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Optimal and robust control for linear state-delay systems

Michael Basin[a,*], Jesus Rodriguez-Gonzalez[a], Leonid Fridman[b]

[a]Department of Physical and Mathematical Sciences, Autonomous University of Nuevo Leon, Apdo postal 144-F, C.P. 66450, San Nicolas de los Garza, Nuevo Leon, Mexico

[b]Department of Engineering, National Autonomous University of Mexico, Edificio "Bernardo Quintana", Circulo Exterior, Cd. Universitaria, C.P. 04510, Mexico, D.F., Mexico

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Abstract

This paper presents the optimal regulator for a linear system with state delay and a quadratic criterion. The optimal regulator equations are obtained using the maximum principle. Performance of the obtained optimal regulator is verified in the illustrative example against the best linear regulator available for linear systems without delays. Simulation graphs demonstrating better performance of the obtained optimal regulator are included. The paper then presents a robustification algorithm for the obtained optimal regulator based on integral sliding mode compensation of disturbances. The general principles of the integral sliding mode compensator design are modified to yield the basic control algorithm oriented to time-delay systems, which is then applied to robustify the optimal regulator. As a result, the sliding mode compensating control leading to suppression of the disturbances from the initial time moment is designed. The obtained robust control algorithm is verified by simulations in the illustrative example.

Keywords: Optimal control; Robust control; Integral sliding mode; Linear time-delay system

*Corresponding author. Tel.: +52 81 83294030; fax: +52 81 83522954.

E-mail addresses: mbasin@fcfm.uanl.mx (M. Basin), jgrg17@yahoo.com.mx (J. Rodriguez-Gonzalez), lfridman@verona.fi-p.unam.mx (L. Fridman).

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1. Introduction

Although the optimal control (regulator) problem for linear system states was solved in 1960s (see [1,2]), the optimal control problem for linear systems with delays is still open, depending on the delay type, specific system equations, criterion, etc. A detailed comment on the up-to-date state of the control theory for time-delay systems is given in [3–5]. Comprehensive reviews of theory and algorithms for time-delay systems can be found in [6–15]. Among many other papers devoted to the general optimal control theory for time-delay systems, the following five could be specially mentioned. The paper [16] establishes the necessary optimality conditions for time-optimal control systems. In [17,18], the linear-quadratic problem is solved for state-delay and state-and-input-delay systems, respectively, where the optimal control is obtained in the form of the integral over the previous system trajectory and depends on the system co-state satisfying a system of partial differential equations. The papers [19,20] develop the generalized Riccati approach, where the optimal control depends on the current system state and is determined by the gain matrices satisfying a set of Riccati-type differential and partial differential equations.

The first part of this paper concentrates on the solution of the optimal control problem for a linear system with state delay and a quadratic criterion. Using the maximum principle [21,22], the solution to the stated optimal control problem is obtained in a closed form, i.e., it is represented as a linear in state control law, whose gain matrix satisfies an ordinary differential (quasi-Riccati) equation, which does not contain time-advanced arguments and does not depend on the state variables. The obtained optimal regulator makes an advance with respect to general optimality results for time-delay systems (such as given in [16–20]), since (a) the optimal control law is given explicitly and not as a solution of a system of integro-differential or PDE equations, and (b) the quasi-Riccati equation for the gain matrix does not contain any time advanced arguments and does not depend on the state variables and, therefore, leads to a conventional two points boundary-valued problem generated in the optimal control problems with quadratic criterion and finite horizon (see, for example, [1]). Thus, the obtained optimal regulator is realizable using two delay-differential equations. Taking into account that the state space of a delayed system is infinite-dimensional [6], this seems to be a significant advantage.

Performance of the obtained optimal control for a linear system with state delay and a quadratic criterion is verified in the illustrative example against the best linear regulators available for the system without delay and the first-order approximation of the original state-delay system. The simulation results show a definitive advantage of the obtained optimal regulator in the criterion value.

The second part of the paper presents an integral sliding mode regulator robustifying the optimal regulator for linear systems with state delay and a quadratic criterion. The idea is to add a compensator to the known optimal control to suppress external disturbances deteriorating the optimal system behavior [23,24]. The integral sliding mode compensator is realized as a relay control in a such way that the sliding mode motion starts from the initial moment, thus eliminating the matched uncertainties from the beginning of system functioning. This constitutes the crucial advantage of the integral sliding modes in comparison to the conventional ones. Note that in the framework of this modified (with respect to [3,23]) integral sliding mode approach, the optimal control is not required to be differentiable and the sliding mode manifold matrix is always invertible. Other original modifications of the sliding mode control technique applicable to disturbance suppression were suggested in [25,26].

The proposed solution to the optimal control problem for linear state-delay system assumes that the system state is completely measured. Nonetheless, the obtained result can be readily extended to the case of unmeasured system state, using the optimal filter for linear state-delay systems [27] and applying the separation principle substantiated for linear time-delay systems in [28].

The paper is organized as follows. Section 2 states the optimal control problem for a linear system with state delay. The solution to the optimal control problem is given in Section 3. The proof of the obtained results, based on the maximum principle [21,22], is given in Appendix. The paper then presents a robustification algorithm for the obtained optimal regulator based on integral sliding mode compensation of disturbances [23]. Section 4 outlines the new general principles of the integral sliding mode compensator design, which yield the basic control algorithm oriented to time-delay systems. This basic algorithm is then applied to robustify the optimal regulator. As a result, the sliding mode compensating control leading to suppression of the disturbances from the initial time moment is designed. Section 5 presents an example illustrating the quality of control provided by the obtained optimal regulator for linear systems with state delay against the best linear regulators available for the system without delay and the first-order approximation of the original state-delay system. Simulation graphs and comparison tables demonstrating better performance of the obtained optimal regulator are included. The example is then continued illustrating the quality of disturbance suppression provided by the obtained robust integral sliding mode regulator against the optimal regulator under the presence of disturbances. Satisfactory results are obtained.

