PERIODIC MOTIONS IN VSS AND SINGULAR PERTURBATIONS

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Abstract

The singularly perturbed relay control systems (SPRCS) which have stable periodic motion in reduced systems are studied. Slow motions integral manifold of such systems consists of parts which correspond to different values of control and the solutions contain the jumps from the one part of slow manifold to the other. Three classes of such systems are considered: systems without sliding modes, systems which contain internal sliding modes and time delay systems. The theorems about existence and stability of the slow periodic solutions are proved. The algorithm of asymptotic representation for this periodic solutions using boundary layer method is suggested.

1 Introduction

There are a wide classes of relay control systems which are working in periodic regimes. For example such regimes arise every time in relay control systems with time delay because time delay does not allow to realize an ideal sliding mode, but resulting periodic oscillations Kolmanovskii and Myshkis [5], Fridman *et al* [3]. In controllers of exhausted gases for fuel injectors automotive control systems (see for example Choi and Hedrick [2], Li and Yurkovich [6]) are the sensors which can measure only the sign of controlled variable with delay. In such systems only oscillations around zero value can occur. In the controllers for stabilization of the underwater manipulator it's possible to realize only oscillations because of the manipulators properties (see in Bartolini *et al* [1]). Such situation can occur in the mechanical systems with dry friction. In the paper by Rumpel [8] was shown that there are the periodic oscillations with the internal sliding modes when we consider the pendulum which has dry friction contact with some inclined uniformly rotating disk. First this

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pendulum is moving together with disc until returned point in which he will returned back.

In this paper we will investigate the existence and stability of periodic solutions for singularly perturbed relay control systems (SPRCS). SPRCS could describe for example the complete model of fuel injector systems taking into account the influence of the additional dynamics (the car motor). The knowledge of properties of SPRCS it is necessary in the controllers for stabilization of the underwater manipulator fingers to take into account the influence of the elasticity of these fingers. In pendulums systems contacting with dry friction with inclined uniformly rotated disc SPRCS allows to take into account the presence of the second small pendulum (see figure 1). For

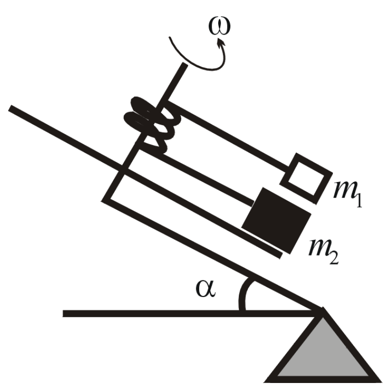

Figure 1. Two pendulums on the inclined uniformly rotated disc.

the smooth singularly perturbed systems there are two main classes of periodic solutions. The slow periodic solutions of the smooth singularly perturbed systems "without jumps" are situated on slow motions integral manifold (see for example Wasov [10]). The other important class of such solutions are the relaxation solutions (see Mishchenko and Rosov [7]), which contain the "jumps" from the neighbourhood of the one stable branch of slow motions manifold to the neighbourhood of another one.

The slow motions integral manifold of relay systems is discontinuous and consists at least of two parts which are the corresponding to the different values of control (see fig. 2). This means that the corresponding periodic solution of SPRCS should have the jumps from the small neighbourhood of the one sheet of integral manifold to the neighbourhood of another one. From this viewpoint the qualitative behavior of this periodic solution will be more near to relaxation solution.

But the main specific feature of systems with relaxation oscillations is the following: at the time moment corresponding to the jump from the neighbourhood of the one branch of the stable integral manifold to the neighbourhood of another one, the value of the right hand side is small. That's why to find the asymptotic representation of relaxation solution it was necessary to make the special asymptotic representations.

The situation with SPRCS is different. The right hand side of SPRCS immediately after the switching moment the right hand side of fast equations in (1) is very big. It allows to use the standard boundary layer functions method (see Vasil'eva *et al* [9]) for asymptotic representation of slow periodic solution of system (1).

2 Problem statement

Consider the system

$$\begin{array}{c} \mu dz/dt = f(z,s,x,u), \quad ds/dt = g(z,s,x,u), \\ dx/dt = h(z,s,x,u), \end{array} \tag{1}$$

where $z \in R^m, s \in R, x \in R^n, u(s)$ is relay control depending on s; $f, g, h \in \mathbf{C}^2(\bar{Z}), Z \subset \mathbf{R}^m \times \mathbf{R} \times \mathbf{R}^n \times [-1,1]$; μ is a small parameter.

Suppose that ignoring additional dynamics, having accepted $\mu = 0$ and expressing z_0 from the equation

$$f(z_0, s, x, u) = 0$$

according the formula $z_0 = \varphi(s,x,u)$, we obtain the system

$$\begin{array}{c} ds/dt = g(\varphi(s,x,u),s,x,u) = G(s,x,u), \\ dx/dt = h(\varphi(s,x,u),s,x,u) = H(s,x,u). \end{array} \tag{2}$$

For this the sufficient conditions for existence of the orbitally asymptotically stable isolated periodic solution are held. It turns out that the desired periodic solution of the original system (1) contains internal boundary layers describing the jumps from the one part of the slow motions manifold to the another one. We will find sufficient conditions under which the original system (1) has an orbitally asymptotically stable isolated periodic solution which corresponds to the periodic solutions of the reduced system. These conditions consist of the next three blocks:

I. Stability of fast motions.

(i) function $z_0 = \varphi(s,x,u)$ at all $(s,x,u) \in \bar{S}, S \subset R \times R^n \times [-1,1]$ is the uniformly asymptotically stable isolated equilibrium point of system

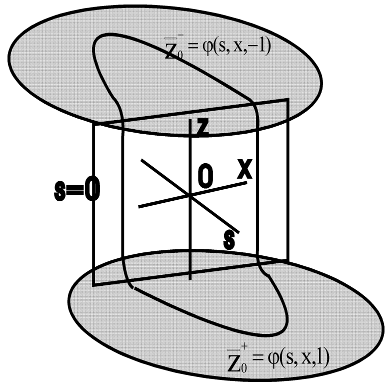

Figure 2. Two sheets of slow motions integral manifold.

