Differential Equations, Vol. 39, No. 2, 2003, pp. 256–266. Translated from Differentsial'nye Uravneniya, Vol. 39, No. 2, 2003, pp. 236–245.

Application of the Boundary Function Method to Finding Slow Periodic Solutions of Singularly Perturbed Bang-Bang Systems

L. M. Fridman

Samara State Academy of Architecture and Civil Engineering, Samara, Russia

Received April 8, 1999

Numerous systems with bang-bang control algorithms and mechanical systems with dry friction operate in oscillatory modes. Such systems include, for example, fuel injection control systems [1], in which the sensor shows only the sign of the deviation of the exhaust parameters from given values and the basic operation mode is oscillations in a neighborhood of the given values. Another example of such systems is given by oscillations of a pendulum on an inclined uniformly rotating disk in the presence of dry friction [2]. Such motions have the specific feature that, for some part of the period, the pendulum moves together with the disk (in a sliding mode) and then returns to the original position.

At the same time, real control systems always contain actuators, sensors, and other devices whose operation is described by differential equations with small parameters multiplying the derivatives (these parameters correspond to the time constants of these devices); consequently, the complete model of such a system is described by singularly perturbed bang-bang systems (SPBBS). In fuel injection control systems [1], SPBBS can describe, say, the influence of the engine on the injector operation. In the system with a rotating disk, dealing with SPBBS is necessitated if one considers the problem of perturbations induced by an additional pendulum elastically connected to the pendulum lying on the disk in the case of dry friction [2].

It was shown in [3–5] that slow periodic solutions of smooth singularly perturbed systems lie on slow integral manifolds. The integral manifold of slow motions of a SPBBS consists of leaves corresponding to different values of the bang-bang control. We show that slow periodic solutions of SPBBS have interior boundary layers appearing in the transition from a neighborhood of one leaf of the integral manifold into a neighborhood of another. Slow periodic solutions of SPBBS are also characterized by the fact that, unlike relaxation oscillations (see the bibliography in [6]), the right-hand sides of the equations describing rapid motions are nonzero at control switching points, which permits one to use the boundary function method [7] for describing the breakaway of a solution from a neighborhood of one leaf of the integral manifold into a neighborhood of another leaf.

1. STATEMENT OF THE PROBLEM

In the present paper, we consider the SPBBS

$$\mu\, dz/dt = g(z,s,x,u), \qquad ds/dt = h_1(z,s,x,u), \qquad dx/dt = h_2(z,s,x,u), \tag{1}$$

where $z \in R^m$, $s \in R$, $x \in R^n$, and $u(s) = \operatorname{sgn} s$. Here g, h_1, and h_2 are smooth functions of their arguments, and μ is a small parameter. In control systems, the vector z usually describes the behavior of actuators, the variables s and x describe the behavior of the controlled object, and μ characterizes the time constant of the actuators.

We set $\mu = 0$ and express z from the equation

$$g\left(z_0, s, x, u(s)\right) = 0 \tag{2}$$

by the formula $z_0 = \varphi(s,x,u)$; then, instead of (1), we obtain the system

$$\begin{aligned} d\bar{s}_0/dt &= h_1\left(\varphi\left(\bar{s}_0,\bar{x}_0,u\right),\bar{s}_0,\bar{x}_0,u\right) = H_1\left(\bar{s}_0,\bar{x}_0,u\right), \\ d\bar{x}_0/dt &= h_2\left(\varphi\left(\bar{s}_0,\bar{x}_0,u\right),\bar{s}_0,\bar{x}_0,u\right) = H_2\left(\bar{s}_0,\bar{x}_0,u\right). \end{aligned} \tag{3}$$

0012-2661/03/3902-0256$25.00 © 2003 MAIK "Nauka/Interperiodica"

Suppose that this system satisfies sufficient conditions for the existence of an orbitally asymptotically stable periodic solution.

Note that the slow integral manifold of system (1) consists of two leaves corresponding to $\bar{z}_0^+ = \varphi(s, x, 1)$ and $\bar{z}_0^- = \varphi(s, x, -1)$.

We shall show that, under some natural conditions, system (1) can have stable limit cycles whose slow coordinates are close to periodic solutions appearing in (3). These limit cycles contain boundary layers describing jumps from a neighborhood of one leaf of the slow integral manifold into a neighborhood of another leaf at control sign switching times.

2. THE EXISTENCE OF PERIODIC SOLUTIONS

We denote the domains where the variables (z, s, x) and (s, x) range by Z and X, respectively. Suppose that the following conditions are satisfied:

(1^0) $h_1, h_2, g \in C^2\left[\bar{Z} \times [-1, 1]\right]$;

(2^0) the function $z_0 = \varphi(s, x, u)$ is an isolated solution of Eq. (2) for all $(s, x, u) \in \Omega = \bar{X} \times [-1, 1]$;

(3^0) the equilibrium $z_0 = \varphi(s, x, u)$ of the system $dz/d\tau = g(z, s, x, u)$ is uniformly asymptotically stable with respect to Ω; moreover, $\operatorname{Re}\operatorname{Spec}(\partial g/\partial z)(z_0, s, x, u) < -\alpha < 0$ for all $(s, x, u) \in \Omega$.

The right-hand sides of systems (1) and (3) are discontinuous; consequently, one cannot use the ordinary variational equations in the stability analysis of their periodic solutions. Conditions for the existence of periodic solutions of systems (1) and (3) and their orbital asymptotic stability will be stated in terms of the point mappings of the surface $s = 0$ into itself determined by these systems.

Let us first define the point mapping of the surface $s = 0$ into itself determined by system (3). To this end, we consider the solution of system (3) with $u = 1$,

$$d\bar{s}_0^+/dt = H_1\left(\bar{s}_0^+, \bar{x}_0^+, 1\right), \qquad d\bar{x}_0^+/dt = H_2\left(\bar{s}_0^+, \bar{x}_0^+, 1\right) \tag{3^+}$$

with the initial conditions

$$\bar{s}_0^+(0) = 0, \qquad \bar{x}_0^+(0) = \xi, \qquad \xi \in V \subset S^+ = \{\xi : \ H_1(0, \xi, 1) > 0\}. \tag{4}$$

Suppose that, for all $\xi \in V$, there exists a least positive root $t = \theta(\xi)$ of the equation $\bar{s}_0^+(\theta(\xi)) = 0$ such that $H_1\left(0, \bar{x}_0^+(\theta(\xi)), \pm 1\right) > 0$. Consequently, the function u jumps from -1 to $+1$ at $t = \theta(\xi)$, and the behavior of the solution of the Cauchy problem for $t > \theta(\xi)$ until the next switching is described by the equations

$$d\bar{s}_0^-/dt = H_1\left(\bar{s}_0^-, \bar{x}_0^-, -1\right), \qquad d\bar{x}_0^-/dt = H_2\left(\bar{s}_0^-, \bar{x}_0^-, -1\right) \tag{3^-}$$

with the initial conditions [sewing conditions for the solutions of system (3)]

$$\bar{s}_0^-(\theta(\xi)) = \bar{s}_0^+(\theta(\xi)), \qquad \bar{x}_0^-(\theta(\xi)) = \bar{x}_0^+(\theta(\xi)). \tag{4^-}$$

Now we suppose that, for all $\xi \in V$, for the first coordinate $\bar{s}_0^-(t)$ of the solution of this Cauchy problem, there exists a least root $T(\xi)$ of the equation $\bar{s}_0^-(T(\xi)) = 0$ such that $\bar{x}_0^-(T(\xi)) \in V$ and $H_1\left(0, \bar{x}_0^-(T(\xi)), -1\right) > 0$. Then $\Psi(\xi) : \xi \to \bar{x}_0^-(T(\xi))$ is the point mapping of the surface $s = 0$ into itself determined by system (3).

