sqrt{\dot{F}_1^2 + \dot{F}_2^2}} & \frac{\dot{F}_2}{\sqrt{\dot{F}_1^2 + \dot{F}_2^2}} \\ -\frac{\dot{F}_2}{\dot{F}_1^2 + \dot{F}_2^2} & \frac{\dot{F}_1}{\dot{F}_1^2 + \dot{F}_2^2} \end{bmatrix} \begin{bmatrix} \ddot{F}_1 \\ \ddot{F}_2 \end{bmatrix}, \quad \det \left(\frac{\partial u}{\partial \ddot{F}} \right) = \frac{1}{\sqrt{\dot{F}_1^2 + \dot{F}_2^2}}$$

The locally non-singular input coordinate transformation:

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \frac{\dot{F}_1}{\sqrt{\dot{F}_1^2 + \dot{F}_2^2}} & \frac{\dot{F}_2}{\sqrt{\dot{F}_1^2 + \dot{F}_2^2}} \\ -\frac{\dot{F}_2}{\dot{F}_1^2 + \dot{F}_2^2} & \frac{\dot{F}_1}{\dot{F}_1^2 + \dot{F}_2^2} \end{bmatrix} \begin{bmatrix} \vartheta_1 \\ \vartheta_2 \end{bmatrix}$$

transforms the system into the following set of two decoupled integrator chains in Brunovsky form,

$$\ddot{F}_1 = \vartheta_1, \quad \ddot{F}_2 = \vartheta_2$$

Suppose it is desired to follow a smooth desired trajectory $(x^*(t), y^*(t))$, for the position variables (x, y) . The sliding surface coordinate functions,

$$\begin{aligned} \sigma_1 &= \dot{F}_1 - \dot{x}^*(t) + \lambda_1(F_1 - x^*(t)), \quad \lambda_1 > 0 \\ \sigma_2 &= \dot{F}_2 - \dot{y}^*(t) + \lambda_2(F_2 - y^*(t)), \quad \lambda_2 > 0 \end{aligned}$$

impose independent exponentially asymptotically stable motions towards the desired trajectories upon permanently enforcing the ideal sliding conditions: $\sigma_1 = 0$, $\sigma_2 = 0$, by means of the auxiliary control inputs ϑ_1 and ϑ_2 . The time derivatives of the sliding surface coordinates σ_1 and σ_2 are given by:

$$\begin{aligned}\dot{\sigma}_1 &= \vartheta_1 - \ddot{x}^*(t) + \lambda_1 (\dot{F}_1 - \dot{x}^*(t)) \\ \dot{\sigma}_2 &= \vartheta_2 - \ddot{y}^*(t) + \lambda_2 (\dot{F}_2 - \dot{y}^*(t))\end{aligned}$$

The choices

$$\vartheta_1 = -W_1 \operatorname{sign} \sigma_1, \quad \vartheta_2 = -W_2 \operatorname{sign} \sigma_2$$

with $W_i = \delta_i + \eta_i$, $i = 1, 2$ and $\eta_i > 0$, $i = 1, 2$, where δ_1, δ_2 satisfy:

$$\begin{aligned}\delta_1 &> |-\ddot{x}^*(t) + \lambda_1 (\dot{F}_1 - \dot{x}^*(t))| > 0 \quad \forall t \\ \delta_2 &> |-\ddot{y}^*(t) + \lambda_2 (\dot{F}_2 - \dot{y}^*(t))| > 0 \quad \forall t\end{aligned}$$

yield, $\frac{d}{dt}(\sigma_i^2) \leq -\eta_i |\sigma_i|$, $i = 1, 2$. Thus, the prescribed discontinuous feedback control laws locally create sliding motions on the intersection of the manifolds, $\sigma_1 = 0$ and $\sigma_2 = 0$.

However, due to the fact that the actual control, u_2 , is a velocity, which evidently cannot be discontinuous, we use a "smoothed" sign function for the specification of the auxiliary control input, ϑ_2 , in the form: $\vartheta_2 = \sigma_2 / (\epsilon_2 + |\sigma_2|)$ with ϵ_2 being a small strictly positive number. Similarly, for the choice of the control input ϑ_1 , we take $\vartheta_1 = \sigma_1 / (\epsilon_1 + |\sigma_1|)$. Note that this input is, in fact, an acceleration and can, in principle, be chosen as a discontinuous function. Evidently these "high gain" choices compromise the steady state values of the tracking errors in the presence of unmodeled perturbations.

Figure 1 shows the performance of the *high gain* multivariable controller solving the task of having the two wheeled car follow a *three petals rose* figure in the plane. Such a graph is defined in the plane, (x, y) , by the polar coordinates equation, $\rho(t) = R \cos(3\psi(t))$ with (ρ, ψ) being, respectively, the radius vector and the angular position of the radius vector, with respect to the x axis, characterizing a point located on the rose graph. We adopt the following time parameterization for the angular position coordinate, $\psi(t) = \omega t$, with ω being a constant parameter. This yields the following desired trajectories for the flat outputs

$$x^*(t) = R \cos(3\omega t) \cos(\omega t), \quad y^*(t) = R \cos(3\omega t) \sin(\omega t)$$

The design parameters for the digital computer simulations were set to be: $\lambda_1 = \lambda_2 = 1$, $W_1 = W_2 = 10$, $\epsilon_1 = \epsilon_2 = 0.1$, $R = 4$, $\omega = 0.2$.

2.3 Flatness of time-invariant, linear, multivariable systems

Consider the multivariable controllable linear system

$$\dot{x} = Ax + Bu, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m$$

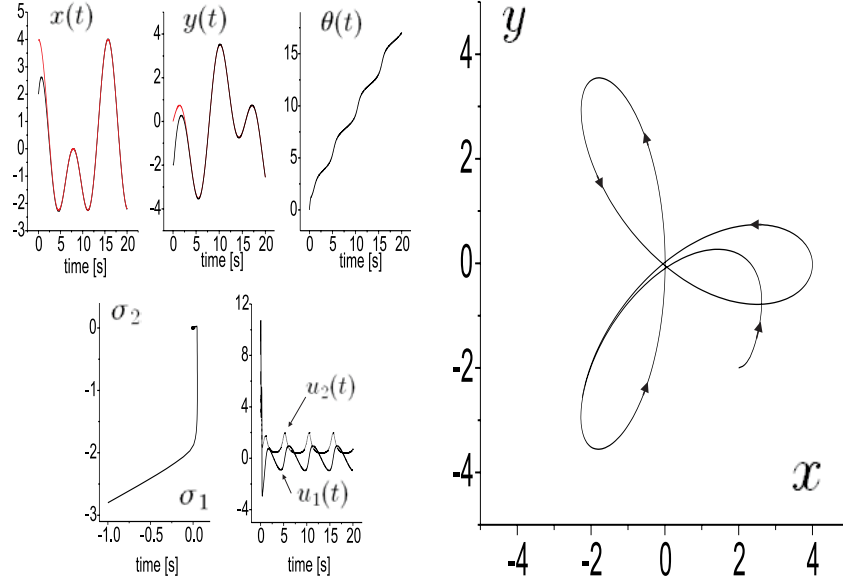


Fig. 1. Closed loop performance for trajectory tracking of a high-gain controlled non-holonomic two wheeled car

where the matrix B is full rank m and constituted by the column vectors $B = [b_1, \dots, b_m]$. The system being controllable implies that the $n \times nm$, Kalman controllability matrix

$$\mathcal{K}_C = [B, AB, \dots, A^{n-1}B]$$

is full rank n .

Controllability of the system implies then that we may extract the following full rank $n \times n$ matrix, $\mathcal{C}$, from the Kalman controllability matrix,

$$\mathcal{C} = [b_1, Ab_1, \dots, A^{\gamma_1-1}b_1, b_2, Ab_2, \dots, A^{\gamma_2-1}b_2, \dots, b_m, Ab_m, \dots, A^{\gamma_m-1}b_m]$$

with γ_i , $i = 1, \dots, m$, being the *Kronecker controllability indices* of the system, which, evidently, satisfy: $\sum_i \gamma_i = n$. In the construction of $\mathcal{C}$ we assume that a column vector of the form $A^{\gamma_j}b_j$, for any j , is eliminated from the collection whenever $A^{\gamma_j}b_j \in \text{range } \mathcal{K}_C$.

Under the above assumptions, the flat outputs are given by the following m quantities

$$F = \begin{bmatrix} \phi_1 \\ \vdots \\ \phi_m \end{bmatrix} \mathcal{C}^{-1}x$$

with ϕ_i , $i = 1, \dots, m$ being n -dimensional row vectors of the form $\phi_j = [0, \dots, 0, \dots, 0, 1, 0, \dots, 0]$ with the 1 in the $(\sum_{i=1}^j \gamma_i)$ -th position.

The proof of this fact is quite straightforward by considering the non-singular state coordinate transformation, $z = \mathcal{C}^{-1}x$, which results in the following set of decoupled sub-systems.

$$\begin{aligned} \dot{z}_{j1} &= \alpha_1^{jj} z_{j\gamma_j} + \left(\sum_{i=1, i \neq j}^m \alpha_1^{ij} z_{i\gamma_i} \right) + u_j \\ \dot{z}_{j2} &= z_{j1} + \alpha_2^{jj} z_{j\gamma_j} + \left(\sum_{i=1, i \neq j}^m \alpha_2^{ij} z_{i\gamma_i} \right) \\ &\vdots \\ \dot{z}_{j\gamma_j} &= z_{j(\gamma_j-1)} + \alpha_{\gamma_j}^{jj} z_{j\gamma_j} + \left(\sum_{i=1, i \neq j}^m \alpha_{\gamma_j}^{ij} z_{i\gamma_i} \right) \end{aligned} \quad j = 1, 2, \dots, m$$

It should be clear that *all* the state variables in the j -th subsystem and the control input, u_j , can be differentially parameterized by the state variables $z_{j\gamma_j}$, $j = 1, \dots, m$. These are precisely the components gathered in the vector F .

Note that for a single-input controllable linear systems, a flat output F is given by the linear combination, or any constant multiple thereof, of the original states obtained by using the last row of the inverse of the controllability matrix.

Example 2. Sliding mode control of the linearized PVTOL example

Consider the normalized nonlinear multivariable system model representing the Planar Vertical Take-Off and Landing aircraft or better known as the PVTOL aircraft shown in Figure 2.

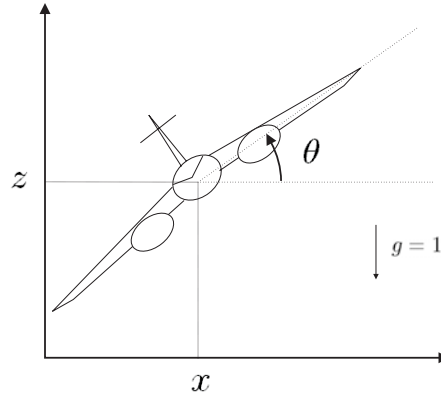


Fig. 2. PVTOL aircraft

The PVTOL model constitutes a typical example of a multivariable non-minimum phase nonlinear system which is linearizable by means of dynamic endogenous feedback.

$$\ddot{x} = -u_1 \sin \theta + \epsilon u_2 \cos \theta$$

$$\begin{aligned}\ddot{z} &= u_1 \cos \theta + \epsilon u_2 \sin \theta - 1 \\ \ddot{\theta} &= u_2\end{aligned}$$

The normalized system model exhibits, as constant equilibrium states, the following values

$$x = \bar{x}, \quad z = \bar{z}, \quad \theta = \bar{\theta} = 0, \quad u_1 = \bar{u}_1 = 1, \quad u_2 = \bar{u}_2 = 0$$

The tangent linearization of the system around this constant equilibrium point is given by:

$$\ddot{x}_\delta = -\theta_\delta + \epsilon u_{2\delta}, \quad \ddot{z}_\delta = u_{1\delta}, \quad \ddot{\theta}_\delta = u_{2\delta}$$

The controllability matrix of the system is given by,

$$\mathcal{C} = [b_1, Ab_1, b_2, Ab_2, \dots, A^3 b_2] = \begin{bmatrix} 0 & 0 & 0 & \epsilon & 0 & 1 \\ 0 & 0 & \epsilon & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

The control u_1 has controllability index, $\gamma_1 = 2$, while u_2 has controllability index, $\gamma_2 = 4$. The inverse of the controllability matrix is readily found to be,

$$\mathcal{C}^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -\epsilon \\ 1 & 0 & 0 & 0 & -\epsilon & 0 \end{bmatrix}$$

The flat outputs of the linearized system are then given by the linear combinations of the original states obtained with the *second* and *last* row of the controllability matrix. i.e.

$$F_1 = z_\delta, \quad F_2 = x_\delta - \epsilon \theta_\delta$$

The differential parameterization of the states and control inputs in terms of the flat outputs is obtained, after a few differentiations, as,

$$x_\delta = F_2 - \epsilon \ddot{F}_2, \quad z_\delta = F_1, \quad \theta_\delta = -\ddot{F}_2, \quad u_{1\delta} = \ddot{F}_1, \quad u_{2\delta} = -F_2^{(4)}$$

The control objective is to bring the system to the constant equilibrium position. This may be achieved by forcing the flat outputs F_1 and F_2 to zero in an exponentially asymptotic manner. Thus a pair of sliding surface coordinate functions may be prescribed as follows

$$\begin{aligned}\sigma_1 &= \dot{F}_1 + \lambda_1 F_1 = \dot{z}_\delta + \lambda z_\delta \\ \sigma_2 &= F_2^{(3)} + k_3 \ddot{F}_2 + k_2 \dot{F}_2 + k_1 F_2 \\ &= -\dot{\theta}_\delta - k_3 \theta_\delta + k_2 (\dot{x}_\delta - \epsilon \dot{\theta}_\delta) + k_1 (x_\delta - \epsilon \theta_\delta)\end{aligned}$$

with $\lambda_1 > 0$ and the coefficients k_1, k_2, k_3 chosen so that the polynomial in the complex variable s , given by

$$s^3 + k_3 s^2 + k_2 s + k_1$$

is a Hurwitz polynomial.

The discontinuous incremental feedback control laws

$$u_{1\delta} = -W_1 \text{sign } \sigma_1 \quad u_{2\delta} = W_2 \text{sign } \sigma_2$$

locally create a sliding regime on the sliding surfaces $\sigma_1 = 0$ and $\sigma_2 = 0$.

The sliding mode control to be applied to the nonlinear system is then obtained as

$$u_1 = 1 - W_1 \text{sign } \sigma_1, \quad u_2 = W_2 \text{sign } \sigma_2$$

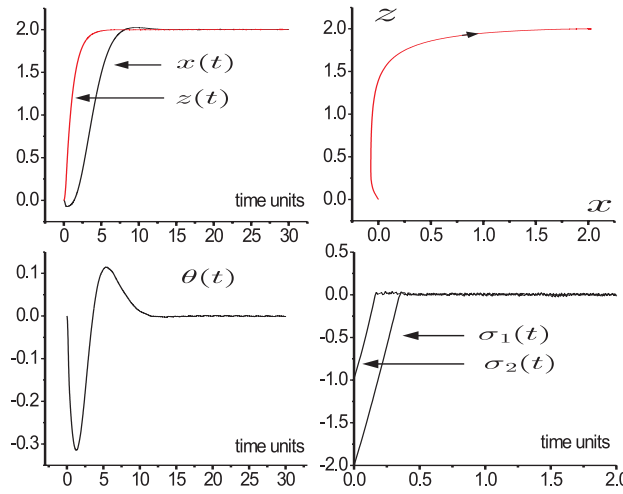


Fig. 3. Sliding mode controlled response of nonlinear PVTOL model

Figure 3 shows the controlled responses of the nonlinear system to the actions of the designed multivariable sliding mode controller in a typical rest-to-rest stabilization maneuver. The normalized desired equilibrium positions were set to be: $\bar{x} = 2$, $\bar{z} = 2$. The only system parameter, ϵ , was set to be 0.5. The sliding mode controller design parameters were chosen as follows:

$$\lambda = 1, \quad W_1 = 5, \quad W_2 = 5,$$

$$k_3 = 2\xi\omega_n + \beta, \quad k_2 = 2\xi\omega_n\beta + \omega_n^2, \quad k_1 = \omega_n^2\beta,$$

$$\beta = 1, \omega_n = 0.7, \quad \xi = 0.8$$

This particular choice places the poles of the incremental ideal equivalent dynamics at the same location of the roots of the polynomials: $s + \lambda$ and $(s + \beta)(s^2 + 2\xi\omega_n s + \omega_n^2)$.

2.4 Flatness of Linear Time-Varying Uniformly Controllable systems

Consider the linear SISO time-varying system

$$\begin{aligned}\dot{x} &= A(t)x + b(t)u, & x \in R^n, u \in R \\ y &= c(t)x & y \in R\end{aligned}\tag{2}$$

We assume the system to be *uniformly controllable* ([11]). In other words, the matrix

$$\mathcal{C}(t) = [b(t), (A(t) - \frac{d}{dt})b(t), \dots, (A(t) - \frac{d}{dt})^{(n-1)}b(t)]$$

is uniformly full rank n over a given finite interval of time $[t_0, t_1]$.

Uniform controllability implies, according to the results of [6], that the system is equivalent to a system in Brunovsky canonical form (a chain of n integrators) after a static change of coordinates and a state-dependent redefinition of the control input. The system is thus differentially flat. The flat output, or Brunovsky output, is directly obtained as a time-varying linear combination of the original states as indicated in the next proposition.

Proposition 1. *A flat output of the uniformly controllable system, $\dot{x} = A(t)x + b(t)u$, is given by the following time-varying linear combination of the components of the state vector x ,*

$$F(t) = [0, \dots, 0, 1]^T \mathcal{C}^{-1}(t)x$$

Moreover, any uniformly non-zero scalar, time-varying, multiple of this linear combination is also a flat output.

Proof

Consider the non-singular time-varying state coordinate transformation,

$$z = \mathcal{C}^{-1}(t)x$$

of the system (2). It is easy to see, that this transformation takes the system into the following time-varying linear system

$$\begin{aligned}\dot{z}_1 &= -\alpha_1(t)z_n + u \\ \dot{z}_2 &= z_1 - \alpha_2(t)z_n \\ &\vdots \\ \dot{z}_{n-1} &= z_{n-2} - \alpha_{n-1}(t)z_n \\ \dot{z}_n &= z_{n-1} - \alpha_n(t)z_n\end{aligned}$$

Evidently, under such circumstances, z_n is the flat output since the control input u and all the components of the transformed vector z , and consequently all the components of the original state vector x , can be differentially parameterized in terms of the variable z_n and a finite number of its time derivatives.

Example 3. Consider the system

$$\dot{x}_1 = tu, \quad \dot{x}_2 = u$$

In this case $A(t) = 0$ and $b(t) = [t; 1]^T$. The controllability matrix and its inverse matrix are given by

$$\mathcal{C}(t) = \begin{bmatrix} t & -1 \\ 1 & 0 \end{bmatrix} \quad ; \quad \mathcal{C}^{-1}(t) = \begin{bmatrix} 0 & 1 \\ -1 & t \end{bmatrix}$$

The controllability matrix rank is 2, independently of t and, hence, the system is uniformly controllable over any finite time interval of time i.e. the system is *totally* controllable. The state coordinate transformation, $z_1 = x_2$, $z_2 = -x_1 + tx_2$ yields,

$$\dot{z}_1 = u, \quad \dot{z}_2 = z_1$$

A flat output is given by $z_2 = F = -x_1 + tx_2$. The differential parameterization of the system variables is given by

$$x_1 = -F + t\dot{F}, \quad x_2 = \dot{F}, \quad u = \ddot{F}$$

The previous results may be extended to linear time-varying multivariable systems, in a manner similar to that used in the time-invariant case. We proceed by working out an example which illustrates the extension.

2.5 Control around a nominal trajectory of the non-holonomic two-wheeled car

Consider the simpler model of a non-holonomic two wheeled car,

$$\dot{x} = v \cos \theta, \quad \dot{y} = v \sin \theta, \quad \dot{\theta} = \omega$$

where x and y denote the position coordinates of the middle point of the wheels' axis, θ is the orientation angle with respect to the horizontal axis x . The control inputs are represented by the forward velocity v and the turning angular velocity ω .

The system is differentially flat, with flat outputs given by the coordinates x and y . The differential parameterization of the system variable in terms of x , y and a finite number of their time derivatives is given by,

$$\theta = \arctan \left(\frac{\dot{y}}{\dot{x}} \right), \quad v = \sqrt{\dot{x}^2 + \dot{y}^2}, \quad \omega = \frac{\ddot{y}\dot{x} - \dot{y}\ddot{x}}{\dot{x}^2 + \dot{y}^2} \quad (3)$$

This differential parameterization (3) is most useful in obtaining a *nominal* state and input trajectory. For instance, given the time parameterized expressions for a circular trajectory on the plane $x - y$, as follows,

$$x^*(t) = R \cos \gamma t, \quad y^*(t) = R \sin \gamma t \quad (4)$$

we immediately obtain:

$$\theta^*(t) = \arctan(-\cot \gamma t) = \frac{\pi}{2} + \gamma t, \quad v^*(t) = R\gamma, \quad \omega^*(t) = \gamma \quad (5)$$

We define the incremental variables $x_\delta = x - x^*(t)$, $y_\delta = y - y^*(t)$ and $\theta_\delta = \theta - \theta^*(t)$. Consider the Jacobian linearization of the non-holonomic car equations along the given nominal trajectory, (4)-(5). The system in the matrix form:

$$\dot{x}_\delta = A(t)x_\delta + B(t)u_\delta \quad (6)$$

is obtained as:

$$\begin{bmatrix} \dot{x}_\delta \\ \dot{y}_\delta \\ \dot{\theta}_\delta \end{bmatrix} = \begin{bmatrix} 0 & 0 & -v^*(t) \sin \theta^*(t) \\ 0 & 0 & v^*(t) \cos \theta^*(t) \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_\delta \\ y_\delta \\ \theta_\delta \end{bmatrix} + \begin{bmatrix} \cos \theta^*(t) & 0 \\ \sin \theta^*(t) & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_\delta \\ \omega_\delta \end{bmatrix}$$

The linearized time-varying system is uniformly controllable on any given finite time interval (i.e. the system is *totally* controllable). Indeed, the controllability matrix is given by the array of vectors:

$$\mathcal{C}(t) = \left[b_1(t), \left(A(t) - \frac{d}{dt} \right) b_1(t), b_2(t) \right] = \begin{bmatrix} \cos \theta^*(t) & \omega^*(t) \sin \theta^*(t) & 0 \\ \sin \theta^*(t) & -\omega^*(t) \cos \theta^*(t) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The system has controllability indices: $\gamma_1 = 2$ and $\gamma_2 = 1$. The inverse of the controllability matrix reveals that the incremental flat outputs are given by:

$$F_{1\delta} = \cos \theta^*(t)x_\delta + \sin \theta^*(t)y_\delta, \quad F_{2\delta} = \theta_\delta$$

The differential parameterization of the system incremental state variables and incremental control input variables, in terms of the flat outputs $F_{1\delta}$, $F_{2\delta}$, is readily obtained to be

$$\begin{aligned} x_\delta &= \left[\sin \theta^*(t) F_{1\delta} + \cos \theta^*(t) \left(\dot{F}_{1\delta} + v^*(t) F_{2\delta} \right) \right] \\ y_\delta &= \frac{1}{\omega^*(t)} \left[\cos \theta^*(t) F_{1\delta} + \sin \theta^*(t) \left(\dot{F}_{1\delta} + v^*(t) F_{2\delta} \right) \right] \\ \theta_\delta &= F_{2\delta} \\ v_\delta &= \frac{1}{\omega^*(t)} \left[\ddot{F}_{1\delta} + v^*(t) \dot{F}_{2\delta} + [\omega^*(t)]^2 F_{1\delta} + \dot{v}^*(t) F_{2\delta} \right] \\ \omega_\delta &= \dot{F}_{2\delta} \end{aligned}$$

Let ξ , ω_n and λ be strictly positive constants. The following set of time-varying feedback control laws:

$$\begin{aligned} v_\delta &= \frac{1}{\omega^*(t)} \left[-2\xi\omega_n \dot{F}_{1\delta} - (\omega_n^2 - [\omega^*(t)]^2) F_{1\delta} + v^*(t) \dot{F}_{2\delta} + \dot{v}^*(t) F_{2\delta} \right] \\ \omega_\delta &= -\lambda F_{2\delta} \end{aligned}$$

impose, on the closed loop incremental system, the following set of decoupled exponentially asymptotically stable responses which cause the incremental flat outputs to converge to zero.

$$\ddot{F}_{1\delta} + 2\xi\omega_n\dot{F}_{1\delta} + \omega_n^2 F_{1\delta} = 0, \quad \dot{F}_{2\delta} + \lambda F_{2\delta} = 0$$

Figure 4 shows the simulated closed loop responses of the system to the proposed continuous feedback controller.

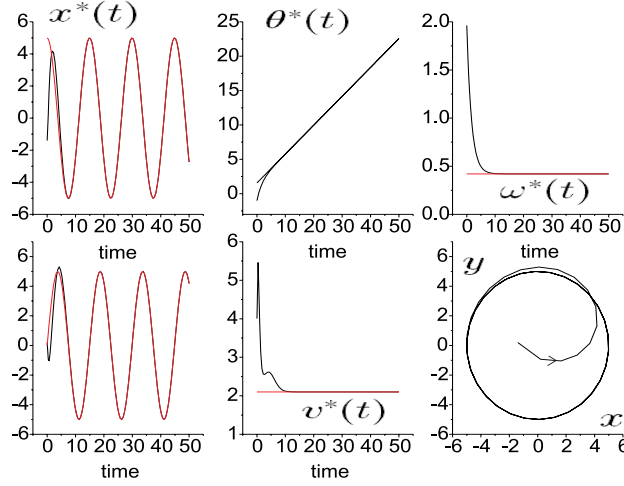


Fig. 4. Closed loop responses of the two wheeled car system

A controller locally creating stabilizing motions on the intersection of the sliding surfaces, can also be synthesized by setting the required sliding surfaces and the high-gain control policies to be,

$$\sigma_{1\delta} = \dot{F}_{1\delta} + \lambda_1 F_{1\delta}, \quad v_\delta = -\frac{W_1}{\omega^*(t)} \left(\frac{\sigma_{1\delta}}{|\sigma_{1\delta}| + \epsilon_1} \right), \quad 0 < \epsilon_1 \ll 1$$

$$\sigma_{2\delta} = F_{2\delta}, \quad \omega_\delta = -W_2 \left(\frac{\sigma_{2\delta}}{|\sigma_{2\delta}| + \epsilon_2} \right), \quad 0 < \epsilon_2 \ll 1$$

3 Sliding mode control without state measurements

Here we first explore, in the context of linear time-invariant systems, the possibilities of achieving stabilizing sliding motions *without* state measurements. The technique, which does not use asymptotic state observers either, relies on measurements of available inputs and outputs only. A state-dependent sliding surface, known to create an asymptotically stable motion towards the

desired equilibrium, is synthesized on the basis of inputs, outputs and iterated integrals of such quantities. The effect of the unknown initial values of the state variables is compensated via integral control actions entitling again iterated integration of the output stabilization error. We briefly develop a simple theoretical basis for such an approach.

3.1 Generalities on Sliding Mode Control of Linear Multivariable Systems

Consider the controllable and observable linear, time-invariant, multivariable system, with m inputs and m outputs

$$\begin{aligned}\dot{x} &= Ax + Bu, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m \\ y &= Cx, \quad y \in \mathbb{R}^m\end{aligned}\tag{7}$$

The unknown initial state vector, at time 0, is denoted by $x(0)$.

Let S be an $m \times n$, full rank m , constant matrix with rows vectors, s_i , $i = 1, \dots, m$, i.e. $S = [s_1; \dots; s_m]$, such that the constant, $m \times m$, matrix SB is full rank m , i.e. it is invertible. Assume that the matrix S is chosen so that the projection matrix, $M = (I - B(SB)^{-1}S)$, is such that the matrix MA exhibits all its non-zero eigenvalues with strictly negative real parts. This means that when restricted to the the *null* space of S , (which we denote by $\mathcal{N}(S) = \{x : Sx = 0\}$), the dynamics: $\dot{x} = MAx$, $x \in \mathcal{N}(S)$ is asymptotically exponentially stable to the zero state.

Consider then, the set of sliding surfaces, represented by the set of equalities, $\sigma = Sx = 0$, i.e. $\sigma_i = s_i x$, $i = 1, \dots, m$. These sets are fully described by the null space of S . Let D be a positive definite $m \times m$ matrix of the form $D = W + \Delta$ which we may choose to be diagonal with entries of the form: $\omega_i + \delta_i$, $\omega_i, \delta_i > 0$ for all i . Away from $\mathcal{N}(S)$, the time derivative of the vector, $\sigma = Sx$, is given by

$$\dot{\sigma} = SAx + (SB)u$$

Let $\text{SIGN}(\cdot)$ be a vector operator of *signum* functions, whose i -th component, $(\text{SIGN } \sigma)_i$ is just given by: $(\text{SIGN } \sigma)_i = \text{sign } \sigma_i$. The choice $u = -(SB)^{-1}D \text{SIGN } \sigma$, yields

$$\begin{aligned}\frac{d}{dt} \left(\frac{1}{2} \sigma^T \sigma \right) &= \sigma^T \dot{\sigma} = \sigma^T SAx - \sigma^T D \text{SIGN } \sigma \\ &= \sigma^T (SAx - D \text{SIGN } \sigma) - \sigma^T W \text{SIGN } \sigma\end{aligned}$$

If we choose an open neighborhood, Σ , of the state space, containing the origin and also containing a non-empty intersection with the subspace $\mathcal{N}(S)$, where the following relations are satisfied for all i ,

$$\delta_i > |s_i Ax| \quad \forall x \in \Sigma$$

then, it follows that, in Σ , the following inequalities are satisfied,

$$\frac{d}{dt} \left(\frac{1}{2} \sigma^T \sigma \right) \leq -\sigma^T W \text{SIGN } \sigma = - \sum_{i=1}^m \omega_i | \sigma_i | < 0$$

As a consequence, a sliding regime is locally achieved on the sliding surface, $\mathcal{N}(S)$, which asymptotically exponentially stabilizes the motions of the system to the origin of the state space.

We remark that the choice: $u = -(SB)^{-1}SAx - (SB)^{-1}W \text{SIGN } \sigma$, globally achieves a sliding motion on $\mathcal{N}(S)$.