2. Optimal control problem for linear state-delay system

Consider a linear system with time delay in the state

$$\dot{x}(t) = a_0(t) + a(t)x(t-h) + B(t)u(t), \tag{1}$$

with the initial condition $x(s) = \varphi(s)$, $s \in [t_0 - h, t_0]$, where $x(t) \in R^n$ is the system state, $u(t) \in R^m$ is the control variable, and $\varphi(s)$ is a piecewise continuous function given in the interval $[t_0 - h, t_0]$. Existence of the unique solution of Eq. (1) is thus assured by the Carathéodory theorem (see, for example, [29]). The quadratic cost function to be minimized is defined as follows:

$$J = \frac{1}{2}[x(T)]^{\mathrm{T}}\psi[x(T)] + \frac{1}{2}\int_{t_0}^{\mathrm{T}} u^{\mathrm{T}}(s)R(s)u(s)\,\mathrm{d}s + \frac{1}{2}\int_{t_0}^{\mathrm{T}} x^{\mathrm{T}}(s)L(s)x(s)\,\mathrm{d}s, \tag{2}$$

where R is a positive definite symmetric matrix, ψ and L are nonnegative definite symmetric matrices, and $T > t_0$ is a certain time moment.

The optimal control problem is to find the control $u^*(t)$, $t \in [t_0, T]$, that minimizes the criterion J along with the trajectory $x^*(t)$, $t \in [t_0, T]$, generated upon substituting $u^*(t)$ into the state equation (1). The solution to the stated optimal control problem is given in the next section and then proved using the maximum principle [21,22] in Appendix.

3. Optimal control problem solution

The solution to the optimal control problem for the linear system with state delay (1) and the quadratic criterion (2) is given as follows. The optimal control law is given by

$$u^*(t) = (R(t))^{-1} B^{\mathrm{T}}(t) Q(t) x(t), \tag{3}$$

where the matrix function $Q(t)$ satisfies the matrix equation

$$\dot{Q}(t) = L(t) - Q(t)M_1(t)a(t) - a^{\mathrm{T}}(t)M_1^{\mathrm{T}}(t)Q(t) - Q(t)B(t)R^{-1}(t)B^{\mathrm{T}}(t)Q(t), \tag{4}$$

with the terminal condition $Q(T) = -\psi$. The auxiliary matrix $M_1(t)$ is defined as $M_1(t) = (\partial x(t-h)/\partial x(t))$, whose value is equal to zero, $M_1(t) = 0$, if $t \in [t_0, t_0 + h)$, and is determined as $M_1(t) = \Phi^{-1}(t, t-h) = \Phi(t-h, t) = \exp(-\int_{t-h}^{t} B(s)R^{-1}(s)B^{\mathrm{T}}(s)Q(s)\,\mathrm{d}s)$, if $t \geqslant t_0 + h$, where $\Phi(t, \tau)$ satisfies the matrix equation

$$\frac{\mathrm{d}\Phi(t,\tau)}{\mathrm{d}t} = B(t)R^{-1}(t)B^{\mathrm{T}}(t)Q(t)\Phi(t,\tau),$$

with the initial condition $\Phi(t, t) = I$, and I is the identity matrix.

Upon substituting the optimal control (3) into the state equation (1), the optimally controlled state equation is obtained

$$\dot{x}(t) = a_0(t) + a(t)x(t-h) + B(t)R^{-1}(t)B^{\mathrm{T}}(t)Q(t)x(t), \tag{5}$$

with the initial condition $x(s) = \varphi(s)$, $s \in [t_0 - h, t_0]$.

The results obtained in this section by virtue of the duality principle are proved in Appendix using the general equations of the Pontryagin maximum principle [21,22].

It should also be noted that the obtained optimal regulator makes an advance with respect to general optimality results for time-delay systems (such as given in [16–20]), since (a) the optimal control law is given explicitly and not as a solution of a system of integro-differential or PDE equations, and (b) the quasi-Riccati equation (4) for the gain matrix does not contain any time advanced arguments and does not depend on the state variables and, therefore, leads to a conventional two points boundary-valued problem generated in the optimal control problems with quadratic criterion and finite horizon (see, for example, [1]). Thus, the obtained optimal regulator is realizable using two delay-differential equations. Taking into account that the state space of a delayed system (1) is infinite-dimensional [6], this seems to be a significant advantage.

4. Robust control problem

Consider a nominal control system with state delay, which for generality is assumed to be nonlinear with respect to the state x,

$$\dot{x}(t) = f(x(t-h)) + B(t)u(t), \tag{6}$$

where $u(t) \in R^m$ is the control input, the rank of matrix $B(t)$ is complete and equal to m for any $t > 0$, and the pseudoinverse matrix of B is uniformly bounded:

$$\|B^{+}(t)\| \leqslant b^{+}, \; b^{+} = const > 0, \quad B^{+}(t) := [B^{\mathrm{T}}(t)B(t)]^{-1}B^{\mathrm{T}}(t), \quad \text{and} \quad B^{+}(t)B(t) = I,$$

where I is the m-dimensional identity matrix.

Suppose that there exists a state feedback control law $u_0(x(t), t)$, such that the dynamics of the nominal closed loop system takes the form

$$\dot{x}_0(t) = f(x_0(t-h)) + B(t)u_0(x_0(t), t), \tag{7}$$

and has certain desired properties. However, in practical applications, system (6) operates under uncertainty conditions that may be generated by parameter variations and external disturbances. Let us consider the real trajectory of the disturbed closed loop control system

$$\dot{x}(t) = f(x(t-h)) + B(t)u(t) + g_1(x(t), t) + g_2(x(t-h), t), \tag{8}$$

where g_1, g_2 are smooth uncertainties presenting perturbations and nonlinearities in the system (6). For g_1, g_2, the standard matching and conditions are assumed to be held: $g_1, g_2 \in \operatorname{span} B$, or, in other words, there exist smooth functions γ_1, γ_2 such that

$$g_1(x(t), t) = B(t)\gamma_1(x(t), t),$$

$$g_2(x(t-h), t) = B(t)\gamma_2(x(t-h), t),$$

$$\|\gamma_1(x(t), t)\| \leqslant q_1\|x(t)\| + p_1, \quad q_1, p_1 > 0,$$

$$\|\gamma_2(x(t-h), t)\| \leqslant q_2\|x(t-h)\| + p_2, \quad q_2, p_2 > 0.$$

The last two conditions provide reasonable restrictions on the growth of the uncertainties.