To a periodic solution of system (3), there corresponds a fixed point ξ_0 of the mapping $\Psi(\xi_0) = \xi_0$; the period of this solution and the time of intersection with the discontinuity surface are determined by the relations $T(\xi_0) = T_0$ and $\theta(\xi_0) = \theta_0$; the periodicity and sewing conditions for the solution acquire the form $\bar{s}_0^-(T_0) = \bar{s}_0^+(0) = 0$, $\bar{x}_0^-(T_0) = \bar{x}_0^+(0)$, $\bar{s}_0^-(\theta_0) = \bar{s}_0^+(\theta_0) = 0$, and $\bar{x}_0^-(\theta_0) = \bar{x}_0^+(\theta_0)$.

Suppose the following:

(4^0) system (3) has an isolated T_0-periodic solution $(\bar{s}_0(t), \bar{x}_0(t))$, $\bar{s}_0(0) = 0$, such that the corresponding fixed point ξ_0 of the mapping $\Psi(\xi)$ satisfies

$$\det(\partial\Psi/\partial\xi)(\xi_0) \neq 0;$$

(5^0) the spectrum of the matrix $(\partial\Psi/\partial\xi)(\xi_0)$ lies inside the unit disk;

(6^0) the points $\varphi(0,\xi_0,-1)$ and $\varphi(0,\bar{x}_0(\theta_0),1)$ lie in the influence domains of the stable equilibria $\varphi(0,\xi_0,1)$ and $\varphi(0,\bar{x}_0(\theta_0),-1)$, respectively.

By $\mathscr{L}_0(t)$ we denote the polygon

$$\mathscr{L}_0(t)=\begin{cases}\varphi(\bar{s}_0(t),\bar{x}_0(t),1) & \text{for}\quad t\in(0,\theta_0)\\ \varphi(\bar{s}_0(t),\bar{x}_0(t),-1) & \text{for}\quad t\in(\theta_0,T_0)\\ (1-\gamma)\varphi(0,\bar{x}_0(\theta_0),1)+\gamma\varphi(0,\bar{x}_0(\theta_0),-1) & \text{for}\quad \gamma\in[0,1]\ \text{if}\ t=\theta_0\\ (1-\gamma)\varphi(0,\xi_0,-1)+\gamma\varphi(0,\xi_0,1) & \text{for}\quad \gamma\in[0,1]\ \text{if}\ t=0.\end{cases}$$

3. THE EXISTENCE OF SLOW PERIODIC MOTIONS

Suppose that the norm in $\mathbf{R}^n$ is chosen to ensure that Ψ is a contraction mapping in some neighborhood of the point ξ_0, and moreover, $\|(\partial\Psi/\partial\xi)(\xi_0)\|<q<1$. By analogy with the investigation of a periodic solution of system (3), to prove the existence and analyze the stability of a periodic solution of system (1), we consider the point mapping $\Phi(z,x,\mu)$ of the surface $s=0$ into itself determined by system (1).

Lemma 1. *Under conditions* (1^0)–(6^0), *there exists a neighborhood* Γ *of the point* $(\varphi(0,\xi_0,1),\xi_0)$ *on the surface* $s=0$ *such that, for any* $(\eta,\xi)\in\Gamma$ *and for sufficiently small* μ, *there exists a solution* $(z(t,\mu),s(t,\mu),\ x(t,\mu))$ *of system* (1) *with the initial conditions*

$$z(0,\mu)=\eta;\qquad s(0,\mu)=0,\qquad x(0,\mu)=\xi \tag{5}$$

and there exist $0<\theta(\eta,\xi,\mu)<T(\eta,\xi,\mu)$ *such that* $s(\theta(\eta,\xi,\mu),\mu)=0$ *and* $s(T(\eta,\xi,\mu),\mu)=0$, *the solution is defined and is unique on* $[0,T(\eta,\xi,\mu)]$, *and its value for* $t=T(\eta,\xi,\mu)$ *belongs to* Γ.

Proof. It follows from the implicit function theorem, theorems on the continuous dependence of solutions of systems of differential equations on parameters, and the continuity of the function φ that, in the space R^n, there exists a closed ball $\bar{U}(\alpha)\in V$ with radius α and center ξ_0 on the surface $s=0$ such that if $\xi\in\bar{U}(\alpha)$, then $\|(\partial\Psi/\partial\xi)(\xi)\|<q'<1$ and the following assertions are valid:

(a_0) for the solution $\left(\bar{s}_0^+(t),\bar{x}_0^+(t)\right)$ of the Cauchy problem (3^+), (4), there exists a $\theta(\xi)$ such that $\bar{s}_0^+(\theta(\xi))=0$ and $\left(d\bar{s}_0^+/dt\right)(\theta(\xi))=H_1\left(0,\bar{x}_0^+(\theta(\xi)),1\right)<0$; moreover, $H_1\left(0,\bar{x}_0^+(\theta(\xi)),-1\right)<0$;

(b_0) $\varphi\left(0,\bar{x}_0^+(\theta(\xi)),1\right)$ is an interior point of the influence domain of the stable equilibrium $\varphi\left(0,\bar{x}_0^+(\theta(\xi)),\ -1\right)$;

(c_0) for the solution $\left(\bar{s}_0^-(t),\bar{x}_0^-(t)\right)$ of problem (3^-), (4^-), there exists a $T(\xi)$ such that $\bar{s}_0^-(T(\xi))=0$ and

$$\left(d\bar{s}_0^-/dt\right)(T(\xi))=H_1\left(0,\bar{x}_0^-(T(\xi)),1\right)>0,\qquad \bar{x}_0^-(T(\xi))\in U(q'\alpha)$$

and the set $\bar{W}=\operatorname{co}\varphi\left(0,\bar{U}(\alpha),-1\right)$ lies in the influence domain of the stable equilibrium

$$\varphi\left(0,\bar{x}_0^-(T(\xi)),1\right).$$

Then the Tikhonov theorem [8], together with the implicit function theorem, implies that, for each point $(\eta,\xi)\in\bar{W}\times\bar{U}(\alpha)$, there exists a $\mu_0(\eta,\xi)$ such that the following assertions are valid for all $\mu\in[0,\mu_0(\eta,\xi)]$:

(a_μ) for the solution $(z^+(t,\mu),s^+(t,\mu),x^+(t,\mu))$ of system (1) with $u=1$ and with the initial conditions (5), there exists a least positive root $\theta(\eta,\xi,\mu)$ of the equation $s^+(\theta(\eta,\xi,\mu),\mu)=0$; moreover, $h_1\left(z^\pm(\theta(\eta,\xi,\mu),\mu),s^\pm(\theta(\eta,\xi,\mu),\mu),x^\pm(\theta,\mu),\pm1\right)<0$;

(b_μ) the point $z^+(\theta(\eta,\xi,\mu),\mu)$ lies in the influence domain of

$$\varphi\left(\bar{s}_0^+(\theta(\eta,\xi,\mu),\mu),\bar{x}_0^+(\theta(\eta,\xi,\mu),\mu),-1\right);$$

(c_μ) for the solution $(z^-(t,\mu), s^-(t,\mu), x^-(t,\mu))$ of system (1) with $u=-1$ and with the initial conditions

$$z^-(\theta(\eta,\xi,\mu),\mu) = z^+(\theta(\eta,\xi,\mu),\mu), \qquad s^-(\theta(\eta,\xi,\mu),\mu) = s^+(\theta(\eta,\xi,\mu),\mu),$$

and $x^-(\theta(\eta,\xi,\mu),\mu) = x^+(\theta(\eta,\xi,\mu),\mu)$, there exists a least positive root $T(\eta,\xi,\mu) > \theta(\eta,\xi,\mu)$ of the equation $s^-(T(\eta,\xi,\mu),\mu)=0$; moreover,

$$h_1\left(z^-(T,\mu), s^-(T,\mu), x^+(T,\mu), -1\right) > 0, \qquad \left(z^+(t,\mu), s^+(t,\mu), x^+(t,\mu)\right)$$

exists and is unique on $[\theta(\eta,\xi,\mu), T(\eta,\xi,\mu)]$, and $(z^-(T(\eta,\xi,\mu),\mu), x^-(T(\eta,\xi,\mu),\mu))$ lies in the set $\left(\varphi(0,\, U\left((1+q')\,\alpha/2\right), -1), U\left((1+q')\,\alpha/2\right)\right)$.

