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Numerical method for weights adjustment in minimax multi-model LQ-control

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SUMMARY

The minimax linear quadratic problem, where 'max' is taken over a finite set of indices (models) and 'min' is taken over the set of admissible controls, is considered. The solution is obtained by the robust optimal control application. The control turns out to be a linear combination of the controls optimal for each individual model. This paper develops a numerical method for the optimal weights adjustment. An example shows a quick convergence of the proposed procedure. Copyright © 2007 John Wiley & Sons, Ltd.

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KEY WORDS: minimax control; linear quadratic problem; numerical methods

1. INTRODUCTION

Dealing with designing of control for some uncertain systems, there exist situations when the model of the system cannot be defined exactly since the more adequate model can depend on several possible scenarios. In this case, the control can be designed as a *multi-model control* (see, e.g. [1–4]). To design such control, the minimax approach has been suggested (see [5, 6]) where 'max' is taken over all possible models (scenarios) and 'min' is taken over all admissible controls. Evidently that such control does not depend on an individual model and serves simultaneously for the set of possible scenarios. The minimax approach was generalized in [7, 8]. Specifically the version of the robust maximum principle for a linear quadratic (LQ) problem was considered in details. The minimax problem was considered in the following aspect. One has N linear state dynamics equations (ODE) each of them corresponding a possible model. The same control law should be applied to all of them in such a way that the worst LQ-index would be minimal over

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all admissible controls. Such *robust optimal control* is shown to be a weighted combination of the controls optimal for each individual model. Hence, the problem is reduced to a finite dimensional optimization problem since this robust optimal control depends on a weighting vector belonging to the N-dimensional simplex which should be selected providing a minimal value for the original worst LQ-functional. Finding an analytical expression for this function as a function of the weights seems to be very difficult task. In the simplest cases with two ($N=2$) and three ($N=3$) models such expression can easily obtained in a graphic form using standard PC. However, for more complex situations ($N\geqslant 4$) such approach cannot be realized and any numerical procedure for the corresponding weight adjustment (optimization) is welcomed.

In [9–11] was considered a similar (to the first glance) problem. But, in fact, it differs in many aspects with the problem which we are dealing with. First, the dynamics, considered in the references above, does not contain any exogenous input. Second, they considered only a single plant with a multi-criterion cost functional. In our paper, we are dealing with a really multi-plant (or multi-model dynamic system) but having the same criterion for each possible scenario. Evidently, these two problems have a different philosophical treatment, but, sure, they may be attacked by closed mathematical methods. The main difference between these two considerations consists in the corresponding differential Riccati equations used in the min–max feedback realization: we deal with different matrices A_i and B_i in the differential equation, governing the considered dynamics, but with the same matrix R (the control-cost matrix); in the papers, cited above, the authors have only a single matrix A and a single matrix B but several R_i. The corresponding weight dependence in both cases is absolutely different since we have the right-hand side of the Riccati ODE which is linearly dependent on weights, and in the cited papers this dependence is essentially nonlinear that significantly complicates the analysis of the weight-adjustment procedure. Moreover, there is no any convergence analysis of the suggested iterative weight-adjustment procedure in the papers referred above (only [11] contains some brief scheme of the convergence analysis, but not a proof).

In this paper, we suggest a new numerical (iterative) method which provides a quick convergence of the weighting vector to its optimal value. This numerical method practically makes workable the realization of the robust optimal control suggested in [7, 8] and complements the results given in [12–14].

1.1. Structure of the paper

In Section 2, the model description is presented and the purpose of the control law is formulated. A preliminary result, needed for the following consideration, is given in Section 3. Then the iterative numerical procedure is suggested. The convergence analysis of this numerical procedure is given in Section 4. An example, illustrating a quick convergence of the method, is presented in Section 5.