3.2 Integral reconstructors of the state

Consider again the linear time-invariant multivariable system (7). Integrating the system equations we obtain

$$x(t) = A \int_0^t x(\sigma_1) d\sigma_1 + B \int_0^t u(\sigma_1) d\sigma_1 + x(0)$$

Substituting the expression for x , back into the same equation, we obtain:

$$\begin{aligned} x(t) = & A^2 \int_0^t \int_0^{\sigma_1} x(\sigma_2) d\sigma_2 d\sigma_1 + AB \int_0^t \int_0^{\sigma_1} u(\sigma_2) d\sigma_2 d\sigma_1 \\ & + B \int_0^t u(\sigma_1) d\sigma_1 + x_0 + A \int_0^t x_0 d\sigma_1 \end{aligned}$$

We adopt the following notation for iterated integration

$$\left(\int_0^t \phi(t) dt \right)^{[j]} = \int_0^t \int_0^{\sigma_1} \cdots \int_0^{\sigma_{j-1}} \phi(\sigma_j) d\sigma_j d\sigma_{j-1} \cdots d\sigma_1$$

with $\left(\int_0^t \phi(\sigma) d\sigma \right)^{[0]} = \phi(t)$.

Carrying out the indicated substitution process, repeatedly, we obtain for some integer ρ satisfying: $n > \rho > 1$:

$$\begin{aligned} x(t) = & A^{\rho-1} \left(\int_0^t x(t) dt \right)^{[\rho-1]} + \sum_{i=1}^{\rho-1} A^{i-1} B \left(\int_0^t u(t) dt \right)^{[i]} \\ & + \sum_{i=1}^{\rho-2} A^{i-1} \left(\int_0^t x_0 dt \right)^{[i-1]} \\ = & A^{\rho-1} \left(\int_0^t x(t) dt \right)^{[\rho-1]} + \sum_{i=1}^{\rho-1} A^{i-1} B \left(\int_0^t u(t) dt \right)^{[i]} \\ & + \sum_{i=1}^{\rho-2} A^{i-1} x_0 \frac{t^{i-1}}{(i-1)!} \end{aligned} \quad (8)$$

Consider now the nm -dimensional column vector of successive derivatives of the outputs and its relation with the states and the inputs

$$\begin{bmatrix} y \\ \dot{y} \\ \vdots \\ y^{(n-1)} \end{bmatrix} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} x + \begin{bmatrix} 0 & 0 & \cdots & 0 \\ CB & 0 & \cdots & 0 \\ CAB & CB & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ CA^{n-2}B & CA^{n-3}B & \cdots & CB \end{bmatrix} \begin{bmatrix} u \\ \dot{u} \\ \vdots \\ u^{(n-2)} \end{bmatrix}$$

Let the matrix C be constituted by m independent row vectors denoted by $C = [c_1; \dots; c_m]$. The observability of the system implies that the Kalman observability matrix $\mathcal{K}_O = [C; CA; \dots; CA^{n-1}]$ is an $nm \times n$ matrix of rank n . Observability means that we may choose n independent row vectors of the matrix $\mathcal{K}_O$ by eliminating those row vectors of the form: $c_j A^{\rho_j}$, which are in the rank of $\mathcal{K}_O$ for, and beyond, some integer ρ_j for each j . We obtain:

$$\begin{bmatrix} y_1 \\ \vdots \\ y_1^{(\rho_1-1)} \\ \vdots \\ y_m \\ \vdots \\ y_m^{(\rho_m-1)} \end{bmatrix} = \begin{bmatrix} c_1 \\ \vdots \\ c_1 A^{\rho_1-1} \\ \vdots \\ c_m \\ \vdots \\ c_m A^{\rho_m-1} \end{bmatrix} x + M \begin{bmatrix} u \\ \dot{u} \\ \vdots \\ u^{(\rho-2)} \end{bmatrix}, \quad \rho = \max_j \{ \rho_j \}$$

where M is a matrix depending on A , B , C and the observability indices ρ_j . We denote by $\mathcal{O}$ the matrix $\mathcal{O} = [c_1; c_1 A; \dots; c_1 A^{\rho_1-1}; \dots; c_m; \dots; c_m A^{\rho_m-1}]$. From the above relations we obtain

$$x = \mathcal{O}^{-1} \left\{ \begin{bmatrix} y_1 \\ \vdots \\ y_1^{(\rho_1-1)} \\ \vdots \\ y_m \\ \vdots \\ y_m^{(\rho_m-1)} \end{bmatrix} - M \begin{bmatrix} u \\ \dot{u} \\ \vdots \\ u^{(\rho-2)} \end{bmatrix} \right\}$$

This last relation implies that we may write the quantity, $\left(\int_0^t x(t) dt \right)^{[\rho-1]}$, solely in terms of outputs and iterated integrals of inputs and outputs.

$$\left(\int_0^t x(t) dt \right)^{[\rho-1]} = \sum_{i=0}^{\rho-1} \left[P_i \left(\int_0^t y(t) \right)^{[i]} + Q_i \left(\int_0^t u(t) \right)^{[i+1]} \right] \quad (9)$$

Substituting the expression (9) in (8) we obtain that x may be written as

$$x(t) = \sum_{i=0}^{\rho} \left[\Phi_i \left(\int_0^t y(t) dt \right)^{[i]} + \Theta_i \left(\int_0^t u(t) dt \right)^{[i+1]} \right] + \sum_{i=1}^{\rho-2} A^{i-1} x_0 \frac{t^{i-1}}{(i-1)!}$$

For appropriate matrices Φ_i and Θ_i .

We refer to the following quantity as an *integral reconstructor* of the state vector x

$$\hat{x}(t) = \sum_{i=0}^{\rho} \left[\Phi_i \left(\int_0^t y(t) dt \right)^{[i]} + \Theta_i \left(\int_0^t u(t) dt \right)^{[i+1]} \right]$$

Example 4. Consider the circuit shown in figure 5 consisting of two LC oscillators controlled by the same external input voltage source.

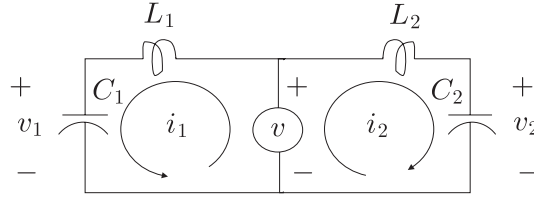


Fig. 5. two input-coupled oscillators

The system is described by the following set of differential equations

$$\begin{aligned} C_1 \frac{dv_1}{dt} &= i_1, & L_1 \frac{di_1}{dt} &= -v_1 + v \\ C_2 \frac{dv_2}{dt} &= i_2, & L_2 \frac{di_2}{dt} &= -v_2 + v \end{aligned}$$

Let E is an arbitrary positive constant voltage acting as a *per unit* reference value. We define new (normalized) state variables as: $x_1 = v_1/E$, $x_2 = (i_1/E)\sqrt{L_1/C_1}$, $x_3 = v_2/E$, $x_4 = (i_2/E)\sqrt{L_2/C_2}$, and we also set the new input variable as: $u = v/E$. One, hence, obtains the following set of second order equations for the normalized capacitor voltages, $x = x_1$ and $y = x_3$,

$$\ddot{x} = -\omega_1^2(x - u), \quad \ddot{y} = -\omega_2^2(y - u)$$

where $\omega_1 = 1/\sqrt{L_1 C_1}$ and $\omega_2 = 1/\sqrt{L_2 C_2}$.

It is clear that the system is controllable if and only if $\omega_1 \neq \omega_2$. The flat output, which may also be found by inspection, is given by

$$F = \omega_2^2 x - \omega_1^2 y$$

The system is also found to be observable when the output of the system is taken to be the flat output F . This means that we can provide, modulo initial conditions, an integral input-output parameterization of the variables x , y and $\dot{x}$, $\dot{y}$.

$$\begin{aligned}
 x &= -\omega_1^2 \left[\frac{1}{\omega_2^2(\omega_2^2 - \omega_1^2)} \left(F + \omega_2^2 \left(\int F \right)^{[2]} \right) - \left(\int u \right)^{[2]} \right] \\
 y &= -\omega_2^2 \left[\frac{1}{\omega_1^2(\omega_2^2 - \omega_1^2)} \left(F + \omega_1^2 \left(\int F \right)^{[2]} \right) - \left(\int u \right)^{[2]} \right] \\
 \dot{x} &= \left[\frac{\omega_1^4}{\omega_2^2(\omega_2^2 - \omega_1^2)} \left(\left(\int F \right) + \omega_2^2 \left(\int F \right)^{[3]} \right) - \omega_1^4 \left(\int u \right)^{[3]} \right] \\
 &\quad + \omega_1^2 \left(\int u \right) \\
 \dot{y} &= \left[\frac{\omega_2^4}{\omega_1^2(\omega_2^2 - \omega_1^2)} \left(\left(\int F \right) + \omega_1^2 \left(\int F \right)^{[3]} \right) - \omega_2^4 \left(\int u \right)^{[3]} \right] \\
 &\quad + \omega_2^2 \left(\int u \right)
 \end{aligned}$$

This parameterization allows a similar parameterization of $\dot{F} = \omega_2^2 \dot{x} - \omega_1^2 \dot{y}$, $\ddot{F} = \omega_1^2 \omega_2^2 (y - x)$, $F^{(3)} = \omega_1^2 \omega_2^2 (\dot{y} - \dot{x})$, which are needed for any flatness based controller.

3.3 Sliding mode control based on integral reconstructors of the state

It should be clear from the previous developments that the relation linking the integral reconstructor, $\hat{x}$, and the actual value of the state, x , is given by,

$$\begin{aligned}
 x(t) &= \hat{x}(t) + \sum_{i=1}^{\rho-2} A^{i-1} x_0 \frac{t^{i-1}}{(i-1)!} \\
 &= \hat{x}(t) + \sum_{i=1}^{\rho-2} A^{i-1} \left(\int_0^t x_0 dt \right)^{[i-1]}
 \end{aligned} \tag{10}$$

Our main concern is how to appropriately compensate for the effects of the unknown initial conditions, x_0 , when the actual value of the state is replaced by its integral reconstructor in a given state-based feedback controller design. In particular, when a set of sliding surface coordinate functions of the form: $\sigma = Sx$, are to be synthesized in terms of the integral reconstructor for the state x , as $\hat{\sigma} = S\hat{x}$.

Consider then the following *compensated set of sliding surface coordinate functions*, synthesized in terms of the integral reconstructor of the state,

$$\sigma_c = S\hat{x} + \xi(t)$$

This set of sliding functions, in light of (10), is clearly equivalent to the following set of perturbed sliding surface coordinates functions,

$$\begin{aligned}\sigma_c &= Sx + \xi(t) - S \sum_{i=1}^{\rho-2} A^{i-1} x_0 \frac{t^{i-1}}{(i-1)!} \\ &= Sx + \xi(t) - \sum_{i=1}^{\rho-2} SA^{i-1} \left(\int_0^t x_0 dt \right)^{[i-1]}\end{aligned}$$

Let Γ_i , $i = 1, \dots, \rho-2$ be an $n \times m$ matrix of constant entries. We choose ξ to be given by $\xi(t) = \sum_{i=1}^{\rho-2} SA^{i-1} \left(\int_0^t \Gamma_i \zeta(t) dt \right)^{[i]}$ with $\dot{\zeta}(t) = y(t)$ and $\zeta(0) = 0$. The compensated sliding surface coordinate functions are thus expressed as

$$\sigma_c = Sx - \sum_{i=1}^{\rho-2} SA^{i-1} \left(\int_0^t (x_0 - \Gamma_i \zeta) dt \right)^{[i-1]}$$

The fundamental idea behind GPI control is to appropriately choose the gain matrices, Γ_i , so as to guarantee that asymptotically, $\sigma_c \rightarrow \sigma$ for any x_0 . Certainly, the class of time-varying functions that need to be compensated in the above scheme, corresponds to those that can be satisfactorily matched by means of iterated integral control actions based, solely, on the outputs of the system (namely: constants, ramps, parabolic ramps, etc). That this is certainly possible, in a stable manner, stems from the controllability of the original system and of its associated integral-based extensions. We shall not specifically demonstrate the generality of this remark here, and, rather, refer the reader to Fliess *et al*, [4] for a complete theoretical support. We proceed to illustrate the proposed approach by means of several examples.

Example 5. GPI Sliding Mode Control of the PVTOL

Consider again the tangent linearization of the PVTOL model around a given equilibrium point,

$$\ddot{x}_\delta = -\theta_\delta + \epsilon u_{2\delta}, \quad \ddot{z}_\delta = u_{1\delta} \quad \ddot{\theta}_\delta = u_{2\delta}$$

Suppose that the system incremental outputs are given by

$$y_{1\delta} = x_\delta \quad y_{2\delta} = z_\delta$$

We obtain the following integral reconstructor, or integral parameterization, of the states variables of the system, and of the flat outputs $F_1 = z_\delta$, $F_2 = x_\delta - \epsilon \theta_\delta$

$$\begin{aligned}\hat{\theta}_\delta &= \left(\int u_{2\delta} \right)^{[2]}, \quad x_\delta = y_{1\delta}, \quad z_\delta = y_{2\delta} \\ \hat{\dot{\theta}}_\delta &= \left(\int u_{2\delta} \right), \quad \hat{x}_\delta = - \left(\int u_{2\delta} \right)^{[3]} + \epsilon \left(\int u_{2\delta} \right), \quad \hat{z}_\delta = \left(\int u_{1\delta} \right)\end{aligned}$$

$$\begin{aligned}\widehat{F}_2 &= y_{1\delta} - \epsilon \left(\int u_{2\delta} \right)^{[2]}, & \widehat{F}_1 &= y_{2\delta} \\ \widehat{\dot{F}}_2 &= - \left(\int u_{2\delta} \right)^{[3]}, & \widehat{\dot{F}}_1 &= \left(\int u_{1\delta} \right) \\ \widehat{\ddot{F}}_2 &= - \left(\int u_{2\delta} \right)^{[2]}, & \widehat{F}_2^{(3)} &= - \left(\int u_{2\delta} \right)\end{aligned}$$

The exact relations between the integral reconstructors of the flat outputs, and the corresponding reconstructors of their time derivatives, and the actual values of such variables are clearly given by

$$\begin{aligned}F_2 &= \widehat{F}_2 - \epsilon \dot{\theta}_{\delta 0} t - \epsilon \theta_{\delta 0}, & \dot{F}_2 &= \widehat{\dot{F}}_2 - \frac{1}{2} \dot{\theta}_{\delta 0} t^2 - \theta_{\delta 0} t + \dot{x}_{\delta 0} - \epsilon \dot{\theta}_{\delta 0} \\ \ddot{F}_2 &= \widehat{\ddot{F}}_2 - \dot{\theta}_{\delta 0} t - \theta_{\delta 0}, & F_2^{(3)} &= \widehat{F}_2^{(3)} - \dot{\theta}_{\delta 0} \\ F_1 &= \widehat{F}_1 = y_{2\delta}, & \dot{F}_1 &= \widehat{\dot{F}}_1 + \dot{z}_{\delta 0}\end{aligned}$$

We recall the following flatness based sliding surfaces used in the previous section:

$$\begin{aligned}\sigma_1 &= \dot{F}_1 + \lambda_1 F_1 = \dot{z}_{\delta} + \lambda z_{\delta} \\ \sigma_2 &= F_2^{(3)} + k_3 \ddot{F}_2 + k_2 \dot{F}_2 + k_1 F_2 \\ &= -\dot{\theta}_{\delta} - k_3 \theta_{\delta} + k_2 (\dot{x}_{\delta} - \epsilon \dot{\theta}_{\delta}) + k_1 (x_{\delta} - \epsilon \theta_{\delta})\end{aligned}$$

The corresponding integral reconstructors of the sliding surface coordinate functions are thus given by

$$\widehat{\sigma}_1 = \widehat{\dot{F}}_1 + \lambda_1 \widehat{F}_1, \quad \widehat{\sigma}_2 = \widehat{F}_2^{(3)} + k_3 \widehat{\ddot{F}}_2 + k_2 \widehat{\dot{F}}_2 + k_1 \widehat{F}_2$$

Using the above relations we readily obtain the form of the actual relations between the integral reconstructors of the sliding surface coordinate functions and their actual values.

$$\sigma_1 = \widehat{\sigma}_1 + \dot{z}_{\delta 0}, \quad \sigma_2 = \widehat{\sigma}_2 + a + bt + ct^2$$

where a , b and c are unknown constants which depend on the initial conditions of, θ_{δ} , $\dot{\theta}_{\delta}$, x_{δ} and $\dot{x}_{\delta}$.

We thus propose the following integrally compensated sliding surface coordinate functions

$$\begin{aligned}\sigma_{1c} &= \widehat{\sigma}_1 + \gamma_{11} \int_0^t y_{2\delta} d\tau \\ &= \int_0^t (u_{1\delta} + \gamma_{11} y_2) d\tau + \lambda y_2 \\ \sigma_{2c} &= \widehat{\sigma}_2 + \gamma_{21} \int_0^t y_{1\delta} d\tau + \gamma_{22} \int_0^t \int_0^{\tau} y_{1\delta} d\tau d\rho + \gamma_{23} \int_0^t \int_0^{\tau} \int_0^{\rho} y_{1\delta} d\tau d\rho d\alpha\end{aligned}$$

$$\begin{aligned}
&= k_1 y_1 + \int_0^t (\gamma_{21} y_1 - u_{2\delta}) d\tau + \int_0^t \int_0^\tau (\gamma_{22} y_1 - (k_1 \epsilon + k_3) u_{2\delta}) d\tau d\rho \\
&\quad + \int_0^t \int_0^\tau \int_0^\rho (\gamma_{23} y_1 - k_2 u_{2\delta}) d\tau d\rho d\alpha
\end{aligned}$$

To show the effectiveness of the proposed integrally compensated sliding surface coordinate functions, suppose, first, that the sliding mode condition: $\sigma_{1c} = 0$, is permanently achieved by some appropriate discontinuous control action. Then we have

$$\begin{aligned}
\sigma_{1c} &= \sigma_1 - \dot{z}_{\delta 0} + \gamma_{11} \int_0^t z_\delta d\tau \\
&= \dot{F}_1 + \lambda F_1 - \left(\dot{z}_{\delta 0} - \gamma_{11} \int_0^t z_\delta d\tau \right) = 0
\end{aligned}$$

Taking one time derivative, and using the fact that $z_{1\delta} = F_1$, this last expression is seen to be equivalent to the following expression

$$\ddot{F}_1 + \lambda \dot{F}_1 + \gamma_{11} F_1 = 0$$

Evidently constants λ and γ_{11} can always be found such that the imposed sliding condition creates an asymptotically stable motion of the variable F_1 towards zero.

Similarly, setting $\sigma_{2c} = 0$ we obtain, after taking three time derivatives of the resulting equality, and letting $y_{1\delta} = x_\delta = F_2 - \epsilon \ddot{F}_2$,

$$\begin{aligned}
&F_2^{(6)} + k_3 F_2^{(5)} + (k_2 - \gamma_{21} \epsilon) F_2^{(4)} + (k_1 - \epsilon \gamma_{22}) F_2^{(3)} + (\gamma_{21} - \epsilon \gamma_{23}) \ddot{F}_2 \\
&\quad + \gamma_{22} \dot{F}_2 + \gamma_{23} F_2 = 0
\end{aligned}$$

The Laplace transform of this linear differential equation is a polynomial in the complex variable s that can be made into a Hurwitz polynomial by appropriate choice of the constant design parameters k_1, k_2, k_3 , and $\gamma_{21}, \gamma_{22}, \gamma_{23}$. The constrained motions for the flat output F_2 can be made asymptotically exponentially stable to zero.

Figure 6 shows the performance of the closed loop nonlinear PVTOL system to the GPI sliding mode controller, based on the tangent linearization, given by:

$$u_1 = 1 - W_1 \operatorname{sign} \sigma_{1c}, \quad u_2 = W_2 \operatorname{sign} \sigma_{2c}$$

The closed loop decoupled “characteristic polynomial” for F_2 was set to be: $(s^2 + 2\xi_2 \omega_{n2} s + \omega_{n2}^2)^3$ with $\xi_2 = 0.8$ and $\omega_{n2} = 0.6$, while that corresponding to F_1 was chosen as : $s + 2\xi_1 \omega_{n1} s + \omega_{n1}^2$, with $\xi_1 = 0.8$ and $\omega_{n1} = 1$.

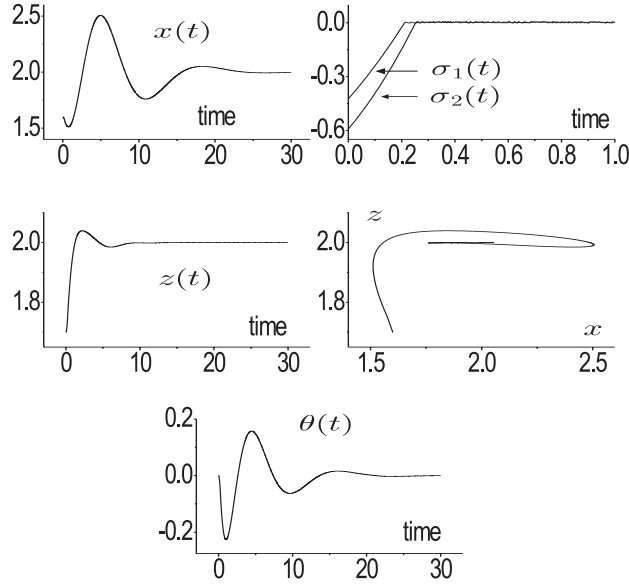


Fig. 6. GPI Sliding mode controlled responses of nonlinear PVTOL model

4 GPI sliding mode control of an Inertia-spring DC motor system

4.1 The inertia-spring-DC motor system model

Consider the electro-mechanical system shown in Figure 7. The mathematical model of this system is readily obtained as:

$$L\dot{I} = -RI - k_e\omega + u, \quad J\dot{\omega} = -B\omega - k\theta + k_m I, \quad \dot{\theta} = \omega \quad (11)$$

where I is the DC motor armature circuit current, θ is the angular displacement of the motor axis, measured with respect to a fixed but arbitrary reference position, and ω is the corresponding angular velocity of the motor axis. The control input, denoted by u , represents the external variable voltage applied to the armature circuit terminals. The parameters L , R , k_e represent, respectively, the armature circuit inductance, the circuit resistance and the back electro-motive force constant of the DC motor. The parameters J , B , k and k_m denote, respectively, the combined rotor and load inertia, the viscous friction coefficient, the torsion spring coefficient and the DC motor torque constant.

Note that the equilibrium of the system, corresponding to a constant desired value of the inertia load position $\theta = \bar{\theta}$, can be obtained as

$$\bar{y} = \frac{k}{k_m} \bar{\theta}, \quad \bar{u} = \frac{Rk}{k_m} \bar{\theta}$$

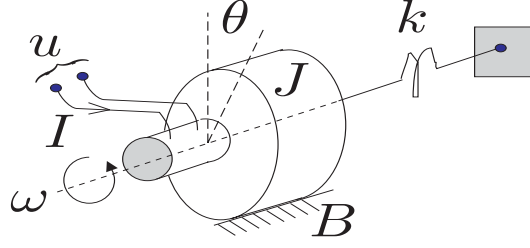


Fig. 7. Inertia-spring DC motor system

The system is controllable and, hence, differentially flat, with flat output given by the inertia load, or motor axis, angular position θ . The flat output satisfies the following differential polynomial relation with the input,

$$J\theta^{(3)} + \left(B + \frac{RJ}{L}\right)\ddot{\theta} + \left(k + \frac{k_m k_e + RB}{L}\right)\dot{\theta} + \frac{Rk}{L}\theta = \frac{k_m}{L}u \quad (12)$$

As it can be easily determined from (11), the system model is observable for the electrical output $y = I$. This fact establishes the *constructibility* of the system, which, in turn, implies that *all* system state variables are parameterizable in terms of inputs, outputs and iterated integrals of the input and the output variables (See Fliess *et al* [4]). Such an integral input-output parameterization of the system state variables is given, modulo initial conditions, by

$$\begin{aligned} \hat{\theta} &= -\frac{L}{k_e}y + \frac{1}{k_e} \int_0^t [u(\tau) - Ry(\tau)]d\tau \\ \hat{\dot{\theta}} &= -\frac{k}{J} \int_0^t \hat{\theta}(\tau)d\tau + \frac{k_m}{J} \int_0^t y(\tau)d\tau - \frac{B}{J}\hat{\theta} \\ \hat{\ddot{\theta}} &= -\frac{k}{J}\hat{\dot{\theta}} + \frac{k_m}{J}y - \frac{B}{J}\hat{\ddot{\theta}} \\ \hat{I} &= I = y \end{aligned} \quad (13)$$

The first expression in (13) is obtained by integration of the first equation in (11). The second expression is obtained by integration of the second equation in (11). The third relation is just the second equation in (11). Note that, for non-zero initial states, the relations linking the actual values of the angular position derivatives to the structural estimates in (13) are given by

$$\begin{aligned} \theta &= \hat{\theta} + \theta_0, \quad \dot{\theta} = \hat{\dot{\theta}} + \dot{\theta}_0 - \frac{k}{J}\theta_0 t - \frac{B}{J}\theta_0 \\ \ddot{\theta} &= \hat{\ddot{\theta}} - \frac{B}{J}\dot{\theta}_0 + \frac{Bk}{J^2}\theta_0 t + \left(\frac{B^2}{J^2} - \frac{k}{J}\right)\theta_0 \end{aligned} \quad (14)$$

where θ_0 and $\dot{\theta}_0$ denote the initial mass position and the initial mass velocity.

We remark that a similar experimental example, dealing with an inertia-DC-motor system, has been completely worked out by Marquez *et al* in [7] for the flatness-based regulation of the *angular velocity*. Their approach uses pole placement and it also requires no mechanical sensors.

4.2 GPI Sliding mode control

A sliding surface coordinate function, σ , that ideally induces, by the sliding invariance condition $\sigma = 0$, an asymptotic exponential stabilization of the mass position, θ , towards a constant desired value, $\bar{\theta}$, is given by,

$$\sigma = \ddot{\theta} + k_3\dot{\theta} + k_2(\theta - \bar{\theta})$$

for suitable (Hurwitz) choices of k_3 and k_2 . We propose, nevertheless, the following modified sliding surface coordinate function, which does not use the otherwise required measurements of the angular position, angular velocity and angular acceleration, but instead uses the structural estimates of these variables, previously defined in (13).

$$\hat{\sigma} = \hat{\ddot{\theta}} + k_3\hat{\dot{\theta}} + k_2(\hat{\theta} - \bar{\theta}) + k_1\xi + k_0\eta \quad (15)$$

with

$$\dot{\xi} = y - \frac{k}{k_m}\bar{\theta}, \quad \xi(0) = 0; \quad \dot{\eta} = \xi, \quad \eta(0) = 0$$

The added iterated integral control action suitably compensates the constant and the linearly growing structural estimate errors with respect to the actual values of the flat output and its time derivatives. The underlying equivalent (but never used) expression of the sliding surface in terms of the actual (unmeasured) values of the load angular position variables, and its time derivatives, is obtained by replacing (14) into (15). The obtained expression is of the form:

$$\begin{aligned} \hat{\sigma} = & \ddot{\theta} + k_3\dot{\theta} + k_2(\theta - \bar{\theta}) + \left(k_1 \int_0^t \left[y - \frac{k}{k_m}\bar{\theta} \right] d\tau - \alpha \right) \\ & + \int_0^t \left[k_0 \int_0^\tau \left(y - \frac{k}{k_m}\bar{\theta} \right) d\rho - \beta \right] d\tau \end{aligned}$$

where the constant parameters α and β depend on the initial conditions for θ and $\dot{\theta}$. The ideal sliding condition, $\hat{\sigma} = 0$, is easily seen to be equivalent to the following forth order closed loop system, which is completely independent of any initial condition values.

$$\begin{aligned} \theta^{(4)} + (k_3 + k_1 \frac{J}{k_m})\theta^{(3)} + (k_2 + \frac{k_1 B}{k_m} + \frac{k_0 J}{k_m})\ddot{\theta} \\ + \left(\frac{k_1 k}{k_m} + \frac{k_0 B}{k_m} \right) \dot{\theta} + \frac{k_0 k}{k_m}(\theta - \bar{\theta}) = 0 \end{aligned}$$

where, as before, use has been made of the flatness-based differential parameterization linking the system output, $y = I$, to the flat output θ . It is clear that a suitable choice of the design parameter set $\{k_3, k_2, k_1, k_0\}$ yields an exponentially asymptotically stable closed loop dynamics for the mass position θ .

We propose the following discontinuous sliding mode feedback controller:

$$u = \bar{u} - W \operatorname{sign} \hat{\sigma}, \quad W > 0$$

4.3 Experimental results

The desired objective was to stabilize the system motions to $\bar{\theta} = 0$ [rad] from an initial position of $\theta_0 = 0.03$ [rad] and an initial current $I(0) = 0.183$ [Amp]. The design constants were chosen to be the coefficients of a fourth order polynomial, in the complex variable s , of the form: $(s^2 + 2\zeta\omega_n s + \omega_n^2)^2$, with $\zeta = 0.707$, $\omega_n = 80$. We also set, $W = 10$. Figure 8 depicts the system variables responses obtained from the used experimental set up.

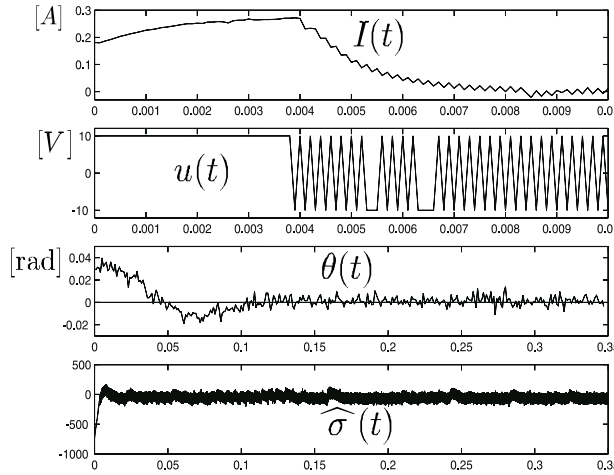


Fig. 8. Experimental results for the sliding mode based GPI controller

5 Conclusions and Suggestions for Future Work

In this chapter, we have studied in a rather tutorial fashion the implications of differential flatness in the design of sliding mode control of dynamic systems. The flatness property was seen to largely trivialize the sliding mode

controller design problem in several classes of dynamic systems. Linear, nonlinear, SISO or Multivariable, time-invariant or time-varying systems, all can be easily treated thanks to the flatness property. We have also explored the possibilities of incorporating the ideas of Generalized PI controllers into the sliding surface coordinate function synthesis task. This implies, at least in the linear case, that for observable, or constructible, systems there is no real need to measure the state vector of the system, nor to build asymptotic state observers, in order to effectively synthesize a sliding surface with desired, closed loop, stability features. As demonstrated by several examples, integral reconstructors, based on the availability of inputs and outputs, can be used for the sliding surface coordinate function synthesis. The basic advantage of GPI-based sliding mode control over traditional observer-based sliding surface synthesis lies in the enhanced robustness of the integral reconstructors, with their associated integral output error compensation loops, to sudden parameter variations and other external un-modeled uncertainties (see [4] for a preliminary example).

