— Brief Paper —

ESTIMATION OF AMPLITUDE OF OSCILLATIONS IN SLIDING MODE SYSTEMS CAUSED BY TIME DELAY

P. Acosta, A. Polyakov, L. Fridman and V. Strygin

ABSTRACT

Time delay does not allow realizing ideal sliding mode but implies oscillations in the state variable space. An estimation technique is developed for an upper bound of oscillation amplitude induced by bounded uncertain time delay presence.

KeyWords: Sliding modes, variable structure systems, time delay.

I. INTRODUCTION

Relay control systems are widely used due to the following main reasons:

- Relay controllers allow rejection of some bounded uncertainties [1].
- There are control systems in which only signs of variables are observable [2,3].

Time delay, that usually takes place in relay and sliding mode control systems, must be taken into account in system analysis and design (see, for example, [1]). Moreover, time delay does not allow the design of sliding mode control in the space of state variables. In [4] and [5], it was shown that even in the simplest one-dimensional delayed relay control system, only oscillatory solutions can occur.

The following main research directions in relay delay control are currently being followed:

1.1 Time delay compensation

Reducing the relay delay output tracking problem using the Pade approximation of delay was suggested in [6] for sliding mode control applied to nonminimum phase systems. In [3] and [7], sliding mode control in the predictor variable space was introduced, but in [8] and [9], two major restrictions of this approach were noted:

- In the general case, matching conditions for uncertainties in the state variable space do not hold in the predictor space.
- In the case of square systems, where the dimensions of the state space and control are the same, sliding mode design in the space of predictors can remove the uncertainties in the space of predictor variables but can not guarantee robustness with respect to uncertainties in the space of state variables.

1.2 Sliding mode control design via memoryless control

For systems that include delays in inputs and states but are controllable by at least one input without delay, the problem can be reduced to the delay free case (see, for example, [10], [11], and [12]).

1.3 Estimation of the amplitude of oscillations

Instead of trying to compensate for the delay, we can try to estimate the magnitude of oscillations based on a given relation between system and control parameters and delay. If a relation of this kind is known, then limits can be imposed on delay and control gain for a given system to achieve satisfactory performance. A method that uses Lyapunov techniques and Taylor series

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to estimate the sliding mode boundary layer and region of attraction for systems with a small delay in the input was proposed in [11].

This paper focuses on the amplitude estimation of stable oscillations in sliding mode systems with time delay. The paper is organized as follows: In subsection II, a lemma for amplitude estimation of oscillations in a scalar case is proved. A lemma for oscillation amplitude estimation in the second order complex conjugate case is presented in subsection II. In subsection III, we apply the results obtained in subsections 2.1 and 2.2 to a vectorial case. The last section presents to numerical examples illustrating the obtained results.

II. ESTIMATION OF AMPLITUDES

2.1 Scalar case

Consider the following system:

$$\dot{x}(t) = -\lambda\, x(t) + f(t,x) + u(t)\,, \tag{1}$$

where $x(t) \in R$; $\lambda > 0$; $f(t, x)$ is either an unknown smooth disturbance, a nonlinearity, or a parameter uncertainty with $|\, f(t, x)\,| < K$ and $u(t) = -p\ \mathrm{sign}(x(t))$ with $p = K + \mu$ and $\mu > 0$. For this system, it can be shown that $x(t) = 0$ for any $t \geq t_s$, where $t_s \leq \frac{1}{\lambda}\ln\left(\frac{\sqrt{2}\lambda x(0)+\mu}{\mu}\right)$.

Now, let us consider a delay in the input:

$$\dot{x}(t) = -\lambda x(t) + f(t,x) + u(t-h(t))\,, \tag{2}$$

where $h(t)$ is a bounded variable continuous time delay, $0 \leq h(t) \leq h_0$, and

$$x(t) = \phi(t)\,, \qquad (-h_0 \leq t \leq 0) \tag{3}$$

for $\phi(t) \in C[-h_0, 0]$, where $C[-h_0, 0]$ is the set of all continuous functions whose domain is $[-h_0, 0]$. In this case, the system does not remain in $x = 0$ after t_s but exhibits oscillations. In order to show the existence of such oscillations and obtain an estimate of their amplitude, the following lemma is introduced.

