REGULATION OF THE RTAC SYSTEM VIA FLATNESS AND SECOND ORDER SLIDING

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In this article we illustrate how the property of *differential flatness* can be advantageously joined to the, so called, *second order sliding mode* controller design methodology for the robust and active stabilization of a benchmark nonlinear mechanical vibration system, known as the rotational/translational proof mass actuator (RTAC).

1 Introduction

Differential flatness is a useful structural property exhibited by many nonlinear systems of practical, or realistic, significance. The theoretical background of "flatness" has been established in several articles by Prof. M. Fliess and his colleagues (Fliess *et al*,[4]) from the viewpoints, respectively, of differential algebra and Lie-Bäcklund transformations.

Sliding mode control, on the other hand, represents a quite robust and simple approach which has enjoyed well gained respect and popularity over the years thanks to its theoretical and practical developments (see the book by V. Utkin, [7]). In recent years, an interesting extension of sliding mode control, known as "higher order sliding" (HOS) has been developed (See the articles by Friedman and Levant [5], Levant [6] and Bartolini and his co-workers [1], [2]).

In this article, we use a nontrivial nonlinear, single input, benchmark example to suitably combine a *second order* sliding mode feedback controller with the differential flatness property of the system.

2 The nonlinear RTAC system

The RTAC system is shown in Figure 1. This system consists of a cart, of mass M, connected by means of a linear spring, of stiffness coeficient k, to a fixed reference point. The cart has only one degree of translational freedom in the direction of the coordinate x The proof mass, of value m, is constituted by a horizontal pendulum provided with a link, of length L, whose moment of inertia is I. N is the torque applied to the proof mass, and F is the disturbance force, acting on the cart. We assume this perturbation is bounded with bounded time derivatives up to a second order, as that provided by an eccentric actuator attached to the cart. The angular displacement of the horizontal pendulum is measured by the angle, θ, shown in the figure.

We choose as generalized coordinates $q = [x, \theta]$ and as generalized external forces, the applied torque N and the perturbation force F.

The normalized system dynamics can be derived to be the expressions:

$$\begin{aligned} \ddot{\xi} + \xi + \epsilon\left(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta\right) &= \omega \\ \ddot{\theta} + \epsilon\ddot{\xi}\cos\theta &= u \end{aligned} \tag{1}$$

with ϵ given by,

$$\epsilon = \frac{mL}{\sqrt{(I + mL^2)(M + m)}}$$

Notice that ϵ is strictly smaller than 1.

2.1 Flatness of the unperturbed model

The unperturbed system is obtained from (1) with $\omega = 0$. The resulting system is differentially flat, with flat output given by the normalized Huygens center of oscillation, which we denote by R, given by,

$$R = \xi + \epsilon\sin\theta \tag{2}$$

Indeed, it can be seen that all system variables in the unperturbed system are differentially parameterized by the fictitious output R,

$$\begin{aligned} \theta &= \arcsin\left(\frac{R + \ddot{R}}{\epsilon}\right) \\ \xi &= -\ddot{R} \\ u &= \frac{1}{\sqrt{\epsilon^2 - (R + \ddot{R})^2}}\left\{\left[\frac{(R + R^{(3)})^2(R + \ddot{R})}{\epsilon^2 - (R + \ddot{R})^2} + \ddot{R}\right] + \right. \end{aligned}$$

$$R^{(4)}\left[1-\epsilon^2+(R+\ddot{R})^2\right]\Bigg\} \tag{3}$$

2.2 Equilibrium parameterization, control objective and minimum phase properties

The desired control objective is to bring the normalized variable ξ and hence the actual cart position x, to a stable rest at the value $\overline{\xi}=0$. Suppose that the normalized Huygens center of oscillation is forced to a constant rest position value, given by $\overline{R}$, then the differential parameterization (3) indicates that the corresponding equilibria for ξ, θ and u are given, respectively, by $\overline{\xi}=0$, $\overline{\theta}=\arcsin(\overline{R}/\epsilon)$ and $\overline{u}=0$. This means that if the flat output is stably forced to any constant value, the corresponding position equilibrium $\overline{\xi}$ is forced to adopt the desired value. The control effort will thus be geared towards this objective via an off-line specified trajectory for the center of oscillation R, denoted by $R^*(t)$.

