

# OPTIMAL AND ROBUST SLIDING MODE CONTROL FOR LINEAR SYSTEMS WITH MULTIPLE TIME DELAYS IN CONTROL INPUT

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## ABSTRACT

This paper presents the optimal regulator for a linear system with multiple time delays in control input and a quadratic criterion. The optimal regulator equations are obtained using the duality principle, which is applied to the optimal filter for linear systems with multiple time delays in observations. Performance of the obtained optimal regulator is verified in the illustrative example against the best linear regulator available for linear systems without delays. Simulation graphs and comparison tables demonstrating better performance of the obtained optimal regulator are included. The paper then presents a robustification algorithm for the obtained optimal regulator based on integral sliding mode compensation of disturbances. The general principles of the integral sliding mode compensator design are modified to yield the basic control algorithm oriented to time-delay systems, which is then applied to robustify the optimal regulator. As a result, the sliding mode compensating control leading to suppression of the disturbances from the initial time moment is designed. The obtained robust control algorithm is verified by simulations in the illustrative example.

**Keywords:** Linear system, multiple delays, optimal control, filtering, sliding mode regulator.

## 1. INTRODUCTION

Although the optimal control (regulator) problem for linear system states was solved, as well as the filtering one, in 1960s [15,22], the optimal control problem for linear systems with delays is still open, depending on the delay type, specific system equations, criterion, etc. Such complete reference books in the area as [7,10,20, 21,25] note, discussing the maximum principle [19] or

the dynamic programming method [26] for systems with delays, that finding a particular explicit form of the optimal control function might still remain difficult. A particular form of the criterion must be also taken into account: the studies mostly focused on the time-optimal criterion (see the paper [27] for linear systems) or the quadratic one [9,13,34]. Virtually all studies of the optimal control in time-delay systems are related to systems with delays in the state (see, for example, [1]), although the case of delays in control input is no less challenging, if the control function should be causal, i.e., does not depend on the future values of the state. A considerable bibliography existing for the robust control problem for time delay systems (such as [12,24]) is not discussed here.

The first part of this paper concentrates on the solution of the optimal control problem for a linear system with multiple time delays in control input and a quadratic criterion, which is based on the duality principle in a closed-form situation [3] applied to the optimal filter for linear systems with multiple time delays in observations recently obtained in [5]. Taking into account that the optimal control problem can be solved in the linear case [22] applying the duality principle to the solution of

Manuscript received January 15, 2003; revised April 23, 2003; accepted October 16, 2003.

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The authors thank the Mexican National Science and Technology Council (CONACyT) for financial support in the framework of grant 39388-A.

the optimal filtering problem, this paper exploits the same idea for designing the optimal control in a linear system with multiple time delays in control input, using the optimal filter for linear systems with multiple time delays in observations. In doing so, the optimal regulator gain matrix is constructed as dual transpose to the optimal filter gain one and the optimal regulator gain equation is obtained as dual to the variance equation in the optimal filter. The results obtained by virtue of the duality principle can be rigorously verified through the general equations of the maximum principle [19,29] or the dynamic programming method [6,26] applied to a specific time-delay case, although the physical duality seems obvious: if the optimal filter exists in a closed form, the optimal closed-form regulator should also exist, and vice versa [3]. It should be noted, however, that application of the maximum principle to the present case gives one only a system of state and co-state equations and does not provide the explicit form of the optimal control or co-state vector. So, the duality principle approach actually provides one with the explicit form of the optimal control and co-state vector, which should be then substituted into the equations given by the rigorous optimality tools and thereby verified.

Finally, performance of the obtained optimal control for a linear system with multiple time delays in control input and a quadratic criterion is verified in the illustrative example against the best linear regulator available for linear systems without delays. The simulation results show a definitive advantage of the obtained optimal regulator in both the criterion value and the value of the controlled variable.

The paper then presents a robustification algorithm for the obtained optimal regulator based on integral sliding mode compensation of disturbances. Conventional (non-integral) sliding modes are widely used for uncertainties compensation (see, for example, [35]). On the other hand, time delay effects that take place in relay and sliding mode control systems must be taken into account for the systems analysis and control design [2,35].

It is also known that time delays do not allow to design the sliding mode control in the space of state variables. Moreover, papers [16,17] show that even in the simplest one-dimensional delayed relay control system only oscillatory solutions can occur. That is why there are the two following main research directions in sliding mode uncertainties compensation for delay systems.

### 1.1 Time delay compensation

Pade approximation of delay reducing the relay delay output tracking problem to the sliding mode control for nonminimum phase systems was suggested in [33]. In [11,23,31], the sliding mode control in the space

of predictor variables was realized. However, subsequent research [18,32] has demonstrated that the conventional sliding mode control design in the space of predictor variables

- cannot compensate even for the matching uncertainties;
- in the case of square systems, if the dimensions of state space and control are the same, sliding mode design in the space of predictor variables can remove the uncertainties in the space of predictor variables but cannot guarantee suppression of uncertainties in the space of state variables.

### 1.2. Sliding mode control design via feedback control

In a series of papers (see for example, [30,28]), the conventional sliding modes in the space of state variables of delayed systems are specifically used for elimination of uncertainties. Recently, application of the integral sliding mode to time-delay systems has been initiated: in [4], the integral sliding mode is used for robustification of optimal filters over observations with delay.

