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Discussion on: "Dynamic Sliding Mode Control for a Class of Systems with Mismatched Uncertainty"

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The paper [3] addresses the robust output feedback control of a class of nonlinear systems

$$\dot{x}(t) = Ax(t) + B(u + \Delta g(x, t)) + \Delta f(x, y) + \Phi(x), \qquad (1)$$
$$y(t) = Cx(t),$$

where $x \in \Re^n$ is the state, $u \in \Re^m$ the control and $y \in \Re^p (m \le p < n)$ the output; $\Delta g(x, t)$ is the matched uncertainty and $\Delta f(x, t)$ the unmatched uncertainty; $\Phi(x)$ is a known nonlinear vector function with $\Phi(0) = 0$.

The authors use the ideas of robust sliding mode control and incorporate a sliding surface originally proposed by Edwards and Spurgeon [2]. The discontinuous sliding mode control has been described fully elsewhere (see, e.g. [4,7,9]). Using the estimated states and the system output, a dynamic sliding mode control is developed which is shown to satisfy the well-known reachability condition [7,9]. A nonlinear asymptotic observer is proposed and this yields exponential state estimation error convergence based on the solution to a constrained Lyapunov equation.

Both matched and unmatched system uncertainties are considered. Like all sliding mode control systems, the following important property holds: the system behaviour is invariant to the matched uncertainty during the sliding mode [9].

The authors impose two realistic assumptions that guarantee the existence of the output sliding mode [1,2]. Also assumed are the assumptions that the pair (A, C) is observable and that the nonlinear function

 $\Phi(x)$ is Lipschitz and that there exist known continuous functions ξ_1, ξ_2 and γ such that

$$\begin{aligned} |\Delta g(x,t)|| &\leq \xi_1(y,t)\xi_2(x,t),\\ \Delta f(x,t) &= E\Delta\eta(x,t) \end{aligned}$$

with

 $\|\Delta\eta(x,t)\| \le \gamma(x,t)\|y\|.$

The related technique developed by Zak and Hui [8] has the following stringent assumptions [5,6]; the uncertainty is bounded by a function of the output y, and there exists a matrix M such that SA = MC, where S is the sliding matrix. This paper has eliminated these restrictive assumptions and this is an important development.

The sliding surface

$$\sigma(x) = Sx = 0$$

has a particular structural form [3]. For detailed definitions and the partitioning of linearly transformed variables z and \tilde{x} with $z = \hat{T}\tilde{x}$, see [3].

It is shown that a continuous function χ exists such that

$$\|\Delta \Phi(z_1, t)\| \le \chi(z_1, t)\|z_1\|$$

and the sliding mode is proved to be asymptotically stable in the neighbourhood of the state origin. The matched uncertainty does not affect the sliding mode stability.

However, the unmatched uncertainty does affect the sliding mode dynamics, and hence its stability. So it is necessary to impose a limitation on the bound of the mismatched component to guarantee stability [3].

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A sliding observer is then designed with a number of design parameters so that asymptotic estimation of the state x is achieved. To complete the full control scheme the observer provides the state x to a sliding mode controller [3]. Some theorems are proved using a Lyapunov approach.

The scheme is illustrated by designing the controller/observer for a simplified nonlinear aircraft model. A nonlinear mode of the longitudinal dynamics is considered and the model incorporates the uncertainty caused by the aerodynamic drag and the error present in the modelling of the aircraft lift. Simulation results have been given which demonstrate the practicality of the proposed scheme and its effectiveness in achieving robust closed-loop control. This section clearly indicates the nonlinear functions, the necessary transformations and structural information, the matched and unmatched uncertainties in the model and the choice of design parameters [3].

The main contribution is that the control scheme is less conservative than other comparable work, because the bounds on the uncertainty are utilised fully in the controller and observer design, and the known nonlinearities and the uncertain nonlinearities are processed separately. This is in contrast with earlier work which simply treats all the nonlinearities as disturbances.