Furthermore,

$$\Phi(\eta,\xi,0) = \lim_{\mu\to 0}\Phi(\eta,\xi,\mu) = \left(\varphi\left(0, x^-(T(\xi)), 1\right), x^-(T(\xi))\right),$$

and $\Phi\left(\varphi\left(0,\xi_0,1\right),\xi_0,0\right) = \left(\varphi\left(0,\xi_0,1\right),\xi_0\right)$ for $\xi=\xi_0$. Since $\bar W \times \bar U(\alpha)$ is compact, it follows that there exists a μ_0 such that, for all $\mu\in[0,\mu_0]$, the point mapping

$$\Phi(\eta,\xi,\mu) = (\Phi_1(\eta,\xi,\mu), \Phi_2(\eta,\xi,\mu)) = \left(z^-(T(\eta,\xi,\mu),\mu), x^-(T(\eta,\xi,\mu),\mu)\right)$$

of the surface $s=0$ into itself determined by system (1) is defined on the set $\Gamma = \bar W \times \bar U(\alpha)$ and maps this set into itself. It follows that, for all $\Phi(\eta,\xi,\mu)$, the mapping $\mu\in[0,\mu_0]$ has a fixed point on Γ corresponding to a periodic solution of system (1) in a neighborhood of the polygon $(\mathscr{L}_0(t), \bar s_0(t), \bar x_0(t))$.

4. THE UNIQUENESS OF SLOW PERIODIC MOTIONS AND THEIR STABILITY

Theorem 1. *Under conditions* (1^0)–(6^0), *in a neighborhood of the polygon* $(\mathscr{L}_0(t), \bar s_0(t), \bar x_0(t))$ *for sufficiently small* μ, *there exists an isolated orbitally asymptotically stable* $T(\mu)$*-periodic solution of system* (1) *with boundary layers for* $t=0$ *and in a neighborhood of* $t=\theta_0$. *Moreover,* $\lim_{\mu\to0} T(\mu) = T_0$.

Proof. The derivative of the mapping Φ with respect to the initial data η and ξ is a smooth function of the derivatives of the functions $\theta(\eta,\xi,\mu)$, $T(\eta,\xi,\mu)$, $z^+(\theta(\eta,\xi,\mu),\mu)$, $x^+(\theta(\eta,\xi,\mu),\mu)$, $z^-(T(\eta,\xi,\mu),\mu)$, and $x^-(T(\eta,\xi,\mu),\mu)$ with respect to initial data, whose existence and continuity follows from theorems on the differentiability of solutions of systems of singularly perturbed differential equations with respect to initial data at the endpoint of the interval [9] and from the implicit function theorem.

We introduce the new variable $\chi = \eta - \varphi\left(0, x^-(T(\xi)), -1\right)$. Then, instead of the problem of finding a fixed point of the transformation $\Phi(\eta,\xi,\mu)$, we consider the problem of finding a fixed point of the auxiliary transformation

$$\begin{aligned}\Lambda(\chi,\xi,\mu) &= (\Lambda_1(\chi,\xi,\mu), \Lambda_2(\chi,\xi,\mu))\\ &= \Big(\Phi_1\left(\chi+\varphi\left(0,x^-(T(\xi)),-1\right),\xi,\mu\right) - \varphi\left(0,x^-(T(\xi)),-1\right),\\ &\qquad \Phi_2\left(\chi+\varphi\left(0,x^-(T(\xi)),-1\right),\xi,\mu\right)\Big).\end{aligned}$$

Note that if $\mu=0$, then the point $(0,\xi_0)$ is a fixed point of the transformation Λ, which maps the set $M\left(\beta,\alpha,\bar\mu\right) = \left\{(\chi,\xi,\mu):\ \|\chi\|<\beta,\ x\in\bar U(\alpha),\ \mu\in[0,\bar\mu]\right\}$ into itself for sufficiently small β and $\bar\mu$.

Let us compute the derivative of Λ with respect to χ and ξ. If $\mu=0$, then $\Lambda(\chi,\xi,0)$ is independent of χ and $\Lambda_1(\chi,\xi,0)$ is independent of ξ. It follows that

$$\frac{\partial\Lambda}{\partial(\chi,\xi)} = \begin{pmatrix} O(\mu) & O(\mu) \\ O(\mu) & \left(\partial\Psi/\partial\xi\right)(\xi_0) + O(\mu) \end{pmatrix}.$$

We take $\beta, \bar{\mu} > 0$ such that $\sup_{M(\beta,\alpha,\bar{\mu})} \|\partial\Lambda/\partial(\chi,\xi)\| < q_1 < 1$. Hence $\Lambda(\chi,\xi,\mu)$ is a contraction mapping of $M(\beta,\alpha,\bar{\mu})$ and has a unique isolated fixed point in $M(\beta,\alpha,\bar{\mu})$ corresponding to the desired isolated periodic solution of system (1). The orbital asymptotic stability of this solution follows from the fact that Λ is a contraction mapping on $M(\beta,\alpha,\bar{\mu})$.

5. AN ALGORITHM FOR THE CONSTRUCTION OF THE ASYMPTOTICS OF A PERIODIC SOLUTION

Suppose that $h_1, h_2, g \in C^{k+3}\left[\bar{Z} \times [-1,1]\right]$ and conditions (1^0)–(6^0) are satisfied.

Let y be the vector formed by the coordinates of the variables (z,s,x). We seek the asymptotics of a slow periodic solution, the switching time $\theta(\mu)$, and the period $T(\mu)$ of system (1) on the interval $[0,\Theta(\mu)]$ in the form

$$\begin{aligned} Y_k(t,\mu) &= \sum_{i=0}^{k}\left[\bar{y}_i(t) + \Pi_i^+ y(\tau) + \Pi_i^- y\left(\tau_{k+1}\right)\right]\mu^i, \\ S_k(t,\mu) &= \sum_{i=0}^{k}\left[\bar{s}_i(t) + \Pi_i^+ s(\tau) + \Pi_i^- s\left(\tau_k\right)\right]\mu^i, \end{aligned} \tag{AS}$$

$$\begin{aligned} X_k(t,\mu) &= \sum_{i=0}^{k}\left[\bar{x}_i(t) + \Pi_i^+ x(\tau) + \Pi_i^- x\left(\tau_k\right)\right]\mu^i, \\ \theta(\mu) &= \theta_0 + \mu\theta_1 + \cdots + \mu^k\theta_k + \cdots, \\ T(\mu) &= T_0 + \mu T_1 + \cdots + \mu^k T_k + \cdots, \qquad \Theta(\mu) = T(\mu) - \theta(\mu) \end{aligned} \tag{SM),}$$

respectively, where $\tau = t/\mu$, $\tau_k = \left(t - \tilde{\theta}_{k+1}(\mu)\right)/\mu$, $\tilde{\theta}_{k+1}(\mu) = \theta_0 + \mu\theta_1 + \cdots + \mu^{k+1}\theta_{k+1}$, $\left\|\Pi_i^{\pm} y(\tau)\right\| < Ce^{-\gamma\tau}$, $C, \gamma > 0$, and $\Pi_i^{\pm} y(\tau) \equiv 0$ for $\tau < 0$.