The second part of this paper presents an integral sliding mode regulator robustifying the optimal regulator for linear systems with multiple time delays in control input. The idea is to add a compensator to the known optimal control to suppress external disturbances deteriorating the optimal system behavior [8,35]. The integral sliding mode compensator is realized as a relay control in a such way that the sliding mode motion starts from the initial moment, thus eliminating the external disturbances from the beginning of system functioning. This constitutes the crucial advantage of the integral sliding modes in comparison to the conventional ones.

The paper is organized as follows. Section 2 states the optimal control problem for a linear system with multiple time delays in control input and describes the duality principle for a closed-form situation [3]. For reference purposes, the optimal filtering equations for a linear state and linear observations with multiple time delays [5] are briefly reminded in Section 3. The optimal control problem for a linear system with multiple time delays in control input is solved in Section 4, based on application of the duality principle to the optimal filter of the preceding section. The paper then presents a robustification algorithm for the obtained optimal regulator based on integral sliding mode compensation of disturbances [35]. Section 5 presents the new general principles of the integral sliding mode compensator design, which yield the basic control algorithm oriented to time-delay systems. This basic algorithm is then applied to robustify the optimal regulator. As a result, the sliding mode compensating control leading to suppression of the disturbances from the initial time moment is designed.

Section 6 presents an example illustrating the quality of control provided by the obtained optimal regulator for linear systems with multiple time delays in control input against the best linear regulator available for systems without delays. Simulation graphs and comparison tables demonstrating better performance of the obtained optimal regulator are included. This section then presents an example illustrating the quality of disturbance suppression provided by the obtained robust integral sliding mode regulator against the optimal regulator under the presence of disturbances. Satisfactory results are obtained.

## II. OPTIMAL CONTROL PROBLEM FOR LINEAR SYSTEM WITH MULTIPLE TIME DELAYS IN CONTROL INPUT

Consider a linear system with time delay in control input

$$\dot{x}(t) = a_0(t)x(t) + B(t)u(t) + \sum_{i=1}^p B_i(t)u(t-h_i), \quad (1)$$

with the initial condition  $x(s) = \phi(s)$ ,  $s \in [-h, 0]$ . Here,  $x(t) \in R^n$  is the system state,  $u(t) \in R^m$  is the control variable,  $h = \max\{h_1, \dots, h_p\}$  is the maximum delay shift,  $\phi(s)$  is a piecewise continuous function given in the interval  $[-h, 0]$ , and  $a_0(t)$ ,  $B(t)$ , and  $B_i(t)$  are piecewise continuous matrix functions of appropriate dimensions. Existence of the unique solution of the equation (1) is thus assured by the Caratheodori theorem (see, for example, [14]).

The control function  $u(t)$  regulates the system state by fusing the values of  $u(t)$  at various time moments  $t - h_i$ ,  $i = 1, \dots, p$ , where  $h_i$  are delay shifts, as well as at the current time  $t$ , which assumes that the current system state depends not only on the current value of  $u(t)$  but also on its values after certain time lags  $h_i$ ,  $i = 1, \dots, p$ . This situation is frequently encountered in, for example, network control systems.

The quadratic cost function to be minimized is defined as follows:

$$J = \frac{1}{2}[x(T) - x_1]^T \Psi [x(T) - x_1] + \frac{1}{2} \int_{t_0}^T u^T(s) R(s) u(s) ds + \frac{1}{2} \int_{t_0}^T x^T(s) L(s) x(s) ds, \quad (2)$$

where  $x_1$  is a given vector,  $R$  is positive and  $\Psi$ ,  $L$  are nonnegative definite symmetric matrices, and  $T > t_0$  is a certain time moment.

The optimal control problem is to find the control  $u(t)$ ,  $t \in [t_0, T]$ , that minimizes the criterion  $J$  along with

the trajectory  $x^*(t)$ ,  $t \in [t_0, T]$ , generated upon substituting  $u^*(t)$  into the state equation (1). To find the solution to this optimal control problem, the duality principle [22] can be used. For linear systems without delay, if the optimal control exists in the optimal control problem for a linear system with the quadratic cost function  $J$ , the optimal filter exists for the dual linear system with Gaussian disturbances and can be found from the optimal control problem solution, using simple algebraic transformations (duality between the gain matrices and between the gain matrix and variance equations), and vice versa (see [22]). Taking into account the physical duality of the filtering and control problems, the last conjecture should be valid for all cases where the optimal control (or, vice versa, the optimal filter) exists in a closed finite-dimensional form [3]. This proposition is now applied to the optimal filtering problem for linear system states over observations with multiple time delays, which is dual to the stated optimal control problem (1),(2), and where the optimal filter has already been obtained (see [5]).