2. MOTIVATION AND PROBLEM STATEMENT

Let us consider a set of linear state models given by

$$\dot{x}^{\alpha}(t)=A^{\alpha}(t)x^{\alpha}(t)+B^{\alpha}(t)u(t)+d^{\alpha}(t), \quad x^{\alpha}(0)=x_0^{\alpha} \tag{1}$$

where the index α belongs to a finite set, that is, $\alpha\in\overline{1,N}$, (N is a positive integer), $x^{\alpha}(t), d^{\alpha}(t)\in R^n$, $u(t)\in\mathbb{R}^m$ and $A^{\alpha}(t)$, $B^{\alpha}(t)$, $d^{\alpha}(t)$ are continuous functions on $t\in[0,T]$. Let us define the

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performance index as

$$h^\alpha := \frac{1}{2}(x^\alpha(T), G^\alpha x^\alpha(T)) + \frac{1}{2}\int_{t=0}^{T} [(x^\alpha(t), Q^\alpha x^\alpha(t)) + (u(t), Ru(t))]\,dt \tag{2}$$

where $Q^\alpha \geqslant 0$, $G^\alpha \geqslant 0$, $R > 0$. The minimax LQ control problem was formulated in [8] as

$$\max_{\alpha \in \overline{1,N}} h^\alpha \longrightarrow \min_{u \in \mathbb{R}^m} \tag{3}$$

The solution of this problem, also given in [8], is as follows. Define the following extended system:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(\mathbf{x}, t) + \mathbf{d}$$

where

$$\mathbf{x} := \begin{bmatrix} x^1(t) \\ \vdots \\ x^N(t) \end{bmatrix}, \quad \mathbf{A} := \begin{bmatrix} A^1(t) & 0\cdots & 0 \\ 0 & & \vdots \\ \vdots & \ddots & 0 \\ 0 & \cdots 0 & A^N(t) \end{bmatrix}, \quad \mathbf{B} := \begin{bmatrix} B^1(t) \\ \vdots \\ B^N(t) \end{bmatrix}, \quad \mathbf{d} := \begin{bmatrix} d^1(t) \\ \vdots \\ d^N(t) \end{bmatrix}$$
$$\mathbf{Q} := \begin{bmatrix} Q_1 & 0\cdots & 0 \\ 0 & & \vdots \\ \vdots & \ddots & 0 \\ 0 & \cdots 0 & Q_N \end{bmatrix}, \quad \mathbf{G} := \begin{bmatrix} G_1 & 0\cdots & 0 \\ 0 & & \vdots \\ \vdots & \ddots & 0 \\ 0 & \cdots 0 & G_N \end{bmatrix}, \quad \mathbf{\Lambda} := \begin{bmatrix} \lambda_1 I_{n\times n} & 0\cdots & 0 \\ 0 & & \vdots \\ \vdots & \ddots & 0 \\ 0 & \cdots 0 & \lambda_N I_{n\times n} \end{bmatrix} \tag{4}$$

with $\lambda = (\lambda_1, \ldots, \lambda_N) \in S^N$

$$S^N = \left\{ \lambda \in \mathbb{R}^N : \lambda_i \geqslant 0, \sum_{i=1}^{N} \lambda_i = 1 \right\}$$

Then, the *robust optimal control* realizing (3) is of the form

$$u = -R^{-1}\mathbf{B}^{\mathrm{T}}(\mathbf{P}_\lambda \mathbf{x} + \mathbf{p}_\lambda) \tag{5}$$

where the matrix $\mathbf{P}_\lambda = \mathbf{P}_\lambda^{\mathrm{T}} \in R^{nN\times nN}$ is the solution of the parameterized differential matrix Riccati equation:

$$\dot{\mathbf{P}}_\lambda + \mathbf{P}_\lambda \mathbf{A} + \mathbf{A}^{\mathrm{T}}\mathbf{P}_\lambda - \mathbf{P}_\lambda \mathbf{B} R^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{P}_\lambda + \mathbf{\Lambda}\mathbf{Q} = 0, \quad \mathbf{P}_\lambda(T) = \mathbf{\Lambda}\mathbf{G} \tag{6}$$

and the shifting vector $\mathbf{p}_\lambda$ satisfies

$$\dot{\mathbf{p}}_\lambda + \mathbf{A}^{\mathrm{T}}\mathbf{p}_\lambda - \mathbf{P}_\lambda \mathbf{B} R^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{p}_\lambda + \mathbf{P}_\lambda \mathbf{d} = \mathbf{0}, \quad \mathbf{p}_\lambda(T) = 0$$