We set $\tilde{\Theta}_{k+1}(\mu) = \Theta_0 + \mu\Theta_1 + \cdots + \mu^{k+1}\Theta_{k+1}$, $\tilde{T}_k(\mu) = T_0 + \mu T_1 + \cdots + \mu^k T_k$, and

$$\bar{y}_0(t) = \begin{cases} \bar{y}_0^+(t) = \left(\varphi\left(\bar{s}_0^+(t), \bar{x}_0^+(t), 1\right), \bar{s}_0^+(t), \bar{x}_0^+(t)\right) & \text{for} \quad t \in [0,\theta_0] \\ \bar{y}_0^-(t) = \left(\varphi\left(\bar{s}_0^-(t), \bar{x}_0^-(t), 1\right), \bar{s}_0^-(t), \bar{x}_0^-(t)\right) & \text{for} \quad t \in [\theta_0, T_0]. \end{cases}$$

The function $\Pi_0^+ z(\tau)$ is found from the equations

$$\begin{gathered} d\Pi_0^+ z/d\tau = g\left(\Pi_0^+ z + \varphi\left(0, \bar{x}_0^+(0), 1\right), 0, \bar{x}_0^+(0), 1\right), \\ \Pi_0^+ z(0) = \varphi\left(0, \bar{x}_0^+(0), -1\right) - \varphi\left(0, \bar{x}_0^+(0), 1\right), \end{gathered}$$

and $\Pi_0^- z(\tau)$ is given by the relations

$$\begin{gathered} d\Pi_0^- z/d\tau = g\left(\Pi_0^- z + \varphi\left(0, \bar{x}_0^-\left(\theta_0\right), 1\right), \bar{s}_0^-\left(\theta_0\right), \bar{x}_0^-\left(\theta_0\right), \theta_0\right), \\ \Pi_0^- z(0) = \varphi\left(0, \bar{x}_0^-\left(\theta_0\right), 1\right) - \varphi\left(0, \bar{x}_0^+\left(\theta_0\right), -1\right). \end{gathered}$$

Now for $\bar{s}_1^{\pm}(t)$, $\bar{x}_1^{\pm}(t)$, and $\bar{z}_1^{\pm}(t)$, we obtain the system of linear equations

$$\begin{aligned} \bar{z}_1^{\pm}(t) &= -\left[g_z'^{\pm}\right]^{-1}\left(g_s'^{\pm}\bar{s}_1^{\pm} + g_x'^{\pm}\bar{x}_1^{\pm} + g_1^{\pm}(t)\right), \\ d\bar{s}_1^{\pm}/dt &= h_{1z}'^{\pm}(t)\bar{z}_1^{\pm}(t) + h_{1s}'^{\pm}\bar{s}_1^{\pm}(t) + h_{1x}'^{\pm}\bar{x}_1^{\pm}(t), \\ d\bar{x}_1^{\pm}/dt &= h_{2z}'^{\pm}(t)\bar{z}_1^{\pm}(t) + h_{2s}'^{\pm}\bar{s}_1^{\pm}(t) + h_{2x}'^{\pm}\bar{x}_1^{\pm}(t). \end{aligned} \tag{6}$$

Here the superscript $\pm$ indicates that the value of the corresponding function is computed at the point $\left(\varphi\left(\bar{s}_0^{\pm}(t), \bar{x}_0^{\pm}(t), \pm 1\right), \bar{s}_0^{\pm}(t), \bar{x}_0^{\pm}(t), \pm 1\right)$. For $\Pi_1^{\pm} z$, $\Pi_1^{\pm} s$, and $\Pi_1^{\pm} x$, we have the uniquely solvable system

$$\begin{aligned}
&d\Pi_1^{\pm} z/d\tau = g_z'\Pi_1^{\pm} z + g_s'\Pi_1^{\pm} s + g_x'\Pi_1^{\pm} x + \Pi_1^{\pm} g(\tau),\\
&d\Pi_1^{+} s/d\tau = \Pi_0^{+} h_1 = h_1\left(\bar z_0^{+}(0) + \Pi_0^{+} z, 0, \bar x_0^{+}(0), 1\right) - h_1\left(\bar z_0^{+}(0), 0, \bar x_0^{+}(0), 1\right),\\
&d\Pi_1^{+} x/d\tau = \Pi_0^{+} h_2 = h_2\left(\bar z_0^{+}(0) + \Pi_0 z, 0, \bar x_0^{+}(0), 1\right) - h_2\left(\bar z_0^{+}(0), 0, \bar x_0^{+}(0), 1\right),\\
&d\Pi_1^{-} s/d\tau = \Pi_0^{-} h_1 = h_1\left(\bar z_0^{-}(\theta_0) + \Pi_0^{-} z, 0, \bar x_0^{-}(\theta_0), -1\right) - h_1\left(\bar z_0^{-}(\theta_0), 0, \bar x_0^{-}(\theta_0), -1\right),\\
&d\Pi_1^{-} x/d\tau = \Pi_0^{-} h_2 = h_2\left(\bar z_0^{-}(\theta_0) + \Pi_0^{-} z, 0, \bar x_0^{-}(\theta_0), -1\right) - h_2\left(\bar z_0^{-}(\theta_0), 0, \bar x_0^{-}(\theta_0), -1\right),
\end{aligned}$$

where, $\bar z_0^{+}(0) = \varphi(0, x_0, 1)$, $\bar z_0^{-}(\theta_0) = \varphi\left(0, \bar x_0^{+}(\theta_0), -1\right)$, the $+$ superscript in the computation of the derivatives of the function g indicates that they are computed at the point $(\bar z_0^{+}(0) + \Pi_0^{+} z, 0, \bar x_0, 1)$, and the derivatives with the $-$ superscript are computed at the point $\left(\bar z_0^{-}(\theta_0) + \Pi_0^{-} z, 0, \bar x_0^{-}(\theta_0), -1\right)$. The initial conditions for the boundary functions of the slow coordinates are found from the relations

$$\Pi_1^{\pm} s(0) = \int\limits_{\infty}^{0} \Pi_0^{\pm} h_1(\Theta)d\Theta, \qquad \Pi_1^{\pm} x(0) = \int\limits_{\infty}^{0} \Pi_0^{\pm} h_2(\Theta)d\Theta.$$

Then $\bar s_1^{+}(0) = -\Pi_1^{+} s(0)$ and $\bar s_1^{-}(\theta_0) = -\Pi_1^{-} s(0)$. The functions $\bar x_1^{\pm}(t)$ and $\bar s_1^{\pm}(t)$ can be uniquely found from system (6) once we determine the initial conditions $\bar x_1^{+}(0)$ and $\bar x_1^{-}(\theta_0)$.