Lemma 1. There exists $T > 0$ such that for any $\phi(t) \in C[-h_0, 0]$, the solution $x(t)$ of the system in (2) and (3) is bounded according to

$$|\, x(t)\,| \leq \varepsilon = \frac{K+p}{\lambda}(1-e^{-\lambda h_0})\,, \qquad \forall t > T\,. \tag{4}$$

Proof. First, we will show that there exists a time $t_0 > 0$ such that $x(t_0) = 0$. If this assumption is not correct, we will have $x(t) \neq 0$, $\forall t \geq 0$. Considering this contradiction and without loss of generality, let $x(t) > 0$, $\forall t > -h_0$. Therefore, $\mathrm{sign}(x(t - h(t))) = 1$, and from Eq. (2),

$$\dot{x} = -\lambda x + f(t,x) - p \leq -\lambda x - p + K\,, \qquad \forall t > h_0\,. \tag{5}$$

Define a new variable $v(t)$ such that

$$\dot{v} = -\lambda v - p + K\,, \tag{6}$$

$$v(h_0) \geq x(h_0) > 0\,. \tag{7}$$

It is known, by Gronwall's inequality, that

$$x(t) \leq v(t)\,. \tag{8}$$

It is easy to see that there exists $t^* > h_0$ such that $v(t^*) = 0$. Therefore, since $x(t^*) \leq v(t^*)$, in the interval $[0, t^*]$ there exists a time $0 < t_0 \leq t^*$, where $x(t_0) = 0$.

Now, we will prove that

$$|\, x(t)\,| \leq \varepsilon\,, \qquad \forall t > T. \tag{9}$$

Consider, by contradiction, that there exists a given time $\xi > T$, where $|\, x(\xi)\,| > \varepsilon$.

Without loss of generality, we can suppose that $x(\xi) > \varepsilon$ and $t' \geq T$ such that $x(t') = 0$ and $x(t) > 0$ for $t \in (t', \xi)$.

Let $t = \eta$ be the first time in the interval (t', ξ) when $|\, x(\eta)\,| = \varepsilon$. Now, in the interval (t', η) for some $t < h_0 + t'$, we have

$$\begin{aligned} \dot{x} &= -\lambda x + f(t,x) - p\ \mathrm{sign}(x(t-h(t))) \\ &\leq -\lambda x + f(t,x) + p \\ &\leq -\lambda x + K + p\,. \end{aligned} \tag{10}$$

Since $x(t') = 0$, from the inequality (10), it follows that

$$x(t) \leq -\frac{K+p}{\lambda}e^{-\lambda(t-t')} + \frac{K+p}{\lambda}\,. \tag{11}$$

For $t = \eta$, it turns out due to the assumption that

$$\begin{aligned} x(\eta) = \varepsilon &\equiv \frac{K+p}{\lambda}(1-e^{-\lambda h_0}) \\ &\leq -\frac{K+p}{\lambda}e^{-\lambda(\eta-t')} + \frac{K+p}{\lambda}\,. \end{aligned} \tag{12}$$

Then, for $t \geq \eta$, $\mathrm{sign}(x(t - h(t))) = 1$ and Eq. (2) has the form

$$\dot{x} = -\lambda x + f(t,x) - p \leq -\lambda x - p + K = -\lambda x - \mu < 0\,, \tag{13}$$

so that the solution decreases for $t \in (\eta, \xi)$. Therefore, the equality $|\, x(\xi)\,| = \varepsilon$ is impossible, and we have a contradiction. On the other hand, it can be seen that the tightness of this estimate depends on the tightness of the upper bounds K and h_0 as well as on the particular form of the functions $f(t, x)$ and $h(t)$. ■

2.2 Two dimensional complex eigenvalues case

Consider the system

$$\frac{dx}{dt} = -\alpha x + \beta y + f_1(x, y, t) + u_1 ,$$

$$\frac{dy}{dt} = -\beta x - \alpha y + f_2(x, y, t) + u_2 ,$$

$$x(t) = x_0(t) ,$$

$$y(t) = y_0(t) , \qquad (-h_0 \le t \le 0) , \tag{14}$$

where f_1 and f_2 are unknown functions with $\| f(x, y, t) \| < K$.

Without a control signal, $u_1 = 0$, $u_2 = 0$, the system does not have $x = 0$, $y = 0$ as a stable equilibrium except for very particular forms of f_1 and f_2.