## III. OPTIMAL FILTER FOR LINEAR STATE EQUATION AND LINEAR OBSERVATIONS WITH MULTIPLE DELAYS

In this section, the optimal filtering equations for a linear state equation over linear observations with multiple time delays (obtained in [5]) are briefly reminded for reference purposes. Let the unobservable random process  $x(t)$  be described by an ordinary differential equation for the dynamic system state

$$dx(t) = (a_0(t) + a(t)x(t))dt + b(t)dW_1(t), \quad x(t_0) = x_0, \quad (3)$$

and a delay-differential equation be given for the observation process:

$$dy(t) = (A_0(t) + A(t)x(t) + \sum_{i=1}^p A_i(t)x(t-h_i))dt + F(t)dW_2(t), \quad (4)$$

where  $x(t) \in R^n$  is the state vector,  $y(t) \in R^m$  is the observation process, the initial condition  $x_0 \in R^n$  is a Gaussian vector such that  $x_0$ ,  $W_1(t)$ ,  $W_2(t)$  are independent. The observation process  $y(t)$  depends on delayed states  $x(t-h_i)$ ,  $i = 1, \dots, p$ , where  $h_i$  are delay shifts, as well as non-delayed state  $x(t)$ , which assumes that collection of information on the system state for the observation purposes is made not only at the current time but also after certain time lags  $h_i$ ,  $i = 1, \dots, p$ . The vector-valued function  $a_0(s)$  describes the effect of system inputs (controls and disturbances). It is assumed that  $A(t)$  is a nonzero matrix and  $F(t)F^T(t)$  is a positive definite matrix. All coefficients in (3)-(4) are deterministic functions of ap-

appropriate dimensions.

The estimation problem is to find the estimate of the system state  $x(t)$  based on the observation process  $Y(t) = \{y(s), 0 \leq s \leq t\}$ , which minimizes the Euclidean 2-norm

$$J = E[(x(t) - \hat{x}(t))^T (x(t) - \hat{x}(t))]$$

at each time moment  $t$ . In other words, our objective is to find the conditional expectation

$$m(t) = \hat{x}(t) = E(x(t) | F_t^Y).$$

As usual, the matrix function

$$P(t) = E[(x(t) - m(t))(x(t) - m(t))^T | F_t^Y]$$

is the estimate variance.

The solution to the stated problem is given by the following system of filtering equations, which is closed with respect to the introduced variables,  $m(t)$  and  $P(t)$ :

$$\begin{aligned} dm(t) = & (a_0(t) + a(t)m(t))dt + P(t)[A^T(t) \\ & + \sum_{i=1}^p \exp(-\int_{t-h_i}^t a^T(s)ds)A_i^T(t)] \\ & \times (F(t)F^T(t))^{-1}(dy(t) - (A_0(t) + A(t)m(t) \\ & + \sum_{i=1}^p A_i(t)m(t-h_i))dt). \end{aligned} \quad (5)$$

$$\begin{aligned} dP(t) = & (P(t)a^T(t) + a(t)P(t) + b(t)b^T(t) \\ & - P(t)[A^T(t) + \sum_{i=1}^p \exp(-\int_{t-h_i}^t a^T(s)ds)A_i^T(t)] \\ & \times (F(t)F^T(t))^{-1}[A(t) + \sum_{i=1}^p A_i(t) \exp \\ & (-\int_{t-h_i}^t a(s)ds)]P(t))dt. \end{aligned} \quad (6)$$

The system of filtering equations (5) and (6) should be complemented with the initial conditions  $m(t_0) = E[x(t_0) | F_{t_0}^Y]$  and  $P(t_0) = E[(x(t_0) - m(t_0))(x(t_0) - m(t_0))^T | F_{t_0}^Y]$ . This system is similar to the conventional Kalman-Bucy filter, except the adjustments for delays in the estimate and variance equations, calculated due to the Cauchy formula for the linear state equation.

In the case of a constant matrix  $a$  in the state equation, the optimal filter takes the especially simple form ( $\exp(-\int_{t-h}^t a^T ds) = \exp(-a^T h)$ )

$$\begin{aligned} dm(t) = & (a_0(t) + am(t))dt + P(t)[A^T(t) \\ & + \sum_{i=1}^p \exp(-a^T h_i)A_i^T(t)] \end{aligned}$$

$$\begin{aligned} & \times (F(t)F^T(t))^{-1}(dy(t) - (A_0(t) + A(t)m(t) \\ & + \sum_{i=1}^p A_i(t)m(t-h_i))dt), \end{aligned} \quad (7)$$

$$\begin{aligned} dP(t) = & (P(t)a^T + aP(t) + b(t)b^T(t) \\ & - P(t)[A^T(t) + \sum_{i=1}^p \exp(-a^T h_i)A_i^T(t)] \\ & \times (F(t)F^T(t))^{-1}[A(t) + \sum_{i=1}^p A_i(t) \exp(-ah_i)]P(t))dt. \end{aligned} \quad (8)$$

Thus, the equation (5) (or (7)) for the optimal estimate  $m(t)$  and the equation (6) (or (8)) for its covariance matrix  $P(t)$  form a closed system of filtering equations in the case of a linear state equation and linear observations with multiple time delays.