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Thus, the solution of (3) is reduced to the finding of the optimal weighting vector λ^* which solves the following finite dimensional optimization problem:

$$\lambda^* = \arg\min_{\lambda \in S^N} J(\lambda) \tag{7}$$

$$\begin{aligned} J(\lambda) := & \max_{\alpha=\overline{1,N}} h^\alpha \\ = & \frac{1}{2}\mathbf{x}^{\mathrm{T}}(0)\mathbf{P}_\lambda(0)\mathbf{x}(0) + \mathbf{x}^{\mathrm{T}}(0)\mathbf{p}_\lambda(0) \\ & + \frac{1}{2}\max_{\alpha=\overline{1,N}}\left[x^{\alpha\mathrm{T}}(T)G^\alpha x^\alpha(T) + \int_{t=0}^{\mathrm{T}} x^{\alpha\mathrm{T}}(t)Q^\alpha x^\alpha(t)\,\mathrm{d}t\right] \\ & - \frac{1}{2}\sum_{\alpha=1}^{N}\lambda_i\left[x^{\alpha\mathrm{T}}(T)G^\alpha x^\alpha(T) + \int_{t=0}^{\mathrm{T}} x^{\alpha\mathrm{T}}(t)Q^\alpha x^\alpha(t)\,\mathrm{d}t\right] \\ & + \frac{1}{2}\int_{t=0}^{\mathrm{T}} \mathbf{p}_\lambda^{\mathrm{T}}[2\mathbf{d} - \mathbf{B}R^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{p}_\lambda]\,\mathrm{d}t \end{aligned} \tag{8}$$

Thus, the *goal of this paper* is to develop a numerical method which allows to find the optimal weighting vector λ^* for any finite number N of possible models (or, scenarios).

3. PRELIMINARY RESULTS

Lemma 1

Let λ^* be a minimum point, that is, $J(\lambda^*) \leqslant J(\lambda)$ for all $\lambda \in S^N$. Then, for any active index $\alpha \in \overline{1, N}$ such that $1 \geqslant \lambda_\alpha^* > 0$, the functional $h^\alpha(\lambda^*)$ satisfies the following equality:

$$h^\alpha(\lambda^*) = J(\lambda^*) \tag{9}$$

and for all inactive indices α such that $\lambda_\alpha^* = 0$

$$h^\alpha(\lambda^*) \leqslant J(\lambda^*) \tag{10}$$

Proof

If for some $j \in \overline{1, N}$ we have $h^j(\lambda^*) > J(\lambda^*)$, then

$$J(\lambda^*) = \max_{\alpha \in \overline{1,N}} h^\alpha(\lambda^*) \geqslant h^j(\lambda^*) > J(\lambda^*)$$

that leads to a contradiction. So, for all indices α it follows $h^\alpha(\lambda^*) \leqslant J(\lambda^*)$. Result (9) for active indices follows directly from the complementary slackness condition established in [8]. □

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Corollary 1
The optimal performance index $J(\lambda^*)$ can be represented as

$$J(\lambda^*) = \frac{1}{2}\mathbf{x}^{\mathrm{T}}(0)\mathbf{P}_{\lambda^*}(0)\mathbf{x}(0) + \mathbf{x}^{\mathrm{T}}(0)\mathbf{p}_{\lambda^*}(0) + \frac{1}{2}\int_{t=0}^{\mathrm{T}} \mathbf{p}_{\lambda^*}^{\mathrm{T}}[2\mathbf{d} - \mathbf{B}R^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{p}_{\lambda^*}]\,\mathrm{d}t \tag{11}$$

Proof
Adding and subtracting the integral of $u^{\mathrm{T}}(t)Ru(t)$ in (8), we get

$$J(\lambda) = \frac{1}{2}\mathbf{x}^{\mathrm{T}}(0)\mathbf{P}_{\lambda}(0)\mathbf{x}(0) + \mathbf{x}^{\mathrm{T}}(0)\mathbf{p}_{\lambda}(0) + \left[J(\lambda) - \sum_{i=1}^{N}\lambda_\alpha h^\alpha(\lambda)\right] + \frac{1}{2}\int_{t=0}^{\mathrm{T}} \mathbf{p}_{\lambda}^{\mathrm{T}}[2\mathbf{d} - \mathbf{B}R^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{p}_{\lambda}]\,\mathrm{d}t$$