$dz/d\tau = f(z, s, x, u)$. Moreover for all $(s, x, u) \in \bar{S}$

$$Re\, Spec\, \frac{\partial f(z_0, s, x, u)}{\partial z} < -\alpha < 0.$$

II. Existence and orbital asymptotic stability for the periodic solution of the reduced system. As the conditions of existence for periodic solutions of reduced system (2) we will use the conditions of existence of the fixed point for the corresponding Poincare map, generating by system (2). The periodic solution of system (2) is orbitally asymptotically stable if corresponding Poincare is contractive.

III. Attractivity. This conditions ensure the existence of the periodic solution at the switching time moment of the relay control. This means that at this time moment the coordinates of the points on the periodic solutions for one value of relay control are situated in the interior of the attractive domain for the other part of slow motions integral manifold corresponding to the other value of the relay control.

In section 3 we will formulate the theorem about existence and stability of the slow periodic solution of the original system (1) for the case when $u(s) = sign[s(t)]$ and periodic solution of the reduced system (2) is the solution without internal sliding modes. In section 4 we will consider the case when periodic solution of the reduced system (2) has an internal sliding modes. In section 5 theorem about existence and stability of the slow periodic solution is proved for the singularly perturbed relay control systems with time delay for which $u(s) = sign\,[s(t-1)]$. For the last case the algorithm is suggested for asymptotic representation of periodic solutions for systems with delay using boundary layer method (see for example Vasil'eva *et al* [9]). In section 7 an example of existence of stable asymptotic periodic solution is given and the asymptotic representation of this solution is found.

3 Periodic solutions without internal sliding modes

Suppose that $u(s) = sign[s(t)]$ and consequently systems (1) and (2) are not smooth. Hence it's impossible to use for the investigation of stability of the systems (1) and (2) the spectral methods and equations in variations. For that goal we will use the Poincare maps generating by this systems. Consider

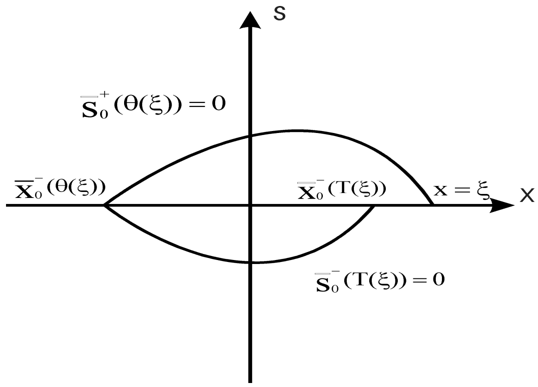

Figure 3. The Poincare map $\Gamma(\xi)$.

the switching surface $s = 0$ in the s, x space and the point ξ on this surface. Denote by $(\bar{s}_0^+(t), \bar{x}_0^+(t))$ the solution of system (2) for $u = 1$ with initial conditions

$$\bar{s}_0^+(0) = 0, \quad \bar{x}_0^+(0) = \xi, \, G(0, \xi, 1) > 0.$$

Suppose that there exists $\theta(\xi)$ being the smallest root of the equation $\bar{s}_0^+(\theta(\xi)) = 0$ and moreover $G(0, \bar{x}_0^+(\theta(\xi)), 1) < 0$. Denote by $(\bar{s}_0^-(t), \bar{x}_0^-(t))$ the solution of system (2) for $u = -1$ with initial conditions

$$\bar{s}_0^-(\theta(\xi)) = 0, \quad \bar{x}_0^-(\theta(\xi)) = x^+(\theta(\xi)).$$

Suppose now that there exists $T(\xi)$ the smallest root of the equation $\bar{s}_0^-(T(\xi)) = 0$ and $G(0, \bar{x}_0^-(T(\xi), -1) > 0$.

Then we can write down the Poincare map $\Gamma(\xi) = \bar{x}_0^-(T(\xi))$ of the switching surface $s = 0$ into itself generating by the system (2). Suppose that:

(1i) the Poincare map $\Gamma(\xi)$ has the isolated fixed point $\xi = x_0$ corresponding to the periodic solution of the system (2).

(1ii) $|Spec\frac{d\Gamma}{dx}(x_0)| < 1$.

(1iii) the switching points

$$\varphi(0, x_0, -1) \,\text{and}\, \varphi(\bar{s}_0^+(\theta(x_0)), \bar{x}_0^+(\theta(x_0)), 1)$$

are situated in the attractive domains of stable equilibrium points $\varphi(0, x_0, 1)$ and $\varphi(\bar{s}_0^+(\theta(x_0)), \bar{x}_0^+(\theta(x_0)), -1)$ respectively.

Theorem 1. *Under conditions* (i) *and* (1i)-(1iii) *system* (1) *has an orbitally asymptotically stable isolated periodic solution in the vicinity of* $(s_0(t), x_0(t))$ *with period* $T(\mu)$ *which tend to* $T(x_0)$ *for* $\mu \to 0$ *and boundary layers near* $t = 0$, $t = \theta(x_0)$.