We introduce the following control law:

$$u = -\frac{p\sqrt{2}}{2}\begin{pmatrix} \cos\beta h_0 & -\sin\beta h_0 \\ \sin\beta h_0 & \cos\beta h_0 \end{pmatrix}\begin{pmatrix} \mathrm{sign}(x(t-h(t))) \\ \mathrm{sign}(y(t-h(t))) \end{pmatrix}, \tag{15}$$

where $p = 3K$.

Without delay, $h(t) = 0$, $h_0 = 0$, it can be shown that, with this control, the system reaches $x = 0$, $y = 0$ in finite time. If the time delay is not zero, there will be oscillations, and an estimate of their amplitude is given in the following lemma.

Lemma 2. Let $\beta \le \pi/(50h_0)$; then, for all initial conditions $x_0, y_0 \in C[-h_0, 0]$, there exists $T > 0$ such that for the solution $x(t)$, $y(t)$ of the system in (14) and (15),

$$| x(t) | \le \varepsilon, \ | y(t) | \le \varepsilon \qquad (t \ge T) , \tag{16}$$

where

$$\varepsilon = 200 h_0 K / 3\pi . \tag{17}$$

Proof. We introduce new coordinates: $x = \rho\cos\varphi$ and $y = \rho\sin\varphi$, and consider $f_1 + if_2 = r(t)e^{i\theta(t)}$. Now, it is obvious that $r(t) \le K$, $f_1 = r(t)\cos(\theta(t))$, and $f_2 = r(t)\sin(\theta(t))$. Then, system (14) with control (15) can be rewritten as

$$\dot{\rho} = -\alpha\rho - p\cos[\beta h_0 + k(t) - \varphi(t)] + r(t)\cos(\varphi(t) - \theta(t)),$$

$$\dot{\varphi} = \beta - \frac{p}{\rho(t)}\sin[\beta h_0 + k(t) - \varphi(t)] + \frac{r(t)}{\rho(t)}\cos(\varphi(t) - \theta(t)), \tag{18}$$

where

$$k(t) = \begin{cases} \dfrac{\pi}{4}, & 0 \le \varphi(t - h(t)) < \dfrac{\pi}{2}, \\ \dfrac{3\pi}{4}, & \dfrac{\pi}{2} \le \varphi(t - h(t)) < \pi , \\ \dfrac{5\pi}{4}, & \pi \le \varphi(t - h(t)) < \dfrac{3\pi}{2}, \\ \dfrac{7\pi}{4}, & \dfrac{3\pi}{2} \le \varphi(t - h(t)) < 2\pi . \end{cases}$$

First Step. We will prove that there exists $t' \ge 0$ such that $\rho(t) < \varepsilon/2$. To achieve a contradiction, we assume that $t' \ge 0$ and $\rho(t) \ge \varepsilon/2$. Integrating the second equation of (18) in an interval $[t - h(t), t]$, we find that

$$\varphi(t) = \beta h(t) + \varphi(t - h(t)) + \xi(t) + x(t) , \tag{19}$$

where

$$\xi(t) = p\int_{t-h(t)}^{t} \frac{\sin(\beta h_0 + k(t) - \varphi(t))}{\rho(t)} dt ,$$

$$x(t) = \int_{t-h(t)}^{t} \frac{r(t)}{\rho(t)}\cos(\varphi(t) - \theta(t))dt .$$

Since $\rho(t) \ge \varepsilon/2$, it turns out that

$$| \xi(t) | \le 2ph_0/\varepsilon ,$$
$$| x(t) | \le 2Kh_0/\varepsilon . \tag{20}$$

Then, substituing (19) into the first equation of (18), we have

$$\dot{\rho} = -\alpha\rho - p\cos(\beta(h_0 - h(t)) + k(t) - \varphi(t - h(t)) - \xi(t) - x(t))$$

$$+ r(t)\cos(\varphi(t) - \theta(t)) . \tag{21}$$

From the definition of $k(t)$, it follows that

$$| k(t) - \varphi(t - h(t)) | \le \pi/4 . \tag{22}$$

Now, since

$$| \beta(h_0 - h(t)) + k(t) - \varphi(t - h(t)) - \xi(t) - x(t) |$$

$$\le | \beta(h_0 - h(t)) | + \frac{\pi}{4} + | \xi(t) | + | x(t) |$$

$$\le | \beta(h_0 - h(t)) | + \frac{\pi}{4} + \frac{2ph_0}{\varepsilon} + \frac{2Kh_0}{\varepsilon}$$