Therefore, taking $\lambda=\lambda^*$, in view of Lemma 1, specifically (9), and since $\sum_{\alpha=1}^{N}\lambda_\alpha = 1$, we find that $J(\lambda^*) = \sum_{i=1}^{N}\lambda_\alpha^* h^\alpha(\lambda^*)$. Hence the performance index $J(\lambda^*)$ is as in (11). □

Corollary 2
If the vector λ^* is a minimum point, then for any $\gamma>0$

$$\lambda^* = \pi\{\lambda^* + \gamma F(\lambda^*)\} \tag{12}$$

where $\pi\{\cdot\}$ is the projector to the simplex S^N, that is,

$$\|\pi\{x\} - x\| < \|\lambda - x\| \quad \text{for any } \lambda\in S^N, \quad \lambda\neq\pi\{x\}$$

and $F(\lambda)\in\mathbb{R}^N$ is the vector whose ith term is the performance functional h^i, i.e.

$$F(\lambda) = \begin{bmatrix} h^1(\lambda) \\ \vdots \\ h^N(\lambda) \end{bmatrix}$$

Proof
Since S^N is a closed convex set, the following property holds:

$$\text{for any } x\in\mathbb{R}^n, \quad \mu=\pi\{x\} \iff (x-\mu, \lambda-\mu)\leqslant 0 \quad \text{for all } \lambda\in S^N \tag{13}$$

Let $\lambda_{i_j}^*$, $j=\overline{1,r}$ be the components of λ^* different from zero and $\lambda_{i_k}^*$ $k=\overline{r+1,N}$ the components of λ^* equal to zero. Thus, taking into account Lemma 1 and, since $\lambda_{i_k} - \lambda_{i_k}^* \geqslant 0$ $(\lambda_{i_k}^*=0)$, we obtain

$$(\lambda^* + \gamma F(\lambda^*) - \lambda^*, \lambda - \lambda^*) = \gamma\left[J(\lambda^*)\sum_{j=1}^{r}(\lambda_{i_j} - \lambda_{i_j}^*) + \sum_{k=r+1}^{N} h^{i_k}(\lambda^*)(\lambda_{i_k} - \lambda_{i_k}^*)\right]$$

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$$\leqslant \gamma J(\lambda^*)\left[\sum_{j=1}^{r}(\lambda_{i_j}-\lambda_{i_j}^*)+\sum_{k=r+1}^{N}(\lambda_{i_k}-\lambda_{i_k}^*)\right]=\gamma J(\lambda^*)\sum_{j=1}^{N}(\lambda_{i_j}-\lambda_{i_j}^*)=0 \tag{14}$$

for all $\lambda\in S^N$. Thus, by (13), (14) implies $\lambda^*=\pi\{\lambda^*+\gamma F(\lambda^*)\}$. □

4. NUMERICAL PROCEDURE FOR THE WEIGHTS ADJUSTMENT

In [8], there was shown that the control $u(\mathbf{x},t)$ designed as in (3) is the combination (where the weights are the components λ_α) of the controls optimal for each individual model. Hence, it seems to be clear that a bigger weight λ_α of the control, optimizing the α-model, implies a better (smaller) performance index $h^\alpha(\lambda)$. This fact may be expressed in the following manner: if $\lambda'_\alpha\neq\lambda''_\alpha$

$$(\lambda'_\alpha-\lambda''_\alpha)[h^\alpha(\lambda')-h^\alpha(\lambda'')]<0 \tag{15}$$

for any $\lambda'\neq\lambda''\in S^N$. Summing (15) on $\alpha\in\overline{1,N}$ leads to the following condition which we will accept as the assumption.