My main reservation is that the structural model (1) and associated bounds and other parameter information, and also the complexity of the transformations and design parameters, required to design the observer and controller, are rather complicated. It is not completely clear to me that all the requirements will be easily determined in certain practical applications. An excellent understanding of this sliding mode control approach will be required by the designer using this design algorithm.

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Discussion on: "Dynamic Sliding Mode Control for a Class of Systems with Mismatched Uncertainty"

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1. Introduction

The paper by Yan et al. uses the standard linear sliding mode procedure – outlined in the books of Edwards and Spurgeon [1] and Utkin [3] – along with an observer, to stabilise a class of systems whose linearisation is stabilisable using the standard sliding mode approach, and whose nonlinearities and

uncertainties have certain structure. This structure as well as modifications to the standard observer and controller design are devised so that the following two criteria are met:

 standard Lyapunov analysis, using the linear Lyapunov function, should serve to determine the stability of the sliding mode dynamics; 2. standard observer estimates should yield exponential convergence to zero of the error between the estimated and actual state.

Thus the design techniques are in the realm of linear sliding mode control, and the principal contribution as concerns nonlinearities and uncertainties is to characterise the sort of these that are allowable by the linear control design.

In this discussion, I would like to discuss few things about sliding mode control in general. Specifically, I would like to compare sliding mode control with standard linear control design techniques, and point out where I think the real advantages of sliding mode control arise. I will not touch upon the issue of combining state estimation with sliding mode control; however, it seems plausible to me that all the comments made below carry over, *mutatis mutandis*, to this situation.

2. Sliding Mode Control for Linear Systems

Let us consider a MIMO linear system $(A, B, C) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m} \times \mathbb{R}^{p \times n}$ where the state comma; input, and output spaces are $\mathbb{R}^n, \mathbb{R}^m$, and \mathbb{R}^p , respectively. Thus the system is governed by the following equations:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t).$$

For simplicity, we suppose that **B** has full rank. Let us first discuss the matter of stabilising this system using sliding mode control, following the conference paper of Hirschorn and Lewis [2]. In linear sliding mode control, one first defines a sliding surface, in this case a subspace S of the state space \mathbb{R}^n . This subspace should be chosen to have the following two properties:

- 1. S is a complement to image(**B**) in \mathbb{R}^n , i.e. $\mathbb{R}^n = \text{image}(\mathbf{B}) \oplus S$;
- 2. the linear map A_S on S defined by $A_S = \operatorname{pr}_S \circ A \circ i_S$ is Hurwitz, where $i_S : S \to \mathbb{R}^n$ is the inclusion and $\operatorname{pr}_S : \mathbb{R}^n \to S$ is the projection along image(**B**) in the decomposition in part 1.

In [2] we show that controllability (or, more generally, stabilisability) of (A, B) suffices to establish the existence of such a subspace S, and we also give a simple algorithm for determining S. Note that property 2 above encodes the usual feature that the sliding dynamics be stable. Once one has the sliding surface S at hand, one then designs a variable structure controller that steers trajectories towards S in finite time. This is possible since a sliding surface has property 1.

Note that the output does not need to participate in stabilisation using sliding mode control. However, the sliding surface design method of Yan et al. does involve the output. At this stage it is required to know whether a certain MIMO system is stabilisable using static output feedback; this problem is NP-hard. This output-based design of the sliding surface also requires that there be at least as many outputs as inputs, and a compatibility condition between B and C. In the paper, the specific algorithm, involving the output, used to design the sliding surface allows the authors to put the system into a certain form after a linear change of coordinates that is advantageous to their computations. I am not sure whether these advantages can be obtained, after some modification, using a sliding surface not designed using outputs. This might be an interesting matter to investigate. Note, however, that the sliding surface in the paper does have both of the properties above.

The description of linear sliding mode control above suggests a comparison with purely linear design via pole placement, using a scheme as follows:

- choose closed-loop eigenvalues λ₁,..., λ_{n-m} with generalised eigenspaces W_{λ1},..., W_{λn-m} such that S ≜ W_{λ1} ⊕··· ⊕ W_{λn-m} is a complement to image(B);
- 2. choose the remaining closed-loop eigenvalues far to the left of the imaginary axis.