Note that this indirect control approach overcomes the critical stability characteristics exhibited by the "hidden dynamics" of the system variables, ξ and θ. Indeed, let $\xi=0$ in the parameterization (3). The corresponding *zero dynamics* is given by the critically stable system: $\ddot{R}=0$. The variable ξ is thus a *weakly minimum phase* output. Also, for $\theta=0$, the corresponding zero dynamics is governed by the undampped oscillator $\ddot{R}+R=0$. The angle θ is, therefore, also a weakly minimum phase output.

2.3 Second order sliding mode control of the RTAC system

2.4 Nominal feedback linearization

The locally invertible input coordinate transformation,

$$u=\frac{1}{\sqrt{\epsilon^2-(R+\ddot{R})^2}}\left\{\left[\frac{(R+R^{(3)})^2(R+\ddot{R})}{\epsilon^2-(R+\ddot{R})^2}+\ddot{R}\right]+v\left[1-\epsilon^2+(R+\ddot{R})^2\right]\right\} \tag{4}$$

reduces the unperturbed normalized nonlinear system (**??**) to a simpler linear system in Brunovsky's canonical form

$$R^{(4)}=v \tag{5}$$

which, as shown in the previous section, greatly facilitates the second order sliding based feedback controller design for trajectory tracking on the part of the output R.

In terms of the (measurable) state variables θ, $\dot{\theta}$ ξ, the nominal input coordinate transformation (4) is given by the expression,

$$u = \dot{\theta}^2 \tan\theta - \frac{\xi}{\epsilon\cos\theta} + v\left(\frac{1-\epsilon^2\cos^2\theta}{\epsilon\cos\theta}\right) \tag{6}$$

Evidently the region of validity of the input coordinate transformation is given by the inequality,

$$-\frac{\pi}{2} \; < \; \theta \; < \; \frac{\pi}{2}$$

Let $T > 0$ be a finite time instant and $R^*(t)$ be a desired trajectory which satisfies $R^*(t) = \overline{R}$ for all times $t \;\geq\; T$. We let the auxiliary variable ρ be defined as,

$$\rho = (\ddot{R} - \ddot{R}^*(t)) + 2\eta\omega_n(\dot{R} - \dot{R}^*(t)) + \omega_n^2(R - R^*(t))$$

with $\eta \;\in\; (0,1]$, representing the damping coefficient of the underlying second order linear system, and ω_n being a strictly positive constant representing the corresponding natural oscillating frequency of the closed loop system.

We choose the "parabolic twisting" algorithm, as an ideal dynamics to be imposed on the auxiliary variable ρ,

$$\ddot{\rho} + \frac{1}{2}\mathrm{sign}(\rho)\left[A + B + (A - B)\mathrm{sign}(\rho\;\dot{\rho})\right] = 0, \quad A > B > 0 \tag{7}$$

The transformed control input v is given by

$$\begin{aligned} v = \frac{d^4}{dt^4}R^*(t) - 2\eta\omega_n\left(R^{(3)} - \frac{d^3}{dt^3}R^*(t)\right) - \omega_n^2\left(\ddot{R} - \ddot{R}^*(t)\right) \\ -\frac{1}{2}\mathrm{sign}(\rho)\left[A + B + (A - B)\mathrm{sign}(\rho\;\dot{\rho})\right] \end{aligned} \tag{8}$$

with

$$\begin{aligned} R^{(3)} &= -\dot{\xi} \\ \ddot{R} &= -\xi \\ \dot{R} &= \dot{\xi} + \epsilon\dot{\theta}\cos\theta \\ R &= \xi + \epsilon\sin\theta \\ \rho &= -\xi - \ddot{R}^*(t) + 2\eta\omega_n(\dot{\xi} + \epsilon\dot{\theta}\cos\theta - \dot{R}^*(t)) + \omega_n^2(\xi + \epsilon\sin\theta - R^*(t)) \end{aligned} \tag{9}$$