4 Periodic solutions with internal sliding modes

Consider the SPRCS in form

$$\begin{array}{c} \mu dz/dt = f(z,s,a,b,u), \quad ds/dt = g(z,s,a,b,u), \\ da/dt = h_1(z,s,a,b,u), \\ db/dt = h_2(z,s,a,b,u), \end{array} \tag{3}$$

where $s, a \in R$, $b \in R^{n-1}$, $x^T = (a, b^T)$, $u(s) = sign[s]$, $f, g, h_1, h_2 \in \mathbf{C}^2(\bar{Z})$, $Z \subset \mathbf{R}^m \times \mathbf{R} \times \mathbf{R} \times \mathbf{R}^{n-1} \times [-1, 1]$. In such case the reduced system for the system (3) has the form

$$z_0 = \varphi(s,a,b,u)$$

$$\begin{array}{c} ds/dt = g(z_0(s,a,b,u),s,a,b,u) = G(s,a,b,u), \\ da/dt = h_1(z_0(s,a,b,u),,s,a,b,u) = H_1(s,a,b,u), \\ db/dt = h_2(z_0(s,a,b,u),,s,a,b,u) = H_2(s,a,b,u). \end{array} \tag{4}$$

Such systems can describe for example the behavior of the coupled pendulums if one of this pendulum has dry friction contact with some inclined uniformly rotating disk (see Fridman and Rumpel [4]). Here the periodic solution has the internal sliding mode and the variable a describes the border of the sliding domain which means that $h_1(z,0,0,b,1) \equiv 0, \quad h_1(z,0,0,b,-1) > 0$. Consider the intersection of the switching surface $s = 0$ and the border of the sliding domain $a = 0$ in the s, a, b space and the point b^0 on this intersection. Denote by $(\bar{s}_0^+(t), \bar{a}_0^+(t), \bar{b}_0^+(t))$ the solution of system (4) for $u = 1$ with initial conditions

$$\bar{s}_0^+(0) = 0,\ \bar{a}_0^+(0) = 0,\ \bar{b}_0^+(0) = b^0,\ H_1(0,0,b^0,1) > 0.$$

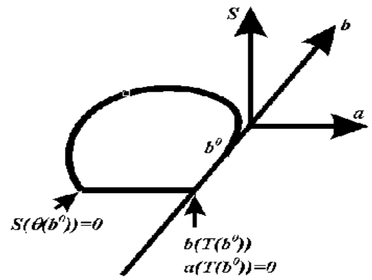

Figure 4. The Poincare map $\mathbf{B}(b^0)$.

Suppose that there exists $\theta(b^0)$ the smallest root of the equation $\bar{s}_0^+(\theta(b^0)) = 0$ and moreover for $(c,d) = (\bar{a}_0^+(\theta(b^0)), \bar{b}_0^+(\theta(b^0)))$,

$$G(0,c,d,1) < 0, \quad G(0,c,d,-1) > 0. \tag{5}$$

Consequently for system (4) the sufficient conditions for the existence of the stable sliding mode holds. Suppose that the motions in this mode are uniquely described by the system

$$\begin{aligned} da^*/dt &= H_1(0,a^*,b^*,u^*(a^*,b^*)), \\ db^*/dt &= H_2(0,a^*,b^*,u^*(a^*,b^*)), \\ G(0,a^*,&b^*,u^*(a^*,b^*)) = 0. \end{aligned} \tag{6}$$

Consider the solution $(\bar{a}_0^*(t), \bar{b}_0^*(t))$ of the system (6) with the initial conditions $\bar{a}_0^*(\theta(b^0)) = \bar{a}_0^+(\theta(b^0))$, $\bar{b}_0^*(\theta(b^0)) = \bar{b}_0^+(\theta(b^0))$. Suppose that there exists $T(b^0)$ the smallest root of the equation $\bar{a}_0^*(T(b^0)) = 0$. Moreover $\frac{d\bar{a}_0^*}{dt}(T(b^0)) > 0$ at all $t \in (\theta_0(b^0), T_0(b^0))$ for all $(c,d) = (\bar{a}_0^*(t), \bar{b}_0^*(t))$ the conditions (5) are held.

Then we can write down the Poincare map $\mathbf{B}(b^0) = \bar{b}_0^*(T(b^0))$ (see the figure 4) of the intersection of the switching surface $s = 0$ and the border of the sliding domain $a = 0$ in the s, a, b space into itself made by the system (4). Suppose that

(2i) the Poincare map $\mathbf{B}(b)$ has the isolated fixed point b_0 corresponding to the periodic solution of the system (4).

(2ii) $|Spec\frac{d\mathbf{B}}{db}(b_0)| < 1$.

(2iii) the point $\varphi(0, \bar{a}_0^+(\theta(b_0)), \bar{b}_0^+(\theta(b_0)), 1)$ is situated in the attractive domain of stable equilibrium point $\varphi(0, \bar{a}_0^+(\theta(b_0)), \bar{b}_0^+(\theta(b_0)), u^*(\bar{a}_0^+(\theta(b_0)), \bar{b}_0^+(\theta(b_0)))$.

Theorem 2. *Under conditions* (i) *and* (2i)-(2iii) *system* (3) *has an orbitally asymptotically stable isolated periodic solution with period* $T(\mu)$ *which tends to* $T(b_0)$ *for* $\mu \to 0$ *and boundary layers near* $t = 0, t = \theta(b_0)$.

Remark. There are no zero order boundary layer functions at the point $t = 0$ in the asymptotic representation of the periodic solution which exists in system (3), because at this point the zero approximation of this solution is continuous (see for example Fridman and Rumpel [4]).

5 Periodic zero frequency steady mode for the relay delay control systems

Let

$$u(s) = -sign[s(t-1)]. \tag{7}$$

Time delay does not allow to realize an ideal sliding mode in the autonomous scalar relay delay control systems (RDCS), but indices periodic oscillations which were called steady modes (Kolmanovskii and Myshkis [5], Fridman *et al* [3].) Only the steady modes with maximal period namely so called zero frequency steady modes are stable. This features of the RDCS was the basic point in design of control systems with time-delayed relay, which stabilize and quench oscillations (see details in Bartolini *et al* [1]). Here we will obtain the sufficient conditions under which there exists the stable zero frequency steady mode for the singularly perturbed RDCS (SPRDCS). Consider the switching surface $s = 0$ in the s, x space and the point ξ on this surface. Denote by $(\bar{s}_0^+(t), \bar{x}_0^+(t))$ the solution of system (2) for $u = 1$ with initial conditions

$$\bar{s}_0^+(0) = 0,\ \bar{x}_0^+(0) = \xi,\ \bar{s}_0^+(t) < 0,\ t \in [-1, 0).$$