$$= | \beta(h_0 - h(t)) | + \frac{\pi}{4} + \frac{2(p + K)h_0}{\varepsilon}$$

$$= | \beta(h_0 - h(t)) | + \frac{\pi}{4} + \frac{6\pi}{50}$$

$$\le \frac{\pi}{50} + \frac{37\pi}{100} = \frac{39\pi}{100} , \tag{23}$$

which is less than $\pi/2$, we can conclude that

$$\cos[\beta(h_0 - h(t)) + k(t) - \varphi(t - h(t)) - \xi(t) - x(t)]$$

$$\geq \cos\left(\frac{39\pi}{100}\right) \geq 1/3 . \quad (24)$$

This is why, for $t \geq h_0$, the inequality $\dot{\rho} \leq -\alpha\rho - K + K = -\alpha\rho$ holds, and it is easy to see that there exists a t' such that $\rho(t') < \varepsilon/2$. Hence, the asumption is not correct.

Second Step. We will prove that $\rho(t) < \varepsilon$ for all $t > T$ by assuming that it is not true. Under this assumption, there exists a first time t_1 such that $\rho(t_1) = \varepsilon$, $t_1 \geq T$.

Since $\rho(t') < \varepsilon/2$, there exists $t_0 \in [t', t_1]$, where $\rho(t_0) = \varepsilon/2$. Moreover, let us suppose that t_0 is the last such instant in $[t', t_1]$.

Therefore, $\varepsilon/2 \leq \rho(t) \leq \varepsilon$ for all $t \in [t_0, t_1]$. Now, the following estimate can be considered:

$$\dot{\rho} \leq -\alpha\rho + p + K \leq 4K, \qquad t \in (t_0, t_1),$$

$$\rho(t_0) = \frac{\varepsilon}{2}, \quad (25)$$

which yields

$$\rho(t) \leq 4K(t - t_0) + \varepsilon/2 . \quad (26)$$

Then for $t = t_1$, the inequality (26) implies that

$$\rho(t_1) = \varepsilon \leq 4K(t_1 - t_0) + \frac{\varepsilon}{2} . \quad (27)$$

Substituting ε from (17) into the last expression, we have $t_1 - t_0 \geq 25h_0/(3\pi) > h_0$.

Now, it is easy to see that the inequality $\rho(t) > \varepsilon$ is incorrect, since after $t - t_0 > h_0$, the system is forced back to the point of origin. ■

Indeed, in a more general case, there exists a tradeoff between β, $h(t)$, p, and ε such that the values presented here comprise only one combination of many possible combinations.

2.3 Vectorial case

One way of applying the results obtained in the previous sections to a vectorial case is as follows:

Consider the vectorial system

$$\dot{x} = Ax + f(t, x) + u(t - h(t)), \qquad t > 0, \quad (28)$$

where $x, u \in R^n$; A is a Hurwitz matrix with different real eigenvalues $\{-\lambda_i\}$ and different complex conjugate eigenvalues $\{-\alpha_j \pm \beta_j\}$; $f(t, x) = (f_1(t, x), \ldots, f_n(t, x))^T$ is a smooth unknown disturbance with $|f(t, x)| \leq K$; and $u(t)$, without delay, is a control that forces the system back to the point of origin in finite time as described below.

Then, there exists a matrix T such that $T^{-1}AT = \Lambda$, where

$$\Lambda = \begin{pmatrix} -\lambda_1 & 0 & \ldots & \ldots & \ldots & \ldots & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & -\lambda_1 & 0 & \ldots & \ldots & \ldots & 0 \\ 0 & \ldots & 0 & -\alpha_1 & -\beta_1 & 0 & \ldots & 0 \\ 0 & \ldots & 0 & \beta_1 & -\alpha_1 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & \ldots & \ldots & \ldots & 0 & -\alpha_v & -\beta_v \\ 0 & \ldots & \ldots & \ldots & \ldots & 0 & \beta_v & -\alpha_v \end{pmatrix}. \quad (29)$$