The flat output tracking error for the closed loop system, after ρ converges to zero in finite time, is thus governed by the second order asymptotically stable dynamics

$$(\ddot{R} - \ddot{R}^*(t)) + 2\eta\omega_n(\dot{R} - \dot{R}^*(t)) + \omega_n^2(R - R^*(t)) = 0$$

2.5 *The perturbed case*

The proposed input coordinate transformation (4) when applied to the perturbed system yields a state-dependent pertubrbed Brunovsky form, given, in this case, by

$$R^{(4)} = v + \vartheta \tag{10}$$

with

$$\vartheta = \ddot{w} - \frac{w}{1 - \epsilon^2 \cos^2 \theta}$$

For sinusoidal perturbations of the form $\omega = M \sin(\Omega t)$, the perturbation $\vartheta(t)$ is bounded by

$$\mid \vartheta \mid \leq \frac{M}{1-\epsilon^2}\left[1 + (1-\epsilon^2)\Omega^2\right]$$

The use of the designed second order sliding mode linearizing controller (8), (9) yields then the following closed loop dynamics for the auxiliary variable ρ

$$\ddot{\rho} + \frac{1}{2}\mathrm{sign}(\rho)\left[A + B + (A - B)\mathrm{sign}(\rho\ \dot{\rho})\right] = \vartheta \tag{11}$$

Since the perturbation term in the right hand side of (11) is assumed to be bounded, it follows, from well known results in second order sliding mode control, that the perturbed closed loop dynamics is guaranteed to converge to a vicinity of the origin where the trajectories remain bounded by a ball centered around the origin of coordinates in the phase plane $(\rho, \dot{\rho})$. It follows that the flat output tracking error is governed by a linear asymptotically stable dynamics subject to external bounded perturbations. Consequently, the closed loop tracking error also remains bounded.

3 Simulation results

Simulations were performed on a RTAC mechanical system characterized by the following set of realistic parameters obtained from the article by Bupp [3]

$$M = 1.3608\,[Kg],\ \ m{-}0.096\,[Kg],\ \ L = 0.0592\,[m],\ \ I = 0.0002175\,[Kg{-}m^2],$$

$$k = 186.3\ [N/m],\ \ \epsilon = 0.200$$

3.1 Control objectives and trajectory planning

The controlled maneuvers were specified as follows: We let the system freely oscillate before a certain time, T_{Ci}. At this moment, we engage the feedback control action, N by means of a "clutch", smoothly increasing the controls amplitudes from zero to its maximum value during a (small) time interval, $[T_{Ci}, T_{Cf}]$. The flat output reference trajectory $R^*(t)$ is planned so that it has a nonzero reference equilibrium value, $\overline{R}$ for all times prior to a certain time T_1, i.e. in the interval, $(-\infty, T_1]$. The control engaging interval is necessarily contained in the infinite interval, i.e. $[T_{Ci}, T_{Cf}] \subset (-\infty, T_1]$. The clutched controller is thus engaged to achieve, right after time T_{Cf}, asymptotic stabilization of the flat output toward the specified constant nonzero equilibrium value, $\overline{R}$. The first stage of the stabilization process, started at T_{Ci}, should not last beyond the time instant $T_1 > T_{Cf}$ (see Figure 2).

At time T_1, the final stabilization maneuver of the flat output towards zero is started. The controller proceeds to drive the flat output R to follow a sufficiently smooth, time-polynomial, trajectory connecting the achieved constant equilibria, $\overline{R}$, with the final rest value of zero for the flat output displacement. This last maneuver is specified to take place in the closed time interval $[T_1, T_2]$. Evidently, the specified control objectives result in a final steady state stable equilibrium at zero for all the system (angular and translational) displacements.