By matching the first-order terms in the asymptotic expansions of the relations $s(T(\mu), \mu) = 0$ and $s(\theta(\mu), \mu) = 0$, respectively, we obtain

$$\Theta_1 H_1(0, \xi_0, 1) + \bar s_1^{-}(T_0) = 0, \qquad \theta_1 H_1\left(0, \bar x_0^{+}(\theta_0), 1\right) + \bar s_1^{+}(\theta_0) = 0. \tag{7}$$

It follows from condition (2^0) that the quantities θ_1 and Θ_1 can be uniquely expressed via $\bar s_1^{+}(0)$ and $\bar s_1^{-}(\theta_0)$ in the form $\Theta_1 = -\left[H_1(0, \xi_0, -1)\right]^{-1} \bar s_1^{-}(T_0)$ and $\theta_1 = -\left[H_1\left(0, \bar x_0^{+}(\theta_0), 1\right)\right]^{-1} \bar s_1^{+}(\theta_0)$. By substituting these expressions into the solution sewing condition $x^{-}(\theta(\mu), \mu) = x^{+}(\theta(\mu), \mu)$, we obtain

$$\bar x_1^{-}(\theta_0) + \Pi_1^{-} x(0) = \bar x_1^{+}(\theta_0) + \theta_1 H_2\left(0, \bar x_0^{+}(\theta_0), 1\right). \tag{8}$$

Likewise, the periodicity condition $x^{+}(T(\mu), \mu) = x^{+}(T(\mu), \mu)$ acquires the form

$$\bar x_1^{+}(0) + \Pi_1^{+} x(0) = \bar x_1^{-}(T_0) + \Theta_1 H_2(0, \xi_0, -1). \tag{9}$$

Then the quantities $\bar x_1^{+}(\theta_0)$ and $\bar x_1^{-}(T_0)$ depend linearly on $\bar x_1^{+}(0)$ and $\bar x_1^{-}(\theta_0)$, and formulas relating them can readily be found from (6). By expressing $\bar x_1^{-}(\theta_0)$ via $\bar x_1^{+}(0)$ from system (9) and by substituting the result into (8), we obtain a system of linear equations for $\bar x_1^{+}(0)$, whose determinant coincides with the Jacobian of the system for finding a periodic solution of system (3). The nondegeneracy of this determinant follows from condition (4^0).

It follows that the initial conditions $\bar s_1^{+}(0)$, $\bar x_1^{+}(0)$, $\bar s_1^{-}(\theta_0)$, and $\bar x_1^{-}(\theta_0)$ are uniquely determined. Now, to find $S_1(t, \mu)$ and $X_1(t, \mu)$, we should define the functions $\bar s_i^{+}(t)$ and $\bar x_i^{+}(t)$ on the closed interval $\left[0, \tilde T_1(\mu)\right]$ as follows:

$$\bar y_i(t) = \begin{cases} \bar y_i^{+}(t) = \left(\bar z_i^{+}(t), \bar s_i^{+}(t), \bar x_i^{+}(t)\right) & \text{for} \quad t \in \left[0, \tilde\theta_1(\mu)\right] \\ \bar y_i^{-}(t) = \left(\bar z_i^{-}(t), \bar s_i^{-}(t), \bar x_i^{-}(t)\right) & \text{for} \quad t \in \left[\tilde\theta_1(\mu), \tilde T_1(\mu)\right], \end{cases} \qquad i = 0, 1.$$

The initial conditions for $\Pi_1^{\pm} z$ are uniquely determined by the relations

$$\begin{aligned}
\bar z_1^{+}(0) + \Pi_1^{+} z(0) &= \bar z_1^{-}(T_0) + \Theta_1 d\bar z_0^{-}(T_0)/dt,\\
\bar z_1^{-}(\theta_0) + \Pi_1^{-} z(0) &= \bar z_1^{+}(\theta_0) + \theta_1 d\bar z_0^{+}(\theta_0)/dt.
\end{aligned}$$

This completes the construction of the asymptotics of the first approximation for the slow components and the period of the desired periodic solution. To complete the construction of the first approximation to the fast variables, one should find θ_2 and substitute it into the function $\Pi_1^- z(\tau_2)$.

Suppose that $z_j^\pm(t)$, $s_j^\pm(t)$, $x_j^\pm(t)$, $\Pi_j^\pm z(\tau)$, $\Pi_j^\pm s(\tau)$, $\Pi_j^\pm x(\tau)$, and the constants θ_j and Θ_j, $j = 1, \ldots, k-1$, have been found. For $\bar{s}_k^\pm(t)$, $\bar{x}_k^\pm(t)$, and $\bar{z}_k^\pm(t)$, we obtain the system of linear equations

$$\begin{gathered}
\bar{z}_k^\pm(t) = -\left[g_z'^\pm\right]^{-1}\left(g_s'^\pm \bar{s}_k^\pm + g_x'^\pm \bar{x}_k^\pm + g_k^\pm(t)\right), \\
d\bar{s}_k^\pm/dt = h_{1z}'^\pm(t)\bar{z}_k^\pm(t) + h_{1s}'^\pm \bar{s}_k^\pm(t) + h_{1x}'^\pm \bar{x}_k^\pm(t) + h_{1k}^\pm(t), \\
d\bar{x}_k^\pm/dt = h_{2z}'^\pm(t)\bar{z}_k^\pm(t) + h_{2s}'^\pm \bar{s}_k^\pm(t) + h_{2x}'^\pm \bar{x}_k^\pm(t) + h_{2k}^\pm(t);
\end{gathered} \tag{6_k}$$

here the superscript $\pm$ indicates that the corresponding function is computed at the point

$$\left(\varphi\left(\bar{s}_0^\pm(t), \bar{x}_0^\pm(t), \pm 1\right), \bar{s}_0^\pm(t), \bar{x}_0^\pm(t), \pm 1\right);$$

furthermore, $g_k^\pm(t)$, $h_{1k}^\pm(t)$, and $h_{2k}^\pm(t)$ are uniquely determined functions depending only on $\bar{z}_j^\pm(t)$, $\bar{s}_j^\pm(t)$, $\bar{x}_j^\pm(t)$, θ_j, and Θ_j, $j = 1, \ldots, k-1$. For $\Pi_k^\pm z$, $\Pi_k^\pm s$, and $\Pi_k^\pm x$, we have the uniquely solvable system

$$\begin{gathered}
d\Pi_k^\pm z/d\tau = g_z'\Pi_k^\pm z + g_s'\Pi_k^\pm s + g_x'\Pi_k^\pm x + \Pi_k^\pm g(\tau), \\
d\Pi_k^\pm s/d\tau = \Pi_{k-1}^\pm h_1, \qquad d\Pi_k^\pm x/d\tau = \Pi_{k-1}^\pm h_2,
\end{gathered}$$

where the $+$ superscript on the derivatives of g indicates that they are computed at the point $\left(\bar{z}_0^+(0) + \Pi_0^+ z, 0, \bar{x}_0, 1\right)$, the derivatives with the $-$ superscript are computed at the point

$$\left(\bar{z}_0^-\left(\theta_0\right) + \Pi_0^- z, 0, \bar{x}_0^-\left(\theta_0\right), -1\right),$$

and $\Pi_{k-1}^\pm h_1$ and $\Pi_{k-1}^\pm h_2$ are functions depending only on $\Pi_j^\pm z(\tau)$, $\Pi_j^\pm s(\tau)$, and $\Pi_j^\pm x(\tau)$, $j = 1, \ldots, k-1$.

The initial conditions for the boundary functions of slow coordinates are found from the relations

$$\Pi_k^\pm s(0) = \int\limits_\infty^0 \Pi_{k-1}^\pm h_1(\Theta)d\Theta, \qquad \Pi_k^\pm x(0) = \int\limits_\infty^0 \Pi_{k-1}^\pm h_2(\Theta)d\Theta.$$

Then $\bar{s}_k^+(0) = -\Pi_k^+ s(0)$ and $\bar{s}_k^-\left(\theta_0\right) = -\Pi_k^- s(0)$. The functions $\bar{x}_k^\pm(t)$ and $\bar{s}_k^\pm(t)$ can be uniquely found from system (6_k) once the initial conditions $\bar{x}_k^+(0)$ and $\bar{x}_k^-\left(\theta_0\right)$ have been determined.