The following initial conditions are assumed for system (6)

$$x(\theta) = \varphi(\theta), \tag{9}$$

where $\varphi(\theta)$ is a piecewise continuous function given in the interval $[t_0 - h, t_0]$.

Thus, the control problem now consists in robustification of control design in system (7) with respect to uncertainties g_1, g_2: to find such a control law that the trajectories of system (8) with initial conditions (9) coincide with the trajectories $x_0(t)$ with the same initial conditions (9). The integral sliding mode technique [3,23,24], enabling one to follow the sliding mode manifold from the initial time moment, is first developed for the general nonlinear state-delay system and then specified for the original linear state-delay system (1) in the next two subsections.

4.1. Design principles

Let us redesign the control law for system (6) in the form

$$u(t) = u_0(x(t), t) + u_1(t), \tag{10}$$

where $u_0(x(t), t)$ is the ideal feedback control designed for (6), and $u_1(t) \in R^m$ is the relay control generating the integral sliding mode in some auxiliary space to reject uncertainties g_1, g_2. Substitution of the control law (10) into the system (6) yields

$$\dot{x}(t) = f(x(t-h)) + B(t)u_0(x(t), t) + B(t)u_1(t) + g_1(x(t), t) + g_2(x(t-h), t). \tag{11}$$

Define the auxiliary function

$$s(t) = z(t) + s_0(x(t), t), \tag{12}$$

where $s_0(x(t),t) = B^+(t)x(t)$, and $z(t)$ is an auxiliary variable defined below. Then,

$$\begin{aligned}\dot{s}(t) &= \dot{z}(t) + G(t)[f(x(t)) + B(t)u_0(x(t),t) \\ &\quad + B(\gamma_1(x(t),t)) + \gamma_2(x(t-h),t) + B(t)u_1(t)] + (\partial s_0(x(t),t)/\partial t),\end{aligned} \tag{13}$$

$G(t) = \partial s_0(x(t),t)/\partial x = B^+(t)$ and $\partial s_0(x(t),t)/\partial t = (\mathrm{d}(B^+(t))/\mathrm{d}t)x(t)$. Note that in the framework of this modified (with respect to [3,23]) integral sliding mode approach, the optimal control $u_0(x(t))$ is not required to be differentiable and the sliding mode manifold matrix $GB = B^+B = I$ is always invertible.

The philosophy of integral sliding mode control is the following: in order to achieve $x(t) = x_0(t)$ at all $t \in [t_0, \infty)$, the sliding mode should be organized on the surface $s(t) = 0$, since the following disturbance compensation should have been obtained in the sliding mode motion

$$B^+(t)B(t)u_{1\mathrm{eq}}(t) = -B^+(t)B(t)\gamma_1(x(t),t) - B^+(t)B(t)\gamma_2(x(t-h),t),$$

that is

$$u_{1\mathrm{eq}}(t) = -\gamma_1(x(t),t) - \gamma_2(x(t-h),t).$$

Note that the equivalent control $u_{1\mathrm{eq}}(t)$ can be unambiguously determined from the last equality and the initial condition for $x(t)$.

Define the auxiliary variable $z(t)$ as the solution to the differential equation

$$\dot{z}(t) = -B^+(t)[f(x(t-h)) + B(t)u_0(x(t),t)] + (\mathrm{d}(B^+(t))/\mathrm{d}t)x(t),$$

with the initial conditions $z(t_0) = -s_0(t_0) = -B^+(t_0)\varphi(t_0)$. Then, the sliding manifold equation takes the form

$$\begin{aligned}\dot{s}(t) &= B^+(t)[B(t)(\gamma_1(x(t),t)) + \gamma_2(x(t-h),t) + B(t)u_1(t)] \\ &= \gamma_1(x(t),t) + \gamma_2(x(t-h),t) + u_1(t) = 0.\end{aligned}$$

Finally, to realize sliding mode, the relay control is designed

$$u_1(t) = -M(x(t), x(t-h), t)sign[s(t)], \tag{14}$$

$$M = q(\|x(t)\| + \|x(t-h)\|) + p,$$

$q > q_1, q_2$, $p > p_1 + p_2$.

The convergence to and along the sliding mode manifold $s(t) = 0$ is assured by the Lyapunov function $V(t) = s^{\mathrm{T}}(t)s(t)/2$ for the system (11) with the control input $u_1(t)$ of Eq. (14):

$$\begin{aligned}\dot{V}(t) &= s^{\mathrm{T}}(t)[\gamma_1(x(t),t) + \gamma_2(x(t-h),t) + u_1(t)] \\ &\leqslant -|s(t)|([q(\|x(t)\| + \|x(t-h)\|) + p] + [\gamma_1(x(t),t) + \gamma_2(x(t-h),t)]) < 0,\end{aligned}$$

where $|s(t)| = \sum_{i=1}^{m} |s_i(t)|$.

The next subsection presents the robustification of the designed optimal control (3). This robust regulator is designed assigning the sliding mode manifold according to (12)–(13) and subsequently moving to and along this manifold using relay control (14).

4.2. Robust sliding mode control design for linear state-delay system

Returning to the original particular linear case, consider the disturbed linear state-delay system (1), whose behavior is affected by uncertainties g_1, g_2 presenting perturbations and nonlinearities in the system

$$\dot{x}(t) = a_0(t) + a(t)x(t-h) + B(t)u(t) + g_1(x(t), t) + g_2(x(t-h), t). \tag{15}$$

It is also assumed that the uncertainties satisfy the standard matching and growth conditions

$$g_1(x(t), t) = B(t)\gamma_1(x(t), t),$$

$$g_2(x(t-h), t) = B(t)\gamma_2(x(t-h), t),$$

$$\|\gamma_1(x(t), t)\| \leqslant q_1\|x(t)\| + p_1, \quad q_1, p_1 > 0,$$

$$\|\gamma_2(x(t-h), t)\| \leqslant q_2\|x(t-h)\|, \quad q_2, p_2 > 0.$$

The quadratic cost function (2) is the same as in Section 2.