The idea is that *S* plays the rôle of the sliding surface, with the eigenvalues $\lambda_1, \ldots, \lambda_{n-m}$ giving the sliding dynamics. The fact that the complementary eigenvalues have large negative real part means that the sliding dynamics will become dominant after a short time. Note that, for linear sliding mode control, the sliding dynamics are achieved after a *finite* time, whereas, for the pole placement scheme above, the sliding dynamics become dominant after a short time, but are only achieved asymptotically. The distinction becomes even less apparent if one chooses to smooth out the sliding mode control law with some sort of saturation function. Typically the effect of this smoothing is to eliminate chatter at the cost of not achieving the sliding dynamics in finite time.

We comment that this idea of relating sliding mode control to pole placement is considered using singular perturbation analysis in an early paper of Young et al. [4].

3. Linear Sliding Mode Control for Nonlinear Systems

A linear sliding mode controller will, just like a purely linear controller, locally stabilise a nonlinear system if it stabilises the system's linearisation. This is well known and a geometric proof appears in my conference paper with Hirschorn. Intuitively, one might expect that the more aggressive control action of the sliding mode control law would be more effective at handling the nonlinearities. This does not appear to be unequivocally true. To investigate this, I considered the example used in the paper. I designed a controller/ observer as follows:

 I designed a purely linear controller, using the pole placement algorithm described in the preceding section. Thus the closed-loop system has two eigenvalues equal to the eigenvalues determined by the sliding dynamics of Yan et al. Moreover, the eigenspace for these eigenvalues is exactly the Yan et al. sliding surface. The remaining two closedloop eigenvalues were chosen to be -20. No attempt was made to compensate for uncertainties or nonlinearities in the controller design.

2. I used a standard Luenberger observer, using the observer gain from the paper of Yan et al. I did not compensate for nonlinearities or uncertainties, although this could easily be done, just as is done in the paper.

In Fig. 1 I show a simulation for this setup with the same initial conditions as in the paper. I also added some unmatched uncertainty that satisfies the conditions in the paper. I did not add matched uncertainty, although this could be accomplished with modification to the observer as in the paper, and also with



Fig. 1. Simulation results for example of Yan et al. with linear controller and observer; $v(t) = (1 + x_1(t))v_0$ with $v_0 = 267.51$.

modifications to the controller. The simulation does not compare at all badly with that of Yan et al., particularly considering that it is the simplest sort of linear controller.

4. Anecdotal Evidence for the Effectiveness of Sliding Mode Control

Sliding mode control is often attributed with having good robustness properties. I believe this may well be true, although I do not know of any theorems that support a claim like "sliding mode control is more robust than control scheme X." (However, I am also not completely familiar with the literature within which such a claim may be supported.) Nonetheless, my (limited) experiences in our control lab at Queen's suggest that there may be something to the robustness claims of sliding mode control, at least in some cases. In our lab we have a (single and double) pendulum/ cart system for which the track on which the cart slides has a significant amount of sticking friction. When performing, say, quadratic regulator design, this sticking friction is not accounted for, and the resulting linear controller, while achieving stabilisation, is quite fragile. Indeed, it is extremely fragile in the double pendulum configuration. Moreover, it takes a significant amount of tuning to get the linear controller to work. By contrast, a linear sliding mode controller implemented on the same system works with almost no fiddling of the system parameters (e.g. location of sliding mode poles), and is also substantially less fragile. In our lab we also have a torsional vibration apparatus whose behaviour is very linear, i.e. the measured system dynamics closely match the linear model. For this system, it is less easy to see, in a fair comparison, an improvement in performance of sliding mode control design over linear control design.

From this (admittedly extremely limited and not thorough) experience with hardware, I would proffer

some rule like, "Sliding mode control is well suited to situations where the aggressive control action is helpful in overcoming the particular character of the unmodelled dynamics." The sticking friction of the inverted pendulum apparatus in our lab is a good example of this.