Assumption 1
For any $\lambda'\neq\lambda''\in S^N$, the following inequality holds:

$$(\lambda'-\lambda'',F(\lambda')-F(\lambda''))<0 \tag{16}$$

and the identity in (16) is possible if and only if $\lambda'=\lambda''$.

Proposition 1
Under Assumption 1, the functional $J(\lambda)$ has a unique minimum point λ^*.

Proof
We will show that if $\tilde\lambda$ differs with λ^*, then $\tilde\lambda$ does not satisfy identity (12). If $\tilde\lambda\neq\lambda^*$, then (14) implies

$$\begin{aligned}&(\tilde\lambda+\gamma F(\tilde\lambda)-\tilde\lambda,\lambda^*-\tilde\lambda)\\&\quad\geqslant\gamma[(F(\tilde\lambda),\lambda^*-\tilde\lambda)+(F(\lambda^*),\tilde\lambda-\lambda^*)]=\gamma(F(\tilde\lambda)-F(\lambda^*),\lambda^*-\tilde\lambda)\end{aligned} \tag{17}$$

On the other hand, Assumption 1 yields

$$\gamma(F(\tilde\lambda)-F(\lambda^*),\lambda^*-\tilde\lambda)=-\gamma(\tilde\lambda-\lambda^*,F(\tilde\lambda)-F(\lambda^*))>0 \tag{18}$$

Thus, both (17) and (18) imply

$$(\tilde\lambda+\gamma F(\tilde\lambda)-\tilde\lambda,\lambda^*-\tilde\lambda)>0 \tag{19}$$

But (19) means that $\tilde\lambda\neq\pi\{\tilde\lambda+\gamma F(\tilde\lambda)\}$ (see (13)). Therefore, by Corollary 2, we deduce that $\tilde\lambda$ is not a minimum point. □

Now, we are ready to present a numerical method for the adjustment of the weight vector λ.

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4.1. Numerical method

Define the sequence of vectors $\{\lambda^k\}$ as

$$
\begin{aligned}
\lambda^{k+1} &= \pi\left\{\lambda^k + \frac{\gamma^k}{J(\lambda^k)+\varepsilon}F(\lambda^k)\right\}, \quad \lambda^0 \in S^N, \quad k = 0, 1, 2, \ldots \\
F(\lambda^k) &= [h^1(\lambda^k) \ \cdots \ h^N(\lambda^k)]^{\mathrm{T}} \\
J(\lambda^k) &:= \max_{\alpha \in \overline{1,N}} h^\alpha(\lambda^k)
\end{aligned} \tag{20}
$$

where ε is an arbitrary strictly positive (small enough) constant.

Theorem 1
Let λ^* be the minimum point for $J(\lambda)$. If

(1) the sequence $\{\lambda^k\}$ is generated by (20);
(2) Assumption 1 holds;
(3) there exists a constant L such that for all $\alpha \in \overline{1, N}$ and for any $\mu, \lambda \in S^N$

$$|h^\alpha(\mu) - h^\alpha(\lambda)| \leqslant J(\lambda) L |\mu - \lambda|$$

(4) the gain sequence $\{\gamma^k\}$ satisfies

$$\gamma^k > 0, \quad \sum_{k=0}^{\infty} \gamma^k = \infty, \quad \sum_{k=0}^{\infty} (\gamma^k)^2 < \infty$$

then

$$\lim_{k\to\infty} \lambda^k = \lambda^* \tag{21}$$

Proof
For $v^k := \lambda^k - \lambda^*$, in view of (12) and the property of projection $\|\pi\{x\} - \pi\{y\}\| \leqslant \|x - y\|$ for all $x, y \in \mathbb{R}^N$, we have

$$
\begin{aligned}
\|v^{k+1}\|^2 &= \left\| \pi\left\{\lambda^k + \frac{\gamma^k}{J(\lambda^k)+\varepsilon}F(\lambda^k)\right\} - \lambda^* \right\|^2 \\
&= \left\| \pi\left\{\lambda^k + \frac{\gamma^k}{J(\lambda^k)+\varepsilon}F(\lambda^k)\right\} - \pi\left\{\lambda^* + \frac{\gamma^k}{J(\lambda^k)+\varepsilon}F(\lambda^*)\right\} \right\|^2 \\
&\leqslant \left\| v^k + \frac{\gamma^k}{J(\lambda^k)+\varepsilon}[F(\lambda^k) - F(\lambda^*)] \right\|^2 \\
&= \|v^k\|^2 + \frac{(\gamma^k)^2}{J^2(\lambda^k) + 2J(\lambda^k)\varepsilon + \varepsilon^2}\|F(\lambda^k) - F(\lambda^*)\|^2
\end{aligned}
$$