Several important issues deserve attention and further development within the context of flatness and integral reconstructor-based sliding surface coordinate functions synthesis. One of them is to extend the presented approaches to deal with a significant class of nonlinear SISO and nonlinear multivariable systems. A second feature is to analyze the central issue of robustness in connection with integral reconstructor-based sliding surfaces when the system is subject to various kinds of uncertainties. A final topic for further study is constituted by a needed extension of some of these kind of controller synthesis results to the case of linear delay differential systems.

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On Robust VSS Nonlinear Servomechanism Problem

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Abstract. The paper deals with the problem of tracking a reference signal while rejecting the effect of a disturbance signal by means of the VSS technique. Analogously to the formulation of the so-called classical servomechanism problem, we define the sliding mode servomechanism problem and derive conditions for the existence of a solution for in general case and for a classes of nonlinear system presented in the Regular Form or in the Nonlinear Block Controllable Form. The effectiveness of the proposed method is demonstrated by the application to the Pendubot system.

1 Introduction

The problem of controlling the output of a uncertain nonlinear system so as to achieve asymptotic tracking of a reference and rejection of unknown admissible disturbances is one of interesting and attractive problem in classical control theory. This problem known as robust output regulation or robust servomechanism problem, has been studied for the nonlinear case (see for example, [2,4–6]), for which a continuous regulator has been proposed.

In this work, a discontinuous regulator is investigated by combining the sliding mode control [10] and block control technique [7,8]. The underlying idea is to design a sliding manifold on which the dynamics of the system is constrained to evolve by means of a discontinuous control law, instead of designing a continuous stabilizing feedback, as in the case of the classical regulator problem. The sliding manifold contains the steady-state manifold, and the dynamics of the systems tends asymptotically, along the sliding manifold, to steady-state behavior.

First, the Sliding Mode Servomechanism Problem is formulated, and the conditions of solution of this problem are investigated. Then a discontinuous regulator is designed for a class of nonlinear systems presented in the so-called Regular Form. Finally, we assume that the class of nonlinear to be considered, might be presented in the Nonlinear Block Controllable Form (NBC-form) with uncertainty, consisting of a set of blocks. In each the block state vector and the block fictitious control vector have the same dimension. Such kind of

representation enables a reduction of the original control law synthesis problem into a sequence of lower-order subproblems which can be solved applying the block control technique. As a result, the nominal sliding mode dynamics can be linearized, and then discontinuous feedback can be used to compensate the matched uncertainty. The effectiveness of the proposed method is demonstrated by the application to the well-known Pendubot system.

2 Problem Statement

Consider the following nonlinear plant subject to perturbation:

$$\dot{x} = f(x, u, w) + \Delta f(x, v), \quad x(0) = x_0 \quad (1)$$

$$y = h(x) \quad (2)$$

with the plant state x , defined in a neighborhood X of the origin of $\mathbb{R}^n$, the m -dimensional plant control input u , the p -dimensional plant regulated output y , and w is the exogenous signal defined in a neighborhood W of the origin in $\mathbb{R}^q$, representing the reference input (to be tracked) produced by an exosystem described by

$$\dot{w} = s(w), \quad w(0) = w_0, \quad (3)$$

The unknown vector Δf represents the model uncertainties, and v is a vector of external disturbances (to be rejected). The output tracking error is defined as

$$e = y - y_{ref}, \quad y_{ref} = q(w) \quad (4)$$

It is assumed that $f(x, u, w)$, $s(w)$, $h(x)$ and $q(w)$ are smooth functions and $f(0, 0, 0) = 0$, $s(0) = 0$, $h(0) = 0$ and $q(0) = 0$.

The systems (1), (2), (4) and (3) are characterized by the following assumptions:

- H1)** The pair $\{f, B\}$ has a stabilizable linear approximation at $x = 0$.
- H2)** The Jacobian matrix $S = \frac{\partial s}{\partial w}(0)$ at the equilibrium point $w = 0$ has all eigenvalues on the imaginary axis.

The Classical State Servomechanism Problem (CSSP) [6] in absence of the uncertainty term Δf , consists on finding a continuous feedback $u = \alpha(x, w)$ that provides the following requirement:

- CS)** The system $\dot{x} = f(x, \alpha(x, 0), 0)$ has a locally exponentially stable equilibrium point at the origin $x = 0$,
- CE)** There exists a neighborhood $U \subset X \times W$ such that for each initial condition $(x_0, w_0) \in U$ the output tracking error $e(t)$ goes asymptotically to zero, i.e. $\lim_{t \rightarrow \infty} [y(t) - y_{ref}(t)] = 0$

In [6] it was shown that the solvability of CSSP can be stated in terms of the existence of a pair of mappings $x = \pi(w)$ and $c(w)$ locally defined in W with $\pi(0) = 0$ and $c(0) = 0$, that solve the partial differential equation

$$f(\pi(w), u(\pi(w), w)) = \frac{\partial \pi(w)}{\partial w} s(w) \quad (5)$$

$$h(\pi(w)) - q(w) = 0 \quad (6)$$

A continuous controller solving the servomechanism problem was obtained by choosing

$$u = c(w) + K(x - \pi(w)) \quad (7)$$

where K is a matrix which places the eigenvalues of the linear approximation of the closed-loop system (1) and (7) at the equilibrium point $x = 0$, namely $(A + BK)$ in C^- .

3 Sliding Mode Servomechanism Problem

Analogously to the CSSP, we may define the *Sliding Mode Servomechanism Problem (SMSP)* as the problem of finding a manifold

$$\sigma(x, w) = 0, \quad \sigma \in \mathfrak{R}^m \quad (8)$$

and a discontinuous controller

$$u_i = \begin{cases} u_i^+(x, w) & \sigma_i(x, w) > 0 \\ u_i^-(x, w) & \sigma_i(x, w) < 0 \end{cases}, \quad i = 1, \dots, m \quad (9)$$

where maps $u_i^+(x, w)$, $u_i^-(x, w)$ and $\sigma_i(x, w)$ are chosen to induce local asymptotic convergence of the state vector to the switching surface $\sigma_i(x, w) = 0$, $i = 1, \dots, m$ so that the following conditions hold:

SS). The equilibrium point $x = 0$ of the closed-loop system (1), (9) and (8), is locally asymptotically stable;

SE). The output tracking error goes asymptotically to zero, i.e.

$$\lim_{t \rightarrow \infty} [y(t) - y_{ref}(t)] = 0$$

Note, that the conditions for sliding motion to occur on $\sigma_i(x, w) = 0$ may be stated in numerous ways. We need

$$\lim_{\sigma_i \rightarrow 0^+} \dot{\sigma}_i < 0$$

and

$$\lim_{\sigma_i \rightarrow 0^-} \dot{\sigma}_i > 0$$

in the neighborhood of $\sigma_i(x, w) = 0$, $i = 1, \dots, m$, [10].

4 Nonlinear Systems Affine in Control

In this section we consider the nonlinear systems that is linear in control and perturbation inputs, in the absence of the uncertainty term, that is

$$\dot{x} = f(x) + B(x)u + D(x)w \quad (10)$$

Let us first investigate the conditions of the solution of SMSP assuming that the initial state vector of the system (68) in the manifold $\sigma(x, w) = 0$ (8), and the sliding mode occurs with the state trajectories confined to this manifold for $t > 0$. The projection motion of the system on the subspace σ is described by the m order system

$$\dot{\sigma} = G(x, w)f(x) + G(x, w)B(x)u + G(x, w)D(x)w + H(x, w)s(w) \quad (11)$$

where $G = \frac{\partial \sigma}{\partial x}$, $H = \frac{\partial \sigma}{\partial w}$.

Following the equivalent control method [10] we put $\dot{\sigma} = 0$, that is

$$G(x, w)f(x) + G(x, w)B(x)u + G(x, w)D(x)w + H(x, w)s(w) = 0$$

Assuming that the matrix $(G(x, w)B(x))$ is nonsingular for any $x, w \in U$, we may find the solution of this algebraic equation, that is, the equivalent control, u_{eq} as

$$u_{eq} = -(G(x, w)B(x))^{-1}[G(x, w)f(x) + G(x, w)D(x)w + H(x, w)s(w)]$$

and substitute it for discontinuous control u in the original system (68):

$$\dot{x} = P(x, w)[f(x) + D(x)w] - M(x, w)H(x, w)s(w) \quad (12)$$

where $P(x, w) = I_n - M(x, w)G(x, w)$, $M(x, w) = B(x)(G(x, w)B(x))^{-1}$.

The systems (70) and (3) can be represented as

$$\dot{x} = PAx + [PD - B(\Sigma B)^{-1}HS]w + \psi_1(x, w) \quad (13)$$

$$\dot{w} = Sw + \psi_2(w) \quad (14)$$

where $P_1 = [I_n - B(\Sigma B)^{-1}\Sigma]$, $A = \frac{\partial f}{\partial x}(0)$, $B = B(0)$, $D = D(0)$, $\Sigma = G(0, 0)$, $H = H(0, 0)$ and $S = \frac{\partial s}{\partial w}(0)$ with functions $\psi_1(x, w)$ and $\psi_2(w)$ vanishing at the origin with its first derivatives. Assume now that the matrix (PA) is stable, and by assumption **H2** the eigenvalues of the matrix S are on the imaginary axis. Therefore, the system (71) and (72) has a central manifold $x = \pi(w)$ at $(0, 0)$ with C^k mapping $\pi(w)$, $(\pi(0) = 0)$ satisfying the condition

$$P(\pi(w), w)[f(\pi(w)) + D(\pi(w))w] = M(\pi(w), w)H(\pi(w), w)s(w) \quad (15)$$

From this it is possible to derive a condition for solution to the SMSP.

Proposition 1. *Under assumptions H1 and H2, if there exist C^k ($k \geq 2$) mapping $\pi(w)$ with $\pi(0) = 0$ which satisfies*

$$P(\pi(w), w)[f(\pi(w)) + D(\pi(w))w] = M(\pi(w), w)\Sigma \frac{\partial \pi}{\partial w} s(w) \quad (16)$$

$$h(\pi(w)) - q(w) = 0 \quad (17)$$

Then the SMSP is solvable.

Proof. Proceeding along the previous discussion, setting

$$\sigma = \Sigma z, \quad z = x - \pi(w), \quad (18)$$

then conditions (15) and (16) are identical. Choosing

$$u = -k(G(x, w)B(x))^{-1} \text{sign}(\sigma) + u_{eq}(x, w) \quad (19)$$

$$u_{eq}(x, w) = -[\Sigma B(x)]^{-1}[\Sigma f(x) + \Sigma D(x)w + \Sigma \frac{\partial \pi}{\partial w} s(w)] \quad (20)$$

the control action (19) with (20) and $k > 0$ guarantees a sliding mode motion on the surface $\sigma = 0$ (73) that described by (70). The Jacobian matrix of $[P(x, w)f(x)]$ in the (70) equal to matrix (PA) , and by assumption **H1** there exists matrix Σ such that $(n - m)$ eigenvalues of matrix (PA) are in C^- . Therefore, by a property of center manifolds, under condition (15) or (16), we have on the sliding manifold $\sigma = 0$ (73) in steady state

$$PAz = 0 \quad (21)$$

$$\Sigma z = 0 \quad (22)$$

It is straightforward to verify that P is a *projection operator* along the range space of B onto the null space of Σ [1] i.e.

$$PB = (I_n - B(\Sigma B)^{-1}\Sigma)B = 0 \quad (23)$$

and

$$Pz = z \quad \forall z \in \mathcal{N}, \quad \mathcal{N} = \{z \in R^n \mid \Sigma z = 0\} \quad (24)$$

Therefore by conditions (21)-(23) we have $z = 0$ that means $x = \pi(w)$. Thus the first requirement **SS**) is fulfilled. So, by continuity, if condition (17) holds, then the second requirement **SE**) is also fulfilled.

The later result provides an existence condition for deriving a sliding manifold on which the output tracking error is zeroed. In contrast to the classical problem it does not need to calculate the steady state value for discontinuous control. Indeed, the equivalent control in sliding mode tends to its steady state

$$u_{eq}(\pi(w), w) = -[\Sigma B(\pi(w))]^{-1}[\Sigma f(\pi(w)) + \Sigma D(\pi(w))w + \Sigma \frac{\partial \pi}{\partial w} s(w)]$$

automatically.

The obtained result can be viewed more clearly in the case of nonlinear systems in regular form.

5 Nonlinear Systems in Regular Form

Consider the nonlinear system (68), (2) and (4), and assume that the system (68), by a diffeomorphism $x' = \varphi_1(x)$ can be transformed in the regular form [9]

$$\dot{x}'_1 = f_1(x'_1, x'_2) + D_1(x'_1, x'_2)w \quad (25)$$

$$\dot{x}'_2 = f_2(x'_1, x'_2) + B_2(x'_1, x'_2)u + D_2(x'_1, x'_2)w \quad (26)$$

where $x' = (x'_1, x'_2)^T$, $x'_1 \in \mathbb{R}^{n-m}$, $x'_2 \in \mathbb{R}^m$ and $\text{rank} B_2(x'_1, x'_2) = m \forall x \in \mathbb{R}^n$.

Let the sliding manifold (8) be described by

$$\sigma = x'_2 + \sigma_1(x'_1, w) = 0 \quad (27)$$

where $\sigma_1(x_1, w)$ is a C^k ($k \geq 2$) mapping with $\sigma_1(0, 0) = 0$, and setting

$$u = -kB_2^{-1}(x'_1, x'_2)\text{sign}(\sigma) + u_{eq} \quad (28)$$

$$u_{eq} = -B_2^{-1}(x'_1, x'_2) \left[f_2(x'_1, x'_2) + \frac{\partial \sigma_1}{\partial x'_1} f_1(x'_1, x'_2) \right. \quad (29)$$

$$\left. + \left(D_2(x'_1, x'_2) + \frac{\partial \sigma_1}{\partial x'_1} D_1(x'_1, x'_2) \right) w + \frac{\partial \sigma_1}{\partial w} s(w) \right] \quad (30)$$

then the control (28) and (29) guarantees a sliding motion on $\sigma = 0$ given by (27) in a finite time. On this surface, the dynamics of the closed-loop system (25), (26) and (28) are given by the reduced order equation

$$\dot{x}'_1 = f_1(x'_1, \sigma_1(x'_1, w)) + D_1(x'_1, \sigma_1(x'_1, w))w \quad (31)$$

$$e = h'(x'_1, \sigma_1(x'_1, w)) - q(w) \quad (32)$$

with $h'(x'_1, x'_2) = h(\varphi_1^{-1}(x))$. At this point, the system (31) and (3) can be represented as

$$\dot{x}'_1 = (A_{11} + A_{12}\Sigma_1)x'_1 + (A_{12}L_1 + D_1)w + \psi_1(x'_1, w) \quad (33)$$

$$\dot{w} = Sw + \psi_2(w) \quad (34)$$

where $A_{11} = \frac{\partial f_1}{\partial x'_1}(0, 0)$, $A_{12} = \frac{\partial f_1}{\partial x'_2}(0, 0)$, $D_1 = D_1(0)$, $\Sigma_1 = \frac{\partial \sigma_1}{\partial x'_1}(0, 0)$, $L_1 = \frac{\partial \sigma_1}{\partial w}(0, 0)$ and $S = \frac{\partial s}{\partial w}(0)$, with functions $\psi_1(x'_1, w)$ and $\psi_2(w)$ vanishing at the origin with its first derivatives. Under assumption **H1** there exists matrix Σ_1 such that matrix $(A_{11} + A_{12}\Sigma_1)$ is Hurwitz. Therefore the system (33) and (34) under assumption **H2** has a central manifold $x'_1 = \pi_1(w)$ at $(0, 0)$ with C^k mapping $\pi_1(w)$, $(\pi_1(0) = 0)$ satisfying the condition

$$\frac{\partial \pi_1}{\partial w} s(w) = f_1(\pi_1(w), \sigma_1(\pi_1(w), w)) + D_1(\pi_1(w), \sigma_1(\pi_1(w), w))w. \quad (35)$$

From this it is possible to deduce a condition for the solution sliding mode servomechanism problem for a nonlinear system in regular form (25) and (26).

Proposition 2. *Under assumptions H1 and H2, if there exist C^k ($k \geq 2$) mappings $c_1(w)$ and $\pi_1(w)$ with $c_1(0) = 0$ and $\pi_1(0) = 0$ which satisfy*

$$f_1(\pi_1(w), c_1(w)) + D_1(\pi_1(w), c_1(w))w = \frac{\partial \pi_1}{\partial w} s(w) \quad (36)$$

$$h'(\pi_1(w), c_1(w)) - q(w) = 0 \quad (37)$$

Then the SMSP is solvable.

Proof. Proceeding along the previous discussion, setting

$$\sigma_1(x'_1, w) = c_1(w) + \Sigma_1(x'_1 - \pi_1(w)) \quad (38)$$

then the Jacobian matrix of $f_1(x'_1, \sigma_1(x'_1, 0))$ in the (31) equal to matrix $(A_{11} + A_{12}\Sigma_1)$ which, by assumption **H1**, can be made stable by appropriate choice of Σ_1 . Therefore, by a property of center manifolds, $x'_1(t) \rightarrow \pi_1(w(t))$, and thus $\sigma_1(\pi_1(w), w) = c_1(w)$, in the manifold $x'_1(t) = \pi_1(w(t))$, and condition (36) reduces to condition (35), so, by continuity, if condition (37) holds, then the output tracking error converges to zero.

The proposed controller (28) and (29) with (27) and (38) provides only local stability of the equilibrium point $x = 0$. In the next section we consider a globally stabilizing discontinuous feedback.

6 Nonlinear System in NBC-Form

In this section a discontinuous recontrol strategy will be investigated for a class of nonlinear system (1) linear with respect to inputs signal

$$\dot{x} = f(x) + B(x)u + D(x)w + F(x)v \quad (39)$$

where $\Delta f(x, v) = F(x)v$, using the block control linearizing technique [8]. The essential feature of the proposed method is the transformation of equation (39) to the NBC-form consisting of r blocks:

$$\dot{x}_r = f_r(x_r) + B_r(x_r)x_{r-1} + D_r(x_r)w + F_r(x_r)v \quad (40)$$

$$\begin{aligned} \dot{x}_i &= f_i(x_i, \dots, x_r) + B_i(x_i, \dots, x_r)x \\ &\quad + D_i(x_i, \dots, x_r)w_{i-1} + F_i(x_i, \dots, x_r)v, \quad i = 2, \dots, r-1 \end{aligned} \quad (41)$$

$$\dot{x}_1 = f_1(\tilde{x}) + B_1(\tilde{x})u + D_1(\tilde{x})w + F_1(\tilde{x})v \quad (42)$$

with output error $e = h(\tilde{x}) - q(w)$ where the transformed vector $\tilde{x}$ is decomposed as $\tilde{x} = (x_1, x_2, \dots, x_r)^T$ and x_i is a $n_i \times 1$ vector. In each block, the vectors x_{i-1} are regarded as fictitious control vectors, and $\text{rank} B_i = n_i, \forall x \in X$. The integers $(n_1, n_2, \dots, n_r)$ define the plant structure, and $\sum_{i=1}^r n_i = n$.

6.1 Block Decomposition of Nonlinear System

The procedure of reducing the system (39) to the NBC-form (??) consists of a series of steps. The main technique is the transformation of an affine control system into the regular form by means of the integral surface method [9].

Step 1. Assume that $\text{rank } B(x, t) = n_1 \leq m$, $\forall x \in X$, and, possibly after reordering, there exists an $(n_1 \times m)$ block $B_1(x_1, x_{12})$ such that

$$B(x) = \begin{bmatrix} B_{12}(x_1, x_{12}) \\ B_1(x_1, x_{12}) \end{bmatrix}, \quad f(x) = \begin{bmatrix} f_{12}(x_1, x_{12}) \\ f_1(x_1, x_{12}) \end{bmatrix},$$

with $\text{rank } B_1(x_1, x_{12}) = n_1$, $\forall x \in X$, where x_1 and x_{12} are $n_1 \times 1$ and $(n - n_1) \times 1$ vectors, respectively. At this point, we introduce the following instrumental assumptions which will be carried for each step of the procedure.

A11) The Pfaffian system

$$dx_{12} + A_1(x)dx_1 = 0, \quad A_1(x) = -B_{12}(x)B_1^+(x) \quad (43)$$

is completely integrable, that is the condition

$$\frac{\partial a_\alpha^i}{\partial x_\beta} - \sum_{j=1}^{n-n_1} a_\alpha^i \frac{\partial a_\alpha^i}{\partial x_j} = \frac{\partial a_\beta^i}{\partial x_\alpha} - \sum_{j=1}^{n-n_1} a_\beta^i \frac{\partial a_\beta^i}{\partial x_j}$$

where $A_1(x) = \{a_\alpha^i(x)\}$, $i = 1, \dots, n - n_1$, α and $\beta \in \{(n - n_1 + 1), \dots, n\}$, and B_1^+ is the right pseudo inverse of B_1 holds.

A12) The mappings $F(x)$ and $D(x)$ can be decomposed as

$$\begin{aligned} D(x) &= D^m(x_1, x_{12}) + D^u(x_{12}) \\ F(x) &= F^m(x_1, x_{12}) + F^u(x_{12}) \end{aligned}$$

where $F^m(x)$ and $D^m(x)$ satisfy the matching conditions, namely

$$F^m(x), D^m(x) \in \text{span } B(x)$$

Under assumption A11, it is possible to show that a solution of the equation (43) is given by $x_{12} = \bar{\varphi}_1(x_1, c)$ where $c = (c_1, c_2, \dots, c_{n-n_1})^T$ is an integration constants vector. From this solution, the vector c can be obtained, namely, $c = \varphi_1(x_1, x_{12})$, and taken as a local change of coordinates given by $x'_2 = \varphi_1(x_1, x_{12})$.

Now, using assumption A12, and partitioning D and F as

$$\begin{aligned} D &= \begin{pmatrix} D_{12}(x) \\ D_1(x) \end{pmatrix} = \begin{pmatrix} D_{12}^m(x_1, x_{12}) + D_{12}^u(x_{12}) \\ D_1^m(x_1, x_{12}) + D_1^u(x_{12}) \end{pmatrix} \\ F &= \begin{pmatrix} F_{12}(x) \\ F_1(x) \end{pmatrix} = \begin{pmatrix} F_{12}^m(x_1, x_{12}) + F_{12}^u(x_{12}) \\ F_1^m(x_1, x_{12}) + F_1^u(x_{12}) \end{pmatrix} \end{aligned}$$

it is possible to show (see appendix) that the original system (39) under the new coordinates is described by

$$\dot{x}'_2 = f'_2(x_1, x'_2) + D'_2(x'_2)w + F'_2(x'_2)v \quad (44)$$

$$\dot{x}_1 = f_1(x_1, x'_2) + B_1(x_1, x'_2)u + D_1(x_1, x'_2)w + F_1(x_1, x'_2)v \quad (45)$$

Note that in the terminology of [3], equation (45) exhibits a controlled dynamics, and (44) exhibits an uncontrolled one with fictitious control input x_1 . The following assumption is fundamental to derive the NBC-form:

A13) The mapping $f'_2(x_1, x'_2)$ in system (44) is affine in its first argument, namely

$$f'_2(x_1, x'_2) = f''_2(x'_2) + B'_2(x'_2)x_1 \quad (46)$$

At this point, we may consider three cases:

- a) rank $B'_2(x'_2) = 0, \forall x'_2 \in X$. This is equivalent to having an uncontrollable system. For the purposes of this work, we will assume in the following that the original system is controllable.
- b) rank $B'_2(x'_2) = n - n_1 \forall x'_2 \in X$. In this case, after defining $x_2 = x'_2$, $B_2 = B'_2$, $f_2 = f'_2$, $D_2 = D'_2$, $F_2 = F'_2$, the NBC-form is:

$$\dot{x}_2 = f_2(x_2) + B_2(x_2)x_1 + D_2(x_2)w + F_2(x_2)v$$

$$\dot{x}_1 = f_1(x_1, x_2) + B_1(x_1, x_2)u + D_1(x_1, x_2)w + F_1(x_1, x_2)v$$

- c) rank $B'_2(x'_2) = n_2 < n - n_1$. In this case, a subsequent step is necessary, and the subsystem (44) and (46) with state x'_2 and input x_1 is further decomposed and transformed similarly to Step 1.

Step k. Consider the system obtained at $(k-1)^{th}$ step

$$\dot{x}'_k = f'_k(x'_k) + B'_k(x'_k)x_{k-1} + D'_k(x'_k)w + F'_k(x'_k)v \quad (47)$$

$$\begin{aligned} \dot{x}_i &= f_i(x_i, \dots, x'_k) + B_i(x_i, \dots, x'_k)x_{i-1} \\ &\quad + D_i(x_i, \dots, x'_k)w + F_i(x_i, \dots, x'_k)v, \quad i = 2, \dots, k-1 \end{aligned} \quad (48)$$

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, \dots, x'_k) + B_1(x_1, \dots, x'_k)x_{r-1} \\ &\quad + D_1(x_1, \dots, x'_k)w + F_1(x_1, \dots, x'_k)v \end{aligned} \quad (49)$$

where rank $B_i(x_i, \dots, x'_k) = n_i, i = 1, \dots, k-1$. If $n_k = \text{rank } B_k(x'_k)$ is such that $n_k = n - \sum_{j=1}^{k-1} n_j$, we define $x_k = x'_k$, $f_k = f'_k$, $D_k = D'_k$, $F_k = F'_k$, and the algorithm terminates giving the equations (47)-(49) as the desired NBC-form. If $n_k < n - \sum_{j=1}^{k-1} n_j$, the subsystem (47) is partitioned as

$$\dot{x}_{k2} = f_{k2}(x_k, x_{k2}) + B_{k2}(x_k, x_{k2})x_{k-1} + D_{k2}(x_k, x_{k2})w + F_{k2}(x_k, x_{k2})v$$

$$\dot{x}_k = f_k(x_k, x_{k2}) + B_k(x_k, x_{k2})x_{k-1} + D_k(x_k, x_{k2})w + F_k(x_k, x_{k2})v$$

where rank $B_k = n_k$, $x'_k = (x_k, x_{k2})^T$, and x_k and x_{k2} are $n_k \times 1$ and $(n - \sum_{j=1}^{k-1} n_j - n_k) \times 1$ vectors, respectively. For this step, we generalize assumptions A11 and A12 as follows:

Ak1) The Pfaffian system

$$dx_{k2} - B_{k2}B_k^+ dx_k = 0 \quad (50)$$

is completely integrable.

Ak2) The mappings F'_k and D'_k can be decomposed in the form

$$\begin{aligned} D'_k(x'_k) &= D_k^m(x_k, x_{k2}) + D_k^u(x_{k2}) \\ F'_k(x'_k) &= F_k^m(x_k, x_{k2}, v) + F_k^u(x_{k2}) \end{aligned}$$

where F_k^m and D_k^m satisfy the matching conditions, namely

$$F_k^m, D_k^m \in \text{span } B_k'(x'_k)$$

Proceeding as in the first step, under the previous assumptions, we may find a local change of coordinates given by $x'_{k+1} = \varphi_k(x_k, x_{k2})$ where φ_k is computed from the solution of (50) such that the system is described by

$$\begin{aligned} \dot{x}'_{k+1} &= f'_{k+1}(x_k, x'_{k+1}) + D'_{k+1}(x'_{k+1})w + F'_{k+1}(x'_{k+1})v \\ \dot{x}_k &= f_k(x_k, x'_{k+1}) + B_k(x_k, x'_{k+1})x_{k-1} \\ &\quad + D_k(x_k, x'_{k+1})w + F_k(x_k, x'_{k+1})v \\ \dot{x}_i &= f_i(x_i, x_{i+1}, \dots, x'_{k+1}) + B_i(x_i, x_{i+1}, \dots, x'_{k+1})x_{i-1} \\ &\quad + D_i(x_i, x_{i+1}, \dots, x'_{k+1})w + F_i(x_i, x_{i+1}, \dots, x'_{k+1})v, \quad i = 2, \dots, k-1 \\ \dot{x}_1 &= f_1(x_1, x_2, \dots, x'_{k+1}) + B_1(x_1, x_2, \dots, x'_{k+1})x_{r-1} \\ &\quad + D_1(x_1, x_2, \dots, x'_{k+1})w + F_1(x_1, x_2, \dots, x'_{k+1})v \end{aligned}$$

with $\text{rank } B_i = n_i, \quad i = 1, \dots, k$.

In the same way, assumption A13 for step k is stated as:

Ak3) The mapping $f'_{k+1}(x_k, x'_{k+1})$ is affine in its first argument, namely

$$f'_{k+1}(x_k, x'_{k+1}) = f'_{k+1}(x'_{k+1}) + B'_{k+1}(x'_{k+1})x_k$$

From the previous algorithm, we may state the following result:

Theorem 1. Assume that the system (39) is controllable and at each step of the NCB-form algorithm assumptions Ak1, Ak2 and Ak3 hold. Then, there exists an integer $r \leq n$ such that the system (39) takes the form (40)–(42).

6.2 Block Linearization of Nominal System

In this section, following the block control design technique [8], a nonlinear transformation linearizing the nominal part of uncontrolled dynamics is derived. Based on this transformation, a nonlinear sliding manifold will be proposed in the next section.