5. Conclusions

In my discussion, I contend that the matter of comparing sliding mode control with a certain sort of pole placement algorithm is one that is worthy of investigation. At the very least, I think that understanding the relationship between sliding mode control and pole placement is helpful in that it places linear sliding mode control design in a context that is easily understandable. I do believe that sliding mode control can be advantageous in certain sorts of applications. I think that a precise quantification of the sort of application would be interesting. It also seems reasonable to me that, if one adds a standard smoothing function to the sliding mode controller off the sliding surface, then the things that one can do with a linear sliding mode controller are essentially the same as those that can be done with a linear controller, using the scheme of Section 2.

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Discussion on: "Dynamic Sliding Mode Control for a Class of Systems with Mismatched Uncertainty"

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1. Contribution

The sliding mode control technique has long been recognized as a powerful tool to counteract matched disturbances. In the paper under discussion, the authors make a step beyond this and demonstrate that a certain degree of robustness pertains against mismatched uncertainties as well. A good job is done at specifying a class of mismatched disturbances (see Assumptions 1-5) and proving Theorem 3, according to which any disturbance of this class is attenuated by the proposed sliding mode control synthesis. To appreciate better the significance of this result let us recall that the utility of the sliding mode approach is only well known when no mismatched disturbances affect the system. In turn, it is stereotype to think that sliding modes are weak or even incapable of providing acceptable performance under mismatched disturbances.

2. Related Publications

It should be pointed out that the authors' paper is not the only work so far on the subject. Various robustness aspects of different sliding mode control algorithms against mismatched disturbances have recently been studied. In order to pertain robustness against mismatched disturbances, sliding mode control approach has been proposed to be combined with other methods. Here we present closely related publications, not mentioned by Yan et al. In [1] a multi-model strategy, coupled to an optimality criterion, uses ISM (see [2] for definition of integral sliding modes). In [3] the Lyapunov-based synthesis of ISM is developed. In [4] the sliding surface is defined using linear matrix inequalities LMI. In [5] the use of conventional sliding modes is coupled to Lyapunov methods. In [6] sliding mode control algorithms are

developed for linear time delay systems and their robustness against a class of mismatched disturbances is provided.

3. Perspective

There are other interesting aspects to this paper. Here we focus on an issue of potential interest for further investigation.

In order to solve the output feedback problem the authors utilize a sliding mode control approach, typically consisting of two steps. The first step is to design a manifold (referred to as a sliding manifold) such that the motion of the system, being restricted to this manifold, is stable. The second step is to synthesize a control law which ensures that the closed-loop system is driven to the sliding manifold and stays there forever, regardless of which admissible disturbance affect the system. It is worth noting that at the first step there is some flexibility in constructing the sliding manifold. This flexibility can be used at the second step to enhance the system performance, e.g. by constructing the sliding manifold such that the so-called sliding motion (i.e. the system dynamics along the sliding manifold) satisfies an \mathcal{H}_{∞} -criterion (cf. that of [7]). While being affected by mismatched disturbances only, this sliding motion would additionally go with robustness properties against mismatched disturbances due to advantages of the \mathcal{H}_{∞} design.

Another attractive feature of the mixed sliding mode/ \mathcal{H}_{∞} -controller, thus constructed, is that the corresponding \mathcal{H}_{∞} -problem is of reduced order because it is confined to the sliding manifold. An open problem then arises as to if the mixed sliding mode/ \mathcal{H}_{∞} -synthesis is capable of guaranteing \mathcal{H}_{∞} -optimality (or quasioptimality) of the over-all system. While realizing that standard sliding modes are hardly possible to ensure \mathcal{H}_{∞} -optimality outside the sliding manifold, we deeply hope that applying integral sliding modes, that occur in the closed loop system from

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the very initial time moment, would be satisfactory in the sense of the \mathcal{H}_{∞} -methodology. With mentioning this appealing problem we would like to conclude our discussion.