The problem is to robustify the obtained optimal control (3), using the method specified by (12)–(13). Define this new control in the form (10): $u(t) = u_0(x(t), t) + u_1(t)$, where the optimal control $u_0(x(t), t)$ coincides with Eq. (3) and the robustifying component $u_1(t)$ is obtained according to Eq. (14)

$$u_1(t) = -M(x(t), x(t-h), t)sign[s(t)],$$

$$M = q(\|x(t)\| + \|x(t-h)\|) + p,$$

$q > q_1, q_2$, $p > p_1 + p_2$. Consequently, the sliding mode manifold function $s(t)$ is defined as

$$s(t) = z(t) + s_0(x(t), t), \tag{16}$$

where

$$s_0(x(t), t) = B^+(t)x(t), \tag{17}$$

and the auxiliary variable $z(t)$ satisfies the delay-differential equation

$$\dot{z}(t) = -B^+(t)[a_0(t) + a(t)x(t-h) + B(t)u_0(x(t), t)], \tag{18}$$

with the initial conditions $z(t_0) = -B^+(t_0)\varphi(t_0)$.

5. Example

This section presents an example of designing the optimal regulator for a system (1) with a criterion (2), using the scheme (3)–(5), and comparing it to the regulator where the matrix Q is selected as in the optimal linear regulator for a system without delays, disturbing the obtained regulator by a noise, and designing a robust sliding mode compensator for that disturbance, using the scheme (16)–(18).

Consider a scalar linear system

$$\dot{x}(t) = 10x(t - 0.25) + u(t), \tag{19}$$

with the initial conditions $x(s) = 1$ for $s \in [-0.1, 0]$. The control problem is to find the control $u(t)$, $t \in [0, T]$, $T = 0.5$, that minimizes the criterion

$$J = \frac{1}{2}\left[\int_0^{\mathrm{T}} u^2(t)\,\mathrm{d}t + \int_0^{\mathrm{T}} x^2(t)\,\mathrm{d}t\right]. \tag{20}$$

In other words, the control problem is to minimize the overall energy of the state x using the minimal overall energy of control u. Since the initial criterion value is zero, the criterion is quadratic, and the state initial condition is positive, the criterion value would necessarily increase: in the part of the state, if the control is small, or in the part of control, if the control is large. Thus, it is required to find such a balanced value of the control input that the total system energy, i.e., state energy plus control one, would increase in the interval $[0, T]$ as minimally as possible. Note that it is not assumed to maintain the state at a given point, such as $x = 0$; both, the state and absolute control values, are permitted to increase while keeping the total system energy at the minimal possible level.

Let us first construct the regulator where the control law and the matrix $Q(t)$ are calculated in the same manner as for the optimal linear regulator for a linear system without delays, that is $u(t) = R^{-1}(t)B^{\mathrm{T}}(t)Q(t)x(t)$ (see Ref. [1]). Since $B(t) = 1$ in Eq. (19) and $R(t) = 1$ in Eq. (20), the optimal control is actually equal to

$$u(t) = Q(t)x(t), \tag{21}$$

where $Q(t)$ satisfies the Riccati equation

$$\dot{Q}(t) = -a^{\mathrm{T}}(t)Q(t) - Q(t)a(t) + L(t) - Q(t)B(t)R^{-1}(t)B^{\mathrm{T}}(t)Q(t),$$

with the terminal condition $Q(T) = -\psi$. Since $a(t) = 10$, $B(t) = 1$ in Eq. (19), and $L(t) = 1$ and $\psi = 0$ in Eq. (20), the last equation turns to

$$\dot{Q}(t) = 1 - 20Q(t) - Q^2(t), \quad Q(0.5) = 0. \tag{22}$$

Upon substituting the control (21) into Eq. (19), the controlled system takes the form

$$\dot{x}(t) = 10x(t - 0.25) + Q(t)x(t). \tag{23}$$

The results of applying the regulator (21)–(23) to the system (19) are shown in Fig. 1, which presents the graphs of the criterion (20) $J(t)$ and the control (21) $u(t)$ in the interval $[0, T]$. The value of criterion (20) at the final moment $T = 0.5$ is $J(0.5) = 15.94$.

Let us now apply the optimal regulator (3)–(5) for linear states with time delay to the system (19). The control law (3) takes the same form as Eq. (21)

$$u^*(t) = Q^*(t)x(t), \tag{24}$$

where $Q^*(t)$ satisfies the equation

$$\dot{Q}^*(t) = 1 - 20Q^*(t)M_1(t) - Q^{*2}(t), \quad Q^*(0.5) = 0, \tag{25}$$

where $M_1(t) = 0$ for $t \in [0, 0.25)$ and $M_1(t) = \exp(-\int_{t-0.25}^{t} Q^*(s)\,\mathrm{d}s)$ for $t \in [0.25, 0.5]$. Since the solution $Q^*(t)$ of Eq. (25) is not smooth, it has been numerically solved with the approximating terminal condition $Q^*(0.5) = 0.04$, in order to avoid chattering.

Upon substituting the control (24) into Eq. (19), the optimally controlled system takes the same form as Eq. (23)