Suppose that there exists the smallest root θ^0 of the equation $\bar{s}_0^-(\theta^0) = 0$ and $d\bar{s}_0^-(\theta^0)/dt < 0$, here $(\bar{s}_0^-(t), \bar{x}_0^-(t))$ is the solution of system (2) for $u = -1$ with the initial conditions

$$\bar{s}_0^-(1) = \bar{s}_0^+(1), \quad \bar{x}_0^-(1) = \bar{x}_0^+(1).$$

Moreover suppose that there exists $T(\xi)$ being the smallest root of equation $\bar{s}_0^+(T(\xi)) = 0$ and $T(\xi) > \theta + 1$, $d\bar{s}_0^+(T(\xi))/dt > 0$, where $(s_0^+(t), x_0^+(t))$ is the solution of system (2) with $u = 1$ and initial conditions $(s_0^+(\theta_0 + 1), x_0^+(\theta_0 + 1)) = (s_0^-(\theta_0+1), x_0^-(\theta_0+1))$. Bellow we will use the properties of the Poincare map $\Psi(\xi) = \bar{x}_0^+(T(\xi))$ (fig. 5) of the surface $s = 0$ into itself generating by reduced system (2). Suppose that

(3i) the Poincare map $\Psi(\xi)$ has an isolated fixed point $\xi = x_0$ corresponding to the periodic solution of the system (2) $(\bar{s}_0(t), \bar{x}_0(t))$ with the

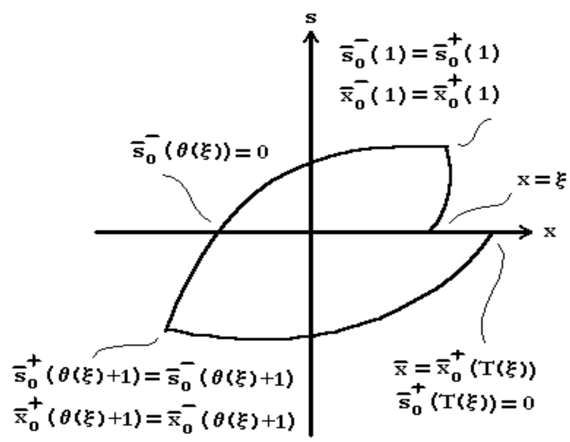

Figure 5. The point mapping $\Psi(\xi)$.

initial conditions

$$\bar{s}_0(0) = 0,\ \bar{x}_0(0) = x_0,\ \bar{s}_0^+(t) < 0,\ t \in [-1, 0).$$

(3ii) $|Spec\frac{d\Psi}{dx}(x_0)| < 1$.

(3iii) the points $\varphi(s_0(1), x_0(1), 1)$ and $\varphi(s_0(\theta_0+1), x_0(\theta_0+1), -1)$ are situated in the attractive domains of stable equilibrium points $\varphi(s_0(1), x_0(1), -1)$ and $\varphi(s_0(\theta_0+1), x_0(\theta_0+1), 1)$ respectively.

Theorem 3. *Under conditions* (i) *and* (3i)-(3iii) *system* (1),(7) *has an orbitally asymptotically stable isolated periodic solution in the vicinity of* $(s_0(t), x_0(t))$ *with period* $T(\mu)$ *which tend to* $T(x_0)$ *for* $\mu \to 0$ *and boundary layers near* $t = 1$, $t = \theta(x_0) + 1$.

6 Asymptotic representation of the periodic steady mode

Suppose that $f, g, h \in C^{k+3}[\bar{Z} \times [-1, 1]]$, conditions of theorem 3 are held and

$(3iv)$ $\ det|\frac{d\Psi}{dx}(x_0)| \neq 0$.

We will find the asymptotic expansion for switching moment $\theta(\mu)$, period $T(\mu)$ of the periodic solution of system (1) on the time interval $[-\Theta(\mu) + 1, \theta(\mu) + 1]$, $\quad (T(\mu) = \theta(\mu) + \Theta(\mu))$ in form

$$Y_k(t, \mu) = \sum_{i=0}^{k} [\bar{y}_i(t) + \Pi_i^- y(\tau^-) + \Pi_i^+ y(\tau_{k+1}^+)]\mu^i;$$

$$S_k(t,\mu)=\sum_{i=0}^{k}[\bar{s}_i(t)+\Pi_i^- s(\tau^-)+\Pi_i^+ s(\tau_k^+)]\mu^i;$$

$$X_k(t,\mu)=\sum_{i=0}^{k}[\bar{x}_i(t)+\Pi_i^- x(\tau^-)+\Pi_i^+ x(\tau_k^+)]\mu^i;$$

$$\theta(\mu)=\theta_0+\mu\theta_1+\cdots+\mu^k\theta_k+\cdots;$$

$$T(\mu)=T_0+\mu T_1+\cdots+\mu^k T_k+\cdots;$$

$$\Theta(\mu)=T(\mu)-\theta(\mu);$$

where

$$\tilde{\theta}_{k+1}(\mu)=\theta_0+\mu\theta_1+\cdots+\mu^{k+1}\theta_{k+1};$$

$$\tilde{\Theta}_{k+1}(\mu)=\Theta_0+\mu\Theta_1+\cdots+\mu^{k+1}\Theta_{k+1};$$

$$\tau^-=(t-1)/\mu,\ \tau_{k+1}^+=(t-(1-\tilde{\Theta}_{k+1}(\mu)))/\mu,$$

$$\tilde{T}_k(\mu)=T_0+\mu T_1+\cdots+\mu^k T_k;$$

$$\| \Pi_i^- y(\tau^-)\| < Ce^{-\gamma\tau^-}, C,\gamma>0,$$

$$\Pi_i^- y(\tau^-)\equiv 0 \quad for \quad \tau^-<0;$$

$$\| \Pi_i^- y(\tau_{k+1}^+)\| < Ce^{-\gamma\tau_{k+1}^+}, \tau_{k+1}^+>0.$$

$$\Pi_i^+ y(\tau_{k+1}^+)\equiv 0 \ for \quad \tau_{k+1}^+<0.$$

Let us denote

$$\bar{y}_0(t)=\begin{cases}\bar{y}_0^+(t)=(\varphi(\bar{s}_0^+(t),\bar{x}_0^+(t),1),\bar{s}_0^+(t),\bar{x}_0^+(t)) \\ \text{for} \quad t\in[-\Theta_0+1,1]; \\ \bar{y}_0^-(t)=(\varphi(\bar{s}_0^-(t),\bar{x}_0^-(t),1),\bar{s}_0^-(t),\bar{x}_0^-(t)) \\ \text{for} \quad t\in[1,\theta_0+1].\end{cases}$$