#### IV. OPTIMAL CONTROL PROBLEM SOLUTION

Let us return to the optimal control problem for the linear state (1) with multiple time delays in linear control input and the quadratic cost function (2). This problem is dual to the filtering problem for the linear state (3) and linear observations with multiple time delays (4). Since the optimal filter gain matrix in (5) is equal to

$$\begin{aligned} K_f = & P(t)[A^T(t) \\ & + \sum_{i=1}^p \exp(-\int_{t-h_i}^t a^T(s)ds)A_i^T(t)](F(t)F^T(t))^{-1}, \end{aligned}$$

the gain matrix in the optimal control problem takes the form of its dual transpose

$$K_c = (R(t))^{-1}[B^T(t) + \sum_{i=1}^p B_i^T(t) \exp(\int_{t-h_i}^t a^T(s)ds)]Q(t),$$

and the optimal control law is given by

$$\begin{aligned} u_{opt}(t) = & K_c x = (R(t))^{-1}[B^T(t) \\ & + \sum_{i=1}^p B_i^T(t) \exp(\int_{t-h_i}^t a^T(s)ds)]Q(t)x(t), \end{aligned} \quad (9)$$

where the matrix function  $Q(t)$  is the solution of the following equation dual to the variance equation (6)

$$\begin{aligned} \dot{Q}(t) = & -a^T(t)Q(t) - Q(t)a(t) + L(t) \\ & - Q(t)[B(t) + \sum_{i=1}^p \exp(\int_{t-h_i}^t a^T(s)ds)B_i(t)]R^{-1}(t) \\ & \times [B^T(t) + \sum_{i=1}^p B_i^T(t) \exp(\int_{t-h_i}^t a^T(s)ds)]Q(t), \end{aligned} \quad (10)$$

with the terminal condition  $Q(T) = \psi$ .

Upon substituting the optimal control (9) into the state equation (1), the optimally controlled state equation is obtained

$$\begin{aligned}\dot{x}(t) = & a_0(t) + a(t)x(t) + B(t)(R(t))^{-1}[B^T(t) \\ & + \sum_{i=1}^p B_i^T(t) \exp(\int_{t-h_i}^t a^T(s)ds)]Q(t)x(t) \\ & + \sum_{i=1}^p B_i(t)(R(t))^{-1}[B^T(t) \\ & + \sum_{i=1}^p B_i^T(t) \exp(\int_{t-h_i}^t a^T(s)ds)]Q(t-h_i)x(t-h_i), \\ x(t_0) = & x_0.\end{aligned}$$

The results obtained in this section by virtue of the duality principle could be proved using the general equations of the Pontryagin maximum principle [19,29]. (Bellman dynamic programming [6,26] could serve as an alternative verifying approach). It should be noted, however, that application of the maximum principle to the present case gives one only a system of state and co-state equations and does not provide the explicit form of the optimal control or co-state vector. So, the duality principle approach actually provides one with the explicit form of the optimal control and co-state vector, which should be then substituted into the equations given by the rigorous optimality tools and thereby verified.

## V. ROBUSTIFICATION OF MOTIONS IN THE DELAY CONTROL SYSTEMS VIA INTEGRAL SLIDING MODES

For given control system with multiple delays

$$\dot{x}(t) = f(x(t)) + B(t)u(t) + \sum_{i=1}^p B_i(t)u(t-h_i), \quad (11)$$

where  $x \in R^n$  is the state vector,  $u(t) \in R^m$  is the control input, the ranks of matrices  $B$ ,  $B_i$  are complete and equal to  $m$ , and  $B_i \in \text{Span}(B)$ , suppose that there exists a differentiable in  $x$  state feedback control law  $u_0(x(t))$ , such that the dynamics of the ideal closed loop system takes the form

$$\dot{x}_0(t) = f(x_0(t)) + B(t)u_0(x_0(t)) + \sum_{i=1}^p B_i(t)u_0(x_0(t-h_i)), \quad (12)$$

and has certain desired properties.

However, in practical applications, system (11) operates under uncertainty conditions that may be generated by parameter variations and external disturbances. Let us consider the real trajectory of the closed loop

control system

$$\begin{aligned}\dot{x}(t) = & f(x(t)) + B(t)u(x(t)) + \sum_{i=1}^p B_i(t)u(x(t-h_i)) \\ & + g(x(t), x(t-h_1), \dots, x(t-h_p), t),\end{aligned} \quad (13)$$

where  $g$  is a smooth uncertainty presenting perturbations and nonlinearities in system (11). For  $g$ , the standard matching conditions are assumed to be held:  $g \in \text{Span}(B)$ , or, in other words, there exists a smooth function  $\gamma$  such that

$$\begin{aligned}g(x(t), x(t-h_1), \dots, x(t-h_p), t) \\ = B(t)\gamma(x(t), x(t-h_1), \dots, x(t-h_p), t), \\ \|\gamma(x(t), x(t-h_1), \dots, x(t-h_p), t)\| \\ \leq q_0\|x(t)\| + \sum_{i=1}^p q_i\|x(t-h_i)\| + p_1, q_0, q_1, \dots, q_p, p_1 > 0,\end{aligned}$$

The following initial conditions are assumed for system (11)

$$x(\theta) = \varphi(\theta), \quad (14)$$

where  $\varphi(\theta)$  is a piecewise continuous function given in the interval  $[-h, 0]$ .

Thus, the control problem now consists in robustification of control design in system (12) with respect to uncertainties  $g$ : to find such a control law that the trajectories of system (13) with initial conditions (14) coincide with the trajectories  $x_0(t)$  with the same initial conditions (14).