$$\dot{x}(t) = 10x(t - 0.25) + Q^*(t)x(t). \tag{26}$$

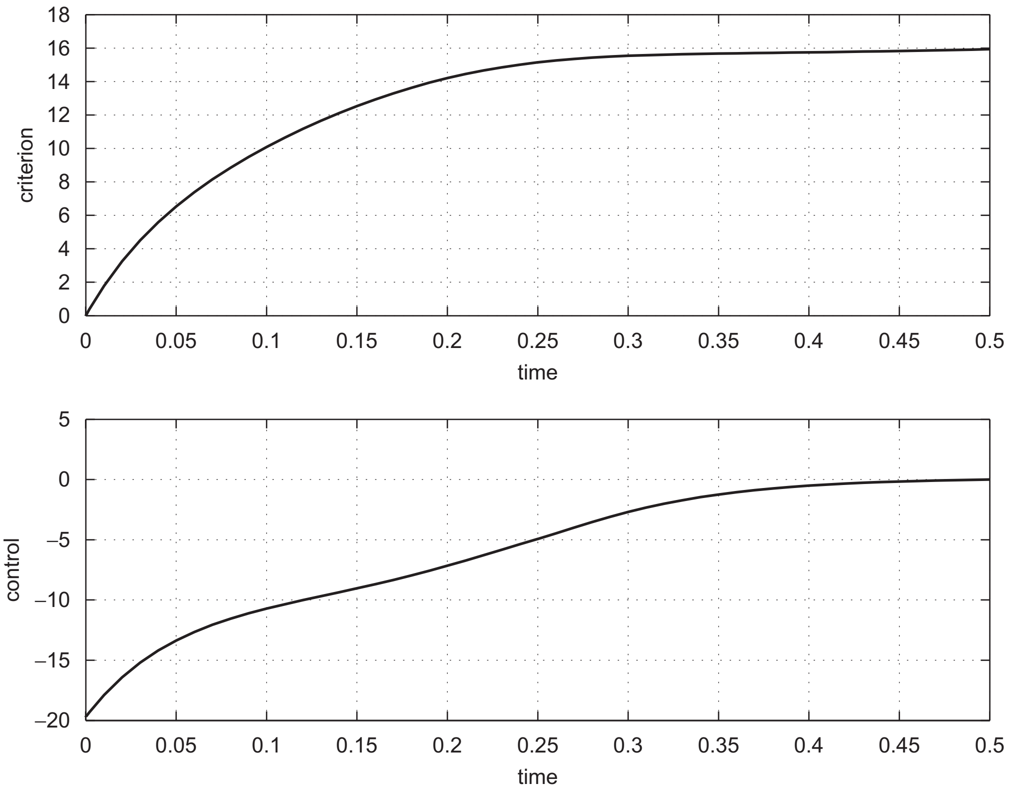

Fig. 1. Best linear regulator available for linear systems without state delay. Graphs of the criterion (20) $J(t)$ and the control (21) $u(t)$ in the interval $[0, 0.5]$.

The results of applying the regulator (24)–(26) to the system (19) are shown in Fig. 2, which presents the graphs of the criterion (20) $J(t)$ and the control (24) $u^*(t)$ in the interval $[0, T]$. The value of the criterion (20) at the final moment $T = 0.5$ is $J(0.5) = 4.63$. There is a definitive improvement (three and half times) in the values of the criterion to be minimized in comparison to the preceding case, due to the optimality of the regulator (3)–(5) for linear states with time delay.

Let us also compare the optimal regulator (3)–(5) to the best linear regulators based on linear approximation of the original time-delay system (19). The input-state transfer function of Eq. (19), $G(s) = (s - 10\exp(-sh))^{-1}$, $h = 0.25$, is approximated by a rational function up to the first order of h: $G^{-1}(s) = s(1 + 10h) - 10 + O(h^2)$ $(h = 0.25)$

$$\dot{x}(t) = (20/7)x(t) + (2/7)u(t), \tag{27}$$

with the initial condition $x(0) = 1$. The control law is calculated as the optimal control for the linear system without delays (27):

$$u_1(t) = (2/7)Q_1(t)x(t), \tag{28}$$

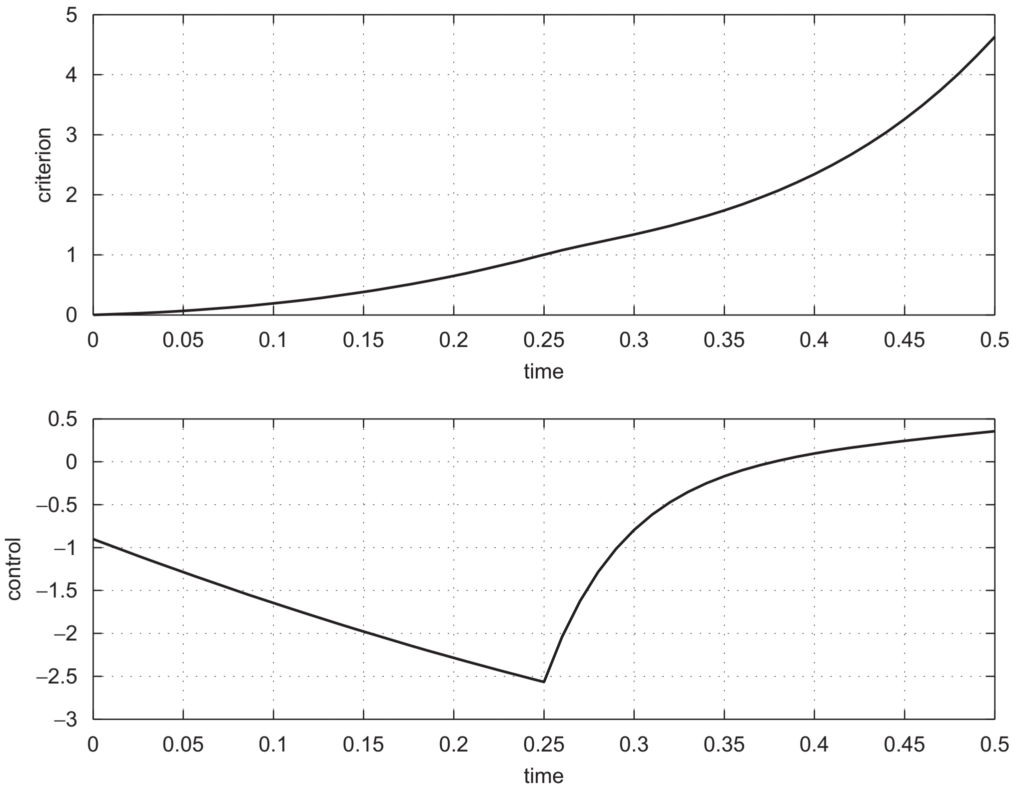

Fig. 2. Optimal regulator obtained for linear systems with state delay. Graphs of the criterion (20) $J(t)$, and the optimal control (24) $u^*(t)$ in the interval $[0, 0.5]$.

and $Q_1(t)$ satisfies the Riccati equation

$$\dot{Q}_1(t) = 1 - (40/7)Q_1(t) - (2/7)^2 Q_1^2(t), \tag{29}$$

with the terminal condition $Q_1(0.5) = 0$. The control (28) is then substituted into the original time-delay system (19).

The results of applying the regulator (27)–(29) to the system (19) are shown in Fig. 3, which presents the graphs of the criterion (20) $J(t)$ and the control (28) $u_1(t)$ in the interval $[0, T]$. The value of the criterion (20) at the final moment $T = 0.5$ is $J(0.5) = 9.77$. Thus, the simulation results show that application of the regulator (27)–(29), based on the first-order approximation, yields still unsatisfactory values of the cost function in comparison to the optimal regulator (24)–(26).