Considering $x = Ty$ in the system (28), we obtain

$$\dot{y} = \Lambda y + g(y, t) + v(t - h(t)), \qquad t > 0, \quad (30)$$

where $g(t, y) = T^{-1} f(t, Ty)$ and $v(t - h(t)) = T^{-1}u(t - h(t))$. It is obvious that

$$|g_i(y, t)| \leq \| g(t, y) \| \leq \| T^{-1} \| \cdot K . \quad (31)$$

We now introduce a control that forces y and, hence, x back to the point of origin in finite time according to the discussions in subsections 2.1 and 2.2, given the following expression:

$$\dot{y} = \Lambda y - r \cdot P \cdot \text{sign}[y(t - h(t))] + g(y, t) , \quad (32)$$

where $r = 3/2 \| T^{-1} \| \cdot K \cdot \sqrt{2}$ and

$$p = \begin{pmatrix} 1 & 0 & \ldots & \ldots & \ldots & \ldots & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 1 & 0 & \ldots & \ldots & \ldots & 0 \\ 0 & \ldots & 0 & \cos(\beta_1 h_0) & -\sin(\beta_1 h_0) & 0 & \ldots & 0 \\ 0 & \ldots & 0 & \sin(\beta_1 h_0) & \cos(\beta_1 h_0) & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & \ldots & \ldots & \ldots & 0 & \cos(\beta_1 h_0) & -\sin(\beta_1 h_0) \\ 0 & \ldots & \ldots & \ldots & \ldots & 0 & \sin(\beta_1 h_0) & \cos(\beta_1 h_0) \end{pmatrix}. \quad (33)$$

Hence, with $h(t) = 0$, $h_0 = 0$, the system is forced back to the point of origin. On the other hand, with $h(t) \neq 0$, $h_0 \neq 0$, using Lemmas 1 and 2, we have for the states with real eigenvalues that

$$|y_i(t)| \leq \frac{(1 + 3\sqrt{2}/2) \| T^{-1} \| K}{\lambda_i}(1 - e^{-\lambda_i h_0}), \; (i = 1, 2, \ldots, 1),$$

$$\text{using } \mu = \frac{3\sqrt{2} - 2}{2} \| T^{-1} \| K, \quad (34)$$

and for those states with complex eigenvalues,

$$|y_{l+j}(t)| = 200h_0 \| T^{-1} \| K / 3\pi, \quad (j = 1, 2, \ldots, v). \quad (35)$$

Returning to the original states, we find that

$$|x| \le \|T\| \, |y| \, . \tag{36}$$

2.4 Remark

In the case of states with real eigenvalues, for sufficiently small h_0,

$$(1 - e^{-\lambda_i h_0}) \approx \lambda_i h_0 ; \tag{37}$$

therefore,

$$|y_i(t)| \le h_0 (1 + 3\sqrt{2}/2) \, \|T^{-1}\| \, K. \tag{38}$$

III. NUMERICAL EXAMPLES

For the scalar system

$$dx/dt = -x + \sin(t) - 1.5 \operatorname{sign}(x(t-1)) \, , \tag{39}$$

we have $p = 1.5$ and $h = 1$; then, the expected oscillation amplitude is less than 1.581. In Fig. 1, the resulting oscillation is observed. The maximum value for x in the interval $t = [2, 40]$ is 1.5503, and the minimum value is −1.5539.

If the delay is reduced to $h = 0.2$, then the resulting oscillation is that shown in Fig. 2. In Figs. 3 and 4, the results of simulations for $h(t) = 0.15 + 0.5 \sin(0.3t)$ and $h(t) = 0.5 + 0.5 \sin(0.3t)$, respectively, are given. From Figs. 3 and 4, it can be verified that the maximum amplitudes of oscillation for these systems with varying delay are the same as that for a constant time delay corresponding to $h(t) = 0.2$ and $h(t) = 1$, respectively.

Now, for the second order system

$$\begin{aligned} \frac{dx}{dt} &= -3x + 2y + (1 - e^{-t}) \cos 3t + u_1 , \\ \frac{dy}{dt} &= 0.5x - 3y + \sin t + u_2 , \end{aligned} \tag{40}$$

the matrix A eigenvalues are $\lambda_{1,2} = -2, -4$.

The diagonal transformation matrix is $T = \begin{bmatrix} 2 & -2 \\ 1 & 1 \end{bmatrix}$.