At this point we introduce the following assumption

H3) The uncertainty term $\Delta f(x)v$ in (1) satisfies the matching condition. If condition H3 holds, then in (40)–(41)

$$F_r(x_1)v = 0$$

$$F_i(x_i, x_{i+1}, \dots, x_r)v = 0, \quad i = 2, \dots, r-1$$

It is more convenient to renumber the state variables of (40)–(41) in the reverse form and represent this system as

$$\dot{x}_1 = f_1(x_1) + B_1(x_1)x_2 + D_1(x_1)w \quad (51)$$

$$\dot{x}_2 = f_2(\bar{x}_2) + B_1(\bar{x}_2)x_3 + D_2(\bar{x}_2)w \quad (52)$$

$$\dot{x}_i = f_i(\bar{x}_i) + B_i(\bar{x}_i)x_{i+1} + D_i(\bar{x}_i)w, \quad i = 3, \dots, r-1 \quad (53)$$

$$\dot{x}_r = f_r(\tilde{x}) + B_r(\tilde{x})u + D_r(\tilde{x})w + F_r(\tilde{x})v \quad (54)$$

with output error $e = h(\tilde{x}) - q(w)$ where $\bar{x}_i = (x_1, \dots, x_i)^T$, $i = 2, \dots, r-1$. The integers $(n_1, n_2, \dots, n_r)$ satisfy

$$n_1 \leq n_2 \leq \dots \leq n_r \leq m. \quad (55)$$

The condition $n_{i-1} \leq n_i$ (55) means $n_{i-1} = n_i$ or $n_{i-1} < n_i$. In the sequel we consider the system (51)–(54) with the structure

$$n_1 = n_2 < n_2 \dots < n_r = m \quad (56)$$

that includes both of the cases.

A nonlinear linearizing state transformation can be derived considering the state x_{i+1} , $i = 1, \dots, r-1$ as a fictitious control vector in the i^{th} block. This procedure is outlined in the following steps.

Step1. Since $n_1 = n_2$, the matrix $B_1(x_1)$ is square and the inverse matrix $B_1^{-1}(x_1)$ exists. Setting $z_1 = x_1 := \alpha_1(x_1)$, the fictitious control x_2 in (51) can be chosen as

$$x_2 = x_2^c(x_1, w) + B_1^{-1}(x_1)(K_1 z_1 + z_2), \quad (57)$$

where z_2 is $n_2 \times 1$ vector of new variables, K_1 is a matrix with desired eigenvalues, and $x_2^c(x_1, w)$ is calculated from the equation $\dot{z}_1 = 0$ along the trajectories of (51), namely,

$$x_2^c = -B_1^{-1}(x_1)[f_1(x_1) + D_1(x_1)w] \quad (58)$$

The transformed 1st block with new coordinates z_1 , z_2 and input (57) and (58) has the form

$$\dot{z}_1 = K_1 z_1 + z_2 \quad (59)$$

The variable z_2 can be obtained using (57) and (58) as

$$z_2 = B_1(x_1)x_2 + f_1(x_1) - K_1\alpha_1(x_1) + D_1(x_1)w := \alpha_2(\bar{x}_2, w) \quad (60)$$

Step 2. Taking the derivative of (60) along the trajectories of the system (??) and (3) yields

$$\dot{z}_2 = \bar{f}_2(\bar{x}_2, w) + \bar{B}_2(\bar{x}_2)x_3 \quad (61)$$

where $\bar{f}_2(\bar{x}_2, w) = \frac{\partial \alpha_2}{\partial x_1} [f_1(x_1) + B_1(x_1)x_2 + D_1(x_1)w] + \frac{\partial \alpha_2}{\partial x_2} [f_2(\bar{x}_2) + D_2(\bar{x}_2)w] + \frac{\partial \alpha_2}{\partial w} s(w)$, $\bar{B}_2 = B_1 B_2$. Note that $\text{rank } \bar{B}_2 = \text{rank } B_2 = n_2$. Since $n_2 < n_3$ the matrix B_2 as well as $\bar{B}_2$ are not square. As on the first step, the fictitious input vector x_3 in (61) is chosen similar to (57) and (58):

$$x_3 = x_3^c(\bar{x}_2, w) + \bar{B}_2^+(\bar{x}_2)[K_2 z_2 + E_{2,1} z_3] \quad (62)$$

where z_3 is $n_3 \times 1$ vector, K_2 is a desired matrix, $\bar{B}_2^+$ denotes the right pseudo inverse of $\bar{B}_2$, $E_{2,1} = [I_{n_2} \ 0]$, and again $x_3^c(\bar{x}_2, w)$ is found from the equation $\dot{z}_2 = 0$ (61) being

$$x_3^c(\bar{x}_2, w) = -\bar{B}_2^+(\bar{x}_2)\bar{f}_2(\bar{x}_2, w) \quad (63)$$

Thus, equation (61) with (62) and (63) takes the same form of equation (59), namely

$$\dot{z}_2 = K_2 z_2 + E_{2,1} z_3.$$

Now, we establish the following assumption.

B2) The elements of matrix $\bar{B}_2(\bar{x}_2)$ can be ordered such that the square matrix

$$\tilde{B}_3(\bar{x}_2) := \begin{bmatrix} \bar{B}_2(\bar{x}_2) \\ E_{2,2} \end{bmatrix}$$

with $E_{2,2} = [0 \ I_{n_3-n_2}]$, has rank n_3 .

Based on this assumption, the variable z_3 can be obtained using (62), (63) and (60) as

$$z_3 = \tilde{B}_3(\bar{x}_2)x_3 + \begin{bmatrix} \bar{f}_2(\bar{x}_2, w) + K_2 \alpha_2(\bar{x}_2, w) \\ 0 \end{bmatrix} := \alpha_3(\bar{x}_3, w)$$

Step i . At this stage, it is possible to show that if we have, after the $(i-1)^{th}$ step, the transformed blocks of the system (51)–(54) with new variables $z_1, z_2, \dots, z_{i-1}$ (under assumption $n_j < n_{j+1}$, $i = 3, \dots, r-1$) of the form

$$\dot{z}_1 = K_1 z_1 + z_2 \quad (64)$$

$$\dot{z}_2 = K_2 z_2 + E_{2,1} z_3 \quad (65)$$

$\vdots$

$$\dot{z}_{i-1} = K_{i-1} z_{i-1} + E_{i,1} z_i \quad (66)$$

with

$$z_i = \tilde{B}_i(\bar{x}_{i-1})x_i + \begin{bmatrix} \bar{f}_{i-1}(\bar{x}_{i-1}, w) + K_{i-1} \alpha_{i-1}(\bar{x}_{i-1}, w) \\ 0 \end{bmatrix} := \alpha_i(\bar{x}_i, w) \quad (67)$$

$\tilde{B}_i(\bar{x}_{i-1}) := \begin{bmatrix} \bar{B}_{i-1}(\bar{x}_{i-1}) \\ E_{i-1,2} \end{bmatrix}$; and $\text{rank } \tilde{B}_i = n_i$. Then, on the i^{th} step of the transformation procedure, we will have the transformed i^{th} block with new

state vector z_i similar to (66). To carry out this, take the derivative of (67) along (51)–(54)

$$\dot{z}_i = \bar{f}_i(\bar{x}_i, w) + \bar{B}_i(\bar{x}_i)x_{i+1} \quad (68)$$

where $\bar{f}_i(\bar{x}_i, w) = \sum_{j=1}^{i-1} \frac{\partial \alpha_i}{\partial x_j} [f_j(\bar{x}_j) + B_j(\bar{x}_j)x_{i+1} + D_j(\bar{x}_j)w] + \frac{\partial \alpha_i}{\partial x_i} [f_i(\bar{x}_i) + D_i(\bar{x}_i)w] + \frac{\partial \alpha_i}{\partial w} s(w)$, $\bar{B}_i = \tilde{B}_i B_i$, and $\text{rank } \bar{B}_i = n_i$. The fictitious control input x_{i+1} in (68) can be selected similar to (63) as

$$x_{i+1} = x_{i+1}^c(\bar{x}_i, w) + \bar{B}_i^+(\bar{x}_i)(K_i z_i + E_{i,1} z_{i+1}), \quad x_{i+1}^c = -\bar{B}_i^+(\bar{x}_i) \bar{f}_i(\bar{x}_i, w) \quad (69)$$

where z_{i+1} is an $n_{i+1} \times 1$ vector of new variables, K_i is a desired matrix, $E_{i,1} = [I_{n_i} \ 0]$, $E_{i,1} \in \mathbb{R}^{n_i \times n_{i+1}}$, with x_{i+1}^c calculated from the equation $\dot{z}_i = 0$ (68). Thus, equation (68) with (69) takes the same form as equation (66), namely $\dot{z}_i = K_i z_i + E_{i,1} z_{i+1}$. For this step, we generalize assumption **B2** as follows:

Bi) The elements of matrix $\bar{B}_i(\bar{x}_i)$ can be ordered such that

$$\text{rank } \tilde{B}_{i+1}(\bar{x}_i) = n_{i+1}, \quad \tilde{B}_{i+1}(\bar{x}_i) = \begin{bmatrix} \bar{B}_i(\bar{x}_i) \\ E_{i,2} \end{bmatrix}$$

where $E_{i,2} = [0 \ I_{n_{i+1}-n_i}] \in \mathbb{R}^{(n_{i+1}-n_i) \times n_{i+1}}$.

Under this assumption, we can obtain from (69) the recursive transformation for z_{i+1} as

$$\begin{aligned} z_{i+1} &= \tilde{B}_{i+1}(\bar{x}_i)x_{i+1} + \begin{bmatrix} \bar{f}_i(\bar{x}_i, w) + K_i \alpha_i(\bar{x}_i, w) \\ 0 \end{bmatrix} \\ &: = \alpha_{i+1}(\bar{x}_{i+1}, w), \quad i = 3, \dots, r-1 \end{aligned} \quad (70)$$

On the last step, after calculating the time derivative of $z_r = \alpha_r(\tilde{x}, w)$ (70), the original system (51)–(54) is represented in the new coordinates $z_1, z_2, \dots, z_r$ as

$$\dot{z}_1 = K_1 z_1 + z_2 \quad (71)$$

$$\dot{z}_i = K_i z_i + E_{i,1} z_{i+1}, \quad i = 2, \dots, r-1 \quad (72)$$

$$\dot{z}_r = \bar{f}_r(\tilde{x}, w) + \bar{B}_r(\tilde{x})u + \Delta f_r(\tilde{x}, w, v) \quad (73)$$

where $z = (z_1, z_2, \dots, z_r)^\top$, $\bar{f}_r(x, w) = \sum_{j=1}^{r-1} \frac{\partial \alpha_r}{\partial x_j} [f_j(\bar{x}_j) + B_j(\bar{x}_j)x_{i+1} + D_j(\bar{x}_j)w] + \frac{\partial \alpha_r}{\partial x_r} [f_r(\bar{x}_r) + D_r(\bar{x}_r)w] + \frac{\partial \alpha_r}{\partial w} s(w)$, $\Delta f_r(\tilde{x}, w, v) = \frac{\partial \alpha_r}{\partial x_r} F_r(\tilde{x})v$, $\bar{B}_r = \tilde{B}_r B_r$, and $\text{rank } \bar{B}_r = n_r$.

Remark 2. Assumption **(Bi)** is not necessary. It is sufficient to use the matrix $E_{i,2}$ with constant parameters. It is clear that if this assumption is not satisfied, then it will be possible to use for instance the matrix $\bar{B}_i^\perp$ orthogonal to $\bar{B}_i$, instead of $E_{i,2}$.

Note the equations (71)–(72) present uncontrolled dynamics while equation (73) presents the controlled one.

6.3 Discontinuous Feedback Design

In the present setting, a natural choice for a switching function σ for system (71)–(73) is taking

$$\sigma = z_r - c_r(w) \quad (74)$$

where

$$z_r = \tilde{B}_r(\bar{x}_{r-1})x_r + \begin{bmatrix} \bar{f}_{r-1}(\bar{x}_{r-1}, w) + K_{r-1}\alpha_{r-1}(\bar{x}_{r-1}, w) \\ 0 \end{bmatrix} := \alpha_r(\tilde{x}, w) \quad (75)$$

To generate sliding mode in (73), we use the fact that the term $\bar{f}_r(\tilde{x}, w)$ in the controlled block (73) is known, and matrix $\bar{B}_r(\tilde{x})$ has full rank, to achieve a sliding mode on the manifold $z_r - c_r(w) = 0$ (74) by using the following combined control law:

$$u = u^c(\tilde{x}, w) - k_r \bar{B}_r^{-1}(\tilde{x}) \text{sign}(\sigma) \quad (76)$$

where $k_r > 0$, and $u^c(\tilde{x}, w)$ is the continuous control component chosen to cancel the known terms in (73), namely

$$u^c(\tilde{x}, w) = -\bar{B}_r^{-1}(\tilde{x}) \left[\bar{f}_r(\tilde{x}, w) - \frac{\partial c_r(w)}{\partial w} s(w) \right]. \quad (77)$$

Note that in the absence of uncertainty, the control component $u^c(\tilde{x}, w)$ coincides with the equivalent control calculated as the solution of equation $\dot{\sigma} = 0$. Substitution of (76)–(77) into (73) yields

$$\dot{\sigma} = -k_r \text{sign}(\sigma) + \Delta f_r(\tilde{x}, w, v) \quad (78)$$

Then the controller (76)–(77) under the condition

$$k_r > \|\Delta f_r(\tilde{x}, w, v)\| \quad (79)$$

guarantees a sliding mode on the surface $z_r - c_r(w) = 0$.

6.4 SMSP Solution Conditions

For the system constrained to the sliding manifold $\sigma = 0$, the system ((71)–(72)) reduces to the following quasi-linear system of $(n - n_r)$ order

$$\dot{z}_1 = K_1 z_1 + E_{1,1} z_2 \quad (80)$$

$$\dot{z}_i = K_i z_i + E_{i,1} z_{i+1}, \quad i = 2, \dots, r-2, \quad (81)$$

$$\dot{z}_{r-1} = K_{r-1} z_{r-1} + E_{r-1,1} c_r(w), \quad (82)$$

with the output error

$$e = \bar{h}(z_1, \dots, z_{r-1}, c(w)) - q(w) \quad (83)$$

We can establish the following result.

Proposition 3. *If*

- 1) *Assumption **H2** and **H3** hold*
- 2) *There exist C^k ($k \geq 2$) mappings $\pi_i(w)$, $i = 1, \dots, r-1$, $c(w)$ with $\pi_i(0) = 0$ and $c(0) = 0$ which satisfy*

$$K_i \pi_i(w) + E_{i,1} \pi_{i+1} = \frac{\partial \pi_i(w)}{\partial w} s(w), \quad i = 1, \dots, r-2 \quad (84)$$

$$K_{r-1} \pi_{r-1}(w) + E_{r-1,1} c_r(w) = \frac{\partial \pi_{r-1}(w)}{\partial w} s(w) \quad (85)$$

$$\bar{h}[\pi_1(w), \dots, \pi_{r-1}(w), c_r(w)] - q(w) = 0 \quad (86)$$

Then the SMFSP is solvable.

Proof. The systems (80)–(82) can be represented as

$$\dot{\eta} = A_\eta \eta + A_w w + \psi_1(\eta, w) \quad (87)$$

$$\dot{w} = S w + \psi_2(w) \quad (88)$$

where $\eta = (z_1, \dots, z_{r-1})$, and

$$A_\eta = \begin{bmatrix} K_1 & I_{n_1} & \cdots & 0 & 0 \\ 0 & K_2 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & K_{r-2} & E_{r-2,1} \\ 0 & 0 & \cdots & 0 & K_{r-1} \end{bmatrix}, \quad A_w = \begin{bmatrix} 0 \\ 0 \\ \cdots \\ 0 \\ C_r \end{bmatrix}$$

$C_r = \frac{\partial c}{\partial w}(0)$ and $S = \frac{\partial s}{\partial w}(0)$, with functions $\psi_1(w)$ and $\psi_2(w)$ vanishing at

the origin with its first derivatives. It is easy to see that there are matrices $K_1, \dots, K_{r-1}$ such that matrix A_η is Hurwitz. Therefore, the systems (87) and (88) under assumption **H2** have a central manifold $\eta = \pi(w)$ at $(0, 0)$ with C^k mapping $\pi(w)$, $\pi(w) = (\pi_1(w), \dots, \pi_{r-1}(w))^T$, $\pi(0) = 0$ satisfying the condition (84)–(85). Since matrix A_η is Hurwitz, by a property of center manifolds, $\eta(t) \rightarrow \pi(w(t))$. So, by continuity, if condition (86) holds, then the output tracking error converges to zero.

Remark. From the previous result we observe that, in the case of unmatched uncertainty, the high gain technique can be applied to an exponentially stable motion.

7 Application to the Pendubot

In this section we will apply the scheme proposed in section 6 to a model of an underactuated system, the well-known *Pendubot* (Figure 1).

The dynamics of the Pendubot are described by the following equations

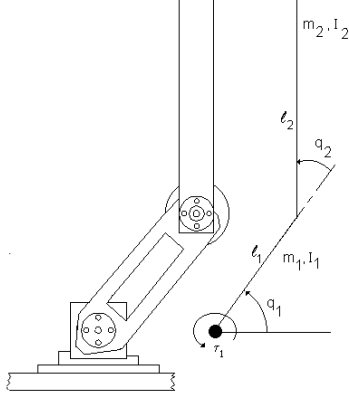


Fig. 1. Figure 1. Schematic diagram of the Pendubot

$$\begin{pmatrix} D_{11}(q_2) & D_{12}(q_2) \\ D_{12}(q_2) & D_{22}(q_2) \end{pmatrix} \begin{pmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{pmatrix} + \begin{pmatrix} C_1(q_2, \dot{q}_1, \dot{q}_2) \\ C_2(q_2, \dot{q}_1) \end{pmatrix} + \begin{pmatrix} G_1(q_1, q_2) \\ G_2(q_1, q_2) \end{pmatrix} = \begin{pmatrix} \tau_1 \\ 0 \end{pmatrix}$$

where q_1 and q_2 represent the generalized coordinates of the actuated and the unactuated joints respectively. In our case, we have

$$\begin{aligned} D_{11} &= m_1 l_{c1}^2 + m_2 (l_1^2 + l_{c2}^2 + 2l_1 l_{c2} \cos(q_2)) + I_1 + I_2, \\ D_{12} &= m_2 (l_{c2}^2 + 2l_1 l_{c2} \cos(q_2)) + I_2 \\ D_{22} &= m_2 l_{c2}^2 + I_2 \\ C_1 &= -2m_2 l_1 l_{c2} \dot{q}_1 \dot{q}_2 \sin(q_2) - m_2 l_1 l_{c2} \dot{q}_2^2 \sin(q_2), \\ C_2 &= m_2 l_1 l_{c2} \dot{q}_2^2 \sin(q_2), \\ G_1 &= m_1 g l_{c1} \cos(q_1) + m_2 g l_1 \cos(q_1) + m_2 g l_{c2} \cos(q_1 + q_2) \\ G_2 &= m_2 g l_{c2} \cos(q_1 + q_2) \end{aligned}$$

where the parameters values are

m_1	m_2	l_1	l_2	l_{c1}	l_{c2}	g	I_1	I_2
0.5289	0.3346	0.26987	0.38417	0.13494	0.19208	9.81	0.13863	0.016749

Choosing $x = (x_1, x_2, x_3, x_4)^T = (q_2, \dot{q}_2, q_1, \dot{q}_1)^T$ as state vector, $u = \tau$ as control input, and $y = q_2$ as output, the description of the system can be given in the state space form as:

$$\dot{x} = f(x) + b(x)u \quad (89)$$

$$y = h(x) \quad (90)$$

where

$$f = (f_1, f_2, f_3, f_4)^T, \quad b = (b_1, b_2, b_3, b_4)^T, \quad h(x) = x_1,$$

$$\begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} = \begin{pmatrix} x_2 \\ \frac{-D_{12}}{D_{11}D_{22}-D_{12}^2} - c \\ x_4 \\ \frac{D_{22}}{D_{11}D_{22}-D_{12}^2} p_1 \end{pmatrix}; \quad \begin{pmatrix} b_1 \\ b_2(x_1) \\ b_3 \\ b_4(x_1) \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{-D_{12}}{D_{11}D_{22}-D_{12}^2} \\ 0 \\ \frac{D_{22}}{D_{11}D_{22}-D_{12}^2} \end{pmatrix}$$

with $c = \frac{C_2}{D_{22}} - \frac{G_2}{D_{22}}$, $p_1 = \frac{D_{12}C_2}{D_{22}} - \frac{D_{12}G_2}{D_{22}} - C_1 - G_1$.

In order to apply the control scheme described in the previous sections to the model of the Pendubot, suppose we are interested in tracking a reference signal produced by the exosystem

$$\begin{aligned} \dot{w}_1 &= \alpha w_2 \\ \dot{w}_2 &= -\alpha w_1 \end{aligned}$$

with $\alpha = 1$ and $w(0) = \text{col}(w_1(0), w_2(0))$, and $e = x_1 - w_2$.

The corresponding Pfaffian system (43) in this case has the form

$$dx_1 - a_{14}(x_1)dx_4 = 0 \quad (91)$$

$$dx_2 - a_{24}(x_1)dx_4 = 0 \quad (92)$$

$$dx_3 - a_{34}(x_1)dx_4 = 0 \quad (93)$$

with $a_{14}(x_1) = b_1 b_4^{-1}(x_1) = 0$, $a_{24}(x_1) = b_2(x_1) b_4^{-1}(x_1) = -D_{12}(x_1) D_{22}^{-1}(x_1)$ and $a_{34}(x_1) = b_3 b_4^{-1}(x_1) = 0$. Since $a_{14}(x_1) = a_{24}(x_1) = 0$ the equations (91) and (93) have integrals $x_1 = C_1$ and $x_3 = C_3$. Therefore x_1 can be considered in (92) as a parameter and after integrating we have $x_2 - a_1(x_1)x_4 = C_2$ where C_1, C_2 and C_3 are the integration constants.

Choosing the transformation $x'_1 = x_1$, $x'_2 = x_2 - a_1(x_1)x_4$, $x'_3 = x_3$ and $x'_4 = x_4$ system (89) is transformed into the regular form $\dot{x} = f'(x) + b'(x)u$ where

$$\begin{pmatrix} f'_1 \\ f'_2 \\ f'_3 \\ f'_4 \end{pmatrix} = \begin{pmatrix} x'_2 - D_{12}D_{22}^{-1}x'_4 \\ -(c + G_2)D_{22}^{-1} + D_0x'_4 \\ x'_4 \\ \frac{D_{22}}{D_{11}D_{22}-D_{12}^2} p_1 \end{pmatrix}; \quad \begin{pmatrix} b'_1 \\ b'_2 \\ b'_3 \\ b'_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{D_{22}}{D_{11}D_{22}-D_{12}^2} \end{pmatrix}$$

with $D_0 = -m_2 l_1 l_{c2} \sin(x'_1) [x'_2 - D_{12}D_{22}^{-1}x'_4]$.

The system cannot be transformed to the NBC form, therefore we can apply the control scheme described for the regular form in section 5. Set

$$\begin{aligned} \pi &= \text{col}(\pi_1, \pi_2, \pi_3) \\ z_1 &= x'_1 - \pi_1(w) \\ z_2 &= x'_2 - \pi_2(w) \\ z_3 &= x'_3 - \pi_3(w) \end{aligned}$$

and $\sigma = x_4 - \sigma_1(z, w)$, $\sigma_1 = c_1(w) - k_1 z_1 - k_2 z_2 - k_3 z_3$ where $\pi_1(w) = w_2$; k_1, k_2 and k_3 are constant parameters, and $\pi_2(w)$, $\pi_3(w)$ and $c_1(w)$ are derived

from

$$\begin{aligned} -w_1 &= f'_1(\pi(w), c_1(w)) \\ \frac{\partial \pi_2(w)}{\partial w} s(w) &= f'_2(\pi(w), c_1(w)) \\ \frac{\partial \pi_3(w)}{\partial w} s(w) &= c_1(w) \end{aligned}$$

as a polynomial solution

$$\begin{aligned} \pi_2(w) &= -w_1 + c_1(w) \\ \pi_3(w) &= -0.974377w_2 + 0.0145446w_1^2w_2 - 0.00921993w_2^3 \\ c_1(w) &= 0.974374w_1 - 0.0145446w_1^3 - 0.05674899w_1^2w_2. \end{aligned}$$

Finally, the control

$$\begin{aligned} u &= u_{eq} - kb_4^{-1}\text{sign}(\sigma), \\ u_{eq} &= -b_4^{-1} \left(f'_4 + k_1f'_1 + k_2f'_2 + k_3f'_3 - \frac{\partial c_1(w)}{\partial w} \right), \quad k = 10 \end{aligned}$$

is chosen to maintain sliding mode motion on $\sigma = 0$. To stabilize the linear approximation of the sliding mode equation

$$\begin{aligned} \dot{z}_1 &= 1.5961k_1z_1 + (1 + 1.5961k_2)z_2 + 1.5961k_3z_3 \\ \dot{z}_2 &= 21.6707z_1 + 21.6707z_3 \\ \dot{z}_3 &= -k_1z_1 - k_2z_2 - k_3z_3, \end{aligned}$$

the set of parameters $[k_1, k_2, k_3] = [-34.8401, -7.4834, -40.6080]$ were used.

In all the simulations, the initial conditions of the system were chosen near the equilibrium point $x_1 = \frac{\pi}{2}$ and $x_3 = 0$ while variations in the nominal values of m_i and l_i were introduced.

Figures 2 and 4 show the output responses of the closed loop systems with nominal parameters, while Figures 3 and 5 show the behavior when a 20% variation of the mass m_2 is introduced. As we can see from Figure 5, the sliding mode based controller guarantees a small output tracking error, despite variation on the systems parameters.

8 Conclusion

The servomechanism problem for the variable structure system is introduced, and the solution conditions are derived for different classes of nonlinear systems including systems in the so-called regular and NBC-forms. In particular, the combination of VSS and block control techniques allow the solution conditions to be obtained in a simpler way with respect to the classical setting. In addition, the sliding mode based controller achieves robustness with respect to uncertainties.

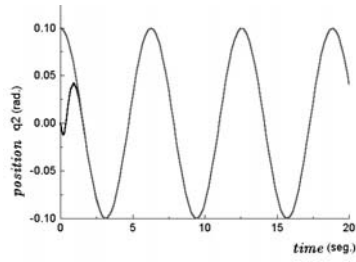


Fig. 2. Output signal response for the classical controller in absence of parameter perturbation.

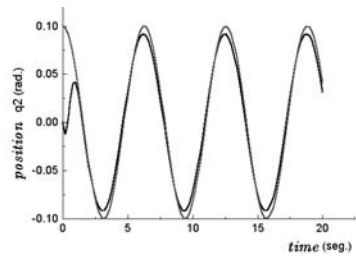


Fig. 3. Output signal response for the classical controller under parameter variations.

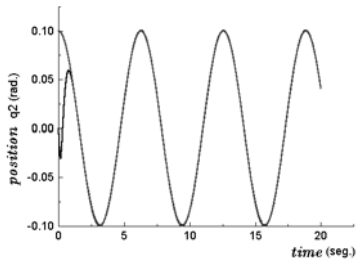


Fig. 4. Output signal response for the sliding mode controller in absence of parameter perturbation.

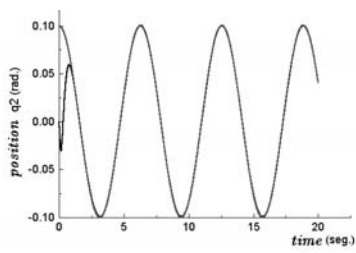


Fig. 5. Output signal response for the sliding mode controller under parameter variations.

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9 Appendix

To obtain equation (44), at first, because the matrix $B(x, t)$ does not have full rank, we impose a constraint on the components of the control vector u , $u = B_1^+(x, t)v_o$, such that

$$Bu = BB_1^+v_o = \begin{bmatrix} B_{12}B_1^+ \\ I_{n_1} \end{bmatrix} v_o \quad (94)$$

where v_o is a $n_1 \times 1$ input vector. It is easy to see, that , if $\varphi_1(x_1, x_{12}) = c$ is an integral of the Pfaffian system (43), the following identity

$$\frac{\partial \varphi_1}{\partial x_1} + \frac{\partial \varphi_1}{\partial x_{12}} B_2 B_1^+ \equiv 0 \quad (95)$$

holds. Indeed, a differential $d\varphi_1$ calculated using the Pfaffian system (43), of the form

$$d\varphi_1 = \frac{\partial \varphi_1}{\partial x} dx = \frac{\partial \varphi_1}{\partial x_1} dx_1 + \frac{\partial \varphi_1}{\partial x_{12}} dx_{12} = \left(\frac{\partial \varphi_1}{\partial x_1} + \frac{\partial \varphi_1}{\partial x_{12}} B_2 B_1^+ \right) dx_1 \quad (96)$$

is identically equal to zero. From this result, the identity (95) follows immediately.

Under assumption A12 there exist $n_1 \times 1$ vectors $\lambda_0(x, t)$ and $\rho_0(x, t)$ such that

$$\begin{bmatrix} F f_{12}^m \\ F_1^m \end{bmatrix} = \begin{bmatrix} B_{12} B_1^+ \\ I_{n_1} \end{bmatrix} \lambda_0 \text{ and } \begin{bmatrix} D_{12}^m \\ D_1^m \end{bmatrix} = \begin{bmatrix} B_{12} B_1^+ \\ I_{n_1} \end{bmatrix} \rho_0$$

and next expression under condition (95)

$$\begin{aligned} \frac{\partial \varphi_1}{\partial x_1} F_1^m + \frac{\partial \varphi_1}{\partial x_{12}} F_{12}^m &= \left(\frac{\partial \varphi_1}{\partial x_1} + \frac{\partial \varphi_1}{\partial x_{12}} B_2 B_1^+ \right) \lambda_0 = 0 \\ \frac{\partial \varphi_1}{\partial x_1} D_1^m w(t) + \frac{\partial \varphi_1}{\partial x_{12}} D_{12}^m w(t) &= \left(\frac{\partial \varphi_1}{\partial x_1} + \frac{\partial \varphi_1}{\partial x_{12}} B_2 B_1^+ \right) \rho_0 = 0 \end{aligned}$$

hold. Differentiating the new variable $x'_2 = \varphi_1(x_1, x_{12})$ with respect to time along trajectories of the system (39), gives

$$\begin{aligned} \dot{x}'_2 &= \frac{\partial \varphi_1}{\partial x_1} (f_1 + B_1 u + F_1^m v + F_1^u v + (D_1^m + D_1^u) w) \\ &\quad + \frac{\partial \varphi_1}{\partial x_{12}} (f_{12} + B_{12} u + (F_{12}^m + F_{12}^u) v + (D_{12}^m + D_{12}^u) w) + \frac{\partial \varphi_1}{\partial t} \\ &= \frac{\partial \varphi_1}{\partial x_1} f_1 + \frac{\partial \varphi_1}{\partial x_{12}} f_{12} + \left(\frac{\partial \varphi_1}{\partial x_1} + \frac{\partial \varphi_1}{\partial x_{12}} B_2 B_1^+ \right) v_0 + \left(\frac{\partial \varphi_1}{\partial x_1} + \frac{\partial \varphi_1}{\partial x_{12}} B_2 B_1^+ \right) \lambda_0 \\ &\quad + \left(\frac{\partial \varphi_1}{\partial x_1} + \frac{\partial \varphi_1}{\partial x_{12}} B_2 B_1^+ \right) \rho_0 w + (F_1^u + F_{12}^u) v + (D_1^u + D_{12}^u) w \end{aligned}$$

Defining

$$\begin{aligned} f'_2 &= \left(\frac{\partial \varphi_1}{\partial x_1} f_1 + \frac{\partial \varphi_1}{\partial x_{12}} f_{12} + \frac{\partial \varphi_1}{\partial t} \right)_{x_2 = \varphi_1^{-1}(x_1, x_{12}, t)} \\ D'_2 &= (D_1^u + D_{12}^u)_{x_2 = \varphi_1^{-1}(x_1, x_{12}, t)} \\ F'_2 &= (F_1^u + F_{12}^u)_{x_2 = \varphi_1^{-1}(x_1, x_{12}, t)} \end{aligned}$$

we obtain equation (44).