The function $\Pi_0^+ z(\tau)$ is defined by system

$$d\Pi_0^+ z/d\tau=g(\Pi_0^+ z+\varphi(\bar{s}_0^+(-\Theta_0+1),$$

$$\bar{x}_0^+(-\Theta_0+1),1),\bar{s}_0^+(-\Theta_0+1),\bar{x}_0^+(-\Theta_0+1),1),$$

$$\Pi_0^+ z(0) = \varphi(\bar{s}_0^+(-\Theta_0+1), \bar{x}_0^+(-\Theta_0+1), -1) -$$

$$-\varphi(\bar{s}_0^+(-\Theta_0+1), \bar{x}_0^+(-\Theta_0+1), 1).$$

The boundary layer function $\Pi_0^- z(\tau)$ we will find from the system

$$d\Pi_0^- z/d\tau = g(\Pi_0^- z + \varphi(\bar{s}_0^-(1), \bar{x}_0^-(1), 1), \bar{s}_0^-(1), \bar{x}_0^-(1), 1),$$

$$\Pi_0^- z(0) = \varphi(\bar{s}_0^-(1), \bar{x}_0^-(1), 1) - \varphi(\bar{s}_0^-(1), \bar{x}_0^+(1), -1).$$

To find the functions $\bar{s}_1^\pm(t), \bar{x}_1^\pm(t), \bar{z}_1^\pm(t)$ we will have the system

$$\bar{z}_1^+(t) = -[g_z']^{-1}(g_s'\bar{s}_1^+ + g_x'\bar{x}_1^+ + g_1^+(t));$$

$$d\bar{s}_1^+/dt = h_{1z}'(t)\bar{z}_1^+(t) + h_{1s}\bar{s}_1^+(t) + h_{1x}'\bar{x}_1^+(t),$$

$$d\bar{x}_1^+/dt = h_{2z}'(t)\bar{z}_1^+(t) + h_{2s}\bar{s}_1^+(t) + h_{2x}'\bar{x}_1^+(t),$$

$$\bar{z}_1^-(t) = -[g_z']^{-1}(g_s'\bar{s}_1^- + g_x'\bar{x}_1^- + g_1^-(t));$$

$$d\bar{s}_1^-/dt = h_{1z}'(t)\bar{z}_1^-(t) + h_{1s}\bar{s}_1^-(t) + h_{1x}'\bar{x}_1^-(t),$$

$$d\bar{x}_1^-/dt = h_{2z}'(t)\bar{z}_1^-(t) + h_{2s}\bar{s}_1^-(t) + h_{2x}'\bar{x}_1^-(t),$$

where index $\pm$ means that corresponding functions are computed at the points $(\varphi(\bar{s}_0^\pm(t), \bar{x}_0^\pm(t), \pm 1), \bar{s}_0^\pm(t), \bar{x}_0^\pm(t), \pm 1))$. For the first order boundary layer functions $\Pi_1^\pm z, \Pi_1^\pm s, \Pi_1^\pm x$ we have the system

$$d\Pi_1^+ z/d\tau = g_z'\Pi_1^+ z + g_s'\Pi_1^+ s + g_x'\Pi_1^+ x + \Pi_1^+ g(\tau),$$

$$d\Pi_1^+ s/d\tau = \Pi_0^+ h_1 =$$

$$h_1(\bar{z}_0^+(-\Theta_0+1) + \Pi_0^+ z, 0, \bar{x}_0^+(-\Theta_0+1), 1) -$$

$$-h_1(\bar{z}_0^+(-\Theta_0+1), 0, \bar{x}_0^+(-\Theta_0+1), 1),$$

$$d\Pi_1^+ x/d\tau = \Pi_0^+ h_2 =$$

$$= h_2(\bar{z}_0^+(-\Theta_0+1) + \Pi_0 z, 0, \bar{x}_0^+(-\Theta_0+1), 1) -$$

$$-h_2(\bar{z}_0^+(-\Theta_0+1), 0, \bar{x}_0^+(-\Theta_0+1), 1),$$

$$d\Pi_1^- z/d\tau = g_z'\Pi_1^- z + g_s'\Pi_1^- s + g_x'\Pi_1^- x + \Pi_1^- g(\tau),$$

$$d\Pi_1^- s/d\tau = \Pi_0^- h_1 =$$

$$= h_1(\bar{z}_0^-(1) + \Pi_0^- z, 0, \bar{x}_0^-(1), -1) -$$

$$-h_1(\bar{z}_0^-(1), 0, \bar{x}_0^-(1), -1),$$

$$d\Pi_1^- x/d\tau = \Pi_0^- h_1 =$$

$$= h_2(\bar{z}_0^-(1) + \Pi_0^- z, 0, \bar{x}_0^-(1), -1) -$$

$$-h_2(\bar{z}_0^-(1), 0, \bar{x}_0^-(1), -1),$$

where index $\pm$ means that corresponding functions are calculated at the points

$$(\bar{z}_0^+(-\Theta_0 + 1) + \Pi_0^+ z, 0, \bar{x}_0^+(-\Theta_0 + 1), 1)$$

and

$$(\bar{z}_0^-(1) + \Pi_0^- z, 0, \bar{x}_0^-(1), -1)$$

respectively.