By matching the kth-order terms in the asymptotic expansions of the relations $s(T(\mu), \mu) = 0$ and $s(\theta(\mu), \mu) = 0$, respectively, we obtain

$$\Theta_k H_1\left(0, \xi_0, -1\right) + \bar{s}_k^-\left(T_0\right) + \mathscr{S}_k^- = 0, \qquad \theta_k H_1\left(0, \bar{x}_0^+\left(\theta_0\right), 1\right) + \bar{s}_k^+\left(\theta_0\right) + \mathscr{S}_k^+ = 0. \tag{7_k}$$

Just as with the first approximation, we express θ_k and Θ_k from (7_k) and substitute them into the sewing condition for solutions under the jump of the control sign from $+$ to $-$ and into the periodicity condition; then we obtain

$$\bar{x}_k^-\left(\theta_0\right) + \Pi_k^- x(0) = \bar{x}_k^+\left(\theta_0\right) + \theta_k H_2\left(0, \bar{x}_0^+\left(\theta_0\right), 1\right) + \mathscr{X}_k^+, \tag{8_k}$$

$$\bar{x}_k^+(0) + \Pi_k^+ x(0) = \bar{x}_k^-\left(T_0\right) + \Theta_k H_2\left(0, \xi_0, -1\right) + \mathscr{X}_k^-, \tag{9_k}$$

respectively, where $\mathscr{S}_k^\pm$ and $\mathscr{X}_k^+$ are functions depending on $\bar{s}_j^+\left(\theta_0\right)$, $\bar{x}_j^+\left(\theta_0\right)$, $\bar{s}_j^-\left(T_0\right)$, and $\bar{x}_j^-\left(T_0\right)$, $j = 1, \ldots, k-1$. System (8_k), (9_k) has the same structure as system (8), (9). Consequently, the initial conditions $\bar{s}_k^+(0)$, $\bar{x}_k^+(0)$, $\bar{s}_k^-\left(\theta_0\right)$, and $\bar{x}_k^-\left(\theta_0\right)$ are uniquely determined. Now, to find $S_k(t, \mu)$ and $X_k(t, \mu)$, one should complete the definition of the functions $\bar{s}_i^+(t)$ and $\bar{x}_i^+(t)$ on the closed interval $\left[0, \tilde{\theta}_k(\mu)\right]$ as follows:

$$\bar{y}_i(t) = \begin{cases} \bar{y}_i^+(t) = \left(\bar{z}_i^+(t), \bar{s}_i^+(t), \bar{x}_i^+(t)\right) & \text{for} \quad t \in \left[0, \tilde{\theta}_k(\mu)\right] \\ \bar{y}_i^-(t) = \left(\bar{z}_i^-(t), \bar{s}_i^-(t), \bar{x}_i^-(t)\right) & \text{for} \quad t \in \left[\tilde{\theta}_k(\mu), \tilde{T}_k(\mu)\right], \end{cases} \qquad i = 0, \ldots, k.$$

The initial conditions for $\Pi_k^{\pm} z$ are uniquely determined by the relations

$$\bar{z}_k^+(0) + \Pi_k^+ z(0) = \bar{z}_k^-(T_0) + \Theta_k d\bar{z}_0^-(T_0)/dt + \mathscr{Z}_k^-,$$
$$\bar{z}_k^-(\theta_0) + \Pi_k^- z(0) = \bar{z}_k^+(\theta_0) + \theta_k d\bar{z}_0^+(\theta_0)/dt + \mathscr{Z}_k^+,$$

where the $\mathscr{Z}_k^{\pm}$ are functions depending only on $\bar{z}_j^+(\theta_0)$ and $\bar{z}_j^-(T_0)$, $j = 1, \ldots, k-1$. Now, to complete the construction of the first approximation for the rapid variables, one should find the quantity θ_{k+1} and substitute it into the function $\Pi_k^- z(\tau_{k+1})$.

Theorem 2. *Under conditions* (1^0)–(6^0), *the estimates*

$$\left|\tilde{T}_k(\mu) - T(\mu)\right| < C\mu^{k+1}, \qquad \|y(t,\mu) - Y_k(t,\mu)\| < C\mu^{k+1},$$
$$\|(s(t,\mu), x(t,\mu)) - (S_k(t,\mu), X_k(t,\mu))\| < C\mu^{k+1} \tag{10}$$

are valid uniformly with respect to $\left[0, \hat{T}(\mu)\right]$, *where* $\hat{T}(\mu) = \max\left\{T(\mu); \tilde{T}_{k+1}(\mu)\right\}$.

Theorem 2 follows from Theorem 1 and Lemma 2, which will be proved in Section 7.

6. EXAMPLE

Let us consider the singularly perturbed bang-bang system

$$\mu\, dz/dt = -z - u, \qquad ds/dt = x + u/2, \qquad dx/dt = -x + z, \qquad u = \operatorname{sgn}[s(t)], \tag{11}$$

where $z, s, x \in R$ and μ is a small parameter. Let us prove the existence and uniqueness of a slow periodic solution of system (11) and construct the zero and first approximations to its asymptotics.

Let us first show that system (11) satisfies the assumptions of Theorems 1 and 2. If $\mu = 0$, then system (11) acquires the form

$$\bar{z}_0 = -u, \qquad d\bar{s}_0/dt = \bar{x}_0 + u/2, \qquad d\bar{x}_0/dt = -\bar{x}_0 - u, \tag{12}$$

and its solution with the initial conditions $\bar{x}_0^+(0) = \xi > 0$, $\bar{s}_0^+(0) = 0$ has the form

$$\bar{x}_0^+(t,\xi) = e^{-t}(\xi + 1) - 1, \qquad \bar{s}_0^+(t,\xi) = \left(1 - e^{-t}\right)(\xi + 1) - t/2.$$

System (12) is symmetric around the origin. Consequently, θ_0 is the half-period of the desired periodic solution, and the fixed point ξ_0 of the transformation $\Psi(\xi)$ corresponding to the desired periodic solution is found from the relations

$$\bar{s}_0^+(\theta_0, \xi_0) = 0, \qquad \bar{x}_0^+(\theta_0, \xi_0) = -\xi_0;$$

therefore, $\xi_0 = \left(1 - e^{-\theta_0}\right)/\left(1 + e^{-\theta_0}\right)$, $\xi_0 = \theta_0/4$. By solving this system of algebraic equations, we obtain $\theta_0 \cong 3.83$ and $\xi_0 \cong 0.96$. Moreover,

$$(\partial\Psi/\partial\xi)^{1/2}(\xi_0) = \left(1 - e^{-\theta_0} - \theta_0\right)/\left(e^{\theta_0} - \theta_0 - 1\right) \cong -0.07,$$

which provides the validity of the assumptions of Theorems 1 and 2.