Variable Structure Systems Theory in Computational Intelligence

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Abstract. Intelligence in the form of well-organized solutions to the ill-posed problems has been the primary focus of many engineering applications. The ever-increasing developments in data fusion, sensor technology and high-speed micro-processors made the design in digital domain with high performance. A natural consequence of the progression during the last few decades is the emergence of computationally intelligent systems. Neural networks and fuzzy inference systems constitute the core approaches of computational intelligence, whose methods have extensively been used in the applications extending from image/pattern recognition to identification and control of nonlinear systems. This chapter is devoted to the analysis and design of learning strategies in the context of variable structure systems. Several approaches are discussed in detail with special emphasis on the sliding mode control of nonlinear systems.

1 Introduction

Twentieth century has witnessed widespread innovations in all disciplines of engineering sciences. Two snapshots from early 1900s and late 1990s differ particularly in terms of the active role of humans in performing complicated tasks. The trend during the last century had the goal of implementing systems having some degrees of intelligence to cope with the problem specific difficulties that are likely to arise during the normal operation of the system. Today, it is apparent that the trend towards the development of autonomous machinery will maintain its importance as the tasks and the systems are getting more and more complicated. A natural consequence of the increase in the complexity of the task and physical hardware is to observe an ever-widening gap between the mathematical models and the physical reality to which the models correspond. Having this picture in front of us, what now becomes evident is the need for research towards the development of approaches having the capability of self-organization under the changing conditions of the task and the environment. Computational Intelligence (CI) is a framework offering various solutions to handle the complexity and the difficulties

of information-limited operating environments. The diversity in the solution space is a remarkable advantage that the designer utilizes either in the sense of algorithm-oriented manner or in the sense of architecture-oriented manner, hence, the result is an autonomous system exploiting these advantages.

Autonomy is one of the most important characteristics required from a computationally intelligent system. A basic requirement in this context is the ability to refresh and to refine the information content of the dynamics of the system. It therefore requires a careful consideration in the realm of engineering practice. From a systems and control engineering point of view, the designer is motivated by the time-varying nature of structural and environmental conditions to realize controllers that can accumulate the experience and improve the mapping precision [1-2]. Methodologies imitating the inference mechanism of the human brain are good in achieving the former and those imitating the massively interconnected structure of the human brain are good in achieving the latter. In the literature, the linguistic aspects of intelligence are discussed in the area Fuzzy Logic (FL) while the connectionist aspects are scrutinized in the area Neural Networks (NN). The integration of these methodologies that exploit the strength of each collectively and synergistically is a driving force to synthesize hybrid intelligent systems. Being not limited to what is mentioned, methods mimicking the process of evolution, which are discussed under the title Genetic Algorithms (GA), and those adapted from artificial intelligence constitute other branches of CI and fall beyond the focus of the approaches presented in this chapter.

NN are well known for their property of representing complex nonlinear mappings. Earlier works on the mapping properties of these architectures have shown that NN are universal approximators [3-5]. The mathematical power of intelligence is commonly attributed to the neural systems because of their structurally complex interconnections and fault tolerant nature. Various architectures of neural systems are studied in the literature. Feedforward and Recurrent Neural Networks (FNN, RNN) [6], Radial Basis Function Neural Networks (RBFNN) [1,6], dynamic neural networks [7], and Runge-Kutta neural networks [8] constitute typical topologically different models.

FL is the most popular constituent of the CI area since fuzzy systems are able to represent human expertise in the form of IF *antecedent* THEN *consequent* statements. In this domain, the system behavior is modeled through the use of linguistic descriptions. Although the earliest work by Prof. Lotfi Zadeh on fuzzy systems [9] has not been paid as much attention as it deserved in the early 1960s, since then the methodology has become a well-developed framework. The typical architectures of Fuzzy Inference Systems (FIS) are those introduced by Wang [10], Takagi and Sugeno [11] and Jang, Sun and Mizutani [1]. In [10], a fuzzy system having Gaussian membership functions, product inference rule and weighted average defuzzifier is constructed and has become the standard method in most applications. Takagi and Sugeno [11] change the defuzzification procedure where dynamic systems are used in

the defuzzification stage. The potential advantage of the method is that under certain constraints, the stability of the system can be studied. Jang et al [1] propose an adaptive neuro-fuzzy inference system, in which polynomials are used in the defuzzifier. This structure is commonly referred to as ANFIS in the related literature.

When the applications of NN and FL are considered the process of learning gains a vital importance. Although there is not a standard definition, the process of improving the future performance of the structures of CI by tuning the parameters can be described as learning. The approaches existing in the literature employ various techniques in achieving the desired parameter set (which is unknown), and require an iteratively evolving search mechanism. It should be noted that the most common technique that can be used in performing a suitable search operation in a multidimensional parameter space is based on the use of an appropriately defined cost function. Alternatively, the search procedure can be implemented without using the derivative information; such as is done by the use of methods adapted from the evolutionary computation, e.g. GAs, or random search methods [1].

Error Backpropagation (EBP) technique [12] and Levenberg-Marquardt (LM) optimization technique [13] are the frequently used techniques used for parameter adaptation in CI. Both approaches are based on the utilization of gradient information and necessitate the differentiability of the nonlinear activation functions existing in the architecture with respect to the parameter to be updated, and frequently utilize some heuristics for improved realization performance. These typically concern the selection of learning rate, momentum coefficient, and adaptive learning rate strategies in EBP or step-size considerations in LM technique. However, the problem of convergence or that of maintaining the bounded parameter evolution is an open problem associated with these approaches. More explicitly, the learning strategy is not protected against disturbances, which may excite the undesired internal modes of EBP or LM approaches. The multidimensionality of the problem is another difficulty in coming up with a thorough analysis distinguishing the useful training information and disturbance-related excitation signals. Since the ultimate goal of the design is to meet the performance specifications, reducing the adverse effects of the disturbances requires that the adopted learning dynamics should be robustified. This steers the designer to seek for methods known in the conventional design framework. From this point of view, a learning strategy based on Variable Structure Systems (VSS) theory constitutes a good candidate for eliminating the adverse effects of disturbances.

VSS with sliding modes were first proposed in early 1950s [14-15]. However, due to the implementation difficulties of high speed switching, it was not until 1970s that the approach received the attention it deserved. Sliding Mode Control (SMC) technique nowadays enjoys a wide variety of application areas; such as in general motion control applications and robotics, in

process control, in aerospace applications, and in power converters [16-19]. The main reason for this popularity is the attractive properties that SMC have, such as good control performance for nonlinear systems, applicability to Multi-Input-Multi-Output (MIMO) systems, and well established design criteria for discrete time systems. The most significant property of a sliding mode control system is its robustness. Loosely speaking, when a system is in a sliding mode, it is insensitive to parameter changes or external disturbances.

From a systems and control theoretic point of view, the primary characteristic of variable structure control is that the feedback signal is discontinuous, switching on one or more manifolds in the state space. When the state crosses each discontinuity surface, the structure of the feedback system is altered. Under certain circumstances, all motions in the neighborhood of the manifold are directed towards the manifold and thus a sliding motion on a predefined subspace of the state-space is established in which the system state repeatedly crosses the switching surface [20]. This mode has useful invariance properties in the face of uncertainties in the plant model and therefore is a good candidate for tracking control of uncertain nonlinear systems. The theory is well developed, especially for single-input systems in controller canonical form.

The theory of VSS with sliding modes has been studied intensively by many researchers. A recent comprehensive survey is given in [17] and various aspects of latest developments in VSS can be found in the chapters of this book. Motion control, especially in robotics, has been an area that has attracted particular attention and numerous reports have appeared in the literature [21-25]. One of the first experimental investigations that demonstrates the invariance property of a motion control system under a sliding mode is due to Kaynak et al [26].

In practical applications, a pure SMC approach suffers from the following disadvantages. Firstly, there is the problem of chattering, which is the high frequency oscillations of the controller output, brought about by the high speed (ideally at infinite frequency) switching necessary for the establishment of a sliding mode. In practical implementations, chattering is highly undesirable because it may excite unmodeled high frequency plant dynamics and this can result in unforeseen instabilities.

Secondly, a SMC based feedback loop is extremely vulnerable to measurement noise since the control input depends tightly on the sign of a measured quantity that is very close to zero [27]. Thirdly the SMC may employ unnecessarily large control signals to overcome the parametric uncertainties. Last but not least, there exists appreciable difficulty in the calculation of what is known as the equivalent control. A complete knowledge of the plant dynamics is required for this purpose [28]. To alleviate these difficulties, several modifications to the original sliding control law have been proposed [29], the most popular being the boundary layer approach, which is, in essence, the application of a high gain feedback when the motion of the system reaches

ϵ -vicinity of a sliding manifold [22,28]. This approach is based on the idea of the equivalence of the high gain systems and the systems with sliding modes [30]. Another variation of the scheme is called provident control that combines variable structure control and variable structure adaptation and performs hysteretic switching between the structures so as to avoid a sliding mode [31-32]. Both approaches are based on the calculation of the equivalent control, requiring a good mathematical model of the plant.

The essence of the discussion presented in this chapter is to integrate VSS technique and CI in an appropriate manner such that the difficulties of VSS approach are alleviated by intelligence and the mathematical intractability of intelligence is alleviated by VSS technique. Such a hybrid approach, particularly operating as the learning mechanism of CI architectures, is therefore a good candidate to represent the autonomous behavior of intelligent systems with a robustified learning performance.

2 A Functional Overview of Computationally Intelligent Architectures

2.1 Adaptive Linear Elements (ADALINEs)

Being categorized as the basic operation in all architectures of CI, ADALINE performs an inner product of two vectors, which is The output is a net sum in the case of NNs, or the response of the system in the cases of RBFNN, SFS, ANFIS. The vectors of interest are the adjustable parameter vector and the excitation input denoted by ϕ and u respectively. The input-output relation can now be described as $\tau = \phi^T u$, where τ is the scalar output. Clearly, the applications requiring multiple outputs τ would be a vector while ϕ would be a matrix of appropriate dimensions.

2.2 Feedforward Neural Networks (FNNs)

FNNs constitute a class of NN structures in which the data flow is from input to the output and no feedback connections are allowed. Because of the structural diversity of neural models, this discussion is devoted to the architecture and the mathematical representation of FNN structure, which is discussed from the point of control engineering. The architecture of a typical FNN is illustrated in Figure 1, in which the neural network has three layers implying the sufficiency for realizing any continuous mapping to a desired degree of accuracy as long as the hidden layer contains sufficiently many neurons [3-5]. The number of neurons in the hidden layer is a design variable and is mostly determined either by trial and error or by empirical results.

Functionally, $\mathbf{o} = \psi_h(W_h \mathbf{u})$ and $\boldsymbol{\tau} = \psi_0(W_0 \mathbf{o})$, where ψ_h and ψ_0 stand for the vectors of nonlinear activation functions for the hidden layer and the output layer respectively. Adaptation is carried out on the adjustable weights

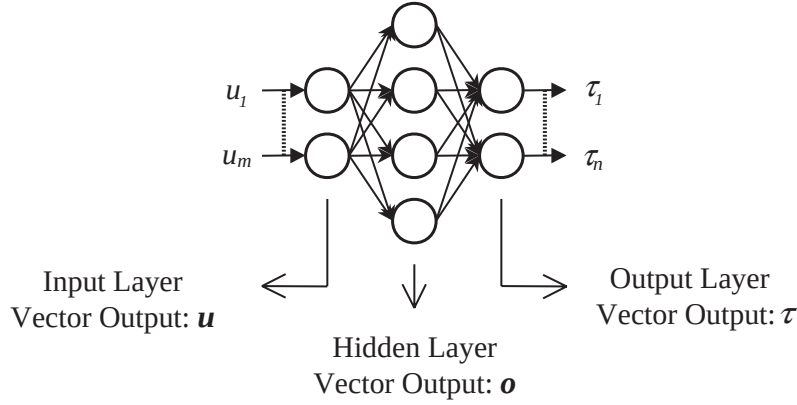


Fig. 1. Structure of a FNN

contained in W_h and W_0 matrices. In most applications of NNs, hyperbolic tangent or sigmoid functions are used in ψ_h whereas the selection of ψ_0 is generally a linear function of its argument, e.g. $\psi_0(x) = x$. The standard approach for tuning the parameters of FNNs is EBP or LM techniques [12-13].

Information contained in such a nonlinear map is distributed over its architectural constituents, i.e. neurons, such that a local failure in the structure can be tolerated because of the parametric redundancy existing in the structure, which is an analogue of the fault tolerance in biological systems. More explicitly, the task can be redistributed upon death of neurons forming a local infrastructure of a massive network.

2.3 Radial Basis Function Neural Networks (RBFNNs)

In the literature, RBFNNs are generally considered as a smooth transition between FL and NNs. Structurally, a RBFNN is composed of receptive units (neurons) which act as the operators providing the information about the class to which the input signal belongs. If the aggregation method, number of receptive units in the hidden layer and the constant terms are equal to those of a FIS, then there exists a functional equivalence between RBFNN and FIS [1]. Although the architectural view of a RBFNN is very similar to that of a FNN illustrated in Figure 1, the hidden neurons of a RBFNN possess basis functions to characterize the partitions of the input space. Each neuron in the hidden layer provides a degree of membership value for the input pattern with respect to the basis vector of the receptive unit itself. The output layer is comprised of linear neurons. NN interpretation makes RBFNN useful in incorporating the mathematical tractability, especially in the sense of propagating the error back through the network, while the FIS interpretation enables the incorporation of the expert knowledge into the

training procedure. The latter is of particular importance in assigning the initial value of the network's adjustable parameter vector to a vector that is to be sought iteratively. Expectedly, this results in faster convergence in parameter space.

Mathematically, $o_i = \prod_{j=1}^m \psi_{ij}(u_j)$ and a common choice for the hidden layer activation function is the Gaussian curve described as $\psi_{ij}(\mathbf{u}) = \exp\{-(u_j - c_{ij})^2/\sigma_{ij}^2\}$, where c_{ij} and σ_{ij} stand for the center and the variance of the i^{th} neuron's activation function qualifying the j^{th} input variable. The output of the network is evaluated through the inner product of the adjustable weight vector denoted by Φ and the vector of hidden layer outputs, i.e. $\tau = \Phi^T \mathbf{o}$, which is just as in the case of output evaluation in ADALINEs. Clearly the adjustable parameter set of the structure is composed of $\{\mathbf{c}, \sigma, \phi\}$ triplet.

2.4 Standard Fuzzy Systems (SFSs)

Contrary to what is postulated in the realm of predicate logic, representation of knowledge by fuzzy quantities can provide extensive degrees of freedom if the task to be achieved can better be expressed in words than in numbers. The concept of fuzzy logic in this sense can be viewed as a generalization of binary logic and refers to the manipulation of knowledge with sets, whose boundaries are unsharp [33]. Therefore the paradigm offers a possibility of designing intelligent controllers operating in an environment, in which the conditions are inextricably intertwined, subject to uncertainties and impreciseness.

Understanding the information content of fuzzy logic systems is based on the subjective judgements, intuitions and the experience of an expert. From this point of view, a suitable way of expressing the expert knowledge is the use of *IF antecedent THEN consequent* rules, which can easily evaluate the necessary action to be executed for the current state of the system under investigation.

Structurally, a fuzzy controller is comprised of five building blocks, namely, fuzzification, inference engine, knowledge base, rule base, and defuzzification. Since the philosophy of the fuzzy models is based on the representation of knowledge in fuzzy domain, the variables of interest are graded first. This grading is performed through the evaluation of membership values of each input variable in terms of several class definitions. According to the definition of a membership function, how the degree of confidence changes over the domain of interest is characterized. This grading procedure is called fuzzification. In the knowledge base, the parameters of membership functions are stored. Rule base contains the cases likely to happen, and the corresponding actions for those cases through linguistic descriptions, i.e. the IF-THEN statements. The inference engine emulates the expert's decision making in interpreting and applying knowledge about how the best fulfillment of the task is achieved. Finally, the defuzzifier converts the fuzzy decisions back onto the crisp domain [34].

SFS architecture that has been proposed by Wang [35] uses algebraic product operator for the aggregation of the rule premises and bell-shaped membership functions denoted by μ . The overall representation of SFS structure is given in (1), in which R and m stand for the number of rules contained in the rule base and the number of inputs of the structure.

$$\tau = \sum_{i=1}^R f_i \left(\frac{\prod_{j=1}^m \mu_{ij}(u_j)}{\sum_{i=1}^R \prod_{j=1}^m \mu_{ij}(u_j)} \right) \quad (1)$$

with i^{th} rule as: IF u_1 is U_{1i} AND u_2 is U_{2i} AND ... AND u_m is U_{mi} THEN $f_i = \phi_i$. In the IF part of this representation, the lowercase variables denote the inputs and the uppercase variables stand for the fuzzy sets corresponding to the domain of each linguistic label. The THEN part is comprised of the prescribed decision in the form of a scalar number denoted by ϕ_i . Clearly, the adjustable parameters of the structure are comprised of the parameters of the membership functions together with the defuzzifier parameters ϕ_i . Another common feature of the representation in (1) is the linearity of the output in the defuzzifier parameters.

2.5 Adaptive Neuro-Fuzzy Inference Systems (ANFIS)

Adaptive neuro-fuzzy inference systems are synthesized by an appropriately integrating the neural and fuzzy system interpretations. The resulting hybrid combination therefore inherits the numeric power of NN as well as the verbal power of FL [1,36]. An ANFIS structure having m -inputs and single output with product inference rule and first order Sugeno model can be described as in (1) with f_i being described as in the rule consequent. The structural view of such a system is illustrated in Figure 2, in which $\hat{N}$ stands for the normalization operator seen as the last term of (1).

The rule structure for an ANFIS utilizing first order Sugeno model has the following representation: IF u_1 is U_{1i} AND u_2 is U_{2i} AND ... AND u_m is U_{mi} THEN $f_i = \phi_{i,1}u_1 + \dots + \phi_{i,m}u_m + \phi_{i,m+1}$. When the consequent part of the rule structure is compared with that of rules in SFS architecture, it is seen that the polynomial representation of the decision introduces higher parametric flexibility extending the realization capability. Being not confined to what is discussed above, depending on the requirements of the problem in hand, the designer can choose higher order polynomials to improve the realization accuracy. When the issue of parameter tuning in ANFIS is considered the well-known gradient approaches as well as the method of least mean squares or VSS based approaches can easily be utilized.

ANFIS structure has been utilized with gradient based training strategies for identification of nonlinear systems [37] and with VSS based training strategies for variable structure control of motion control systems. In [1], an in-depth discussion is given with numerous examples on the use of ANFIS structure.

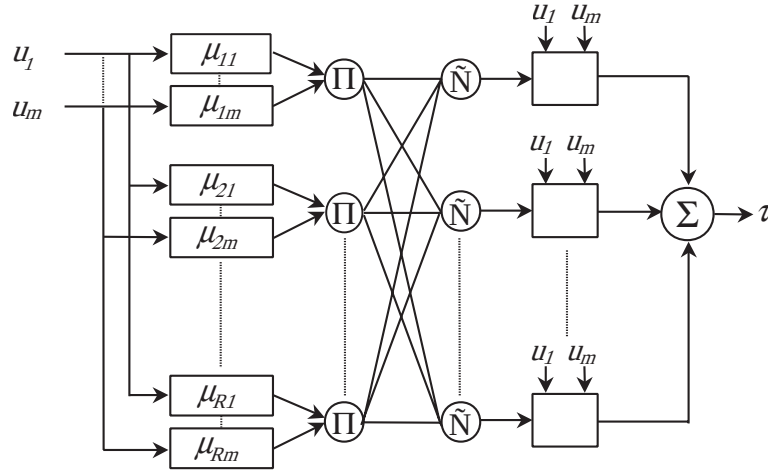


Fig. 2. Structure of an ANFIS

3 VSS Based Parameter Tuning in Intelligent Control Systems

The studies reporting the use of VSS for parameter tuning in CI by Sanner and Slotine [38], and Sira-Ramirez and Colina-Morles [39] have been the stimulants, which proved that the robustness feature of VSS could be exploited in the training of the architectures of CI. These studies pioneered a vast majority of researchers working on VSS and CI. Sanner and Slotine considered the training of GRBFNN which has certain degrees of analytical tractability in explaining the stability issues, and Sira-Ramirez et al have shown the use of ADALINEs with a VSS based learning strategy. As an illustrative example, the inverse dynamics identification of a Kapitsa pendulum has been demonstrated together with a thorough analysis towards the handling of disturbances. Hsu and Real [40-41] demonstrate the use of VSS with Gaussian NNs, Yu et al [42] introduces the dynamic uncertainty adaptation of what is proposed in [39], and demonstrate the performance of the scheme on the Kapitsa pendulum. Parma et al [43] use the VSS technique in parameter tuning process of multilayer perceptron. Latest studies towards the integration of VSS and CI have shown that the tuning can be implemented in dynamic weight filter neurons [44], in parameters of a controller [45]. A different viewpoint towards this integration is due to Efe et al [46-47], which has the goal of reducing the adverse effects of noise driven parameter tuning activity in gradient techniques. The key idea in these works is to mix the two training signals in a weighted average sense. A good deal of review is provided in the recent survey of Kaynak et al [48]. The survey illustrates how

VSS can be used for training in CI as well as how CI can be utilized for the tuning of parameters in conventional VSS.

In what follows, the use of VSS approach for intelligent control of non-linear systems is presented together with the analytical details wherever required. The emphasis is mainly on the works presented in [44-45] with the authors' latest research outcomes towards the direction of control engineering.

3.1 Control System Structure

Consider the feedback loop illustrated in Figure 3, in which a subscript d denotes the desired value of the relevant quantity. Furthermore, it is shown in the figure that if a supervisor provides the desired controller outputs, one might evaluate the error on the control signal denoted by s_c .

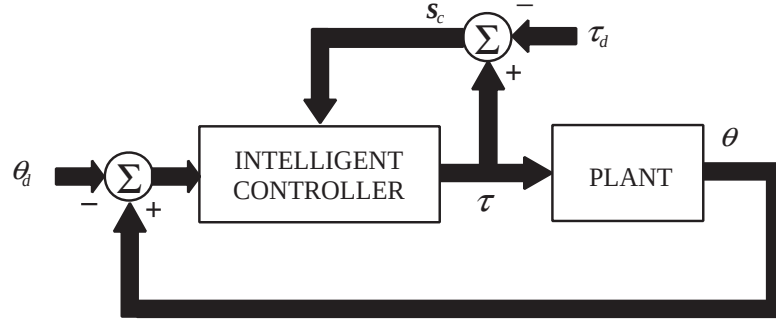


Fig. 3. Block diagram of the control system

The plant shown in Figure 3 is assumed to have the structure described in (2), in which θ and τ are $(r_1 + r_2 + \dots + r_n) \times 1$ -dimensional state vector and $n \times 1$ -dimensional input vector. The system of (2) with these vectors can be restated as $\dot{\theta} = f_p(\theta) + D\tau$.

$$\theta_i^{(r_i)} = f_{p_i}(\theta) + \sum_{j=1}^n d_{ij}\tau_j \quad i = 1, 2, \dots, n \quad (2)$$

The design problem is to enforce the behavior of the system towards the desired response, which is known but the control signal (τ_d) resulting in which is unavailable. Therefore, the solution to this problem is a search towards the synthesis of such a signal iteratively by an intelligent controller. Assuming that the intelligent controller in Figure 3 is composed of n individual controllers, the i^{th} one of which is to construct the i^{th} component of input vector τ , the j^{th} entry of the error vector driving this sub-controller can be

given as $e_i^{(j)} = \theta_i^{(j)} - \theta_{di}^{(j)}$. Apparently, this component is the j^{th} derivative of the relevant state component.

3.2 Conventional VSS Design - An Overview

Consider the vector of sliding surfaces for the system in (2): $\mathbf{s}_p(\mathbf{e}) = G\mathbf{e} = G(\boldsymbol{\theta} - \boldsymbol{\theta}_d)$. The widespread selection of the matrix G is such that the i^{th} sliding surface function has the form

$$s_{p_i}(e_i) = \left(\frac{d}{dt} + \lambda_i \right)^{r_i-1} e_i \quad (3)$$

in which, λ_i is a strictly positive constant. Let V_p be a candidate Lyapunov function given as

$$V_p(\mathbf{s}_p) = \frac{1}{2} \mathbf{s}_p^T \mathbf{s}_p \quad (4)$$

If the prescribed control signal satisfies $\dot{V}_p(\mathbf{s}_p) = -\mathbf{s}_p^T \Xi \text{sgn}(\mathbf{s}_p)$, the negative definiteness of the time derivative of the Lyapunov function in (4) is ensured. In above, Ξ is a positive definite diagonal matrix of dimension $n \times n$. More explicitly, $\mathbf{s}_p^T \dot{\mathbf{s}}_p = -\mathbf{s}_p^T \Xi \text{sgn}(\mathbf{s}_p)$ must hold true to drive the error vector towards the sliding hypersurface. On the other hand, the use of $\dot{\mathbf{s}}_p = -G\dot{\boldsymbol{\theta}}_d + G(\mathbf{f}_p(\boldsymbol{\theta}) + D\boldsymbol{\tau})$ leads to the following control signal:

$$\boldsymbol{\tau} = -(GD)^{-1} (G\mathbf{f}_p(\boldsymbol{\theta}) - G\dot{\boldsymbol{\theta}}_d) - (GD)^{-1} \Xi \text{sgn}(\mathbf{s}_p) \quad (5)$$

in which, the first term is the equivalent control term and the second term is the corrective control term. For the existence of the mentioned components, the matrix GD must not be rank deficient. In the literature, equivalent control is considered as the low frequency (average) component of the control signal. Because of the discontinuity on the sliding surface, the corrective term brings a high rate component [20,25]. If $\mathbf{e}(0) = \mathbf{0}$, the tracking problem can be considered as keeping $\mathbf{e}$ on the sliding surface, however, for nonzero initial conditions, the strategy must enforce the state trajectories towards the sliding surface, which is ensured by the negative definiteness of the time derivative of the Lyapunov function as in (4). For the case of nonzero initial conditions, the phase until the error vector hits the sliding surface is called the reaching mode, the dynamic characteristics of the system during which is determined by the control strategy adopted. Application of the control input formulated in (5) imposes the dynamics described as $\dot{\mathbf{s}}_p = -\Xi \text{sgn}(\mathbf{s}_p)$, which clearly enforces the error vector towards the sliding surface. Once the sliding surface is reached, the value of (3) becomes zero; and this enforces the error vector to move towards the origin.

Aside from the practical difficulties of conventional VSS schemes, the control signal in (5) is applicable if a nominal representation of the system under control is available. In the next subsection, a method for obtaining the error on the control signal is presented for unknown systems of structure (2).

3.3 Calculation of the Control Error

Remark 3.1: The VSS task is achievable if the dynamics of the system in (2) is totally known or if the nominal system is known with the bounds of the uncertainties. It must be noted that to satisfy the matching conditions, the disturbances and uncertainties are always assumed to enter the system through the control channels [17]. When the conventional VSS strategy is applied to the system of (2), we call the resulting behavior as the *target VSS* and the input vector leading to it as the *target control sequence* (τ), which is described in (5). If the functional form of the vector function f_p is not known, it should be obvious that the target control sequence cannot be constructed by following the traditional VSS design approaches.

Definition 3.2: Given an uncertain plant, which has the structure described as in (2), and a command trajectory vector $\theta_d(t)$ for $t \geq 0$, the input sequence satisfying the following vector differential equation is defined to be the *idealized control sequence* denoted by τ_d , and the vector differential equation itself is defined to be the *reference SMC model*.

$$\dot{\theta}_d = f_p(\theta_d) + D\tau_d \quad (6)$$

Mathematically, the existence of such a model and the sequence means that the system of (2) perfectly follows the command trajectory vector if both the idealized control sequence is known and the initial conditions are set as $\theta(t=0) = \theta_d(t=0)$, more explicitly $e(t) \equiv 0$ for $t \geq 0$. Undoubtedly, such an idealized control sequence will not be a norm-bounded signal when there are step-like changes in the vector of command trajectories or when the initial errors are nonzero. It is therefore that the reference SMC model is an abstraction due to the limitations of the physical reality, but the concept of idealized control sequence should be viewed as the synthesis of the command signal θ_d from the time solution of the differential equation set in (6).

Fact 3.3: Based on the Lyapunov stability results of the previous subsection, if the target control sequence formulated in (5) were applied to the system of (2), the idealized control sequence would be the steady state solution of the control signal, i.e. $\lim_{t \rightarrow \infty} \tau = \tau_d$. However, under the assumption of the achievability of the VSS task, the difficulty here is again the unavailability of the functional form of the vector function f_p . Therefore, the aim in this subsection is to discover an equivalent form of the discrepancy between the control applied to the system and its target value by utilizing the idealized control viewpoint. This discrepancy measure is denoted by $s_c = \tau - \tau_d$ and is of $n \times 1$ dimension.