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$$
\begin{aligned}
&+2\frac{\gamma^k}{J(\lambda^k)+\varepsilon}(v^k, F(\lambda^k)-F(\lambda^*)) \\
&\leqslant \|v^k\|^2(1+(\gamma^k)^2L^2)+2\frac{\gamma^k}{J(\lambda^k)+\varepsilon}(v^k, F(\lambda^k)-F(\lambda^*)) \\
&\leqslant \|v^k\|^2(1+(\gamma^k)^2L^2) \qquad (22)
\end{aligned}
$$

In the last inequality of (22) we have used Assumption 1. Define the new variable w^k by

$$w^k := \|v^k\|^2 \prod_{s=k}^{\infty}[1+(\gamma^s)^2L^2]$$

So, (22) implies

$$
\begin{aligned}
w^{k+1} &:= \|v^{k+1}\|^2 \prod_{s=k+1}^{\infty}[1+(\gamma^s)^2L^2] \\
&\leqslant \|v^k\|^2(1+(\gamma^k)^2L^2)\prod_{s=k+1}^{\infty}[1+(\gamma^s)^2L^2]=w^k
\end{aligned}
$$

which means (by the Weierstrass theorem) that the sequence $\{w^k\}$ converges and, hence, there exists the limit

$$w := \lim_{k\to\infty} w^k = \lim_{k\to\infty}\|v^k\|^2$$

But, from (22), we have also the inequality

$$
\begin{aligned}
2\frac{\gamma^k}{J(\lambda^k)+\varepsilon}|(v^k, F(\lambda^k)-F(\lambda^*))| &\leqslant \|v^k\|^2(1+(\gamma^k)^2L^2)-\|v^{k+1}\|^2 \\
&= \frac{w^k-w^{k+1}}{\prod_{s=k+1}^{\infty}[1+(\gamma^s)^2L^2]} \leqslant w^k-w^{k+1}
\end{aligned}
$$

Summation it by k from 0 up to ∞ yields

$$2\sum_{k=0}^{\infty}\gamma^k\frac{|(v^k, F(\lambda^k)-F(\lambda^*))|}{J(\lambda^k)+\varepsilon} \leqslant w^0-w<\infty$$

In view of the property $\sum_{k=0}^{\infty}\gamma^k=\infty$, it follows that there exists a subsequence k_t $(t=1,2,\ldots)$ such that

$$\frac{|(v^{k_t}, F(\lambda^{k_t})-F(\lambda^*))|}{J(\lambda^{k_t})+\varepsilon} \underset{t\to\infty}{\longrightarrow} 0$$

Since $J(\lambda^{k_t})$ is bounded, then, by (16), this implies $\lambda^{k_t} \underset{t\to\infty}{\longrightarrow} \lambda^*$, or, equivalently,

$$\lim_{t\to\infty} w^{k_t} = \lim_{t\to\infty}\|v^{k_t}\|^2=0$$

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But $\{w^k\}$ converges to w. So all its subsequences converges to the same limit that implies $w=0$. Theorem is proven. □