The next task is to introduce a disturbance into the controlled system (26). This deterministic disturbance is realized as a constant: $g(t) = 100$. The matching conditions are valid, because state $x(t)$ and control $u(t)$ have the same dimension: $dim(x) = dim(u) = 1$. The restrictions on the disturbance growth hold with $q_1 = q_2 = p_2 = 0$ and $p_1 = 100$, since $\|g(t)\| = 100$. The disturbed system equation (26) takes the form

$$\dot{x}(t) = 100 + 10x(t - 0.25) + Q^*(t)x(t). \tag{30}$$

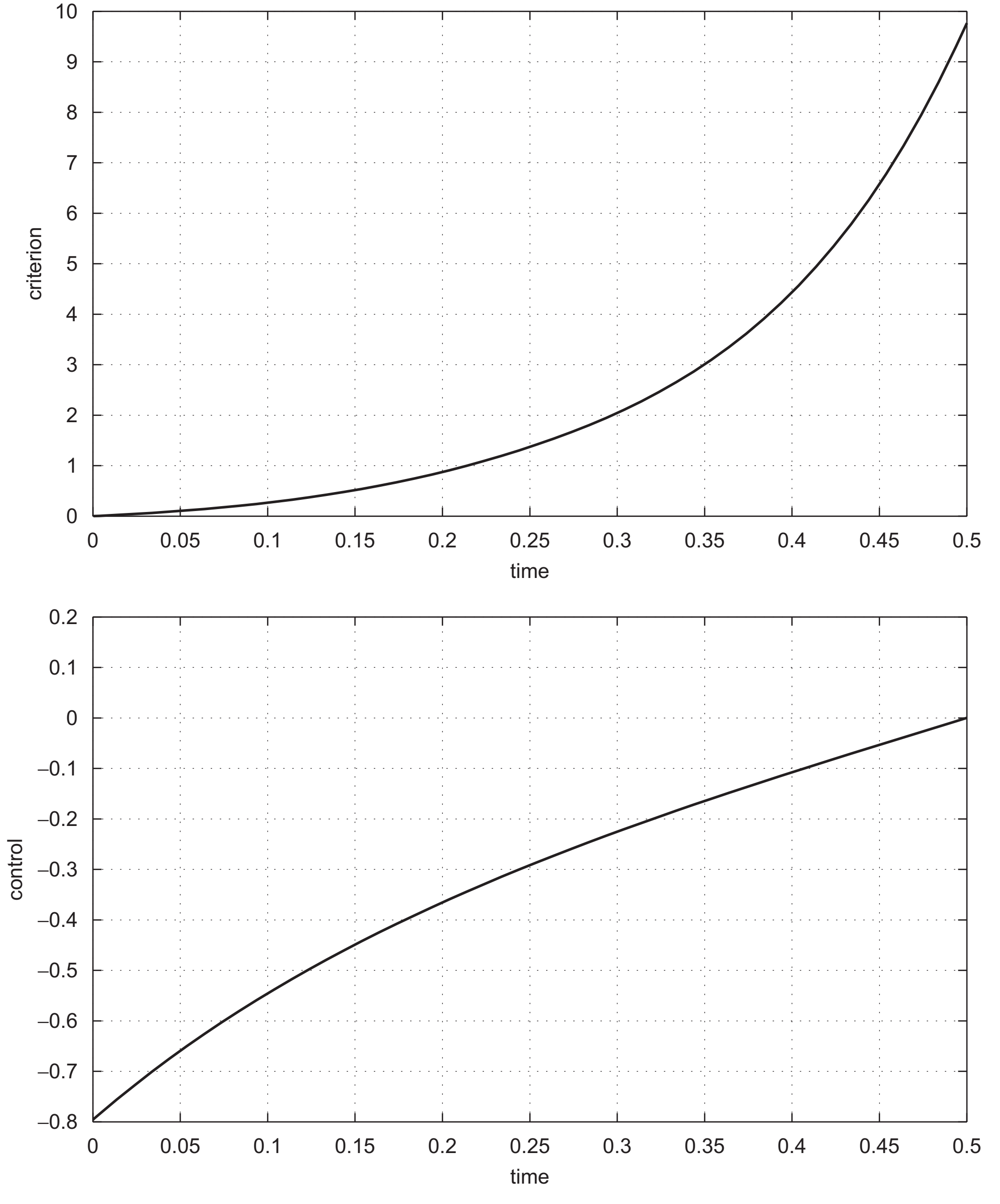

Fig. 3. Best linear regulator based on the first-order approximation of the transfer function of the original time-delay system. Graphs of the criterion (20) $J(t)$, and the control (28) $u_1(t)$ in the interval $[0, 0.5]$.

The system state behavior significantly deteriorates upon introducing the disturbance. Fig. 4 presents the graphs of the criterion (20) $J(t)$ and the control (24) $u(t)$ in the interval $[0, T]$. The value of the criterion (20) at the final moment $T = 0.5$ is $J(0.5) = 398.68$. The deterioration of the criterion value in comparison to that obtained using the optimal regulator (24) is more than 80 times.