If the target control sequence of (5) is rewritten by using (6), one gets

$$\tau = -(GD)^{-1} \left(Gf_p(\theta) - G(f_p(\theta_d) + D\tau_d) + \Xi \operatorname{sgn}(s_p) \right)$$

$$\begin{aligned}
&= -(GD)^{-1} \left(G\mathbf{f}_p(\boldsymbol{\theta}) - G\mathbf{f}_p(\boldsymbol{\theta}_d) + \Xi \operatorname{sgn}(\mathbf{s}_p) \right) + \boldsymbol{\tau}_d \\
&= -(GD)^{-1} \left(G\boldsymbol{\Delta}\mathbf{f}_p(\boldsymbol{\theta}) + \Xi \operatorname{sgn}(\mathbf{s}_p) \right) + \boldsymbol{\tau}_d
\end{aligned} \tag{7}$$

The target control sequence becomes identical to the idealized control sequence, i.e. $\boldsymbol{\tau} \equiv \boldsymbol{\tau}_d$, as long as $G\boldsymbol{\Delta}\mathbf{f}_p(\boldsymbol{\theta}) + \Xi \operatorname{sgn}(\mathbf{s}_p) = \mathbf{0}$ holds true. However, this condition is of no practical importance as we do not have the analytic form of the vector function $\mathbf{f}_p$. Therefore, one should consider this equality as an equality to be enforced instead of an equality that holds true all the time, because its implication is $\mathbf{s}_c = \mathbf{0}$ and is the aim of the design. It is obvious that to enforce this to hold true will let us synthesize the target control sequence, which will ultimately converge to the idealized control sequence by the adaptation algorithm yet to be discussed. Consider the time derivative of the vector of sliding surfaces

$$\begin{aligned}
\dot{\mathbf{s}}_p(\mathbf{e}) &= G\dot{\mathbf{e}} \\
&= G(\dot{\boldsymbol{\theta}} - \dot{\boldsymbol{\theta}}_d) \\
&= G\left(\mathbf{f}_p(\boldsymbol{\theta}) + D\boldsymbol{\tau} - \mathbf{f}_p(\boldsymbol{\theta})_d - D\boldsymbol{\tau}_d\right) \\
&= G\left(\boldsymbol{\Delta}\mathbf{f}_p + D(\boldsymbol{\tau} - \boldsymbol{\tau}_d)\right) \\
&= G\left(\boldsymbol{\Delta}\mathbf{f}_p + D\mathbf{s}_c\right)
\end{aligned} \tag{8}$$

Utilizing $G\boldsymbol{\Delta}\mathbf{f}_p + \Xi \operatorname{sgn}(\mathbf{s}_p) = \mathbf{0}$ in (8) and solving for $\mathbf{s}_c$ yields the following relation:

$$\mathbf{s}_c = (GD)^{-1} \left(\dot{\mathbf{s}}_p + \Xi \operatorname{sgn}(\mathbf{s}_p) \right) = \boldsymbol{\tau} - \boldsymbol{\tau}_d \tag{9}$$

Remark 3.4: The reader must here notice that the application of $\boldsymbol{\tau}_d$ to the system of (2) with zero initial errors will lead to $\mathbf{e}(t) \equiv \mathbf{0}$ for $\forall t \geq 0$, on the other hand, the application of $\boldsymbol{\tau}$ to the system of (2) will lead to $\mathbf{s}_p = \mathbf{0}$ for $\forall t \geq t_h$, where t_h is the hitting time, and the origin will be reached according to the dynamics of the sliding surface. Therefore, the adoption of (9) as the equivalent measure of the control error loosens $\mathbf{e}(t) \equiv \mathbf{0}$ for $\forall t \geq 0$ requirement and introduces all trajectories in the error space to tend to the sliding hypersurface, i.e. $G\boldsymbol{\Delta}\mathbf{f}_p + \Xi \operatorname{sgn}(\mathbf{s}_p) = \mathbf{0}$ is enforced. Consequently, the tendency of the control scheme will be to generate the target VSS sequence of (5) without requiring the analytical details of the plant.

Now consider the ordinary feedback control loop illustrated in Figure 3, and define the following Lyapunov function, which is a measure of how well the controller performs:

$$V_c(\mathbf{s}_c) = \frac{1}{2} \mathbf{s}_c^T \mathbf{s}_c \tag{10}$$

Remark 3.5: An adaptation algorithm ensuring $\dot{V}_c(s_c) < 0$ when $s_c \neq \mathbf{0}$ enforces $G\Delta f_p + \Xi \text{sgn}(s_p) = \mathbf{0}$ and creates the predefined sliding regime after a reaching mode lasting until the hitting time denoted by t_h , beyond which $s_c = \mathbf{0}$ as the system is in the sliding regime. If $\dot{V}_c(s_c) < 0$ when $s_c \neq \mathbf{0}$, then $\lim_{t \rightarrow t_h} V_c = 0 \iff \lim_{t \rightarrow t_h} \|s_c\| = 0 \iff \lim_{t \rightarrow t_h} \|\dot{s}_p + \Xi \text{sgn}(s_p)\| = 0$. Note that the meaning of $s_c = \mathbf{0}$ is now equivalent to $s_p = \mathbf{0}$ by Remark 3.4, therefore the limits above are evaluated as $t \rightarrow t_h$.

3.4 Parameter Tuning based on a Single-Term Lyapunov Function

If the architectures introduced in the second section are utilized for the purpose of control, without loss of generality, the output of the i^{th} controller can be restated as $\tau_i = \phi_i^T \Omega_i$, where Ω_i is the vector of signals exciting the adjustable parameters denoted by ϕ_i . Therefore the algorithm discussed here is applicable to ADALINE, GRBFNN, SFS and ANFIS architectures. Furthermore, the Lyapunov function in (10) constitutes the basis of the design.

In order not to be in conflict with the physical reality, the designer must impose $\|\phi_i\| \leq B_{\phi_i}$, $\|\Omega_i\| \leq B_{\Omega_i}$, $\|\dot{\Omega}_i\| \leq B_{\dot{\Omega}_i}$, and $\|\dot{\tau}_{id}\| \leq B_{\dot{\tau}_{id}}$ the truth of which state that the adjustable parameters of the controller, the time derivative of the signal exciting the adjustable parameter set and the time derivative of the idealized output of the controller remain bounded. Note that in Definition 3.2, we stated that there may not be a finite $B_{\dot{\tau}_{id}} \in \mathbb{R}$ even in some realistic situations like nonzero initial errors, however, the practical meaning of imposing $\|\dot{\tau}_{id}\| \leq B_{\dot{\tau}_{id}}$ will lead us to the construction of an approximation of the idealized control sequence and the requirement of $e(t) \equiv \mathbf{0}$ for $\forall t \geq 0$ must therefore be loosened.

Theorem 3.6: For the i^{th} subsystem of the system described in (2), adopting the controller of structure $\tau_i = \phi_i^T \Omega_i$, the adaptation of the controller parameters as described in (11) enforces the value of the i^{th} component of control discrepancy vector (s_{ci}) to zero.

$$\dot{\phi}_i = -\frac{\Omega_i}{\Omega_i^T \Omega_i} k_i \text{sgn}(s_{ci}) \quad (11)$$

where, k_i is a sufficiently large positive constant satisfying $k_i > B_{\phi_i} B_{\dot{\Omega}_i} + B_{\dot{\tau}_{id}}$. The adaptation mechanism in (11) drives an arbitrary initial value of s_{ci} to zero in finite time denoted by t_{hi} satisfying the inequality in (12).

$$t_{hi} \leq \frac{|s_{ci}(0)|}{k_i - (B_{\phi_i} B_{\dot{\Omega}_i} + B_{\dot{\tau}_{id}})} \quad (12)$$

Proof: See Sira-Ramirez et al [39] and Efe et al [45].

An important feature of this approach is the fact that the controller parameters evolve bounded as assumed initially. The details of the bounded parametric evolution analysis can be found in [42,45].

3.5 Parameter Tuning based on a Two-Term Lyapunov Function

Similar to what is initially designated in the previous subsection, the output of the i^{th} controller is described as $\tau_i = \phi_i^T \Omega_i$. In addition to the stated boundedness conditions the truth of $\|\Omega_i\| \leq B_{\Omega_i}$ is imposed. Consider the Lyapunov function given in (13), in which μ and ρ are the weights to be selected by the designer.

$$V = \mu V_{c_i} + \rho \frac{1}{2} \left\| \frac{\partial V_{c_i}}{\partial \phi_i} \right\|^2 \quad \text{with} \quad V_{c_i} = \frac{1}{2} s_{c_i}^2 \quad (13)$$

Theorem 3.7: If the adaptation strategy for the adjustable parameters of the i^{th} controller is chosen as

$$\dot{\phi}_i = -k_i \left(\mu I + \rho \frac{\partial^2 V_{c_i}}{\partial \phi_i \partial \phi_i^T} \right)^{-1} \text{sgn} \left(\frac{\partial V_{c_i}}{\partial \phi_i} \right) \quad (14)$$

with k_i is a sufficiently large constant satisfying $k_i > (\mu B_{\phi_i} + \rho B_{\Omega_i}) B_{\Omega_i}$, then the negative definiteness of the time derivative of the Lyapunov function in (13) is ensured.

Proof: See Efe [49].

3.6 A Generalization of EBP and LM Techniques in the Context of VSS

A recent contribution towards the generalization of EBP and LM techniques is due to Yu et al [50]. The approach postulated is applicable to all architectures discussed in the second section and is based on the Lyapunov function given in (15).

$$V(J_i, \phi_i) = \mu J_i + \rho \frac{1}{2} \left\| \frac{\partial J_i}{\partial \phi_i} \right\|^2 \quad (15)$$

where $J_i = \gamma^{-1} \int_{t-\gamma}^t s_{c_i}(\sigma) d\sigma$ with γ being the length of a time window to evaluate the training efficiency [51-52].

Theorem 3.8: For a computationally intelligent structure whose input-output relationship is $\tau_i(t) = \mathfrak{F}(\phi_i(t), \mathbf{u}_i(t))$, if

- (a) $\frac{\partial J_i}{\partial t} < 0$ and
- (b) The parameter adaptation rule is

$$\dot{\phi}_i = \begin{cases} -\left(\mu I + \rho \frac{\partial^2 J_i}{\partial \phi_i \partial \phi_i^T}\right)^{-1} \left(\frac{\frac{\partial J_i}{\partial \phi_i^T}}{\left\|\frac{\partial J_i}{\partial \phi_i}\right\|^2}\right)^* \\ \left(\mu \frac{\partial J_i}{\partial t} + \rho \frac{\partial J_i}{\partial \phi_i} \frac{\partial^2 J_i}{\partial t \partial \phi_i^T} + \zeta \left\|\frac{\partial J_i}{\partial \phi_i}\right\|^2 + \eta J_i\right) \text{ if } \left\|\frac{\partial J_i}{\partial \phi_i}\right\| \neq 0 \\ 0 \text{ otherwise} \end{cases} \quad (16)$$

Then $\frac{\partial J_i}{\partial \phi_i}$ tends to zero asymptotically.

Proof: See Yu et al [50].

The formulation of J_i is particularly useful for on line training and continuous time learning. However, for discrete data, since the evaluation of errors can only be done at discrete instants of time, J_i at time t_k can be defined as $J_i(t = t_k) = \lim_{\gamma \rightarrow 0} \gamma^{-1} \int_{t-\gamma}^t s_{c_i}(\sigma) d\sigma = s_{c_i}(t_k)$. The conventional gradient descent learning algorithm can be now obtained by setting $\rho = 0$ and $\eta = 0$. Since $\frac{\partial^2 J_i(t=t_k)}{\partial t \partial \phi_i^T} = 0$ one obtains the law in (17), whose learning rate in the conventional sense is $\eta^{-1}\zeta$.

$$\dot{\phi} = -\eta^{-1}\zeta \frac{\partial J_i(t = t_k)}{\partial \phi^T} \quad (17)$$

The Gauss-Newton algorithm can be obtained by setting $\mu = 0$ and $\eta = 0$. Since $\frac{\partial J_i(t=t_k)}{\partial t} = 0$ and $\frac{\partial^2 J_i(t=t_k)}{\partial t \partial \phi_i^T} = 0$, from (16) one gets the law in (18).

$$\begin{aligned} \dot{\phi}_i &= \left(\sigma \frac{\partial^2 J_i(t = t_k)}{\partial \phi_i \partial \phi_i^T}\right)^{-1} \left(\zeta \frac{\partial J_i(t = t_k)}{\partial \phi_i^T}\right) \\ &= -\sigma^{-1}\zeta \left(\frac{\partial^2 J_i(t = t_k)}{\partial \phi_i \partial \phi_i^T}\right)^{-1} \left(\frac{\partial J_i(t = t_k)}{\partial \phi_i^T}\right) \end{aligned} \quad (18)$$

Similarly, the LM algorithm can be easily obtained by setting $\eta = 0$. Since $\frac{\partial J_i(t=t_k)}{\partial t} = 0$ and $\frac{\partial^2 J_i(t=t_k)}{\partial t \partial \phi_i^T} = 0$, from (16) the law in (19) is obtained.

$$\dot{\phi}_i = -\left(\mu I + \sigma \frac{\partial^2 J_i(t = t_k)}{\partial \phi_i \partial \phi_i^T}\right)^{-1} \left(\zeta \frac{\partial J_i(t = t_k)}{\partial \phi_i^T}\right) \quad (19)$$

3.7 Practical Issues

The analysis and the design approach presented so far have tried to illuminate the VSS based training problem from a theoretical perspective. In this subsection, we discuss several issues related to the practical applications of the discussed methodologies.

Chattering Since the control decision during the sliding mode is tightly dependent to the sign of a measured quantity being noisy and very close to zero, the decision along the sliding manifold exhibits sensitivity to noise on the observations. Among many alternatives available [17,28,53], a common approach to eliminate the chattering is to smooth the sign function, which corresponds to introduce a boundary layer [28]. A widespread choice is the following approximation for the $\text{sgn}(\cdot)$ function.

$$\text{sgn}(x) \cong \frac{x}{|x| + \delta} \quad (20)$$

where δ determines the sharpness around the origin. Since the function in (20) is not discontinuous at the origin, the decision mechanism softly switches inside the boundary layer.

Actuation Speed Another important issue is the actuation speed of the system under control, i.e. the ability to respond to what is imposed timely. Since the details concerning the dynamic model of the plant under control are assumed to be unavailable, what causes a difficulty from a practical point of view is the selection of the matrix Ξ , which characterizes the behavior during the reaching mode. The values of this quantity can only be set by trial-and-error due to the lack of system-specific details.

Obtaining the Equivalent Error from the Observed Data Lastly in this subsection, we focus on the construction of the $\mathbf{s}_c$ of (9), which requires the differentiation of $\mathbf{s}_p$. A suitable approach is to filter the measured values of $\mathbf{s}_p$ and differentiate afterwards. Denote S as the Laplace variable, and use the linear dynamic system given as

$$H(S) = \frac{\alpha S}{Q(S)} \quad (21)$$

where $Q(0) = \alpha > 0$ and $\text{Real}\{\text{roots}(Q(S))\} < 0$. The order of the denominator polynomial and the locations of the roots are left to the designer, because these issues require several trials to refine the selections and are subject to the application together with its operating environment. It should be noted that the cost of the information loss by using such a filter, whose input is $\mathbf{s}_p$ and output is an estimate of $\dot{\mathbf{s}}_p$, is a matter of how robust the devised control algorithm is. More explicitly, the separation of the noise and the actual value of $\mathbf{s}_p$ leads to a corruption on $\mathbf{s}_p$, and when differentiated afterwards, some valuable information is lost together with the elimination of the noise component. Here it is assumed that the mentioned loss causes an uncertainty, which enters the system through the control channels, and which is particularly effective during the sliding mode; and this uncertainty can be alleviated if it falls within the limits allowing the maintenance of the invariance during the sliding mode [17].

Computational Burden One of the factors qualifying the physical implementability of control schemes is the number of computations to be performed by the controller. In this part, a discussion of the computational burden of the tuning mechanisms is presented. It should be noted that the structure of the controller adopted strictly influences the required number of *floating point operations* (flops) between the two consecutive sampling instants. Therefore, the discussion given here focuses on the ADALINE controller, as it constitutes a basis of all structures. If an ANFIS structure is to be used, the designer must consider the extra calculations to generate the vector signal exciting the adjustable parameter set of the defuzzifier.

Another point to clarify is the computational complexity due to the approach postulated in Theorem 3.8, whose practical applications generally subject to the following: the cost function J_i is evaluated at the discrete instants of time and it does not depend explicitly on time, i.e. $\frac{\partial J_i(t=t_k)}{\partial t} = 0$ and $\frac{\partial^2 J_i(t=t_k)}{\partial t \partial \phi_i^T} = 0$.

Figure 4 illustrates a bar graph composed of triplets. The leftmost component represents the flops required to evaluate the ADALINE output and to adjust its parameters once by utilizing the method discussed in the subsection 3.4. The middle and the rightmost components stand for the required number of flops for the methods presented in subsections 3.5 and 3.6 respectively. It is clear from the figure that the complexity due to the first approach is considerably smaller than the other two as the order of the subsystem under control increases. This fact is primarily because of the matrix inversion to be performed at each step. However, the set of criteria qualifying the performance of an intelligent control system is strictly dependent upon the application specific details, which does not give a clue in choosing a tuning mechanism. Therefore, the designer is encouraged to try the alternatives in discovering the one performing the best.

3.8 Summary

What we have discussed so far have illuminated the design considerations at microscopic levels forming the whole picture. When implementing the control system with a VSS based tuning mechanism updating the parameters of an intelligent controller, one has to remember that the plant is in an ordinary feedback loop as illustrated in Figure 3. Having decided on the controller structure, the error vector is processed until the control to be applied is obtained. Since the desired control inputs are unavailable, using the error measure given in (9), the similarity between the applied control and the target control sequence is qualified, then the parameter tuning is performed according to the chosen tuning strategy.

A particular difference in applying the ADALINE structure as the controller with (11) and (14) is that the controller input vector is formed by augmenting the error vector, which is of dimension $r_i \times 1$, with a constant

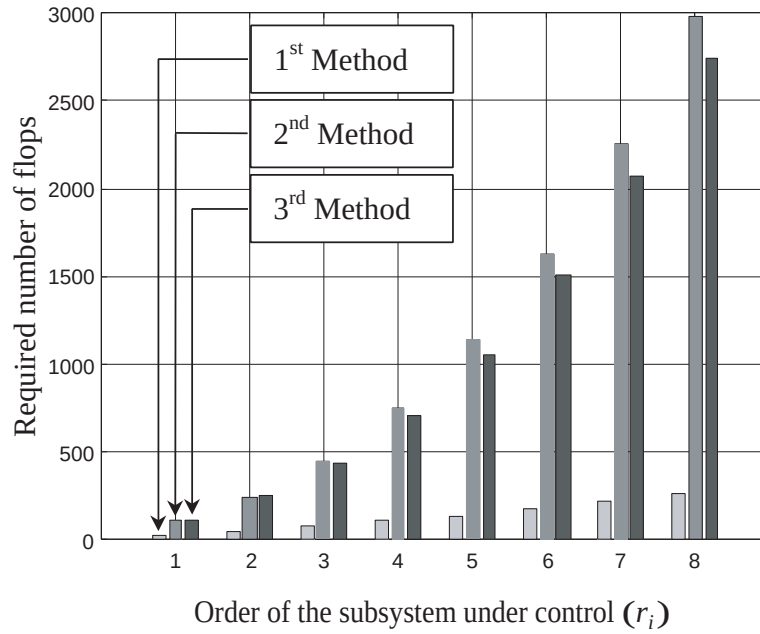


Fig. 4. Computational burden of the discussed schemes

bias of value unity yielding a $(r_i + 1) \times 1$ -dimensional excitation to the controller. The reason for such an augmentation is twofold:

- i) If the denominator of (11) were considered, without such an augmentation the derivative would tend to infinity as the error vector moves towards the origin. However, having such a tendency in the adjustable controller parameters cannot result in convergence. When (14) is considered from the same point of view, together with the open form of matrix inversion, one sees that a convergent behavior enforces the tuning mechanism to behave like gradient descent. Although gradient descent can appropriately be used for controller training purposes, the structural simplicity of ADALINE will not allow the observation of a convergent behavior. This particular structure corresponds to linear time varying state feedback, which is well developed especially for systems whose dynamic representations are known totally or partly with known uncertainty bounds.
- ii) When the sliding mode starts, the error vector rapidly converges to origin and the system starts tracking the desired trajectory precisely. However, since the magnitudes of the entries of the error vector are very close to zero, the corresponding controller parameters do not receive sufficient excitation to maintain the synthesis of target control sequence. In implementing RBFNN,

SFS and ANFIS architectures, the designer will not need such an augmentation since the parameter vector is persistently excited by the hidden neuron outputs of RBFNN or rule outputs of FL based structures.

A last remark here is on the applicability of FNN structure, to which solely the method in subsection 3.6 is applicable among what we have discussed.

4 An Illustrative Example

This section demonstrates the performance of the algorithm discussed in subsection 3.4 for a third order system studied previously by Roy et al [54] and Yilmaz et al [55]. The dynamic equation describing the system is given in (22).

$$\begin{aligned} \theta^{(3)} = & -0.5\theta - 0.5\dot{\theta}^3 - 0.5\ddot{\theta}|\ddot{\theta}| + \left(1 + 0.1\sin\left(\frac{\pi t}{3}\right)\right)\tau + \kappa_1(t) + \kappa_2(t) + \\ & (-0.05 + 0.25\sin(5\pi t))\theta + (-0.03 + 0.3\cos(5\pi t))\dot{\theta}^3 + \\ & (-0.05 + 0.25\sin(7\pi t))\ddot{\theta}|\ddot{\theta}| \end{aligned} \quad (22)$$

where $\kappa_1(t) = 0.2\sin(4\pi t)$ is the disturbance used in [54-55], and $\kappa_2(t)$ is the zero mean Gaussian noise corrupting the state information to be used by the controller additively. The work presented by Roy et al assume that the nominal system dynamics is known and the uncertain part is comprised of what we give as the last three terms in (22). The primary difference between what has been discussed so far and what is assumed in [54] should be stressed as the approaches we discuss only assume the achievability of the VSS task, hence the uncertainties are represented in the system dynamics, whose form is known but the details are not. As the controller, a three input single output ANFIS structure is used and the tuning is performed only on the defuzzifier parameters, which are initially set to zero. The rule base has 27 rules quantifying the relevant input variable as Negative, Zero or Positive. Once the rule outputs are evaluated, the crisp decision of the controller is computed as described in (1).

Parallel to [54], the reference state trajectory, which is described as $\theta_d = 0.5\cos(\pi t/5)$ is used in the simulations. Initially, the states of the system have the following values, $\theta(0) = 1$, $\dot{\theta}(0) = 1$ and $\ddot{\theta}(0) = 1$. One important note here should be on the selection of λ . The value is taken as 5 in [54]; however we use $\lambda = 1$, because the behavior with this value results in a better system response. Figure 5 illustrates the trajectory followed in the phase space. The error vector hits the sliding surface several times and starts moving on it as enforced by the algorithm.

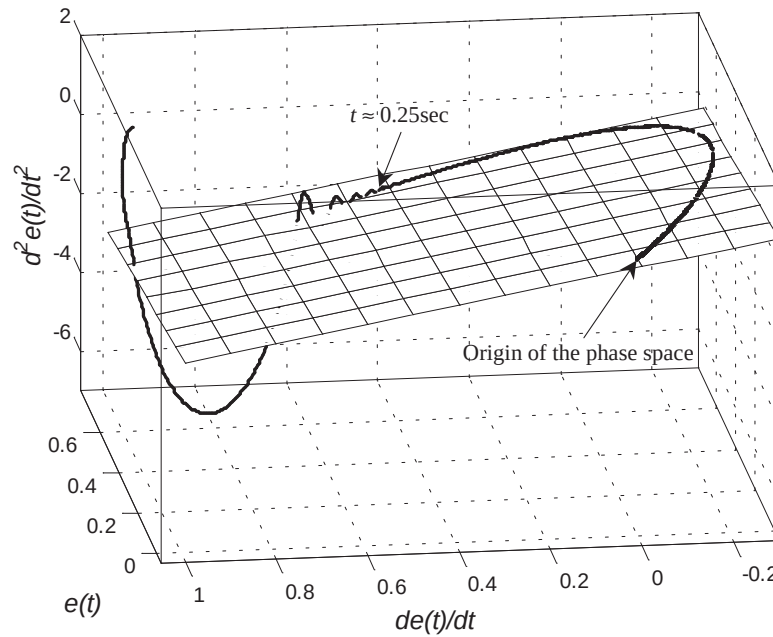


Fig. 5. Behavior in the phase space

5 Computational Intelligence in Variable Structure Control

What we have focused on so far mainly contemplates the use of VSS theory for parameter adaptation in CI. However, the integration of VSS technique with the architectural and algorithmic methods of CI can also be utilized in

- Chattering elimination through filtering [48,56]
- Design of the parameters of a conventional sliding mode controller [48,57-58]
- Modeling of the uncertainties [48,59-61]
- Generating a complementary control action [48,62-63]
- Generating the equivalent control and corrective control actions separately [48,64].

The use of CI in VSS may be a remedy in the situations where the available knowledge is insufficient to produce a safe control action. The selection of the uncertainty bound in this respect constitutes an apparent example. As the value of the uncertainty bound increases, the produced control action is more likely to have high frequency components having high magnitude, which arise through the sign measurement during the sliding mode. In such a situation, CI supported schemes can offer smoothed control signals with a

reasonable uncertainty bound selection. Furthermore, the conventional framework may underestimate the actuation speed of the system under control and may lead to unnecessarily large control inputs. In these cases, the tuning of VSS parameters, e.g. the slope of the sliding line, can be designed using the methods of CI. Being not limited to these, the methods of CI can be used as auxiliary subsystems for improving the control signal, i.e. a complementary control signal is produced so that the undesired effects of conventional sliding mode controller can be reduced. Last but not the least, the components of the control signal driving the system behavior to a predefined sliding regime can separately be realized by a learning system. This can eventually result in a comprehensible way of formulating the equivalent control and the corrective control.

6 Conclusion

This study discusses the design of a VSS theory based training strategies for CI, when the traditional gradient based training approaches are utilized for which, some handicaps arise due to the imperfect modeling, noisy observations or time varying parameters. If the effects of these factors are transformed to the cost hypersurface, whose dimensionality is determined by the adjustable design parameters, it becomes evident that the surface may have directions along which the sensitivity derivatives assume large values. In these cases, gradient based optimization procedures tend to evaluate large parametric displacements, which can eventually lead to a locally divergent behavior. In control engineering practice, such a behavior constitutes a potential danger from a safety point of view. The approaches presented in this work take care of the mentioned difficulties. Since the VSS theory is well known with its robustness property, a training strategy equipped with which retains a high degree of robustness against disturbances and uncertainties. When these approaches are considered for the training of intelligent controllers, under the assumption that the VSS task is achievable, the task is fulfilled without knowing the analytic details describing the plant dynamics. In order to corroborate the performance claims, tracking control of a third order nonlinear system is presented. The behavior in the phase space clearly demonstrates the superior performance despite the unavailability of system-specific details.

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Sliding Mode Control for Systems with Fast Actuators: Singularly Perturbed Approach

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Abstract. Singularly perturbed relay control systems with second order sliding modes are considered for the modeling of sliding mode control systems with fast actuators. For sliding mode control systems with fast actuators, sufficient conditions for the exponential decreasing of the amplitude of chattering and unlimited growth of frequency are found. The connection between the stability of actuators and the stability of the plant on the one hand and the stability of the sliding mode system as the whole on the other hand is investigated. The algorithm for correction of sliding mode equations is suggested for taking into account the presence of fast actuators. Algorithms are proposed to solve the problem of eigenvalues assignment or optimal stabilization for sliding motions using the additional dynamics of fast actuators.

1 Introduction

The chattering phenomenon is one of the major problems in modern sliding mode control (see for example [2], reference in [17], [18]). The presence of fast actuators is one basic reasons for chattering occurring in sliding mode control systems ([3],[17], and [18]). A specific feature of a sliding mode system with fast actuators is the following: a relay control is transmitted to the input of the actuator and the continuous actuator's output is transmitted to the input of the plant (see fig. 1). In [2] it was shown that the behavior of sliding

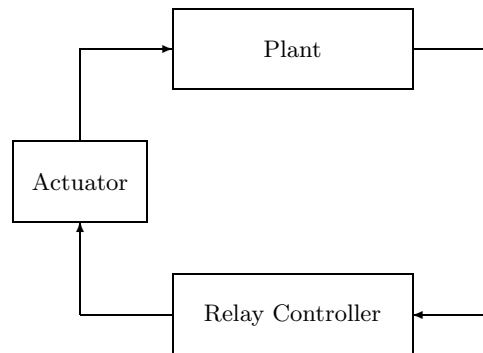


Fig. 1. Control system with actuator

mode systems with fast actuators is described by singularly perturbed relay control systems with higher order sliding modes and the order of sliding is the sum of a relative degrees of the plant and the actuator (see the definition of sliding order in the chapter of Bartolini and Levant in this book).

The main specific features of relay control systems with higher order sliding are the followings (see [1],[6]):

- second order sliding modes could be asymptotically stable;
- sliding modes of the order three and more are unstable, but a stable periodic oscillation can occur.

The chattering phenomenon for sliding mode control systems with fast actuators, whose behaviour is described by singularly perturbed relay control systems with the order of sliding three and more was analyzed in [7],[8] from the view point of averaging. This chapter is devoted to analysis of the chattering phenomenon in sliding mode control systems with fast actuators given by singularly perturbed relay control systems with second order sliding modes (SPRCSSOSM).

These specific features of SPRCSSOSM have determined the motivations of the chapter:

1. Design the mathematical tools for investigating SPRCSSOSM.

The motions in SPRCSSOSM have an infinite number of switches and the time intervals between switches tend to zero. That is why for SPRCSSOSM it is impossible to use classical methods of singular perturbations theory ([11],[19]). In Section 2 the following mathematical tools for investigation of SPRCSSOSM are developed:

- sufficient conditions for the exponential decreasing of the amplitude of chattering and the unlimited growth of frequency are found;
- the reduction principal theorem is proved in which the sufficient conditions of the equivalence for the stability of slow motions of plants and the stability of original systems with an actuator are found;
- it is shown that the asymptotically stable slow-motions integral manifold of a smooth singularly perturbed system, describing the motion of original SPRCSSOSM in the second order sliding domain, is the asymptotically stable slow-motions integral manifold of the original SPRCSSOSM;
- an algorithm for the asymptotical representation of a slow motions integral manifold is suggested.

2. Find the stability conditions for sliding mode control systems with fast actuators.

It is well known for smooth fast actuators systems (see, for example [11]) that the actuator's stability and the stability of plant are not enough for the stability of system at whole. In Section 3 the analysis is made of chattering in sliding mode control systems with fast actuators. In subsection 3.3 the connection between the stability of the actuators and the stability of the plant from the one hand and the stability of the sliding mode system as a whole from the other hand is investigated. It is shown (subsection 3.2) that the definition of the motions in sliding mode, according to the

equivalent control method, corresponds to the presence of fast actuators in the control system.