The initial conditions for the first order boundary layer functions we can find from the equations

$$\Pi_1^+ s(0) = \int_\infty^0 \Pi_0^+ h_1(\Theta) d\Theta,$$

$$\Pi_1^+ x(0) = \int_\infty^0 \Pi_0^+ h_2(\Theta) d\Theta,$$

$$\Pi_1^- s(0) = \int_\infty^0 \Pi_0^- h_1(\Theta) d\Theta,$$

$$\Pi_1^- x(0) = \int_\infty^0 \Pi_0^- h_2(\Theta) d\Theta.$$

For obtaining of the first approximation of the slow coordinates it is necessary to find the initial conditions $\bar{s}_1^\pm(0)$ and $\bar{x}_1^\pm(0)$. For this initial values we will use the following conditions

- intersection with the surface $s = 0$, using the first order approximation of equations $s(T(\mu), \mu) = 0$ and $s(\theta(\mu), \mu) = 0$

$$\Theta_1 H_1(0,\xi_0,1)+\bar{s}_1^+(0)=0, \tag{8}$$

$$\theta_1 H_1(0,\bar{x}_0^-(\theta_0),-1)+\bar{s}_1^-(\theta_0)=0;$$

- continuity of the solution at the point $t=1$

$$\bar{s}_1^-(1)+\Pi_0^- s(0)=\bar{s}_1^+(1)+\Theta_1 H_1(\bar{s}_0^+(1),\bar{x}_0^+(1),1),$$

$$\bar{x}_1^-(1)+\Pi_0^- x(0)=\bar{x}_1^+(1)+\Theta_1 H_2(\bar{s}_0^+(1),\bar{x}_0^+(1),1); \tag{9}$$

- continuity of the solutions of system (1) at the time moment $t=\theta(\mu)+1$, $t=-\Theta(\mu)+1$

$$\bar{s}_1^+(-\Theta_0+1)+\Pi_0^+ s(0)=\bar{s}_1^-(\theta_0+1)+$$

$$+\theta_1 H_1(\bar{s}_0^-(\theta_0+1),\bar{x}_0^-(\theta_0+1),-1), \tag{10}$$

$$\bar{x}_1^+(-\Theta_0+1)+\Pi_0^+ x(0)=\bar{x}_1^-(\theta_0+1)+$$

$$+\theta_1 H_2(\bar{s}_0^-(\theta_0+1),\bar{x}_0^-(\theta_0+1),-1).$$

The values θ_1 and Θ_1 can be found uniquely from equations (8) because according to condition of the (i) and (3i)-(3iv) the periodic solution of system (2) cross the surface $s=0$ without touch. This means that we can express θ_1 and Θ_1 through $\bar{s}_1^+(0)$ and $\bar{s}_1^-(\theta_0)$ in the form

$$\Theta_1=-[H_1(0,\xi_0,1)]^{-1}\bar{s}_1^+(0),$$

$$\theta_1=-[H_1(0,\bar{x}_0(\theta_0),-1)]^{-1}\bar{s}_1^-(\theta_0).$$

Now we can substitute θ_1,Θ_1 into equations (9) and (10), by this we have the system of linear algebraic equations with respect to $\bar{s}_1^+(0),\bar{x}_1^+(0),\bar{s}_1^-(\theta_0),\bar{x}_1^-(\theta_0)$

$$\bar{s}_1^-(1)+\Pi_0^- s(0)=$$

$$=\bar{s}_1^+(1)-[H_1(0,\xi_0,1)]^{-1}\bar{s}_1^+(0)H_1(\bar{s}_0^+(1),\bar{x}_0^+(1),1),$$

$$\bar{x}_1^-(1)+\Pi_0^- x(0)=$$

$$=\bar{x}_1^+(1)-[H_1(0,\xi_0,1)]^{-1}\bar{x}_1^+(0)H_2(\bar{s}_0^+(1),\bar{x}_0^+(1),1),$$

$$\bar{s}_1^+(-\Theta_0+1)+\Pi_0^+ s(0)=\bar{s}_1^-(\theta_0+1)-$$

$$-[H_1(0,\bar{x}_0(\theta_0),-1)]^{-1}\bar{s}_1^-(\theta_0)H_1(\bar{s}_0^-(\theta_0+1),\bar{x}_0^-(\theta_0+1),-1),$$

$$\bar{s}_1^+(-\Theta_0+1)+\Pi_0^+s(0)=\bar{s}_1^-(\theta_0+1)-$$

$$-[H_1(0,\bar{x}_0(\theta_0),-1)]^{-1}\bar{s}_1^-(\theta_0)H_1(\bar{s}_0^-(\theta_0+1),\bar{x}_0^-(\theta_0+1),-1).$$

The determinant of this system coincides with the $det|\frac{d\Psi}{dx}(x_0)|$ and consequently this system has the unique solution.

Hence we have found the initial conditions

$$\bar{s}_1^+(0),\bar{x}_1^+(0),\bar{s}_1^-(\theta_0),\bar{x}_1^-(\theta_0)$$

and it's possible to determine uniquely the functions $\bar{s}_1^{\pm}(t),\bar{x}_1^{\pm}(t)$. Now we can denote

$$\bar{y}_1(t)=\begin{cases}\bar{y}_1^+(t)=(\bar{z}_1^+(t),\bar{s}_1^+(t),\bar{x}_1^+(t))\\ \text{for}\quad t\in[-\tilde{\Theta}_1(\mu)+1,1];\\ \bar{y}_1^-(t)=(\bar{z}_1^-(t),\bar{s}_1^-(t),\bar{x}_1^-(t))\\ \text{for}\quad t\in[1,\tilde{\theta}_1(\mu)+1].\end{cases}$$

The initial conditions for $\Pi_1^{\pm}z$ are uniquely defined by equations

$$\bar{z}_1^-(0)+\Pi_1^-z(0)=\bar{z}_1^+(1)+\Theta_1 d\bar{z}_0^+/dt(1),$$

$$\bar{z}_1^+(-\Theta_0+1)+\Pi_1^+z(0)=\bar{z}_1^-(\theta_0+1)+\theta_1 d\bar{z}_0^-/dt(\theta_0+1).$$