5. EXAMPLE

The following example illustrates the proposed numerical method (20) in the case $N=3$ where the parameters of possible models are as follows:

$$A_1=\begin{bmatrix}-2 & 0.5 & 1\\ 0.5 & 1.2 & -2\\ 1 & 2 & -1.5\end{bmatrix},\quad A_2=\begin{bmatrix}-0.3 & 1.5 & -0.15\\ -1 & 0.12 & 2\\ 1 & 2 & -3\end{bmatrix},\quad A_3=\begin{bmatrix}0.4 & -1 & 0.3\\ 0.5 & -0.4 & 0.3\\ 0.5 & 0.6 & -1\end{bmatrix}$$

$$B_1=\begin{bmatrix}0.5\\ 1\\ 1\end{bmatrix},\quad B_2=\begin{bmatrix}1.5\\ -2\\ 1\end{bmatrix},\quad B_3=\begin{bmatrix}0.5\\ 0.2\\ 1\end{bmatrix}$$

$$d^1=\begin{bmatrix}0.1\\ 0.05\\ 0.01\end{bmatrix},\quad d^2=\begin{bmatrix}0.1\sin(t)\\ 0.2\sin(t/2)\\ 0.1\end{bmatrix},\quad d^3=\begin{bmatrix}0.1\\ 0.05\cos(t)\\ 0.1\end{bmatrix}$$

Table I. Values of λ^k and $h^\alpha(\lambda^k)$.

k	λ_1	λ_2	λ_3	h^1	h^2	h^3	J
1	0.5	0.4	0.1	116.060	109.897	977.037	977.037
2	0.208365	0.102057	0.689576	236.303	1116.18	310.533	1116.18
3	0.065899	0.353738	0.580362	489.798	349.164	503.783	503.783
4	0.093832	0.288619	0.617548	360.415	462.742	460.030	462.742
5	0.057465	0.307535	0.634999	555.476	432.953	451.644	555.476
6	0.084632	0.290587	0.624780	393.174	461.878	455.737	461.878
7	0.068843	0.299589	0.631567	471.404	447.301	453.104	471.404
8	0.073125	0.296569	0.630305	446.649	452.593	453.386	453.386
9	0.071960	0.297042	0.630997	453.083	451.886	452.979	453.083
10	0.072066	0.296855	0.631077	452.488	452.251	452.875	452.875
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
31	0.072036	0.296664	0.631299	452.660	452.656	452.665	452.665
32	0.072036	0.296664	0.631299	452.661	452.657	452.664	452.664
33	0.072036	0.296663	0.631300	452.666	452.658	452.664	452.665
34	0.072036	0.296663	0.631300	452.661	452.658	452.664	452.664
35	0.072036	0.296663	0.631300	452.660	452.658	452.664	452.664
36	0.072035	0.296663	0.631300	452.660	452.660	452.663	452.663
37	0.072035	0.296663	0.631300	452.661	452.659	452.663	452.663
38	0.072035	0.296663	0.631300	452.660	452.660	452.663	452.663
39	0.072035	0.296663	0.631301	452.662	452.659	452.663	452.663
40	0.072035	0.296663	0.631301	452.662	452.659	452.663	452.663

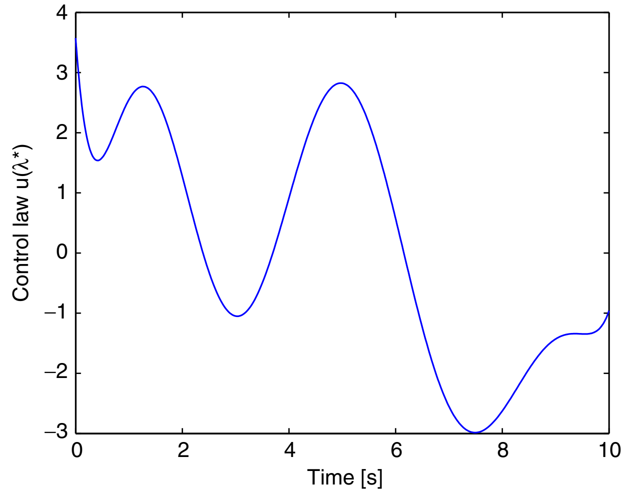

Figure 1. Control law u for λ^*.