3. Show that it's possible to design chattering-free sliding mode control systems with fast actuators in the case when the order of sliding in a complete model of the system is 2. For sliding mode control systems with fast actuators with second order sliding, the sufficient conditions for the exponential decreasing of the amplitude of chattering and unlimited growth of frequency are found in Subsection 2.2.

4. Obtain the algorithms for correction of sliding mode equations for taking into account the presence of fast actuators in control systems. It is well known (see for example [11]) that for a wide class of control systems with fast actuators the correction of slow motion is useful in order to design a control system with better accuracy. The algorithm for correcting the sliding mode equations is suggested in subsection 3.1 for taking into account the presence of fast actuators. In subsection 3.4 it is shown, that whenever the sliding motions of the plant are stable, but not asymptotically stable, *it is obligatory to make a correction* to the sliding mode equations taking into account the presence of fast actuators in the system.

5. Suggest a control algorithm, which allows for ensuring the desired behavior of a system in the sliding domain, using actuator dynamics.

One of the popular control design approaches for smooth systems with fast actuators is the composite control method (see [11]), which guarantees the desired properties of slow or fast motions to be achieved. In Sections 4 and 5 control algorithms are suggested for solving the problems of eigenvalue assignment or optimal stabilization for sliding motions in systems with fast actuators.

These results leads to the following conclusion: *it is possible to save one order of derivative to obtain a sliding mode system without chattering*. In fact, one of the basic methods to avoid chattering is the design of switching surfaces, which include actuator variables and for having the first order sliding mode in the complete model of system (see, for example [17],[18],[14]). But it's not easy to measure the actuator variables and to obtain those values from differentiators. As it was shown in [12], if μ is the actuator time constant, ε is the noise amplitude, and k is the order of sliding for the i -th derivative, we can have an accuracy of not more than $(\frac{\varepsilon}{\mu})^{(k-i)/k}$. That is why if we can save one order of derivative, we improve the accuracy of the system by at least of factor of $(\frac{\varepsilon}{\mu})^{1/k}$.

For electromechanical or hydraulic systems ([18], [10]) typical actuators are of relative degree two. For this case the corresponding complete model of the sliding mode control system can be described by a singularly perturbed systems with order of sliding three. In this case the following scenario is useful:

- Find with a differentiator the value of the input derivative and include it

into the equation of the switching surface;

- Satisfy the sufficient conditions for the exponential decreasing of the chattering amplitude and unlimited growth of frequency for the complete model of the system.

2 Mathematical Tools

2.1 Problem Formulation

The general model of a sliding mode control system with a fast actuator has the following form (see [3],[5])

$$\mu dz/dt = f(t, z, s, x, u(s)), \quad ds/dt = g_1(t, z, s, x), \quad dx/dt = g_2(t, z, s, x), \quad (1)$$

where $s \in R$, $x \in R^n$ are variables describing the behaviour of the plant, $z \in R^m$ is vector describing the behaviour of the actuator, $u(s) = \text{sign}(s)$ is a relay control, f, g_1, g_2 are sufficiently smooth functions of their arguments, μ is the actuator time constant. The specific feature of the system (1) is the following: the equations for plant's variables s, x in (1) don't contain the relay control $u(s)$ but it is included in equations for the fast variable z describing actuator dynamics. This means that there is no first order sliding mode in the system (1) and only the second order sliding mode can occur.

Ignoring the dynamics of the actuator, i.e. having accepted $\mu = 0$ and expressing z from the equation $f(z_0, s, x, u(s)) = 0$ according to the formula $z_0 = \varphi(s, x, u(s))$, we obtain the reduced system

$$\begin{aligned} ds/dt &= g_1(t, \varphi(s, x, u(s)), s, x) = F_1(t, s, x, u(s)), \\ dx/dt &= g_2(t, \varphi(s, x, u(s)), s, x) = F_2(t, s, x, u(s)). \end{aligned} \quad (2)$$

Here we suppose that for system (2) the sufficient conditions for existence of a stable sliding mode

$$F_1(t, 0, x, 1) < 0, \quad F_1(t, 0, x, -1) > 0 \quad (3)$$

hold and the motion into this mode are described by the equations of equivalent control method (see for example [17])

$$dx/dt = F_2(t, 0, x, u_{eq}(t, x)), \quad F_1(t, 0, x, u_{eq}(t, x)) = 0. \quad (4)$$

This means that for the original system (1) and the reduced system (2) two qualitatively different kinds of motion occur:

- in the original system (1) there is no first order sliding mode,
- however in the reduced system (2), the sufficient conditions for its existence are held.

For the relay system (1) it's impossible to use classical methods of singular perturbation based on separation of the spectrum in slow and fast parts, due to the infinite number of switches (see [1], [5]). On the other hand the motion

of (1) in the second order sliding mode are described by the smooth singularly perturbed system. Under some conditions (see for example [11]) this smooth system has an asymptotically stable slow-motion integral manifold. In this section we will show that this manifold is the slow-motion integral manifold for the original system (1) and prove the reduction principle theorem to reduce the problem of investigating stability in the system (1) to the stability problem for (4).

2.2 Decomposition Theorem

System Transformation into a Convenient Form for the Analysis

We shall develop the mathematical tools for the case, when solutions of the relay control system (1) are determined uniquely. It's true (see [11]) for a wide class of such systems in which f is linearly depending on function $U(t, x, u(s))$ which satisfies the inequality

$$U_1|s| < sU(t, x, u(s)) < U_2|s|, (U_2 > U_1 > 0)$$

for all (t, s, x) . Let us make three substitutions of variables in system (1).

1. Here we consider the case when, in system (1), there exists a stable second order sliding mode. In such case $g'_{1_z}(t, z, s, x) \neq 0$. Suppose that z_m is the last coordinate for the vector z and $g'_{1_{z_m}}(t, z, s, x) \neq 0$. Then we can introduce the variable $\sigma = ds/dt = g_1(t, z, s, x)$ instead of z_m in the system (1).

2. It's reasonable to consider sliding mode control systems with a stable fast actuator. From a mathematical viewpoint this means that, according to the boundary layer method (see for example [19]) and conditions (3) ensuring the existence of a stable first order sliding mode, one can conclude that the solution of (1) starting far from switching surface $s = 0$ will reach the neighbourhood of the switching surface with radius $O(\mu)$ after a finite time. It allows us to examine only solutions of (1), starting in the neighbourhood of the switching surface with radius $O(\mu)$. That's why it's possible to introduce the new variable $\xi = s/\mu$ instead of variable s in (1). After this the system (1) takes the form

$$\begin{aligned} \mu d\bar{z}/dt &= f_1(t, \bar{z}, \sigma, \mu\xi, x) + d(t, x)U(t, x, u(\xi)), \\ \mu d\sigma/dt &= f_2(t, \bar{z}, \sigma, \mu\xi, x) + b(t, x)U(t, x, u(\xi)), \quad \mu d\xi/dt = \sigma, \\ dx/dt &= g_2(t, \bar{z}, \sigma, \mu\xi, x). \end{aligned} \quad (5)$$

3. Let us eliminate relay control from the first equation of the system (5). In fact, after the substitution of variables $\bar{\eta} = \bar{z} - d(t, x)\sigma/b(t, x)$, system (1) takes the form

$$\begin{aligned} \mu d\bar{\eta}/dt &= v_1(t, \bar{\eta}, \sigma, \mu\xi, x, \mu), \\ \mu d\sigma/dt &= v_2(t, \bar{\eta}, \sigma, \mu\xi, x, \mu) + b(t, x)U(t, x, u(\xi)), \quad \mu d\xi/dt = \sigma, \\ dx/dt &= v_3(t, \bar{\eta}, \sigma, \mu\xi, x). \end{aligned} \quad (6)$$

The specific feature of system (6) is that only the second order sliding mode can occur in it, and the motion in this mode is determined by the equations

$$\mu d\bar{\eta}/dt = v_1(t, \bar{\eta}, 0, 0, x, \mu), \quad dx/dt = v_3(t, \bar{\eta}, 0, 0, x, \mu). \quad (7)$$

System (7) has an asymptotically stable slow-motion integral manifold $\eta = h(t, x, \mu)$ if the following conditions are held:

I. Equation $v_1(t, \bar{\eta}, 0, 0, x, 0) = 0$ has an isolated solution $\bar{\eta} = h_0(t, x)$ at all $(t, x) \in R \times R^n$.

II. Functions v_i ($i = 1, 3$), h_0 have second order continuous derivatives in the domain

$$\bar{\Omega} = \{(t, \bar{\eta}, x, \mu) \in R \times R^{m-1} \times R^n \times (0, \mu_0) : |\bar{\eta} - h_0(t, x)| < \delta\},$$

where $\delta > 0$, $|\cdot|$ is the euclidean norm.

III. For all $(t, x, \mu) \in R \times R^n \times (0, \mu_0)$

$$Re \operatorname{Spec} \partial v_{\bar{\eta}}(t, h_0(t, x), 0, 0, x, \mu) / \partial \eta < -\kappa \leq 0$$

After the substitution of variables $\eta = \bar{\eta} - h(t, x, \mu)$ and expansion in the series towards to η, σ, ξ degrees at the point $(0, 0, 0)$, the system (6) takes the form

$$\begin{aligned} \mu d\eta/dt &= B_{11}(t, x, \mu)\eta + B_{12}(t, x, \mu)\sigma + \\ &+ \mu B_{13}(t, x, \mu)\xi + \varphi_1(t, \eta, \sigma, \mu\xi, x, \mu), \\ \mu d\sigma/dt &= B_{21}(t, x, \mu)\eta + B_{22}(t, x, \mu)\sigma + \mu B_{23}(t, x, \mu)\xi + \\ &+ \varphi_2(t, \eta, \sigma, \mu\xi, x, \mu) + b(t, x)U(t, x, u(\xi)), \quad \mu d\xi/dt = \sigma, \end{aligned} \quad (8)$$

$$dx/dt = \varphi_3(t, \eta, \sigma, \mu\xi, x), \quad (9)$$

where $\varphi_1(t, \eta, 0, 0, x, \mu) = 0$ and by $y = (\eta, \sigma, \xi) \rightarrow 0$ the following conditions hold $\varphi_2(t, \eta, \sigma, \mu\xi, x, \mu) = \varphi_2(t, 0, 0, 0, x, \mu) + o(|y|)$, $\varphi_1(t, \eta, \sigma, \mu\xi, x, \mu) = o(|y|)$ everywhere in $\Omega = \{(t, \eta, x, \mu) \in R \times R^{m-1} \times R^n \times (0, \mu_0) : |\eta| < \delta\}$.

Exponential Stability of Fast Motions

Let's denote $|y|_* = \sqrt{|\eta|^2 + |\sigma|^2 + |\xi|^2}$. Suppose that for all $(t, x) \in R \times R^n$ conditions I-III are satisfied and, moreover,

IV. $B_{22}(t, x, 0) < -\alpha < 0$, $b(t, x) < -\alpha < 0$, $|\varphi_2(t, 0, 0, 0, x, \mu)| < \alpha U_1$.

In appendix 1 the following lemma is proved.

Lemma 1. *If conditions I-IV are true, there exist constants $K_1 > 0, K_2 > 0, \gamma > 0$ and W some neighbourhood of the origin in the state space of variables $y = (\eta^T, \sigma, \xi)^T$, such that for all $(t_0, y_0, x_0) \in \Omega' = R^+ \times W \times R^n$ the following inequality holds*

$$|y(t, \mu)|_* \leq K_1 |y_0|_* e^{\gamma(t-t_0)/\mu} \leq K_2 e^{-\gamma(t-t_0)/\mu}. \quad (10)$$

Decomposition Theorem

Consider a solution of (8),(9) only starting in Ω' . Then the $x(t, \mu)$ coordinate of the solution (8),(9) will be a solution of the initial problem

$$dx/dt = \Phi(t, y(t, \mu), x, \mu), x = x_0,$$

$$\Phi(t, y(t, \mu), x, \mu) = \varphi_3(t, \eta(t, \mu), \sigma(t, \mu), \mu\xi(t, \mu), x(t, \mu)).$$

Let us represent $x(t, \mu)$ as $x(t, \mu) = \bar{x}(t, \mu) + \pi x(t, \mu)$ such that $\bar{x}(t, \mu), \pi x(t, \mu)$ are solutions of equations

$$d\bar{x}/dt = \Phi(t, 0, \bar{x}, \mu), \bar{x}(t_0) = \bar{x}_0, \quad (11)$$

$$d\pi x/dt = \Phi(t, y(t, \mu), \bar{x} + \pi x, \mu) - \Phi(t, 0, x, \mu), \quad (12)$$

$$\pi x(t) = \pi_0 x, \bar{x}_0 + \pi_0 x = x_0. \quad (13)$$

To define the solutions of the problems (11) - (13) it is necessary to choose $(x_0, \pi_0 x)$. The following theorem shows that $(x_0, \pi_0 x)$ can be chosen in such a way that function $\pi x(t, \mu)$ exponentially decreases.

Theorem 1. *Suppose that for all $(t, y, x), (t, \bar{y}, \bar{x}) \in \Omega'$ conditions I – IV are true, inequality*

$$|\Phi(t, y, x) - \Phi(t, \bar{y}, \bar{x})| < M(|y - \bar{y}| + |x - \bar{x}|),$$

$$M = \sup\{\max_{(t, y, x, \mu) \in \hat{\Omega}} [|d\Phi(t, y, x, \mu)/dx|; |d\Phi(t, y, x, \mu)/dy|]\},$$

where $\hat{\Omega} = \mathbf{R}^+ \times \mathbf{R}^{m+1} \times \mathbf{R}^n \times [0, \mu_0]$, is satisfied and

$$\mu M/\gamma < 1, \quad KM/(\gamma - \mu M) < C. \quad (14)$$

Then for any initial points $(t_0, y_0, x_0) \in \Omega'$ the solutions of the system (8),(9) can be represented as slow and fast parts in the form:

$$(y(t, \mu), x(t, \mu)) = (0, \bar{x}(t, \mu)) + (\pi y(t, \mu), \pi x(t, \mu)).$$

So $\bar{x}(t, \mu)$ is the solution of equation (11) with initial conditions $\bar{x}(0) = \bar{x}_0$ while $x_0 = \bar{x}_0 + O(\mu)$. The fast part of this solution $\{\pi y(t, \mu), \pi x(t, \mu)\}$ satisfies the inequality

$$\mu|\pi y(t, \mu)| + |\pi x(t, \mu)| < \mu(C + K)e^{-\gamma(t-t_0)}. \quad (15)$$

This theorem is proved in Appendix 2.

Reduction Principle

Theorem 1 and inequality (15) yield the following reduction principle theorem.

Theorem 2. *If under the conditions of the theorem 1 the function $\bar{x}(t, \mu)$ is the solution of the system (11) then $(0, 0, 0, \bar{x}(t, \mu))$ is the solution of the system (8), (9) and this solution will be stable (unstable, asymptotically stable) if and only if $\bar{x}(t, \mu)$ is stable (unstable, asymptotically stable).*

3 Analysis of Chattering in Sliding Mode Control Systems with Fast Actuators

3.1 Algorithm of Direct Decomposition

The results of section 2 have justified the following algorithm for correction of equivalent control method for taking into account the presence of fast actuators in sliding mode control systems.

Step 1. Design of algebraic-differential equations for description of motions in (1) in the second order sliding mode.

Suppose that the stable second order sliding mode exists in the system (1). Then motions in this mode are determined by equations of the equivalent control method

$$\begin{aligned}\mu dz/dt &= f(t, z, 0, x, u_{eq}(t, z, x, \mu)) = \hat{f}(t, z, x, \mu) \\ dx/dt &= g_2(t, z, 0, x),\end{aligned}\quad (16)$$

$$g_1(t, z, 0, x) = 0, \quad (17)$$

$$d^2s/dt^2(t, z, 0, x, u_{eq}, \mu) = g'_{1_z} f/\mu + g'_{1_\xi} g_1 + g'_{1_x} g_2|_{(t, z, 0, x, u, \mu)} = 0. \quad (18)$$

Step 2. Design of differential equations for the description of motions in (1) in the second order sliding mode. Let us express one of the vector z coordinates from equation (17). Let it be, for example, its last coordinate z_m , and the corresponding expression has the form $z_m = p(t, \bar{z}, x)$, where $\bar{z} \in R^{m-1}$ is the vector consisting of the first $(m-1)$ coordinates of the vector z . Then system (16) may be represented in the form

$$\mu d\bar{z}/dt = \bar{f}(t, \bar{z}, x, \mu), \quad dx/dt = \bar{g}_2(t, \bar{z}, x), \quad (19)$$

where $\bar{f}$ consists of the first $(m-1)$ coordinates of function $\hat{f}$ at the point $(t, \bar{z}, p(t, \bar{z}, x), x, \mu)$.

Step 3. Design of corrected equations of the equivalent control method.

System (19) is a smooth singularly perturbed system. If in such systems the fast variable are uniformly exponentially stable, there exists the slow-motions integral manifold in the following form: $\bar{z} = h(t, x, \mu)$. Motion on that manifold is described by equations

$$d\bar{x}/dt = \bar{g}_2(t, h(t, \bar{x}, \mu), \bar{x}), \quad \bar{z} = h(t, \bar{x}, \mu) \quad (20)$$

According to theorem 1, the x coordinate of the solutions of (1) will differ from the solutions of equations (20) up to the fast decreasing exponent. In this sense, slow motion in (19) is precisely described by equations (20), and we shall call equations (20) precise equations of the equivalent control method.

Function $h(t, x, \mu)$ could be expressed as an asymptotic series $h(t, x, \mu) = \sum_0^\infty \mu^k h_k(t, x)$ from the equation

$$\mu[h'_t + h'_x g_2(t, h(t, x, \mu), x)] = \bar{f}(t, h(t, x, \mu), x, \mu). \quad (21)$$

Function $h_0(t, x)$ is determined by equation

$$f(t, h_0, x, 0) = 0. \quad (22)$$

This means that for $\mu = 0$, equation (20) coincides with the equivalent control method equation (4). With $\mu = 0$ equation (20) differs from equation (4) only in the terms, which correspond to the presence of fast actuators in the original system (1). From theorem 2 it follows that the problems of investigating the stability for the zero solution of systems (1) and (20) are equivalent.

3.2 The Systems Containing The Relay Control Nonlinearly

Consider the control system

$$ds/dt = -u, \quad dx/dt = (u^2 - 1)x, \quad x, s \in \mathbf{R}, \quad u(s) = \text{sign}(s), \quad (23)$$

containing the relay control $u(s)$ nonlinearly. There is a stable sliding mode in the system (23). Defining solutions in the sliding domain (23) are not unique. For example, on the one hand, extension (23) of this definition of into the sliding mode according A.F. Filippov [4] takes the form $dx/dt = x$ with an unstable zero solution. On the other hand, extension of the definition of (23) into the sliding mode according to the equivalent control method takes the form

$$dx/dt = -x \quad (24)$$

with an asymptotically stable zero solution.

Suppose that a relay control is transmitted to the plant via a fast actuator and a complicated model of a system, taking into account the presence of a fast actuator, has the form

$$\mu dz/dt = -z - u, \quad ds/dt = z, \quad dx/dt = (2z^2 - 1)x, \quad (25)$$

where $z \in \mathbf{R}$ is the actuator variable and μ is the actuator time constant. For system (25) theorems 1 and 2 are true. This means that the fast variables z, s are exponentially decreasing (fig. 2, fig. 3), equation (24) of the equivalent control method is approximately described by the slow motions in system (25) and the zero solution of system (25) is asymptotically stable (fig. 4).

3.3 Stability of Actuators and Absence of Chattering

In this subsection we investigate the correlation between the natural conditions of stability of fast actuators in sliding mode control systems and the existence of the stable first order sliding mode for a reduced system, describing the behaviour of the plant without actuator on the one hand, and sufficient conditions for exponential decreasing of fast oscillations (absence of chattering) in the original system on the other hand.

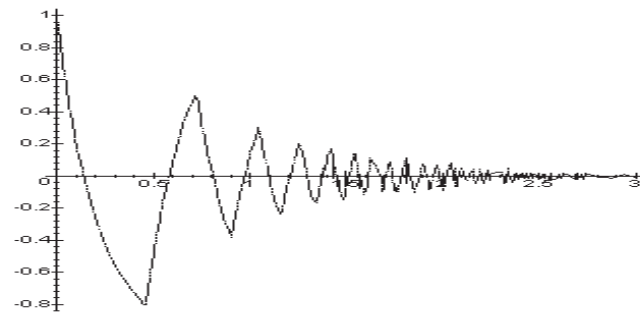


Fig. 2. Exponential decreasing of z

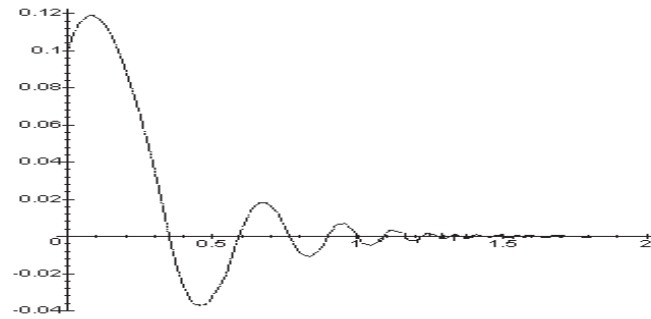


Fig. 3. Exponential decreasing of s

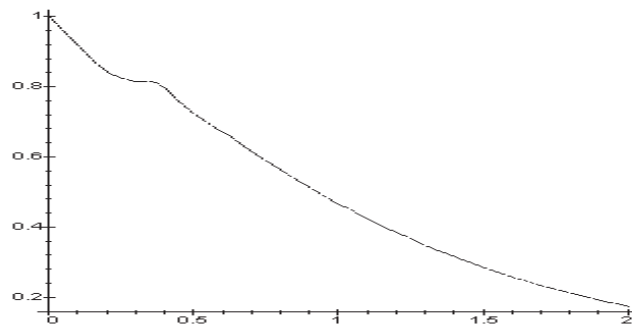


Fig. 4. Stability of slow variable x .

Consider the simplest case, when the behaviour of the plant is described by the equation

$$dx/dt = Ax + Bu, x \in \mathbf{R}^n, u = \text{sign}(s), s = Cx \in \mathbf{R}. \quad (26)$$

Suppose that the relay control u ensures the stable first order sliding mode on the switching surface $s = 0$, and consequently

$$CB < 0. \quad (27)$$

Consider the case when relay control is transmitted to the plant via a fast actuator, with behaviour described by equation

$$\mu dz/dt = Dz + Fx + bu(s), z \in \mathbf{R}^m, \quad (28)$$

where μ is the actuator time constant. This means that the system model taking into account the presence of a fast actuator has the form

$$\mu dz/dt = Dz + Fx + bu(s), dx/dt = Ax + BKz. \quad (29)$$

It is natural to suppose that:

- the actuator is stable which means that

$$\text{Re Spec } D < 0, \quad (30)$$

- the system (28),(29) for $\mu = 0$ turn to equation (26) and consequently

$$KD^{-1}b = -1, \quad -CBKD^{-1}b = CB < 0. \quad (31)$$

Transform the system (28),(29) to the canonic form (see section 2)

$$\begin{aligned} \mu dz_1/dt &= D_{11}z_1 + D_{12}\sigma + F_{11}s + F_{12}x_1, \\ \mu d\sigma/dt &= D_{21}z_1 + D_{22}\sigma + F_{11}s + F_{12}x_1 + du(s), \\ ds/dt &= \sigma, dx_1/dt = B_{11}z_1 + B_{12}\sigma + A_{33}s + A_{34}x_1, \\ z_1 &\in \mathbf{R}^{m-1}, x_1 \in \mathbf{R}^{n-1}, \sigma \in \mathbf{R}. \end{aligned} \quad (32)$$

For system (32) the conditions of stability of the second order sliding mode are

$$d < 0, \quad D_{22} < 0. \quad (33)$$

Inequality

$$\text{Re Spec } D_{11} < 0 \quad (34)$$

ensure exponential decreasing of actuator variables in the second order sliding domain. The following proposition is obvious.

Proposition 1. *When actuator is SISO system ($m = 1, z \in \mathbf{R}$) and the condition of stability of fast actuator (30) and conditions of existence of stable first order sliding mode for the reduced system (26) are held chattering in the system σ and s is exponentially decreasing.*

But it is not true just for $m = 2$. Condition (31) for the system (32) means that

$$(0\ 1)D^{-1}\begin{pmatrix} 0 \\ d \end{pmatrix} = d \frac{\det(D_{11})}{\det D} > 0.$$

Now from condition (30) follows that $\det D > 0$. Conditions (31),(30) mean that d and $\det(D_{11})$ have the same sign. This means, that for $m = 2$, conditions (31),(30) do not ensure exponential decreasing of chattering.

Proposition 2. *Let $m = 2$. If conditions (31), (30), and $D_{11} < 0$ or $d < 0$ are held the chattering in system (29), (28) is absent.*

The stability of the fast actuator (30) and of the second order sliding mode in (33) still does not guarantee the absence of chattering if $\dim z > 1$. In this case fast oscillations may still remain in the 2-sliding mode itself. Consider the system

$$\begin{aligned} \mu dz_1/dt &= z_1 + z_2 + \eta + D_1x, \quad \mu dz_2/dt = 2z_2 + \eta + D_2x, \\ \mu d\eta/dt &= 24z_1 - 60z_2 - 9\eta + D_3x + k \operatorname{sign} s, \\ ds/dt &= \eta, \quad dx/dt = F(z_1, z_2, \eta, s, x), \end{aligned}$$

where z_1, z_2, η, s are scalars, $k < 0$. It is easy to check that the spectrum of the matrix is $\{-1, -2, -3\}$ and condition (33) hold for this system. On the other hand motions in the second order sliding mode are described by the system

$$\begin{aligned} \mu dz_1/dt &= z_1 + z_2 + D_1x, \quad \mu dz_2/dt = 2z_2 + D_2x, \\ dx/dt &= F(z_1, z_2, 0, 0, x). \end{aligned}$$

The fast motion in this system are unstable and the absence of chattering in the original system cannot be guaranteed.

3.4 When correction of the equivalent control method is obligatory?

Suppose that in the sliding mode control system the behaviour of the state vector $s, x \in \mathbf{R}$ is described by equation

$$ds/dt = -u(s), \quad dx_1/dt = x_2, \quad dx_2/dt = u(s) - x_1, \quad u(s) = -\operatorname{sign}(s). \quad (35)$$

There exists a stable first order sliding mode for system (35). The motion in sliding mode are described by equations

$$dx_1/dt = x_2, \quad dx_2/dt = -x_1. \quad (36)$$

It's obvious that the solutions of this system are stable but not asymptotically stable. Suppose that the relay control $u(s)$ is transmitted to the plant with

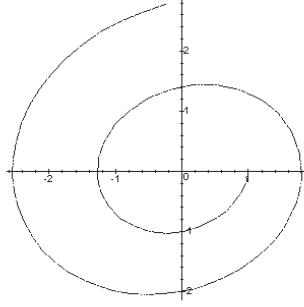


Fig. 5. Unstability of slow variables on $x_2=0$ plane for $a = -0.5$

the help of a fast actuator, whose behaviour is described by variables z_1, z_2 . The complete mathematical model of control system has the form

$$\begin{aligned} \mu dz_1/dt &= -z_1 - x_1, \quad \mu dz_2/dt = -z_2 - \text{sign}(s), \\ ds/dt &= z_2, \quad dx_1/dt = x_2, \quad dx_2/dt = (a+1)z_1 - z_2 + ax_1. \end{aligned} \quad (37)$$

It's easy to see that for system (37) theorems 1 and 2 are true and slow motion for (37) with precision level $o(\mu)$ is described by equations (36). On the other hand, motion in the second order sliding mode for system (37)

$$\mu dz_1/dt = -z_1 - x_1, \quad dx_1/dt = x_2, \quad dx_2/dt = (a+1)z_1 + ax_1. \quad (38)$$

Then the slow-motion manifold of systems (37) and (38) takes the form $z_1 = p_1(\mu)x_1 + p_2(\mu)x_2$, where

$$p_{ij}(\mu) = p_{i0} + p_{i1}\mu + \dots + p_{ik}\mu^k + \dots, \quad i = 1, 2.$$

The functions $p_{ij}(\mu)$ can be found from equation

$$\mu(p_1 \ p_2) \left(\begin{pmatrix} 0 \\ a+1 \end{pmatrix} (p_1 \ p_2) + \begin{pmatrix} 0 & 1 \\ a & 0 \end{pmatrix} \right) = -(p_1 \ p_2) - (1, 0), \quad (39)$$

and consequently $(p_{10} \ p_{20}) = (-1, 0)$, $(p_{11} \ p_{21}) = (0, 1)$. This means that the slow motion in system (37) is described by equations

$$dx_1/dt = x_2, \quad dx_2/dt = -x_1 + \mu(a+1)x_2 + O(\mu^2). \quad (40)$$

From theorems 1 and 2 it follows that variables s and $ds/dt = z_2$ are asymptotically decreasing, but for $a > -1$ the zero solution of the system (37) is unstable (fig. 5) and for the $a < -1$ this solution is asymptotically stable (fig. 6).

This means that in the case, when the spectrum of sliding mode equations is critical, the presence of fast actuators can change the behaviour of a system from stability to instability or asymptotic stability. One can conclude that for the investigation of stability in the critical case, the correction of sliding mode equations is obligatory.

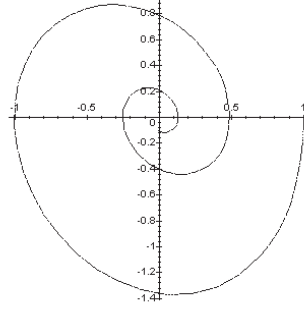


Fig. 6. Stability of slow variables on x_2 - x_1 plane for $a = -2$.