Let us complete the construction of the zero approximation to the periodic solution:

$$\bar{z}_0(t) = \begin{cases} \bar{z}_0^+(t) = 1 & \text{if} \quad 0 \le t < \theta_0 \\ \bar{z}_0^-(t) = -1 & \text{if} \quad \theta_0 \le t \le T_0. \end{cases}$$

Then $d\Pi_0^{\pm} z/d\tau = -\Pi_0^{\pm} z$, $\Pi_0^{\pm} z(0) = \pm 2$, and $\Pi_0^{\pm} z(\tau^{\pm}) = \pm 2e^{-\tau^{\pm}}$. Let us compute terms of first-order smallness with respect to μ. The equations for the regular part of the asymptotics acquire the form $\bar{z}_1^{\pm} = 0$, $d\bar{s}_1^{\pm}/dt = x_1^{\pm}$, $d\bar{x}_1^{\pm}/dt = -\bar{x}_1^{\pm}$; consequently,

$$\bar{x}_1^+\left(t, \bar{x}_1^+(0)\right) = \bar{x}_1^+(0)e^{-t}, \qquad \bar{s}_1^+(t) = \left(1 - e^{-t}\right)\bar{x}_1^+(0) + \bar{s}_1^+(0),$$
$$\bar{x}_1^-\left(t, \bar{x}_1^-(\theta_0)\right) = \bar{x}_1^-(\theta_0)\, e^{-(t-\theta_0)}, \qquad \bar{s}_1^-(t) = \left(1 - e^{-(t-\theta_0)}\right)\bar{x}_1^-(\theta_0) + \bar{s}_1^-(\theta_0).$$

Let us find the first-order boundary terms for the slow variables: $\Pi_1^{\pm}s(\tau) \equiv 0$, $\Pi_1^{\pm}s(0) = 0$, $\Pi_1^{\pm}x(\tau) = \int_{\infty}^{\tau}\Pi_0^{\pm}z(\Theta)d\Theta$, and $\Pi_1^{\pm}x(0) = \mp 2$. Then $\bar{s}_1^{+}(0) = \bar{s}_1^{-}(\theta_0) = 0$.

In this case, relations (7) acquire the form

$$\Theta_1(\xi_0 - 1/2) + \bar{s}_1^{-}(T_0) = 0, \qquad \theta_1\left(\bar{x}_0^{+}(\theta_0) + 1/2\right) + \bar{s}_1^{+}(\theta_0) = 0;$$

therefore, the quantities θ_1 and Θ_1 can be expressed via $\bar{x}_1^{+}(0)$ and $\bar{x}_1^{-}(\theta_0)$ by the formulas

$$\begin{aligned}\theta_1 &= -\left(1 - e^{-\theta_0}\right)\bar{x}_1^{+}(0)/\left(\bar{x}_0^{+}(\theta_0) + 1/2\right),\\ \Theta_1 &= -\left(1 - e^{-\theta_0}\right)\bar{x}_1^{-}(\theta_0)/(\xi_0 - 1/2).\end{aligned}$$

By the symmetry of (12), $\theta_1 = 2\bar{x}_1^{+}(0)\left(1 - e^{-2\theta_0}\right)/\left(1 - 3e^{-\theta_0}\right)$. Relations (8) acquire the form

$$\begin{aligned}-\bar{x}_1^{+}(0) + \Pi_1^{-}(0) &= \bar{x}_1^{+}(0)\left(-3e^{-\theta_0} + e^{-2\theta_0}/\left(1 - 3e^{-\theta_0}\right)\right),\\ \bar{x}_1^{+}(0) &= 2\left(1 - 3e^{-\theta_0}\right)/\left(e^{-2\theta_0} - 6e^{-\theta_0} + 1\right) \cong 2.15,\end{aligned}$$

and $\theta_1 = 4\left(1 - e^{-2\theta_0}\right)/\left(e^{-2\theta_0} - 6e^{\theta_0} + 1\right) \cong 4.60$, and the half-period of the desired periodic solution is given by the relation $T(\mu) \cong 3.83 + 4.60\mu + O\left(\mu^2\right)$.

7. THE ASYMPTOTICS OF SOLUTIONS OF SINGULARLY PERTURBED SYSTEMS WITH FINITELY MANY SWITCHINGS

Consider the Cauchy problem for system (1) with the initial conditions (5). Suppose that conditions (1^0)–(3^0) are satisfied together with the following conditions:

(4^*) the solution $\left(\bar{s}_0^{+}(t), \bar{x}_0^{+}(t)\right)$ of system (3^+) with $u = 1$ and with the initial conditions $\bar{s}_0^{+}(0) = 0$ and $\bar{x}_0^{+}(0) = \xi$ exists and is unique on $[0, \theta_0]$, where θ_0 is the least positive root of the equation $\bar{s}_0^{+}(\theta_0) = 0$; moreover, $\left(d\bar{s}_0^{+}/dt\right)(\theta_0) = H_1\left(0, \bar{x}_0^{+}(\theta_0), 1\right) < 0$ and $H_1\left(0, \bar{x}_0^{+}(\theta_0), -1\right) < 0$;

(5^*) $\varphi\left(0, \bar{x}_0^{+}(\theta_0), 1\right)$ is an interior point of the influence domain of $\varphi\left(0, \bar{x}_0^{+}(\theta_0), -1\right)$;

(6^*) the solution $\left(\bar{s}_0^{-}(t), \bar{x}_0^{-}(t)\right)$ of system (3^-) with $u = -1$ and with the initial conditions $\bar{s}_0^{-}(\theta_0) = 0$ and $\bar{x}_0^{-}(\theta_0) = \bar{x}_0^{+}(\theta_0)$ exists and is unique on $[\theta_0, T]$; moreover, $\operatorname{sgn}\bar{s}_0^{-}(t) < 0$ and $\left(\varphi\left(\bar{s}_0^{-}(t), \bar{x}_0^{-}(t), -1\right), \bar{s}_0^{-}(t), \bar{x}_0^{-}(t)\right) \in Z$ for all $t \in [\theta_0, T]$.

It follows from the Tikhonov theorem [8], condition (4^*), and the implicit function theorem that the solution of the Cauchy problem (1), (5) with $u = 1$ exists, is unique for sufficiently small μ and intersects the discontinuity surface at some time $\theta(\mu)$ $[\theta(\mu) \to \theta_0$ as $\mu \to 0]$. Moreover, condition (5^*) implies that the solution of problem (1), (5) enters the domain $u = -1$, and it follows from the condition (6^*) that $u = 1$ for the solution of problem (1), (5) for $t \in [\theta(\mu), T(\mu)]$. Consequently, the solution of problem (1), (5) can be reduced to the successive solution of the following two Cauchy problems:

(a) the solution $(z^{+}(t,\mu), s^{+}(t,\mu), x^{+}(t,\mu))$ of the Cauchy problem for system (1) with the initial conditions (5) for $u = 1$;

(b) the solution $(z^{-}(t,\mu), s^{-}(t,\mu), x^{-}(t,\mu))$ of the Cauchy problem for system (1) with $u = -1$ and with the initial conditions

$$\begin{aligned}z^{-}(\theta(\mu),\mu) &= z^{+}(\theta(\mu),\mu),\\ s^{-}(\theta(\mu),\mu) &= s^{+}(\theta(\mu),\mu),\\ x^{-}(\theta(\mu),\mu) &= x^{+}(\theta(\mu),\mu).\end{aligned} \tag{13}$$

We construct the asymptotics of the switching time in the form (SM) and the asymptotics of the solution of problem (1), (5) in the form (AS).