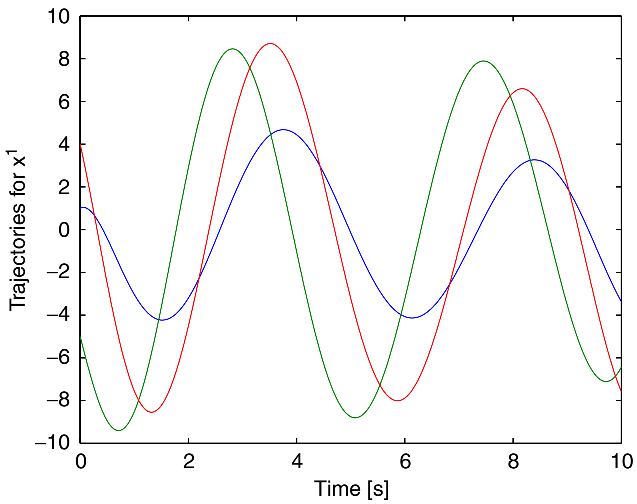

Figure 2. Trajectories of the state corresponding to $\alpha = 1$.

We select the matrices $Q^\alpha = G^\alpha = I$, $R = 1$. Using the gain-step sequence $\{\gamma^k\}$ (20) with $\gamma^k = 1/(k+1)$, $k = 0, 1, 2, \ldots,$ we obtained the results presented in Table I. There are shown the values of the vector λ^k and the performance index $h^\alpha(\lambda^k)$ for each iteration k.

From Table I, one can see that the weights practically converge after 10 iterations. Since all indices are active ($\lambda^*_\alpha > 0$), all performance functional $h^\alpha(\lambda^k)$ practically turn out to be equal after

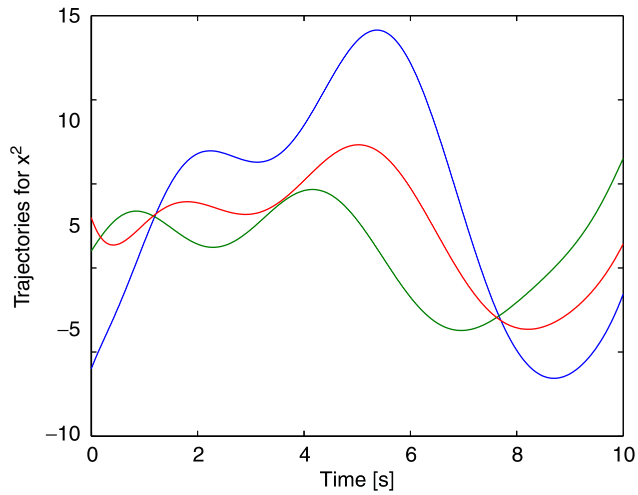

Figure 3. Trajectories of the state corresponding to $\alpha = 2$.

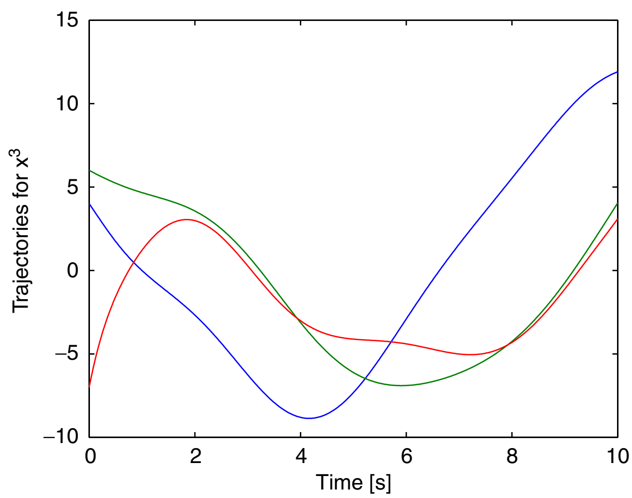

Figure 4. Trajectories of the state corresponding to $\alpha = 3$.

40 iterations. Thus, we have $\lambda^* \approxeq (0.072035, 0.296663, 0.631301)$. The control law $u = u(\lambda^*)$ is depicted in Figure 1. Figures 2–4 show the trajectories of x^α for $\alpha = 1, 2, 3$.

6. CONCLUSIONS

In this paper, the numerical method for finding the optimal weights in the robust optimal control representation is proposed. The suggested procedure is workable without any changes in the

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gain-step sequence and shows a quick convergence. It may be successfully applied also for multi-model stochastic LQ-control [15].

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