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Final comments by the authors X.-G. Yan, S. K. Spurgeon and C. Edwards

First, we would like to thank the reviewers and the other experts for their insightful comments on our paper.

Andrew Lewis discusses some of the design issues associated with the selection of the sliding surface and makes interesting comparisons with purely linear control schemes. Our experience does not indicate, however, that combining state estimation with a statefeedback sliding mode control law is as straightforward as he seems to suggest. The introduction of an observer into a state-feedback sliding mode control scheme can easily destroy the insensitivity properties. The use of asymptotic (Luenberger) observers in conjunction with state-feedback sliding mode controllers is discussed in Chapter 14 of Utkin (1992). Lu and Spurgeon (1999) have analysed the application of a semi-high-gain observer in conjunction with a dynamic sliding mode state feedback controller. The fact that sliding takes place within the space associated with state estimates has under some circumstances beneficial properties and the introduction of an observer has been used in the past as a chatter reduction mechanism because it helps mitigate the effects of measurement delays. However, the effect of uncertainty, i.e. a discrepancy between the real plant and the model used to design the observer usually results in the *non-asymptotic* convergence of the observer states to the plant states with a subsequent loss of robustness. (This is not dissimilar to the LQR situation where it is well known that an unsophisticated choice of observer may destroy the robustness margins of the controller.) There has been work on special 'insensitive observers' (e.g. Breinl and Leitmann, 1987) which tries to overcome these difficulties. Early work combining sliding mode controllers and sliding mode observers showed that for a certain class of systems, all the robustness properties of state-feedback sliding mode controllers with respect to matched uncertainty could be recovered by the design of an appropriate (sliding mode) observer (Edwards and Spurgeon, 1996). The construction of a sliding surface commensurate with a given state feedback linear gain does have some precedence in the literature. The relationship between the eigenvectors associated with a given set of poles (which are closed under the operation of complex conjugation) and the sliding surface is discussed in Edwards and Spurgeon (1998) and is fundamental to some of the output feedback hyperplane design procedures of Zak and Hui (1993). The addition of a non-linear 'switched gain' to provide sliding on a surface associated with a given feedback gain is discussed for a complex process control plant in (Hermann et al., 2003) and in the thesis of Bhatti (1998). Prof Lewis' comments on his practical experience of implementing sliding mode controllers is interesting. It is not surprising perhaps though to find that in essentially linear systems (presumably with few inputs) little benefit is obtained from using sliding mode schemes – after all the raison d'etre for all the analysis and research into sliding mode control schemes is their robustness properties and application to nonlinear systems.

Orlov et al. raise an interesting issue regarding the perceived misapprehension that sliding mode controllers are not capable of dealing with unmatched uncertainty. When dealing with mismatched uncertainty the choice of sliding surface is crucial. Indeed the choice of sliding surface is key to ensuring robustness rather than the introduction of the sliding mode within the closed-loop system per-se. The reviewers have suggested other references not discussed within the original paper. However, most of the

references they cite either deal with the problem of mismatched uncertainty in *state feedback* sliding mode controllers or else references which deal with output feedback sliding mode controllers which assume all the uncertainty is matched. For example, the integral sliding mode references [1,2,3] and the LMI approach of Choi [4] assume all the states are available and so are not output based. The work by Bag et al. described in [5] does consider output feedback schemes but assumes all the uncertainty is matched. The paper under discussion here does, we believe, present some new ideas for a problem which has not been tackled extensively in the existing literature. The mixed sliding mode \mathcal{H}_{∞} scheme based on only output information proposed recently in [7] does represent an interesting area for future research - although the combination of \mathcal{H}_{∞} and sliding mode control has been already considered in (Hermann et al., 2003).

Zinober makes some pertinent comments regarding the complexity of the proposed control scheme (which is also implicit in the comments of Andrew Lewis). Whilst a physical engineering example has been used to demonstrate the technique, it is certainly true that from a design perspective, the synthesis of the controller requires understanding of the different elements and their impact upon the design freedom. Some further research will be required to develop the scheme to the point where it could be easily used by practising engineers.

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