The coefficients of the series (SM) are found from the equation

$$s^{+}\left(\theta_0 + \mu\theta_1 + \cdots + \mu^i\theta_i + \cdots, \mu\right) = 0.$$

Assuming that the coefficients $\bar{s}_0^+(t), \bar{s}_1^+(t), \dots, \bar{s}_k^+(t)$ of the expansion of problem (1), (5) by the method of boundary functions for $u = 1$ are known, one can write out the equations for θ_i in the form

$$\bar{s}_0^+\left(\theta_0 + \dots + \mu^i\theta_i + \dots\right) + \mu\bar{s}_1^+\left(\theta_0 + \dots + \mu^i\theta_i + \dots\right) + \cdots \\ + \mu^i\bar{s}_i^+\left(\theta_0 + \dots + \mu^i\theta_i + \dots\right) + \dots = 0.$$

The terms $\Pi^+ s(\theta(\mu)/\mu)$ are exponentially small; therefore, they can be neglected. By expanding the functions on the left-hand side in the last equation in powers of μ and by matching the coefficients of like powers of μ, we obtain a linear equations for θ_k in the form

$$\theta_i H_1\left(0, \bar{x}_0^+\left(\theta_0\right), 1\right) + p_i\left(\theta_0, \theta_1, \dots, \theta_{i-1}\right) = 0,$$

where p_i is some known function depending only on $\theta_0, \theta_1, \dots, \theta_{i-1}$. It follows from condition (4^*) that the coefficient of θ_i is always nonzero; consequently, the quantities $\theta_0, \theta_1, \dots, \theta_i, \dots$ are uniquely determined.

Suppose that the quantities $\theta_0, \theta_1, \dots, \theta_{k+1}$ and the coefficients $\bar{y}_i^+(t)$ $(i = 1, \dots, k)$ of the regular parts of the asymptotic expansions of the solution of problem (1), (5) with $u = 1$ have been found.

Let us find the asymptotics for the segment

$$\bar{Y}_k^-\left(\tilde{\theta}_k(\mu), \mu\right) = \sum_{i=0}^{k} \bar{y}_i^-\left(\tilde{\theta}_k(\mu), \mu\right)\mu^i$$

of the regular part of the asymptotics of the quantity $y^-\left(\tilde{\theta}_k(\mu), \mu\right)$. To this end, we use the asymptotic expansions of the functions $\bar{y}_i^+\left(\tilde{\theta}_k(\mu)\right)$ in powers of μ and, instead of the segments of the asymptotics $\bar{Y}_k^+\left(\tilde{\theta}_k(\mu), \mu\right)$ we consider the segment $\hat{Y}_k^+\left(\tilde{\theta}_k(\mu), \mu\right)$ of the series

$$\bar{Y}_k^+\left(\tilde{\theta}_k(\mu), \mu\right) = \bar{y}_0^-\left(\theta_0\right) + \mu\left(\bar{y}_1^+\left(\theta_0\right) + \theta_1\left(d\bar{y}_0^+/dt\right)\left(\theta_0\right)\right) + \cdots$$

up to the power μ^k.

To find the asymptotics of the solution of problem (1), (13), we use the asymptotics of the solution of the Cauchy problem for system (1) with $u = 1$ and with the initial conditions

$$y^+\left(\tilde{\theta}_{k+1}(\mu), \mu\right) = \hat{Y}_k^-\left(\tilde{\theta}_k(\mu), \mu\right). \tag{14}$$

Lemma 2. *Under conditions* (1^0)–(3^0) *and* (4^*)–(6^*), *there exist constants* μ_0 *such that, for all* $\mu \in [0, \mu_0]$, *the solution of the Cauchy problem* (1), (5) *exists, is unique on* $t \in [0, T]$, *and satisfies the estimate* (10) *uniformly on* $[0, T]$.

Proof. To be definite, we suppose that $\theta(\mu) < \tilde{\theta}_{k+1}(\mu)$ for sufficiently small μ. There exists a $K_1 > 0$ such that $|\theta(\mu) - \tilde{\theta}_{k+1}(\mu)| < K_1\mu^{k+2}$ and $\|y^+(\theta(\mu), \mu) - \hat{Y}_k^+(\theta(\mu), \mu)\| < K_1\mu^{k+1}$. Then on the closed interval $[\theta(\mu), \tilde{\theta}_{k+1}(\mu)]$, the solution of problem (1), (5) is the solution of problem (1), (14) for $u = 1$; consequently,

$$\left\|z^-(t, \mu) - z^+(\theta(\mu), \mu)\right\| = \left\|\int_{\theta(\mu)}^{t} g\left(z^-(\tau, \mu), s^-(\tau, \mu), x^-(\tau, \mu), 1\right)d\tau/\mu\right\| \\ < M_1\left|\tilde{\theta}_{k+1}(\mu) - \theta(\mu)\right|\Big/\mu,$$

$$\left\|\left(s^-(t, \mu), x^-(t, \mu)\right) - \left(s^+(\theta(\mu), \mu), x^+(\theta(\mu), \mu)\right)\right\| < M_2\left|\tilde{\theta}_{k+1}(\mu) - \theta(\mu)\right|,$$

$$\left\|\bar{Y}_k^+(t, \mu) - \hat{Y}_k^+\left(\tilde{\theta}_k(\mu), \mu\right)\right\| < K_2\mu^{k+1}, \qquad K_2 > 0,$$

$$M_1 = \sup_{\bar{Z}\times[-1,1]}\|g(z, s, x, u)\|, \qquad M_2 = \sup_{\bar{Z}\times[-1,1]}\left\|(h_1(z, s, x, u), h_2(z, s, x, u))\right\|,$$

for all $t \in [\theta(\mu), \tilde{\theta}_{k+1}(\mu)]$. It follows that $\|\hat{Y}_k^-(\tilde{\theta}_k(\mu), \mu) - y^+(\tilde{\theta}_{k+1}(\mu), \mu)\| < K_3\mu^{k+1}$, $K_3 > 0$. These estimates provide the validity of Lemma 2 on $[\theta(\mu), \tilde{\theta}_{k+1}(\mu)]$; next, it follows from the Vasil'eva theorem [7] that the asymptotic expansions of the solutions of the Cauchy problem (1), (14) and (1), (5) for $u = -1$ by the method of boundary functions for $t \in [\tilde{\theta}_{k+1}(\mu), T]$ coincide up to terms of order μ^k, and the estimates (10) are valid on the closed intervals $[0, \theta(\mu)]$ and $[\tilde{\theta}_{k+1}(\mu), T]$.

The proof in the case $\tilde{\theta}_{k+1}(\mu) < \theta(\mu)$ can be performed in a similar way.

REFERENCES

1. Choi, S.B. and Hedrick, K., *ASME Journ. of Dynamic Systems, Measurment, and Control*, 1996, vol. 118, no. 1, pp. 92–98.
2. Fridman, L.M., *Int. J. of Control*, 2002, vol. 75, no. 7, pp. 524–537.
3. Flatto, L. and Levinson, N., *Matematika*, 1958, vol. 2, no. 2, pp. 61–68.
4. Anosov, D.V., *Mat. Sb.*, 1960, vol. 50, no. 3, pp. 299–304.
5. Wazow, W., *Asymptotic Expansions for Ordinary Differential Equations*, New York, 1965. Translated under the title *Asimptoticheskie razlozheniya reshenii obyknovennykh differentsial'nykh uravnenii*, Moscow, 1968.
6. Mishchenko, E.F. and Rozov, N.Kh., *Differentsial'nye uravneniya s malym parametrom i relaksatsionnye kolebaniya* (Differential Equations with a Small Parameter and Relaxation Oscillations), Moscow, 1975.
7. Vasil'eva, A.B. and Butuzov, V.F., *Asimptoticheskie razlozheniya reshenii singulyarno vozmushchennykh uravnenii* (Asymptotic Expansions of Solutions of Singularly Perturbed Equations), Moscow, 1973.
8. Tikhonov, A.N., *Mat. Sb.*, 1952, vol. 31(73), pp. 575–586.
9. Strygin, V.V. and Sobolev, V.A., *Razdelenie dvizhenii metodom integral'nykh mnogoobrazii* (Separation of Motions by the Method of Integral Manifolds), Moscow, 1988.