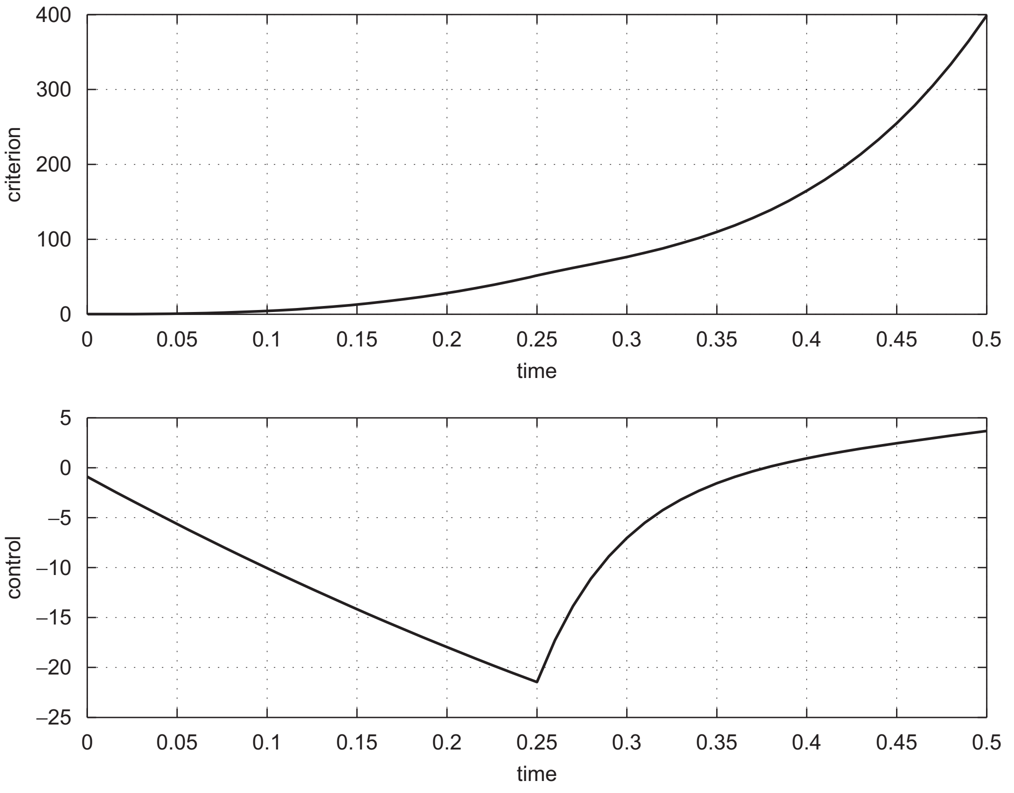

Fig. 4. Controlled system in the presence of disturbance. Graphs of the criterion (20) $J(t)$ and the control (24) $u(t)$ in the interval $[0, 0.5]$.

Let us finally design the robust integral sliding mode control compensating for the introduced disturbance. The new controlled state equation should be

$$\dot{x}(t) = 100 + 10x(t - 0.25) + Q^*(t)x(t) + u_1(t), \tag{31}$$

where the compensator $u_1(t)$ is obtained according to Eq. (14)

$$u_1(t) = -M(x(t), x(t - h), t)sign[s(t)], \tag{32}$$

and $M = 100.4 > p_1 = 100$. The sliding mode manifold $s(t)$ is defined by Eq. (21)

$$s(t) = z(t) + s_0(x(t), t),$$

where

$$s_0(x(t), t) = B^+(t)x(t) = x(t),$$

and the auxiliary variable $z(t)$ satisfies the delay-differential equation

$$\dot{z}(t) = -B^+(t)[10x(t - 0.25) + u_0(t)] = -[10x(t - 0.25) + Q^*(t)x(t)],$$

with the initial conditions $z(0) = -x(0) = -1$.

Upon introducing the compensator (32) into the state equation (31), the system state behavior is much improved. Fig. 5 presents the graphs of the criterion (20) $J(t)$ and the

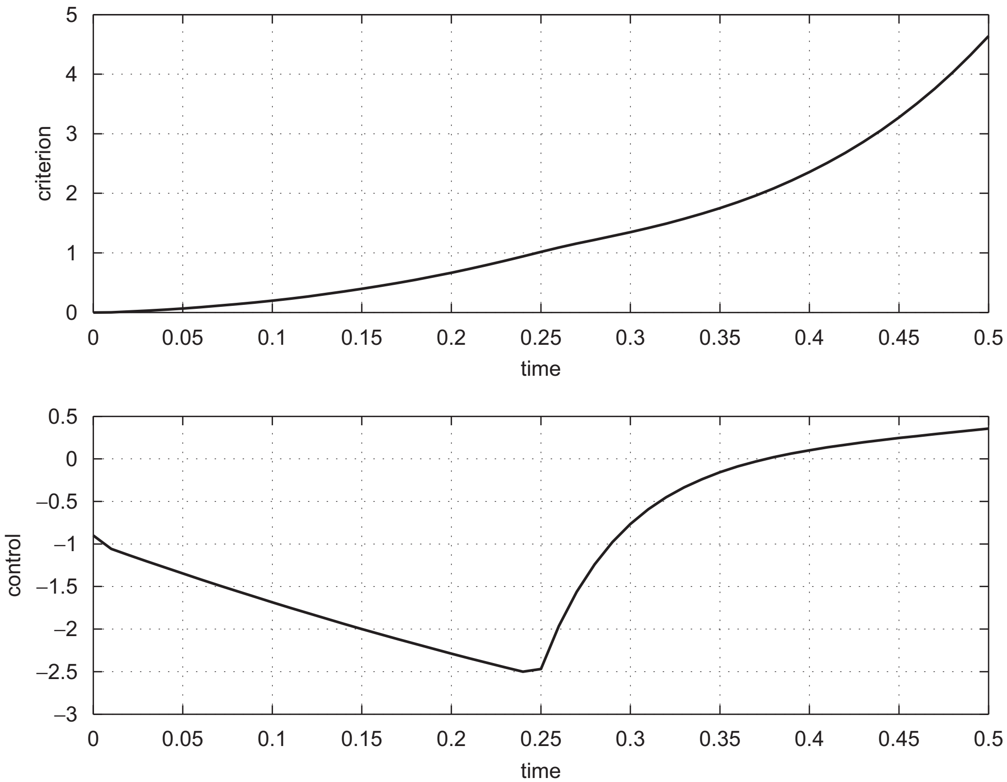

Fig. 5. Controlled system after applying robust integral sliding mode compensator. Graphs of the criterion (20) $J(t)$ and the control (24) $u(t)$ in the interval $[0, 0.5]$.

control (24) $u(t)$, after applying the compensator (32), in the interval $[0, T]$. The value of the criterion (20) at the final moment $T = 0.5$ is $J(0.5) = 4.64$. Thus, the criterion value after applying the compensator (32) is only slightly different from the criterion value given by the optimal regulator (24)–(26) for linear state-delay systems.

6. Conclusions

The optimal regulator for linear system with state delay and a quadratic cost function has been designed in a closed form. It is represented as a real-time feedback control whose gain matrix satisfies a quasi-Riccati equation without time advanced arguments, which provides a significant advantage with respect to previously obtained results in the area of optimal control for time-delay systems. A robustifying control for the obtained optimal regulator has then been designed based on the integral sliding mode technique. The integral sliding mode compensator is realized as a relay control in a such way that the sliding mode motion starts from the initial moment, thus eliminating the matched uncertainties from the beginning of system functioning. This constitutes the crucial advantage of the integral sliding modes in comparison to the conventional ones.