For the proposed control scheme we chose a Hurwitz polynomial of the form $\rho = (p^2 + 2\eta\omega_n p + \omega_n^2)(R - R^*(t))$ with $\eta, \omega_n > 0$. The controller design parameters were set to be,

$$\eta = 0.707, \quad \omega_n = 1, \quad A = 0.1, \quad B = 0.01$$

According to the described control objective we specified the flat output trajectory $R^*(t)$ as follows:

$$R^*(t) = \begin{cases} \overline{R} & \text{for } t \leq T_1 \\ \overline{R}\,[1 - \psi_R(t, T_1, T_2)] & \text{for } T_1 < t < T_2 \\ 0 & \text{for } t \geq T_2 \end{cases}$$

with $\psi_R(t, T_1, T_2)$ being sufficiently differentiable polynomial time functions satisfying: $\psi_R(T_1, T_1, T_2) = 0$ and $\psi_R(T_2, T_1, T_2) = 1$. For the simulations presented we used a polynomial spline of the Bézier type in order to have a sufficiently smooth transfer maneuver between the imposed temporary equilibrium value of the flat output R and the final value zero.

The "clutch" was modeled as a time varying factor, $\mathcal{K}(t)$ multiplying the expression of the feedback control input, as $\mathcal{K}(t)N$. The "clutch" was also specified using polynomial splines of the Bézier type, which smoothly interpolated between the initial value of 0 and the final value of 1.

$$\mathcal{K} = \begin{cases} 0 & \text{for } t \leq T_{Ci} \\ \psi_R(t, T_{Ci}, T_{Cf}) & \text{for } T_{Ci} \ < \ t \ < \ T_{Cf} \\ 1 & \text{for } t \geq T_{Cf} \end{cases}$$

with $\psi_R(T_{Ci}, T_{Ci}, T_{Cf}) = 0$ and $\psi_R(T_{Cf}, T_{Ci}, T_{Ci}) = 1$. The clutch function is given by the following polynomial spline interpolating between 0 and 1.

$$\psi(t, T_{Ci}, T_{Cf}) = \left(\frac{t - T_{Ci}}{T_{Cf} - T_{ci}}\right)^3 \left[r_1 - r_2\left(\frac{t - T_{Ci}}{T_{Cf} - T_{Ci}}\right) + r_3\left(\frac{t - T_{Ci}}{T_{Cf} - T_{Cf}}\right)^2 \right.$$

$$\left. - r_4\left(\frac{t - T_{Ci}}{T_{Cf} - T_{Ci}}\right)^3\right]$$

The constants $r_1, \ldots, r_4$ were suitably chosen to guarantee smooth departures and arrivals i.e., with enough time derivatives being equal to zero at the instants, T_{Ci} and T_{Cf}.

Figure 3 shows the closed loop responses of the unperturbed system's position variables x and θ to the designed feedback controller. The controlled behavior clearly meets the specified objectives. We also show the applied input torque, $N(t)$, the corresponding evolution of the closed loop auxiliary tracking error dynamics in the phase plane $(\rho, \dot{\rho})$ and the controlled behavior of the normalized flat output. In the simulations we set; $T_{Ci} = 0.884$ $[s]$, $T_{Cf} = 3.56$ $[s]$, $T_1 = 4.42$ $[s]$, $T_2 = 6.18$ $[s]$. The intermediate equilibrium value of the flat output was chosen to be $\overline{R} = 0.1$.

3.2 Robustness test

In order to test the robustness of the control scheme with respect to sustained unmodelled oscillatory perturbations. We used a perturbation function for $F(t)$, of the form :

$$F(t) = 0.12 \sin(2\ \pi\ f\ t)$$

The frequency f was set to be $7.203[Hz]$.

Figure 4 shows the performance of the designed controller to the significant unmodelled perturbations.

4 Conclusions

In this article we have shown the benefits, for feedback controller design tasks, of associating a high order sliding mode control scheme with the differential flatness property of a given nonlinear system. The natural candidate output for imposing a discontinuous high order stabilizing dynamics is precisely given by the flat output. This indirect control technique is known to efficiently circumvent any problem associated to the regulation of a non-minimum phase, or weakly minimum phase, output variable while preventing undesirable transient behavior on the closed loop system variables.

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Figure 1. RTAC system

Figure 2. Flat output trajectory and “clutch” function

Figure 3. Controlled behavior of unperturbed RTAC system variables

Figure 4. Controlled behavior of perturbed RTAC system variables