### 5.1 Design principles

Let us redesign the control law for system (11) in the form

$$u(t) = u_0(t) + u_1(t), \quad (15)$$

where  $u_0(t)$  is the ideal feedback control designed for (12), and  $u_1(t) \in R^m$  is the relay control generating the integral sliding mode in some auxiliary space to reject uncertainties  $g$ . Substitution of the control law (15) into the system (13) yields

$$\begin{aligned}\dot{x}(t) = & f(x(t)) + B(t)u_0(t) + \sum_{i=1}^p B_i(t)u_0(t-h_i) \\ & + B(t)u_1(t) + \sum_{i=1}^p B_i(t)u_1(t-h_i) \\ & + g(x(t), x(t-h_1), \dots, x(t-h_p), t).\end{aligned} \quad (16)$$

Define the auxiliary function

$$s(t) = z(t) + s_0(x(t)), \quad (17)$$

where  $s_0(x(t)) \equiv u_0(x(t))$ , and  $z(t)$  is an auxiliary variable defined below. Then,

$$\begin{aligned} \dot{s}(t) &= \dot{z}(t) + G(t)[f(x(t)) + B(t)u_0(t) \\ &\quad + \sum_{i=1}^p B_i(t)u_i(t-h_i) + B(t)u_i(t) \\ &\quad + \sum_{i=1}^p B_i(t)u_i(t-h_i) + B(t)\gamma(x(t), x(t-h_i), \\ &\quad \dots, x(t-h_p), t)], \end{aligned} \quad (18)$$

where  $G(t) = ds_0(x(t))/dx(t) = du_0(x(t))/dx(t)$  and is available for the sliding mode design, because the control gain for  $u_0$  is already known. In addition, assume that matrix  $G(t)B(t)$  is nonsingular, i.e.,  $(G(t)B(t))^{-1}$  exists.

The philosophy of integral sliding mode control is the following: in order to achieve  $x(t) = x_0(t)$  at all  $t \in (-h, \infty)$ , the sliding mode should be organized on the surface  $s(t)$ , since the following disturbance compensation should have been obtained in the sliding mode motion

$$\begin{aligned} B(t)u_{eq}(t) + \sum_{i=1}^p B_i(t)u_{eq}(t-h_i) \\ = -B(t)(\gamma(x(t), x(t-h_1), \dots, x(t-h_p), t)). \end{aligned}$$

Note that the equivalent control  $u_{eq}(t)$  can be unambiguously determined from the last equality and the initial condition for  $x(t)$ , since the ranks of matrices  $B$ ,  $B_i$  are complete and equal to  $m$ ,  $B_i \in \text{Span}(B)$ , and, therefore, the pseudoinversion of  $B$  can be arranged.

Define the auxiliary variable  $z(t)$  as the solution to the differential equation

$$\dot{z}(t) = -G(t)[f(x(t)) + B(t)u_0(t) + \sum_{i=1}^p B_i(t)u_0(t-h_i)],$$

with the initial conditions  $z(\theta) = -s_0(\varphi(\theta))$  for  $\theta \in [-h, 0]$ . Then, the sliding manifold equation takes the form

$$\begin{aligned} \dot{s}(t) &= G(t)[B(t)\gamma(x(t), x(t-h_1), \dots, x(t-h_p), t) \\ &\quad + B(t)u_i(t) + \sum_{i=1}^p B_i(t)u_i(t-h_i)]. \end{aligned}$$

Finally, to realize sliding mode, the relay control is designed

$$\begin{aligned} u_i(t) &= -M(x(t), x(t-h_1), \dots, x(t-h_p), t) \text{SGN}[S(t)], \\ S(t) &= [G(t)B(t)]^T s(t), \quad (19) \\ M &= q(\|x(t)\| + \sum_{i=1}^p \|x(t-h_i)\|) + p_0, \end{aligned}$$

where  $\text{SGN}[S(t)] = [\text{sign}[S_1(t)], \text{sign}[S_2(t)], \dots, \text{sign}[S_m(t)]]^T$ ,  $S_i(t)$ ,  $i = 1, \dots, m$ , are the coordinates of the vector  $S(t)$ , and  $q > q_0$ ,  $q_1, \dots, q_p$ ,  $p_0 > p_1$ . At this point, let us assume that the matrices  $B$ ,  $B_i$ ,  $i = 1, \dots, p$ , and delay shifts  $h_i$ ,  $i = 1, \dots, p$ , satisfy the following stability-ensuring inequality for any  $t \geq 0$ :

$$\left\| \sum_{i=1}^p B_i(t)u_i(t-h_i) \right\| \leq K \|B(t)u_i(t)\|, \quad (20)$$

where  $K$  is a constant or uniformly bounded function of time. Then, the convergence to and along the sliding mode manifold  $s(t) = 0$  is assured by the Lyapunov function  $V(t) = s^T(t)s(t)/2$  for the system (16) with the control inputs  $u_0(t)$  of (9) and  $u_i(t)$  of (19):

$$\begin{aligned} \dot{V}(t) &= s^T(t)G(t)[B(t)u_i(t) + \sum_{i=1}^p B_i(t)u_i(t-h_i) \\ &\quad + B(t)\gamma(x(t), x(t-h_1), \dots, x(t-h_p), t)] \\ &\leq -|S(t)|K(q(\|x(t)\| + \sum_{i=1}^p \|x(t-h_i)\|) + p_0) \\ &\quad + S^T[G(t)B(t)]^{-1}G(t)[B(t)\gamma(x(t), x(t-h_1), \\ &\quad \dots, x(t-h_p), t)] < 0, \end{aligned}$$

where  $|S(t)| = \sum_{i=1}^m |S_i(t)|$ .