Thus we have found the first approximation of asymptotic expansion of the slow variables and period of desired periodic solution. To obtain the first approximation of the fast variables it is necessary to find the value Θ_2 and substitute this constant into the function $\Pi_1^+z(\tau_2^+)$. In such case the initial conditions for $\Pi_1 z^{\pm}$ are uniquely defined by the system

$$\bar{z}_1^+(-\Theta_0+1)+\Pi_1^+z(0)=\bar{z}_1^-(\theta_0+1)+\theta_1 d\bar{z}_0^-/dt(\theta_0+1),$$

$$\bar{z}_1^-(1)+\Pi_0^-z(0)=\bar{z}_1^+(1)+\Theta_1 d\bar{z}_0^+/dt(1).$$

The terms for the higher order approximation can be found analogously.

Theorem 4. *Let's conditions (i) and $(3i)-(3iv)$ hold. Then the following estimates*

$$|\tilde{T}_k(\mu)-T(\mu)|<C\mu^{k+1},$$

$$\| y(t,\mu)-Y_k(t,\mu)\|<C\mu^{k+1},$$

$$\| (s(t,\mu),x(t,\mu))-(S_k(t,\mu),X_k(t,\mu))\|<C\mu^{k+1}$$

take place uniformly on time interval

$$[-\hat{\Theta}(\mu)+1,\hat{\theta}(\mu)+1],$$

where $\hat{\Theta}(\mu)=max\{\Theta(\mu);\tilde{\Theta}_{k+1}(\mu)\}$,

$$\hat{\theta}(\mu)=max\{\theta(\mu);\tilde{\theta}_{k+1}(\mu)\}.$$

7 Example of asymptotic representation for periodic solution

Consider the system

$$\begin{array}{c}\mu dz/dt=-z+u;\quad ds/dt=x;\\ dx/dt=-x+z,\quad u=-sign[s(t-1)].\end{array}\tag{11}$$

For $\mu=0$ system (11) has the form

$$\frac{d\bar{s}_0}{dt}=\bar{x}_0,\ \frac{d\bar{x}_0}{dt}=-\bar{x}_0+u.\tag{12}$$

Then for the solution of (12) with initial conditions

$$\bar{x}_0^+(0)=\xi,\quad \bar{s}_0^+(0)=0,$$

$$\text{sign}\,[\bar{s}_0^+(t-1)]=-1,\quad u=1\quad \text{for}\quad t\in[-1,0]$$

we have

$$\bar{x}_0{}^+(t,\xi)=e^{-t}(\xi-1)+1;\quad \bar{s}_0^+(t,\xi)=(1-e^{-t})(\xi-1)+t;$$

and consequently

$$\bar{x}_0^+(1,\xi)=e^{-1}(\xi-1)+1;\quad \bar{s}_0^+(1,\xi)=(1-e^{-1})(\xi-1)+1.$$

For $t>1,u=-1$ and until switching of sign$[u]$

$$\bar{x}_0^-(t,\xi)=e^{-(t-1)}(\bar{x}_0^+(1,\xi)+1)-1;$$

$$\bar{s}_0^-(t,\xi)=$$

$$=(1-e^{-(t-1)})(\bar{x}_0^+(1,\xi)+1)-(t-1)+(1-e^{-1})(\xi-1)+1.$$

In this case the switching moment $\theta(\xi)$ is defined by equation $s_0^-(\theta(\xi),\xi)=0$. Taking into account the symmetry of system (12) with respect to the point $s=x=0$, we can conclude that the semi-period of the desired periodic solution θ_0 and the fixed point ξ_0 of the point mapping $\Psi(\xi)$ are described by equation

$$\bar{s}_0^-(\theta_0,\xi_0)=0,\quad \bar{x}_0^-(\theta_0,\xi_0)=-\xi_0\ ,$$

hence

$$\xi_0 = 1 - 2\frac{e^{-\theta_0+1}}{1+e^{-\theta_0}}; \quad \theta_0 = 4 - 4\frac{e^{-\theta_0+1}}{1+e^{-\theta_0}}.$$

This system has a solution $\theta_0 \approx 3,75, \xi_0 \approx 0,87$. Here ξ_0 is the fixed point of point mapping $\Psi(\xi)$, corresponding to the $2\theta_0-$ periodic solution of (12) determined by the equations

$$(\bar{s}_0(t), \bar{x}_0(t)) = \begin{cases} (\bar{s}_0^{\,+}(t,\xi_0), \bar{x}_0^{\,+}(t,\xi_0)), & \text{for} \quad -\theta_0+1 \le t \le 1, \\ (\bar{s}_0^{\,-}(t,\xi_0), \bar{x}_0^{\,-}(t,\xi_0)), & \text{for} \quad 1 \le t \le \theta_0+1. \end{cases}$$

Moreover,

$$\frac{d\Psi}{d\xi}(\xi_0) = \left(\frac{dx^-(\theta(\xi),\xi)}{d\xi}(\theta_0,\xi_0)\right)^2 = \left(e^{-\theta_0} - 2\frac{e^{-\theta_0+1}(1-e^{-\theta_0})}{e^{-\theta_0}+1-2e^{-\theta_0+1}}\right)^2 \approx 0,0144.$$

Then the conditions of Theorem 4 hold for system (12), therefore system (11) has an orbitally asymptotically stable periodic zero frequency steady mode at least for the small μ.