Performance of the optimal regulator for linear systems with state delay has been verified in the illustrative example against the best linear regulators available for the system without delay and the first-order approximation of the original state-delay system. The simulation results show a definitive improvement in the values of the criterion in favor of the designed regulator. Subsequent introduction of disturbances significantly affects system behavior in the example: the criterion value to be minimized increases more than 80 times. However, upon applying the robust integral sliding mode compensator, the system behavior is much improved: the criterion value after applying the compensator is insignificantly different from the criterion value given by the optimal regulator. Thus, it can be concluded that the designed optimal regulator and robust integral sliding mode compensator provide together the optimal control technique for linear state-delay systems, which is also resistible to influence of external disturbances.

Appendix

Proof of the optimal control problem solution. Define the Hamiltonian function [21,22] for the optimal control problem (1), (2) as

$$H(x,u,q,t) = \tfrac{1}{2}(u^{\mathrm{T}}R(t)u + x^{\mathrm{T}}L(t)x) + q^{\mathrm{T}}[a_0(t) + a(t)x_1 + B(t)u], \tag{33}$$

where $x_1(x) = x(t-h)$. Applying the maximum principle condition $\partial H/\partial u = 0$ to this specific Hamiltonian function (33) yields

$$\partial H/\partial u = 0 \;\; \Rightarrow \;\; R(t)u(t) + B^{\mathrm{T}}(t)q(t) = 0,$$

and the optimal control law is obtained as

$$u^{*}(t) = -R^{-1}(t)B^{\mathrm{T}}(t)q(t).$$

Taking linearity and causality of the problem into account, let us seek $q(t)$ as a linear function in $x(t)$

$$q(t) = -Q(t)x(t), \tag{34}$$

where $Q(t)$ is a square symmetric matrix of dimension n. This yields the complete form of the optimal control

$$u^{*}(t) = R^{-1}(t)B^{\mathrm{T}}(t)Q(t)x(t). \tag{35}$$

Note that the transversality condition [21,22] for $q(T)$ implies that $q(T) = \partial J/\partial x(T) = \psi x(T)$ and, therefore, $Q(T) = -\psi$.

Using the co-state equation $\mathrm{d}q(t)/\mathrm{d}t = -\partial H/\partial x$ and denoting $(\partial x_1(t)/\partial x) = M_1(t)$ yields

$$-\mathrm{d}q(t)/\mathrm{d}t = L(t)x(t) + a^{\mathrm{T}}(t)M_1^{\mathrm{T}}(t)q(t), \tag{36}$$

and substituting Eq. (34) into Eq. (36), we obtain

$$\dot{Q}(t)x(t) + Q(t)\mathrm{d}(x(t))/\mathrm{d}t = L(t)x(t) - a^{\mathrm{T}}(t)M_1^{\mathrm{T}}(t)Q(t)x(t). \tag{37}$$

Substituting the expression for $\dot{x}(t)$ from the state equation (1) into Eq. (37) yields

$$\dot{Q}(t)x(t) + Q(t)a(t)x(t-h) + Q(t)B(t)u(t) = L(t)x(t) - a^{\mathrm{T}}(t)M_1^{\mathrm{T}}(t)Q(t)x(t). \tag{38}$$

In view of linearity of the problem, differentiating the last expression in x does not imply loss of generality. Upon substituting the optimal control law (35) into Eq. (38), taking into account that $(\partial x(t-h)/\partial x(t)) = M_1(t)$, and differentiating Eq. (38) in x, it is transformed into the quasi-Riccati equation

$$\dot{Q}(t) = L(t) - Q(t)M_1(t)a(t) - a^{\mathrm{T}}(t)M_1^{\mathrm{T}}(t)Q(t) - Q(t)B(t)R^{-1}(t)B^{\mathrm{T}}(t)Q(t) \tag{39}$$

with the terminal condition $Q(T) = -\psi$.

Let us now obtain the value of $M_1(t)$. By definition, $M_1(t) = (\partial x(t-h)/\partial x(t))$. Substituting the optimal control law (35) into Eq. (1) gives

$$\dot{x}(t) = a_0(t) + a(t)x(t-h) + B(t)R^{-1}(t)B^{\mathrm{T}}(t)Q(t)x(t), \tag{40}$$

with the initial condition $x(s) = \phi(s)$, $s \in [t_0 - h, t_0]$. Integrating Eq. (40) yields

$$x(t_0+h) = x(t_0) + \int_{t_0}^{t_0+h} (a_0(s) + a(s)x(s-h))\,\mathrm{d}s + \int_{t_0}^{t_0+h} B(s)R^{-1}(s)B^{\mathrm{T}}(s)Q(s)x(s)\,\mathrm{d}s. \tag{41}$$

Analysis of the formula (41) shows that $x(t)$ does not depend on $x(t-h)$, if $t \in [t_0, t_0+h)$. Therefore, $M_1(t) = 0$ for $t \in [t_0, t_0+h)$. On the other hand, if $t \geqslant t_0 + h$, the following Cauchy formula is valid for the solution $x(t)$ of Eq. (40)

$$x(t) = \Phi(t, t-h)x(t-h) + \int_{t-h}^{t} \Phi(t,s)(a_0(s) + a(s)x(s-h))\,\mathrm{d}s, \tag{42}$$

where $\Phi(t,\tau)$ satisfies the matrix equation

$$\frac{\mathrm{d}\Phi(t,\tau)}{\mathrm{d}t} = B(t)R^{-1}(t)B^{\mathrm{T}}(t)Q(t)\Phi(t,\tau),$$

with the initial condition $\Phi(t,t) = I$, and I is the identity matrix. Expression (39) immediately implies that $M_1(t) = \Phi^{-1}(t, t-h) = \Phi(t-h, t) = \exp(-\int_{t-h}^{t} B(s)R^{-1}(s)B^{\mathrm{T}}(s)$ $Q(s)\,\mathrm{d}s)$ for $t \geqslant t_0 + h$. The optimal control problem solution is proved. $\square$

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