The next section presents the robustification of the designed optimal control (9). This robust regulator is designed assigning the sliding mode manifold according to (17)–(18) and subsequently moving to and along this manifold using relay control (19).

## 5.2 Robustification of optimal control problem solution

Consider again the linear system (1) with multiple time delays in control input, whose behavior is now affected by smooth uncertainties  $g$  presenting perturbations and nonlinearities in the system (1)

$$\begin{aligned} \dot{x}(t) &= a_0(t) + a(t)x(t) + B(t)u(t) + \sum_{i=1}^p B_i(t)u(t-h_i) \\ &\quad + g(x(t), x(t-h_1), \dots, x(t-h_p), t), \end{aligned}$$

with the initial condition  $x(s) = \varphi(s)$ ,  $s \in [-h, 0]$ , where  $\varphi(s)$  is a piecewise continuous function given in the interval  $[-h, 0]$ . It is also assumed that the disturbances satisfy the standard matching conditions

$$\begin{aligned} g(x(t), x(t-h_1), \dots, x(t-h_p), t) \\ = B(t)\gamma(x(t), x(t-h_1), \dots, x(t-h_p), t), \end{aligned}$$

$$\begin{aligned} & \| \gamma_i(x(t), x(t-h_i), \dots, x(t-h_p), t) \| \\ & \leq q_0 \|x(t)\| + \sum_{i=1}^p q_i \|x(t-h_i)\| + p_1, q_0, q_1, \dots, q_p, p_1 > 0, \end{aligned}$$

providing reasonable restrictions on their growth. The quadratic cost function (2) is the same as in Section 2.

The problem is to robustify the obtained optimal control (9), using the method specified by (17)-(18). Define this new control in the form (15):  $u(t) = u_0(t) + u_1(t)$ , where the optimal control  $u_0(t)$  coincides with (9) and the robustifying component  $u_1(t)$  is obtained according to (19)

$$u_1(t) = -M(x(t), x(t-h_1), \dots, x(t-h_p), t) \text{SIGN}[S(t)],$$

$$S(t) = G^T(t)B^T s(t),$$

$$M = q(\|x(t)\| + \sum_{i=1}^p \|x(t-h_i)\|) + p_0,$$

$q > q_0, q_1, \dots, q_p, p_0 > p_1$ , assuming that the condition (20) holds. Consequently, the sliding mode manifold function  $s(t)$  is defined as

$$s(t) = z(t) + s_0(x(t)), \quad (21)$$

where

$$\begin{aligned} s_0(x(t)) &= u_0(t) = (R(t))^{-1}[B^T(t) \\ &+ \sum_{i=1}^p B_i^T(t) \exp(\int_{t-h_i}^t a^T(s)ds)]Q(t)x(t), \end{aligned} \quad (22)$$

and the auxiliary variable  $z(t)$  satisfies the delay differential equation

$$\begin{aligned} \dot{z}(t) &= -G(t)[a_0(t) + a(t)x(t) + B(t)u_0(t) \\ &+ \sum_{i=1}^p B_i(t)u_0(t-h_i)], \end{aligned} \quad (23)$$

with the initial conditions  $z(\theta) = -s_0(\varphi(\theta))$  for  $\theta \in [-h, 0]$ . In accordance with (18), the matrix  $G(t)$  is equal to

$$G(t) = (R(t))^{-1}[B^T(t) + \sum_{i=1}^p B_i^T(t) \exp(\int_{t-h_i}^t a^T(s)ds)]Q(t),$$

where  $Q(t)$  is the solution of the Riccati equation (10).

## VI. EXAMPLE

This section presents an example of designing the optimal regulator for a system (1) with a criterion (2) using the scheme (9)-(10), comparing it to the regulator where the matrix  $Q$  is selected as in the optimal linear regulator for a system without delays, disturbing the obtained optimal regulator by a noise, and designing a ro-

bust sliding mode compensator for that disturbance using the scheme (21)-(23).

Let us start with a scalar linear system

$$\dot{x}(t) = x(t) + u(t-0.1) + u(t), \quad (24)$$

with the initial conditions  $x(s) = 0$  for  $s \in [-0.1, 0)$  and  $x(0) = 1$ . The optimal control problem is to find the control  $u(t)$ ,  $t \in [0, T]$ ,  $T = 0.15$ , that minimizes the criterion

$$J = \frac{1}{2}[x(T) - x^*]^2 + \frac{1}{2} \int_0^T u^2(t)dt, \quad (25)$$

where  $T = 0.15$ , and  $x^* = 25$  is a large value of  $x(t)$  a priori unreachable for time  $T$ . In other words, the optimal control problem is to maximize the state  $x(t)$  using the minimum energy of control  $u$ .