Therefore, $T^{-1} = \begin{bmatrix} 0.25 & 0.5 \\ -0.25 & 0.5 \end{bmatrix}$,

and $\xi = \begin{bmatrix} x \\ y \end{bmatrix} = T\eta$.

In the new coordinates, the system is as follows:

$$\dot{\eta} = \begin{bmatrix} -2 & 0 \\ 0 & -4 \end{bmatrix} \eta + T^{-1} \begin{bmatrix} (1 - e^{-t}) \cos 3t \\ \sin t \end{bmatrix} + T^{-1} u \, , \tag{41}$$

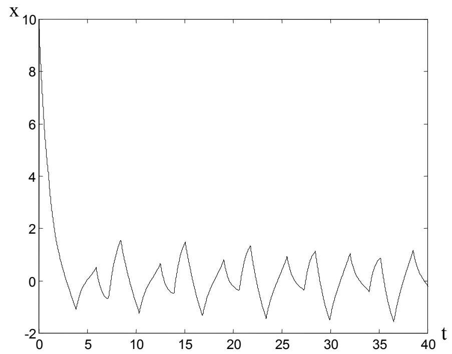

Fig. 1. Amplitude of oscillations for a first order sliding mode system with $h(t) = 1$.

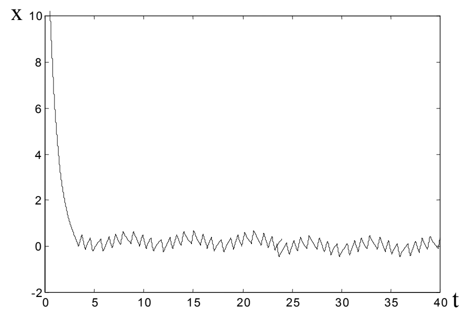

Fig. 2. Amplitude of oscillations for a first order sliding mode system with $h(t) = 0.2$.

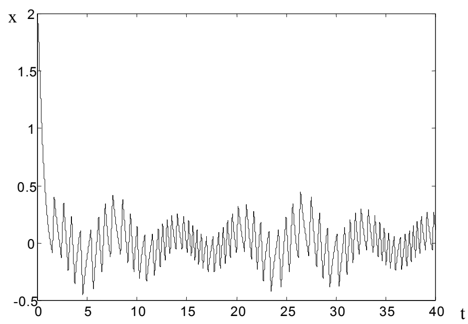

Fig. 3. Amplitude of oscillations for a first order sliding mode system with $h(t) = 0.15 + 0.05 \sin(0.3t)$.

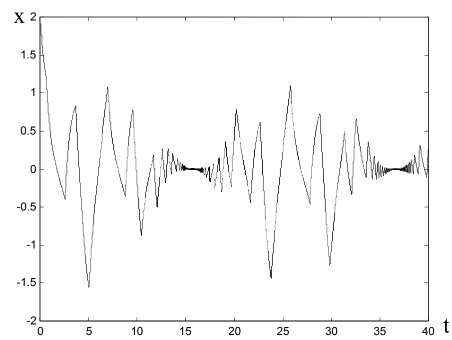

Fig. 4. Amplitude of oscillations for a first order sliding mode system with $h(t) = 0.5 + 0.5 \sin(0.3t)$

$$\dot{\eta} = \begin{bmatrix} -2 & 0 \\ 0 & -4 \end{bmatrix} \eta + \begin{bmatrix} 0.25(1-e^{-t})\cos 3t + 0.5\sin t \\ -0.25(1-e^{-t})\cos 3t + 0.5\sin t \end{bmatrix} + T^{-1}u \,. \tag{42}$$

If we choose

$$T^{-1}u = \begin{bmatrix} -p_1 \mathrm{sign}(\eta_1(t-h)) \\ -p_2 \mathrm{sign}(\eta_2(t-h)) \end{bmatrix} \tag{43}$$

with $p_1 = 0.76\, y$, $p_2 = 0.76$, then

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{44}$$

has the values

$$\begin{aligned} u_1 &= -2p_1 \ \mathrm{sign}(0.25x(t-h) + 0.5y(t-h)) \\ &\quad +2p_2 \ \mathrm{sign}(-0.25x(t-h) + 0.5y(t-h)) \,, \\ u_2 &= -p_1 \ \mathrm{sign}(0.25x(t-h) + 0.5y(t-h)) \\ &\quad -p_2 \ \mathrm{sign}(-0.25x(t-h) + 0.5y(t-h)) \,. \end{aligned} \tag{45}$$