To complete the zero approximation of periodic solution denote

$$\bar{z}_0(t) = \begin{cases} \bar{z}_0^{\,+}(t) = 1 & \text{for} \quad -\theta_0+1 \le t \le 1, \\ \bar{z}_0^{\,-}(t) = -1, & \text{for} \quad 1 \le t \le \theta_0+1. \end{cases}$$

Then

$$d\Pi_0^- z/d\tau = -\Pi_0^- z; \quad \Pi_0^- z(0) = 2;$$

$$\Pi_0^- z(\tau^-) = 2e^{-\tau^-}; \quad \tau^- = (t-1)/\mu;$$

$$d\Pi_0^+ z/d\tau = -\Pi_0^+ z; \quad \Pi_0^+ z(0) = -2;$$

$$\Pi_0^+ z(\tau^+) = -2e^{-\tau^+}; \quad \tau^+ = (t+\Theta(\mu)-1)/\mu.$$

Let's compute the first order terms. Equations for the slow parts have the form

$$\bar{z}_1^{\,\pm} = 0; \quad d\bar{s}_1^{\,\pm}/dt = x_1^{\,\pm}; \quad d\bar{x}_1^{\,\pm}/dt = -\bar{x}_1^{\,\pm},$$

and consequently

$$\bar{x}_1^{\,+}(t, \bar{x}_1^{\,+}(0)) = \bar{x}_1^{\,+}(0)e^{-t}; \bar{s}_1^{\,+}(t) = (1-e^{-t})\bar{x}_1^{\,+}(0) + \bar{s}_1^{\,+}(0);$$

$$\bar{x}_1^{\,-}(t, \bar{x}_1^{\,-}(\theta_0)) = \bar{x}_1^{\,-}(\theta_0)e^{-(t-\theta_0)}; \quad \bar{s}_1^{\,-}(t) = (1-e^{-(t-\theta_0)})\bar{x}_1^{\,-}(\theta_0) + \bar{s}_1^{\,-}(\theta_0).$$

Defining the boundary layer terms of the first order for the slow variables we will have

$$\Pi_1^- s(\tau^-) \equiv 0; \quad \Pi_1^- s(0) = 0; \quad \Pi_1^- x(\tau^-) = \int_\infty^{\tau^-} \Pi_0^- z(\Theta) d\Theta; \quad \Pi_1^- x(0) = -2;$$

$$\Pi_1^+ s(\tau^+) \equiv 0; \quad \Pi_1^+ s(0) = 0; \quad \Pi_1^+ x(\tau^+) = \int_\infty^{\tau^+} \Pi_0^+ z(\Theta) d\Theta; \quad \Pi_1^+ x(0) = 2.$$

Equations for θ_1 and Θ_1 have the form

$$\Theta_1 \xi_0 + \bar{s}_1^+(0) = 0; \quad \theta_1 \bar{x}_0^-(\theta_0) + \bar{s}_1^-(\theta_0) = 0.$$

In this case it's possible to express the variables θ_1 and Θ_1 via $\bar{s}_1^+(0), \bar{s}_1^-(\theta_0)$ according to the formulae

$$\theta_1 = -\bar{s}_1^-(\theta_0)/\bar{x}_0^-(\theta_0); \Theta_1 = -\bar{s}_1^+(0)/\xi_0.$$

Now

$$\begin{gathered}
\bar{s}_1^-(1) = \bar{s}_1^+(1) - \bar{x}_0^+(1)\bar{s}_1^+(0)/\xi_0, \\
\bar{x}_1^-(1) + \Pi_1^- x(0) = \bar{x}_1^+(1) - (-\bar{x}_0^+(1) + 1)\bar{s}_1^+(0)/\xi_0, \\
\bar{s}_1^+(-\Theta_0 + 1) = \bar{s}_1^-(\theta_0 + 1) - \bar{x}_0^-(\theta_0 + 1)\bar{s}_1^-(\theta_0)/\bar{x}_0^-(\theta_0), \\
\bar{x}_1^+(-\Theta_0 + 1) + \Pi_1^+ x(0) = \bar{x}_1^-(\theta_0 + 1) + (\bar{x}_0^-(\theta_0 + 1) + 1)\bar{s}_1^-(\theta_0)/\bar{x}_0^-(\theta_0).
\end{gathered} \tag{13}$$

Taking into account the symmetry of system (11) we will have $\bar{x}_1^+(-\theta_0 + 1) = -\bar{x}_1^-(1)$, $\bar{s}_1^+(-\theta_0 + 1) = -\bar{s}_1^-(1), \theta_0 = \Theta_0$, Consequently

$$\begin{gathered}
-\bar{s}_1^-(1) = (1 - e^{-\theta_0})\bar{x}_1^-(1) + \bar{s}_1^-(1) - \tfrac{[(1-e^{-\theta_0+1})\bar{x}_1^-(1) + \bar{s}_1^-(1)](1-e^{-\theta_0})}{(1-2e^{-\theta_0+1}+e^{-\theta_0})}; \\
-\bar{x}_1^-(1) - 2 = e^{-\theta_0}\bar{x}_1^-(1) - 2\tfrac{[(1-e^{-\theta_0+1})\bar{x}_1^-(1) + \bar{s}_1^-(1)]e^{-\theta_0}}{(1-2e^{-\theta_0+1}+e^{-\theta_0})}.
\end{gathered} \tag{14}$$

Consequently

$$\bar{x}_1^-(1) = 2\frac{1 + 3e^{-\theta_0} - 4e^{-\theta_0+1}}{e^{-2\theta_0} + 2e^{-\theta_0} + 1 - 4e^{\theta_0+1}} \approx -2,06;$$

$$\bar{s}_1^-(1) = 2\frac{e^{-\theta_0} - e^{-\theta_0+1} - e^{-2\theta_0} + e^{-2\theta_0+1}}{e^{-2\theta_0} + 2e^{-\theta_0} + 1 - 4e^{\theta_0+1}} \approx -0,10;$$

$$\theta_1 = 2\frac{(1 + e^{-\theta_0})(1 + e^{-\theta_0} - 2e^{-\theta_0+1})}{e^{-2\theta_0} + 2e^{-\theta_0} + 1 - 4e^{\theta_0+1}} \approx 2.32.$$

Conclusions

Three classes of the singularly perturbed relay control systems are considered. For such systems

- the theorems about existence and stability of the periodic solutions are discussed;
- the algorithm for the asymptotic representation of this periodic solutions using boundary layer method is suggested.

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