Let us first construct the regulator where the optimal control law and the matrix  $Q(t)$  are calculated in the same manner as for the optimal linear regulator for a linear system without delays in control input, that is  $u_{opt}(t) = (R(t))^{-1}B^T(t)Q(t)x(t)$  (see [22] for reference). Since  $B(t) = 1$  in (24) and  $R(t) = 1$  in (25), the optimal control is actually equal to

$$u(t) = Q(t)x(t), \quad (26)$$

where  $Q(t)$  satisfies the Riccati equation

$$\begin{aligned} \dot{Q}(t) &= -a^T(t)Q(t) - Q(t)a(t) + L(t) \\ &- Q(t)B(t)R^{-1}(t)B^T(t)Q(t), \end{aligned}$$

with the terminal condition  $Q(T) = \psi$ . Since  $a(t) = 1$ ,  $B(t) = 1$  in (24), and  $L = 0$  and  $\psi = 1$  in (25), the last equation turns to

$$\dot{Q}(t) = -2Q(t) - (Q(t))^2, \quad Q(0.15) = 1. \quad (27)$$

Upon substituting the optimal control (26) into (24), the controlled system takes the form

$$\dot{x}(t) = x(t) + Q(t-0.1)x(t-0.1) + Q(t)x(t). \quad (28)$$

The results of applying the regulator (26), (27) to the system (24) are shown in Fig. 1, which presents the graphs of the controlled state (28)  $x(t)$  in the interval  $[0, T]$ , the criterion (25)  $J(t)$  in the interval  $[0, T]$ , and the control (26)  $u(t)$  in the interval  $[0, T]$ . The values of the state (28) and the criterion (25) at the final moment  $T = 0.15$  are  $x(0.15) = 1.4923$  and  $J(0.15) = 276.4835$ .

Let us now apply the optimal regulator (9)-(10) for linear systems with multiple time delays in control input to the system (24). Since  $a(t) = 1$ ,  $B(t) = 1$ ,  $B_1(t) = 1$ , and  $h = 0.1$  in (24) and  $\psi = 1$ ,  $R(t) = 1$ , and  $L = 0$  in (25), hence,  $\exp(\int_{t-h}^t a^T(s)ds) = \exp(0.1)$  and the optimal control law (9) takes the form

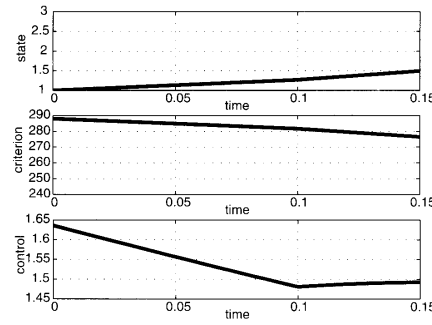


Fig. 1. Best linear regulator available for linear systems without delays. Graphs of the controlled state (28)  $x(t)$  in the interval  $[0, 0.15]$ , the criterion (25)  $J(t)$  in the interval  $[0, 0.15]$ , and the control (26)  $u(t)$  in the interval  $[0, 0.15]$ .

$$u_{opt}(t) = (1 + \exp(0.1))Q(t)x(t), \quad (29)$$

where  $Q(t)$  satisfies the Riccati equation

$$\dot{Q}(t) = -2Q(t) - ((1 + \exp(0.1))Q(t))^2, \quad Q(0.15) = 1 \quad (30)$$

Upon substituting the optimal control (29) into (24), the optimally controlled system takes the form

$$\begin{aligned} \dot{x}(t) = & x(t) + (1 + \exp(0.1))[Q(t-0.1)x(t-0.1) \\ & + Q(t)x(t)]. \end{aligned} \quad (31)$$

The results of applying the regulator (29), (30) to the system (24) are shown in Fig. 2, which presents the graphs of the optimally controlled state (31)  $x(t)$  in the interval  $[0, T]$ , the criterion (25)  $J(t)$  in the interval  $[0, T]$ , and the optimal control (29)  $u_{opt}(t)$  in the interval  $[0, T]$ . The values of the state (31) and the criterion (29) at the final moment  $T = 0.15$  are  $x(0.15) = 2.8944$  and  $J(0.15) = 248.8386$ .

The next task is to introduce a disturbance into the optimally controlled system (31). This deterministic disturbance is realized as a constant:  $g(t) = -10$ . The matching conditions are valid, because state  $x(t)$  and control  $u(t)$  have the same dimension:  $\dim(x) = \dim(u) = 1$ . The restrictions on the disturbance growth hold with  $q_0 = 0$  and  $p_1 = 10$ , since  $\|g(t)\| = 10$ . The disturbed system equation (31) takes the form

$$\begin{aligned} \dot{x}(t) = & -10 + x(t) + (1 + \exp(0.1))[Q(t-0.1)x(t-0.1) \\ & + Q(t)x(t)]. \end{aligned} \quad (32)$$

The system state behavior significantly deteriorates upon introducing the disturbance. Figure 2 presents the graphs of the disturbed state (32)  $x(t)$  in the interval

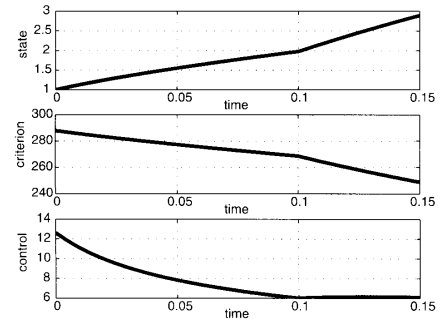


Fig. 2. Optimal regulator obtained for linear systems with multiple time delays in control input. Graphs of the optimally controlled state (31)  $x(t)$  in the interval  $[0, 0.15]$ , the criterion (25)  $J(t)$  in the interval  $[0, 0.15]$ , and the optimal control (29)  $u_{opt}(t)$  in the interval  $[0, 0.15]$ .