For $h(t) = 1$, the estimate of the oscillation amplitude is $\eta_1 \le 0.653$ and $\eta_2 \le 0.3706$. Since $x = 2\eta_1 - 2\eta_2$ and $y = \eta_1 + \eta_2$, it can be expected that

$$\begin{aligned} |x| &\le ((2\eta_1)^2 + (2\eta_2)^2)^{1/2} = 1.5017 \,, \\ |y| &\le ((\eta_1)^2 + (\eta_2)^2)^{1/2} = 0.75084 \,. \end{aligned} \tag{46}$$

Through simulation, we obained: max(x) = 1.0694, min(x) = –1.3045, max(y) = 0.7418, and min(y) = –0.7074, which give max(η_1) = 0.4716, min(η_1) = –0.5293, max(η_2) = 0.338, min(η_2) = –0.330.

For the simulation whose results are shown in Fig. 5, $h(t) = 0.5 + 0.5 \sin(0.3t)$ was used, and it is observed that the maximum amplitude values coincide with those obtained before with $h(t) = 1$, since $h_0 = 1$.

Now, consider a fifth order system with a scalar input:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= x_5 \\ \dot{x}_5 &= -0.03125x_1 - 0.3125x_2 - 1.25x_3 - 2.5x_4 \\ &\quad - 2.5x_5 + 0.03125\, u(t-2) \,, \end{aligned} \tag{47}$$

where $u(t) = -\,p\ \mathrm{sign}(x_1(t))$.

To estimate the oscillation amplitude expected for x_1 in this system, the model was reduced to first order, using the method of Wang *et al*. [13]. Therefore, considering the reduced model for the design

$$\dot{x}_1 = -\frac{1}{6.809}x_1 + f(t,x) + \frac{1}{6.809}u(t-7.26) \,, \tag{48}$$

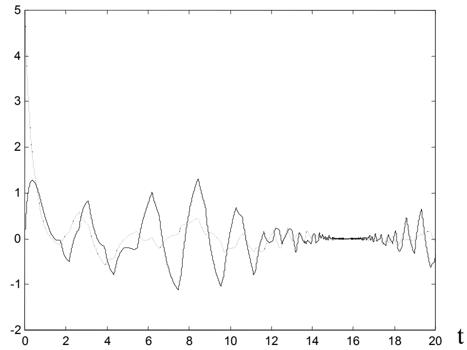

Fig. 5. Bidimensional first order sliding mode system with variable time delay.

if $p = 1$ is chosen in u and if we assume that $\mu \to 0$, then the oscillation amplitude estimate is $x_1 \le 4.47$. Now, assuming that $K \to 0$, the estimate is $x_1 \le 8.93$. The simulation outcome is $|x_1| \le 0.65$ as shown in Fig. 6. The obtained estimate suggests that the reduced model is quite conservative, and that a model with less delay and/or a smaller time constant could perhaps be used as a reduced model. Also, it can be easily shown that if the control is changed to $u = -x_1$, then the origin is asymptotically stable, so the method presented in this paper can also be used to estimate saturation limits in order to preserve asymptotic stability.

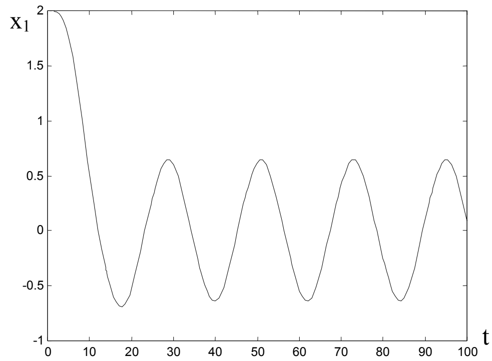

Fig. 6. Chattering in an attempt of a fifth order sliding mode system with a scalar input using p =1.

IV. CONCLUSION

In the present paper, formulas were derived for the amplitude estimation of oscillations arising due to the presence of time delay in sliding mode control systems. The obtained results may be helpful for determining the time delay requirements for actuators, sensors, and controllers as well as boundary layer specifications for quasi sliding mode discrete systems design even for higher sliding modes at least in some cases.

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