4 Additional Dynamics of Actuators in the Problem of Eigenvalue Assignment

4.1 Problem Statement

Consider the control system with the state vector (s, x) ($s \in R, x \in R^n$)

$$ds/dt = A_1s + A_2x + b_1u(s), \quad dx/dt = A_3s + A_4x + b_2u(s), \quad (41)$$

where s is the output of (41). Suppose that the control goal is for the output to go to zero. One of the simplest and most robust methods to reach this goal is to design the relay control in the form $u(s) = \text{sign}(s)$ and ensure the stable first order sliding mode on the surface $s = 0$. In this case the equation for equivalent control has the form $ds/dt = A_2x + b_1u_{eq} = 0$, and for $b_1 \neq 0$ $u_{eq} = -b_1^{-1}A_2x$. Then the sliding motions in system (41) are described by equations

$$dx/dt = (A_4 - b_2b_1^{-1}A_2)x. \quad (42)$$

In [17] two methods to solve the design problem were proposed for the desired equations of sliding mode:

- to extend the state space by using additional dynamics and to solve the problem of eigenvalue assignment in the extended state space;
- to include the derivatives of the variable s into the equation of the switching surface.

In [13], [14] the fast variable describing the behaviour of the fast actuator in the equation for the switching surface was introduced for motion control in singularly perturbed discontinuous control systems. This approach ensures the existence of a first order sliding mode in the overall system. For such systems the composite control method (see [11]) was used [9], [16]. These approaches needed the measuring of fast variables, which is hard in real systems.

In this section control algorithms are designed to ensure:

- the absence of chattering;
- the desired behaviour of system going into the sliding domain via the dynamics of fast actuators.

In the proposed approach only the slow coordinates of the state-vector are used for control design. Moreover, the design problems are solved in the space of sliding mode equations. On the other hand, the proposed algorithm is useful only in the case when the actuator is a MIMO system.

In [7] the problem of eigenvalue assignment is solved with the help of a similar approach for sliding mode systems with the fast actuators described by singularly perturbed relay control systems with the order of sliding three or more. But, in such systems fast periodic oscillations occur. That's why the averaging technique was used and the problem of eigenvalue assignment was solved with precision level $O(\mu)$.

Let us suppose that the complete model of a control system, taking into account the presence of fast actuator, has the form

$$\begin{aligned}\mu dz/dt &= B_1 z + B_2 s + B_3 x + dv \\ ds/dt &= B_4 z + B_5 s + B_6 x, \quad dx/dt = B_7 z + B_8 s + B_9 x\end{aligned}\quad (43)$$

where $z \in R^m, v \in R^l, \mu$ is the actuator time constant. Now we suppose that the conditions

$$\text{rank} \begin{pmatrix} B_4 \\ B_7 \end{pmatrix} \geq 2, \text{rank } d \geq 2, m \geq l \geq 2 \quad (44)$$

are held. Conditions (44) mean that the relay control is transmitted to the plant through an actuator, which is a MIMO system itself.

Ignoring actuator dynamics, having accepted $\mu = 0$ and expressing z according to the formulae $z_0 = -B_1^{-1}(B_2 s + B_3 x + dv)$, we obtain

$$\begin{aligned}ds/dt &= (B_5 - B_4 B_1^{-1} B_2) s + (B_6 - B_4 B_1^{-1} B_3) x - B_4 B_1^{-1} dv, \\ dx/dt &= (B_8 - B_7 B_1^{-1} B_2) s + (B_9 - B_7 B_1^{-1} B_3) x - B_7 B_1^{-1} dv.\end{aligned}\quad (45)$$

Let us suppose that in the case, when the control law has been designed in form $v = Ku(s)$ (K is constant vector), systems (45) and (41) coincide.

The proposed algorithm uses the singular correction of the equivalent control method. Now we propose to use the control law in the form

$$v = Ku(s) + w. \quad (46)$$

The motion on the slow-motion manifold in (43) is described by equation

$$dx/dt = (A_4 - b_2 b_1^{-1} A_2) x - [B_7 - b_2 b_1^{-1} B_4] B_1^{-1} dw. \quad (47)$$

If condition:

D.1. The pair $\{(A_4 - b_2 b_1^{-1} A_2); (B_7 - b_2 b_1^{-1} B_4) B_1^{-1} d\}$ is controllable is true, choosing the control vector for (47) in form $w = Lx$, we can solve the eigenvalue assignment problem for (47). This means, that we have proposed

an algorithm for the design of the control vector w , which allows us to solve the eigenvalue assignment problem for (47). System (47) approximately describes the slow motions in a small neighborhood of the switching surface of the system (43).

To have sufficient conditions for absence of chattering and to obtain the higher approximations, we transform system (43) into the canonical form. Suppose that the control law (46) in the control system (43) has been designed. Let us make three variable changes in system (43) according to Section 2.

1. Consider only solutions of (43) with initial conditions $s(0, \mu) = O(\mu)$ and introduce variable $\xi = s/\mu$ in system (43).
2. Introduce variable $\sigma = d\xi/dt$ instead one of vector z coordinates.
3. Eliminate the relay control $u(s)$ from the system equations except the equation for σ .

Then we obtain system (43) in the canonical form

$$\begin{aligned}\mu d\eta/dt &= F_{11}\eta + F_{12}\sigma + \mu F_{13}\xi + F_{14}x + d_1w, \\ \mu d\sigma/dt &= F_{21}\eta + F_{22}\sigma + \mu F_{23}\xi + F_{24}x + d_2(Ku(s) + w), \\ \mu d\xi/dt &= \sigma, dx/dt = F_{41}\eta + F_{42}\sigma + \mu F_{43}\xi + F_{44}x,\end{aligned}\quad (48)$$

where F_{ij} could only depend on μ . The motion in the second order sliding mode in (48) is described by equations

$$\mu d\eta/dt = F_{11}\eta + F_{14}x + d_1w, dx/dt = F_{41}\eta + F_{44}x. \quad (49)$$

With $\mu = 0$ we have $F_{41}F_{11}^{-1}F_{14} + F_{44} = A_4 - b_2b_1^{-1}A_2$, $-F_{41}F_{11}^{-1}d_1 = (B_7 - b_2b_1^{-1}B_4)B_1^{-1}d$ and following theorem is true.

Theorem 3. *Suppose that condition D.1 is held and for all $\mu \in [0, \mu_0]$ conditions*

D.2. Re Spec $F_{11} < 0$,

D.3. $F_{24} < 0$, $d < 0$

hold. Then the slow motion in system (43), within the accuracy of $O(\mu)$, is described by equation (47) and there exists matrix L_0 , which provides for the desired characteristic polygon of matrix $A_4 - b_2b_1^{-1}A_2 - (B_7 - b_2b_1^{-1}B_4)B_1^{-1}dL_0$.

In the case, when more than $O(\mu)$ precision level is needed, the following algorithm can be used. Suppose that

$$F_{ij} = F_{ij0} + \mu F_{ij1} + \mu^2 F_{ij2} + \dots$$

The behaviour of slow motion in (43) with a precision level up to the fast decreasing exponent is described by equations

$$dx/dt = F_{41}H(\mu) + F_{44}x. \quad (50)$$

Function $H(\mu)$ can be found in the form $H(\mu) = H_0 + \mu H_1 + \mu^2 H_2 + \dots$ from

$$\mu H(F_{41}H + F_{44}) = F_{11}H + F_{14} + d_1L. \quad (51)$$

Let us find matrix $L(\mu) = L_0 + \mu L_1 + \mu^2 L_2 + \dots$, such that the characteristic polynomial of matrix $(F_{41}H(\mu) + F_{44})$ in (50) has the form $\lambda^n + a_1(\mu)\lambda^{n-1} + a_2(\mu)\lambda^{n-2} + \dots + a_n(\mu)$, where $a_i = a_{i0} + \mu a_{i1} + \mu^2 a_{i2} + \dots$, $i = 1, \dots, n$.

Then $H_0 = -F_{110}^{-1}(F_{140} + d_1 L_0)$ and consequently the matrix L can be designed such as in theorem 3. In this case

$$H_1 = -F_{110}^{-1}[F_{141} + d_1 L_1 + F_{111}H_0 - H_0(F_{410}H_0 + F_{440})]$$

and matrix L_1 is designed as the matrix, ensuring that characteristic polynomial of matrix

$$\begin{aligned} & -F_{410}F_{110}^{-1}[F_{141} + F_{111}H_0 - H_0(F_{410}H_0 + F_{440})] + \\ & + F_{411}H_0 + F_{441} - F_{410}F_{110}^{-1}d_1 L_1 \end{aligned}$$

has the form $\lambda^n + a_{11}(\mu)\lambda^{n-1} + a_{12}(\mu)\lambda^{n-2} + \dots + a_{1n}(\mu)$. This means that L_i can be found under the condition

D.1.1. Matrices

$$-F_{410}F_{110}^{-1}[F_{141} + F_{111}H_0 - H_0(F_{410}H_0 + F_{440})] + F_{411}H_0 + F_{441}$$

and $(B_7 - b_2 b_1^{-1} B_4)B_1^{-1}d$ are controllable.

Let us suppose that matrices L_k ($k = 0, \dots, i-1$) have been found. Then we can linearly express H_i through L_i from (51). Substitution of this expression into (51) yields

$$\begin{aligned} dx/dt = & [A_4 - b_2 b_1^{-1} A_2 - (B_7 - b_2 b_1^{-1} B_4)B_1^{-1}dL_0 + \dots \\ & + \mu^i (\Phi_i(L_0, \dots, L_{i-1}) - (B_7 - b_2 b_1^{-1} B_4)B_1^{-1}dL_i)]x, \end{aligned}$$

where the Φ_i is the matrix depends on $L_0, \dots, L_{i-1}$, and, consequently, if matrices Φ_i and $(B_7 - b_2 b_1^{-1} B_4)B_1^{-1}d$ are controllable, then we can design the matrix L_i such that characteristic polynomial of matrix $\Phi_i(L_0, \dots, L_{i-1}) - (B_7 - b_2 b_1^{-1} B_4)B_1^{-1}dL_i$ has the form $\lambda^n + a_{i1}(\mu)\lambda^{n-1} + a_{i2}(\mu)\lambda^{n-2} + \dots + a_{in}(\mu)$. Thus the following theorem is true

Theorem 4. *Suppose that conditions of theorem 3 are held and matrices Φ_k and $(B_7 - b_2 b_1^{-1} B_4)B_1^{-1}d$ ($k = 0, \dots, i$) are controllable. Then there exist matrices L_k ($k = 0, \dots, i$) ensuring the design of a control law in the form $v = Ku(s) + (L_0 + \mu L_1 + \mu^2 L_2 + \dots + \mu^i L_i + \dots)x$, such that the spectrum of the matrix in equation (49) and roots of equation (51) differ by an error $O(\mu^{i+1})$.*

4.2 Example

Let us suppose that the state vector of the control system is described with equations

$$ds/dt = -u(s), dx_1/dt = x_2, dx_2/dt = u(s) - x_1, \quad (52)$$

where $s, x_1, x_2 \in R$ and the relay control $u(s) = -\text{sign}(s)$ has been designed. The motion in the sliding mode in (52) are described by equations

$$dx_1/dt = x_2, \quad dx_2/dt = -x_1. \quad (53)$$

The spectrum of the matrix in (53) is situated on the imaginary axis. Let us suppose that the relay control $u(s)$ is transmitted to the plant with the help of actuators, whose behaviour is described by variables z_1, z_2 . Then the complete model of the system has the form

$$\begin{aligned} \mu dz_1/dt &= -z_1 + v_1 - x_1, \quad \mu dz_2/dt = -z_1 + v_2, \\ ds/dt &= z_2, \quad dx_1/dt = x_2, \quad dx_2/dt = z_1. \end{aligned} \quad (54)$$

It can easily be seen that in this case, when we suppose that $v_1 = v_2 = -\text{sign}(s)$, the slow motion in it is described by system (53) with an error of $O(\mu)$. Now we shall find the control law in the form $v = -\text{sign}(s) + l_1 x_1 + l_2 x_2$. In such case the system (54) can be rewritten in the form

$$\begin{aligned} \mu dz_1/dt &= -z_1 - \text{sign}(s) + (l_1 - 1)x_1 + l_2 x_2, \\ \mu dz_2/dt &= -z_2 - \text{sign}(s), \\ ds/dt &= z_2, \quad dx_1/dt = x_2, \quad dx_2/dt = z_1. \end{aligned} \quad (55)$$

Then the equations describing the system (55) motion in the second order sliding mode have the form

$$\mu dz_1/dt = -z_1 + (l_1 - 1)x_1 + l_2 x_2, \quad dx_1/dt = x_2, \quad dx_2/dt = z_1. \quad (56)$$

System (56) has the slow motion integral manifold $z = h_1 x_1 + h_2 x_2$. Then the equation for h_1, h_2 takes the form

$$\mu(h_1 \ h_2) \left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} (h_1 \ h_2) + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right) = -(h_1 \ h_2) + (l_1 - 1 \ l_2) \quad (57)$$

and slow motion in (54) is described by equations

$$dx_1/dt = x_2, \quad dx_2/dt = h_1 x_1 + h_2 x_2. \quad (58)$$

Let's find such $l_1(\mu), l_2(\mu), h_1(\mu), h_2(\mu)$ as the asymptotic series

$$l_1(\mu) = l_{11} + \mu l_{22} + \dots, \quad l_2(\mu) = l_{21} + \mu l_{22} + \dots,$$

$$h_1(\mu) = h_{11} + \mu h_{12} + \dots, \quad h_2(\mu) = h_{21} + \mu h_{22} + \dots$$

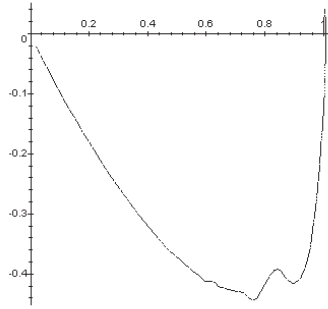


Fig. 7. Desired behavior of slow variables on x_2x_1 plain

such that characteristic polynomial of the system (58) takes the form $\lambda^2 + \alpha(\mu)\lambda + \beta(\mu)$, where

$$\alpha(\mu) = \alpha_0 + \mu\alpha_1 + \dots; \beta(\mu) = \beta_0 + \mu\beta_1 + \dots$$

Substitution of the asymptotic series into (57)

$$(h_{10}, h_{20}) = (l_{10} - 1, l_{20})$$

$$(h_{11}, h_{21}) = (-(l_{10} - 1)l_{20} + l_{11}, -(l_{10} - 1) - l_{20}^2 + l_{21}).$$

This means that choosing

$$l_{10} = 1 - \beta_0, \quad l_{20} = -\alpha_0$$

$$l_{11} = -\alpha_1 + (l_{10} - 1)l_{20}, \quad l_{21} = -\beta_1 + (l_{10} - 1) + l_{20}^2$$

we can provide that the spectrum of the matrix in equations (58) coincides with the roots of the desired polynomial with the accuracy of $O(\mu^2)$. Desired exponential decreasing of variables x_1, x_2 is shown in the figure 7.

5 Optimal Stabilization of Motions into the Sliding Mode

5.1 Problem Statement

Let us suppose that the state vector (s, x) in a control system is described by equations

$$ds/dt = A_1s + A_2x + b_1u(s), \quad dx/dt = A_3s + A_4x + b_2u(s), \quad (59)$$

and $u(s)$ is the relay control, which provides the existence of a stable first order sliding mode. Moreover, we will suppose that the behaviour of the control system, taking into account the presence of a fast actuator, is described by system (43); that conditions (44) are true; and we try to design the control in the form (46) to reach two goals:

- Provide fast exponentially decreasing of chattering to $s = 0$ in (43);
- Find an approximate solution to the optimal stabilization problem for functional

$$\int_0^\infty (x^*(t)Qx(t) + w^*(t)w(t))dt, \quad (60)$$

where Q is symmetrical matrix, for system trajectories of (59).

5.2 Main Theorem

Under conditions D.2,D.3 the difference between systems (43) and (49) is no more than in a fast decreasing exponent. This means that the min. of functional (60) for systems (43) and (49) are the same with a precision level of $O(\mu)$.

Let us follow the algorithm of the optimal stabilization problem (49),(60) provided in [15]. We will try to find the control law in the form

$$w = (d_1^* \ 0) \begin{pmatrix} K_1 & K_2 \\ K_2^* & K_3 \end{pmatrix} \begin{pmatrix} \eta \\ x \end{pmatrix}$$

where K_1, K_2, K_3 are solutions of the following system

$$\begin{aligned} -K_3F_{44} - F_{44}^*K_3 - K_2F_{14} - F_{14}K_2^* + K_2d_1d_1^*K_2 &= Q, \\ -K_1F_{11} - F_{11}^*K_1 + K_2d_1d_1^*K_1 - \mu(K_2^*F_{41} + F_{41}^*K_2) &= 0, \\ -K_3F_{41} - K_2F_{11} - F_{14}^*K_1 + K_2d_1d_1^*K_1 - \mu F_{44}^*K_2 &= 0. \end{aligned} \quad (61)$$

Having accepted $\mu = 0$ in (61), we can obtain $\tilde{K}_1 = 0$, $\tilde{K}_2 = -\tilde{K}_3F_{41}F_{11}^{-1}$. This means that $\tilde{K}_3$ is a solution of Riccati equation

$$\begin{aligned} -\tilde{K}_3(F_{44} - F_{41}F_{11}^{-1}F_{14}) - (F_{44} - F_{41}F_{11}^{-1}F_{14})^*\tilde{K}_3 + \\ + \tilde{K}_3(F_{41}F_{11}^{-1}d_1)(F_{41}F_{11}^{-1}d_1)^*\tilde{K}_3 = Q. \end{aligned}$$

Assume

D.4. The pair $\{q, F_{44} - F_{41}F_{11}^{-1}d_1\}$, where $q^*q = Q$, is observable.

If the condition D.4 holds the unique solution of equation for K exists and the following theorem is true.

Theorem 5. *Let us suppose that for system (41) for all $\mu \in [0, \mu_0]$ the conditions D.1-D.4 are held. Then we can find the solution of the problem of optimal stabilization (43),(60) in the form*

$$w = (d_1^* \ 0) \begin{pmatrix} 0 & -\tilde{K}_3F_{41}F_{11}^{-1} \\ (-\tilde{K}_3F_{41}F_{11}^{-1})^* & \tilde{K}_3 \end{pmatrix} \begin{pmatrix} \eta \\ x \end{pmatrix}$$

and consequently this solution and solution of the problem (47),(60) are the same with precision level up to $O(\mu)$.

5.3 Example

Let us suppose that the state vector of a control system is described by the equations

$$ds/dt = -u(s), dx_1/dt = x_2, dx_2/dt = u(s) - x_1, s, x_1, x_2 \in R \quad (62)$$

and the relay control $u(s) = -\text{sign}(s)$ has been designed. The motion in the sliding mode in (62) are described by equations

$$dx_1/dt = x_2, dx_2/dt = -x_1. \quad (63)$$

The spectrum of matrix in (63) is imaginary. Let us suppose that the relay control $u(s)$ is transmitted to the plant with the help of actuators whose behaviour is described by variables z_1, z_2 and the complete model of system has the following form

$$\begin{aligned} \mu dz_1/dt &= -z_1 + v_1 - x_1, \mu dz_2/dt = -z_2 + v_2, \\ ds/dt &= z_2, dx_1/dt = x_2, dx_2/dt = z_1. \end{aligned} \quad (64)$$

It can be easily seen that in case when we suppose that $v_1 = v_2 = -\text{sign}(s)$, system (64) takes the form

$$\begin{aligned} \mu dz_1/dt &= -z_1 - \text{sign}(s) - x_1, \mu dz_2/dt = -z_2 - \text{sign}(s), \\ ds/dt &= z_2, dx_1/dt = x_2, dx_2/dt = z_1. \end{aligned}$$

The slow motion in it, with an error of $O(\mu)$ is described by system (63).

Let us try to reach two goals:

- Provide the fast exponentially decreasing of chattering in the system (64) to the surface $s = 0$;
- Find an approximate solution of the optimal stabilization problem for functional

$$\int_0^\infty (3(x_1^2(t) + x_2^2(t)) + w^2(t))dt. \quad (65)$$

To solve this problem with high precision, given the control law in the form $v_1 = -\text{sign}(s) + w, v_2 = -\text{sign}(s)$ it is necessary to obtain the minimum of functional (65) on the trajectories of the system

$$dx_1/dt = x_2, dx_2/dt = -x_1 + w. \quad (66)$$

We shall find vector w in the form

$$w = (0 \ 1) \begin{pmatrix} K_{11} & K_{12} \\ K_{12}^* & K_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Then

$$2K_{12} + K_{12}^2 = 3, K_{11} - K_{22} - K_{12}K_{22} = 0, 2K_{12} - K_{22}^2 = -3.$$

This means that choosing the control law in the form

$$w = (0 \ 1) \begin{pmatrix} 2\sqrt{5} & 1 \\ 1 & \sqrt{5} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

we have the solution of the problem (62), (65) with an error of $O(\mu)$.

6 Conclusion

1. The mathematical tools for investigation of SPRCSSOSM are developed.
2. The sufficient conditions under which the oscillations in the sliding mode control systems, with fast actuators, whose behaviour are described by SPRCSSOSM, have the following structure:
 - the oscillations develop in the direction orthogonal to the switching surface designed according to measuring results. An amplitude of these oscillations exponentially decreases, time intervals between switches vanish and the frequency of such oscillations is infinitely increasing;
 - the oscillations in the second order sliding mode which are described by a smooth singularly perturbed system of differential equations, and the slow-motion integral manifold of this system is the stable slow-motion integral manifold of the original system.
- Due to this fact it was shown that it's possible to design chattering-free sliding mode control systems with fast actuators in the case when the order of sliding in a complete model is 2.
3. It is shown that it is possible to save one order of derivative to obtain the sliding mode system without chattering.
4. It is proved that in the general case, when the plant contains the relay control nonlinearly, the equations of the equivalent control method for the sliding motions of the plant are approximately describing the slow motion in the original SPRCSSOSM and correspond to the presence of fast actuators in a sliding mode control system.
5. The connection between the stability of the actuators and the stability of the plant on the one hand and the stability of the sliding mode system as a whole on the other hand is investigated.
6. The algorithm for the correction of the sliding mode equation is proposed. In the case, when the linear part of the sliding mode equations have a critical spectrum, it is obligatory to correct the equations of the sliding motion in order to take into account the presence of fast actuators in the system, because the presence of such devices may cause change to the system behaviour from stability to asymptotic stability or instability.
7. The algorithms are proposed allowing us to solve problems of eigenvalue assignment or optimal stabilization for sliding motion using the additional dynamics of fast actuators.

Appendix 1. Exponential Stability of Fast Motions

For investigating stability of fast motion in (8),(9) it is necessary to introduce in (8) new fast time $\tau = t/\mu$ and rewrite system (8) in form

$$\begin{aligned}
 d\eta/d\tau &= B_{11}(t, x, \mu)\eta + B_{12}(t, x, \mu)\sigma + \\
 &+ \mu B_{13}(t, x, \mu)\xi + \varphi_1(t, \eta, \sigma, \mu\xi, x, \mu), \\
 d\sigma/d\tau &= B_{21}(t, x, \mu)\eta + B_{22}(t, x, \mu)\sigma + \mu B_{23}(t, x, \mu)\xi + \\
 &+ \varphi_2(t, \eta, \sigma, \mu\xi, x, \mu) + b(t, x)U(t, x, u(\xi)), \quad d\xi/d\tau = \sigma,
 \end{aligned} \tag{67}$$

where (t, x) are parameters. To analyze the stability of relay systems (67) let us examine the Lyapunov function

$$E = \eta^* S \eta + \sigma^2 - \xi[2b(t, x)U(t, x, u(\xi)) + \\ + 2\varphi_2(t, \eta, \sigma, \mu\xi, x, \mu) + B_{22}\sigma + 2\eta^* S B_{12} + 2B_{21}\eta].$$

Here $S(t, x, \mu)$ is positive definite solution of equation $S B_{11} + B_{11}^* S = -I_{m-1}$. Then there are some constants $\kappa_2 > \kappa_1 > 0$ such that inequality

$$\kappa_1 |\eta|^2 \leq \eta^* S \eta \leq \kappa_2 |\eta|^2 \quad (68)$$

is true uniformly for $(t, x, \mu) \in \mathbf{R}^+ \times \mathbf{R}^n \times [0, \mu_0]$. Taking into account condition (iv) and inequality (68) one can conclude the following estimation for the Lyapunov function

$$\kappa_3 |y|_*^2 \leq E(t, y, x, \mu) \leq \kappa_4 |y|_*^2 \quad (\kappa_4 > \kappa_3 > 0). \quad (69)$$

which is true for $(t, y, x, \mu) \in \mathbf{R}^+ \times \mathbf{R}^n \times \mathcal{U}_1 \times (0, \mu_0)$, where $\mathcal{U}_1$ is some neighbourhood of zero in the state space of variables $y = (\eta, \sigma, \xi)$. In this case

$$dE/d\tau = -|\eta|^2 + B_{22}\sigma^2 - \xi B_{22}[bU(t, x, u(y), 0) + \\ + \varphi_2(t, 0, 0, 0, x, 0)] + \xi N_1 + \xi N_2 + N_3,$$

where functions B_{ij}, b_j are computed at the point $(t, x, 0)$. For N_1, N_2, N_3 at $y \rightarrow 0, \mu \rightarrow 0$ uniformly on $(t, x, \mu) \in \mathbf{R}^+ \times \mathbf{R}^n \times [0, \mu_0]$ the following asymptotic representations are true

$$N_1(t, y, x, \mu) = o(1), \quad N_2(t, y, x, \mu) = O(\mu), \quad N_3(t, y, x, \mu) = o(|y|^2).$$

Taking into account conditions (iii) – (iv) we have the inequality

$$\kappa_5 |y|_*^2 \leq -dE/d\tau \leq \kappa_6 |y|_*^2 \quad (\kappa_6 > \kappa_5 > 0), \quad (70)$$

which is true at $(t, y, x, \mu) \in \mathbf{R}^+ \times \mathbf{R}^n \times \mathcal{U}_2 \times [0, \mu_0]$, where $\mathcal{U}_2$ some neighbourhood of the origin in the state space of y .

From (69) and (70) one can conclude that there are some constants $\kappa_8 > \kappa_7 > 0$, in the neighbourhood of the origin $W = \mathcal{U}_1 \cap \mathcal{U}_2$ such that for any $(t, y, x, \mu) \in \mathbf{R}^+ \times W \times \mathbf{R}^n \times [0, \mu_0]$ the following inequality

$$\kappa_7 E \leq -dE/d\tau \leq \kappa_8 E. \quad (71)$$

is true. Lemma 1 follows from inequality (71).

Appendix 2. Proof of Decomposition Theorem

Consider the system (11),(12). Let's design an integral manifold of the system (11),(12) in form $\mathcal{S} = \{(t, x, \pi x) \in \mathbf{R}^+ \times \mathbf{R}^n \times \mathbf{R}^n : \pi x = \mathcal{H}(t, x, \mu)\}$, where the function $\mathcal{H}(t, x, \mu)$ is continuous on $\mathbf{R}^+ \times \mathbf{R}^n \times [0, \mu_0]$ and the following inequality is true

$$\sup |\exp(\gamma t/\mu) \mathcal{H}(t, x, \mu)| < \mu d, \quad (t, x) \in \mathbf{R} \times \mathbf{R}^n, \quad (72)$$

The constant $d > 0$ in (72) will be defined later. Denote as $\mathcal{U}$ the metrical space of continuous functions $\mathbf{R}^+ \times \mathbf{R}^n \times [0, \mu_0] \rightarrow \mathbf{R}^n$, satisfying the inequality (72) with the metric $\rho(\mathcal{H}, \bar{\mathcal{H}}) = \sup |\exp(\gamma t/\mu)(\mathcal{H}(t, x, \mu) - \bar{\mathcal{H}}(t, x, \mu))|$, for $(t, x, \mu) \in \mathbf{R}^+ \times \mathbf{R}^n \times [0, \mu_0]$. The space $\mathcal{U}$ is complete metric space.

In this case function $\pi x = \mathcal{H}(t, x, \mu) \in \mathcal{U}$ is the solution of the equation

$$\mathcal{H} = \mathcal{P}(\mathcal{H}), \quad (73)$$

where

$$\begin{aligned} \mathcal{P}(\mathcal{H})(\tau, \xi, \mu) = & - \int_t^\infty [\Phi(\theta, y(\theta, \mu), \phi(\theta, \mu) + \mathcal{H}(\theta, \phi(\theta, \mu), \mu), \mu) - \\ & - \Phi(\theta, 0, \phi(\theta, \mu), \mu)] d\theta, \end{aligned}$$

and $\phi(\theta, \mu)$ is the solution of Cauchy problem $d\phi/d\theta = \Phi(\theta, 0, \phi, \mu)$, $\phi(t) = \xi$.

Let's show that operator $\mathcal{P}$ from (73) transforms $\mathcal{U}$ into itself. Taking into account (72) and (73) one can conclude that

$$\begin{aligned} |\exp(\gamma t/\mu) \mathcal{P}(\mathcal{H})(t, \xi, \mu)| & < M \exp(\gamma t/\mu) \int_t^\infty [|\mathcal{H}(\theta, \phi(\theta, \mu), \mu)| + \\ & + |y(\theta, \mu)|] d\theta < \frac{M}{\gamma} [\mu d + C|y_0|_*]. \end{aligned}$$

Now it's possible to choose such d that for any $y_0 \in W$ (W from lemma 1) the inequality $\frac{M}{\gamma} [\mu d + C|y_0|_*] \leq d$ is true. This means that operator $\mathcal{P}$ transforms the space $\mathcal{U}$ into itself. Similarly,

$$\begin{aligned} \exp(\gamma t/\mu) |\mathcal{P}(\mathcal{H})(t, \xi, \mu) - \mathcal{P}(\bar{\mathcal{H}})(\tau, \xi, \mu)| & \leq \\ \leq \exp(\gamma t/\mu) \int_t^\infty |\Phi(\theta, \phi(\theta, \mu) + \mathcal{H}, y(\theta, \mu), \mu) - \Phi(\theta, \phi(\theta, \mu) + \\ & + \mathcal{H}, y(\theta, \mu), \mu)| d\theta \leq \exp(\gamma t/\mu) \int_t^\infty M |\mathcal{H} - \bar{\mathcal{H}}| d\theta \leq \mu \frac{M}{\gamma} \rho(\mathcal{H}, \bar{\mathcal{H}}), \end{aligned}$$

which means that operator $\mathcal{P}$ is a contraction operator on $\mathcal{U}$. Then, the operator $\mathcal{P}$ has the unique fixed point corresponding to the function $\pi x = \mathcal{H}(t, x, \mu)$. Moreover, from (72) one can conclude that for all $(t, x, \mu) \in \mathbf{R}^+ \times \mathbf{R}^n \times [0, \mu_0]$ the inequality $|\mathcal{H}(t, x, \mu)| < \mu d \exp(\gamma t/\mu)$ is true.

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