$[0, T]$ , the criterion (25)  $J(t)$  in the interval  $[0, T]$ , and the control (29)  $u(t)$  in the interval  $[0, T]$ . The values of the state (32) and the criterion (25) at the final moment  $T = 0.15$  are  $x(0.15) = 0.6472$  and  $J(0.15) = 298.5231$ . The state (32) decreases from its initial value  $x(0) = 1$ , although it should be maximized, and the criterion value increases from its initial value  $J(0) = 288$ , although it should be minimized, and becomes much larger than in the preceding cases.

Let us finally design the robust integral sliding mode control compensating for the introduced disturbance. The new controlled state equation should be

$$\begin{aligned} \dot{x}(t) = & -10 + x(t) + (1 + \exp(0.1))[Q(t-0.1)x(t-0.1) \\ & + Q(t)x(t)] + u_1(t-0.1) + u_1(t), \end{aligned} \quad (33)$$

where the compensator  $u_1(t)$  is obtained according to (19)

$$u_1(t) = -M(x(t), x(t-h), t) \operatorname{sign}[s(t)]. \quad (34)$$

The sliding mode manifold  $s(t)$  is defined by (21)

$$s(t) = z(t) + s_0(x(t)),$$

where

$$s_0(x(t)) = u_0(x(t)) = (1 + \exp(0.1))Q(t)x(t),$$

and the auxiliary variable  $z(t)$  satisfies the delay differential equation

$$\begin{aligned} \dot{z}(t) = & -G(t)[x(t) + u_0(t) + u_0(t-h)] \\ = & -G(t)[x(t) + (1 + \exp(0.1))[Q(t-0.1)x(t-0.1) \\ & + Q(t)x(t)]], \end{aligned}$$



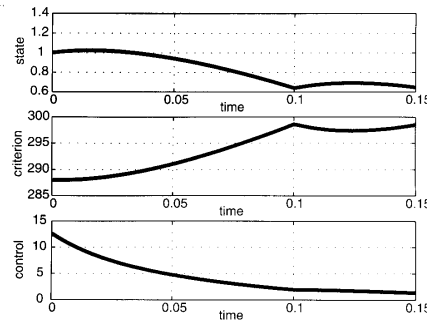


Fig. 3. Controlled system in the presence of disturbance. Graphs of the disturbed state (32)  $x(t)$  in the interval  $[0, 0.15]$ , the criterion (25)  $J(t)$  in the interval  $[0, 0.15]$ , and the control (29)  $u(t)$  in the interval  $[0, 0.15]$ .

with the initial conditions  $z(s) = 0$  for  $s \in [-0.1, 0)$  and  $z(0) = -(1 + \exp(0.1))Q(0)$ . In accordance with (18), the matrix  $G(t)$  is equal to

$$G(t) = ds_0(x(t)) / dx(t) = (1 + \exp(0.1))Q(t),$$

where  $Q(t)$  is the solution of the Riccati equation (30).

Upon introducing the compensator (34) into the state equation (33), the system state behavior is very much improved. Figure 3 presents the graphs of the compensated state (33)  $x(t)$  in the interval  $[0, T]$ , the criterion (25)  $J(t)$  in the interval  $[0, T]$ , and the sum of the control (29) and compensator (34),  $u(t) + u_1(t)$  (equivalent sliding mode control), in the interval  $[0, T]$ . The values of the state (33) and the criterion (25) at the final moment  $T = 0.15$  are  $x(0.15) = 2.893$  and  $J(0.15) = 267.6933$ . Thus, the value of the controlled state after applying the compensator (34) is only indistinguishably less than those values for the optimal regulator (29)-(30) for linear systems with multiple time delays in control input, and much better than the values for the disturbed system (32). Of course, the criterion value here is worse than for the optimal regulator (although it is even better than for the regulator (26),(27)), since an additional control energy is required to suppress the disturbance.

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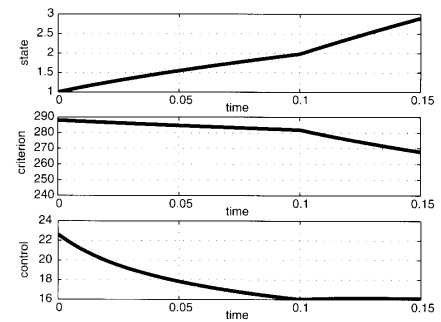
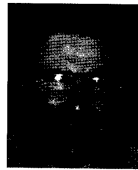


Fig. 4. Controlled system after applying robust integral sliding mode compensator. Graphs of the compensated state (33)  $x(t)$  in the interval  $[0, 0.15]$ , the criterion (25)  $J(t)$  in the interval  $[0, 0.15]$ , and the sum of the control (29) and compensator (34),  $u(t) + u_1(t)$  (equivalent sliding mode control), in the interval  $[0, 0.15]$ .

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