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Higher-order sliding-mode observer for state estimation and input reconstruction in nonlinear systems

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SUMMARY

In this paper, a higher-order sliding-mode observer is proposed to estimate exactly the observable states and asymptotically the unobservable ones in multi-input–multi-output nonlinear systems with unknown inputs and stable internal dynamics. In addition the unknown inputs can be identified asymptotically. Numerical examples illustrate the efficacy of the proposed observer. Copyright © 2007 John Wiley & Sons, Ltd.

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1. INTRODUCTION

State observation and unknown input reconstruction for multi-input–multi-output (MIMO) nonlinear systems is one of the most important problems in modern control theory [1]. The problem of robust state observation continues to be actively studied using sliding modes, see, for example, [2–5]. The corresponding implementation effects were extensively studied in [6]. Sliding-mode observation strategies possess such attractive features as

- insensitivity (more than robustness) with respect to unknown inputs;
- the possibility of using the equivalent output error injection as a further source of information.

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Step-by-step vector-state reconstruction by means of sliding modes has been presented in [2, 7–9]. These observers are based on a transformation to a triangular or the Brunovsky canonical form and successive estimation of the state vector using the equivalent output error injection. The corresponding conditions for linear time-invariant systems with unknown inputs were obtained in [4, 9–11]. Moreover, the above-mentioned observers theoretically ensure finite-time convergence for all system states. Unfortunately, the realization of step-by-step observers is based on conventional sliding modes, leading to filtration at each step due to discretization or non-idealities of the analog devices used to implement the schemes. In order to avoid the necessity for filtration, hierarchical observers were recently developed in [10, 12] iteratively using the continuous super-twisting algorithm, based on second-order sliding-mode ideas [13].

The super-twisting structure is also used in the modified version of the step-by-step observer in [9]. Unfortunately, these observers are also not free of drawbacks: the super-twisting algorithm provides the best possible asymptotic accuracy of the derivative estimation at each single realization step [13]. In particular, the accuracy is proportional to the sampling step δ for the discrete realization in the absence of noise, and to the square root of the input noise magnitude if the discretization error is negligible. The step-by-step and hierarchical observers use the output of the super-twisting algorithm as a noisy input at the next step. As a result, the overall observation accuracy is of the order $\delta^{1/2^{r-1}}$, where r is the observability index of the system. Similarly in the presence of measurement noise with magnitude ε, the estimation accuracy is proportional to $\varepsilon^{1/2^r}$ which requires measurement noises not exceeding 10^{-16} for a fourth-order observer implementation to achieve an accuracy of 10^{-1}.

The use of higher-order sliding-mode differentiators [14] for exact observer design for linear systems with unknown inputs, initially transformed to the Brunovsky canonical form, is suggested in [9]. This work has shown that the accuracies increase to δ and $\varepsilon^{1/(r+1)}$, respectively. In this paper, an exact observer scheme for the nonlinear systems with unknown inputs is proposed based on two steps:

- transformation of the system to the Brunovsky canonical form;
- the application of higher-order sliding-mode differentiators for each component of the output error vector.

The proposed scheme ensures exact finite-time state estimation of the observable variables and asymptotic exact estimation of the unobservable variables for the case when the system has stable internal dynamics. Also the unknown inputs can be identified asymptotically.

2. SYSTEMS DYNAMICS

Consider the following MIMO locally stable system

$$\begin{aligned} \dot{x} &= f(x) + G(x)\varphi(t) \\ y &= h(x) \end{aligned} \tag{1}$$

where $f(x) \in \Re^n$, $h(x) = [h_1, h_2, \ldots, h_m]^{\mathrm{T}} \in \Re^m$, $G(x) = [g_1, g_2, \ldots, g_m] \in \Re^{n\times m}$, $x \in \Re^n$, $y \in \Re^m$, $\varphi \in \Re^m$, and $g_i \in \Re^n$ $\forall i = 1, \ldots, m$ are smooth vector and matrix functions defined on an open set $\Omega \subset \Re^n$.

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Assumptions (*Isidori [15]*)
At a neighbourhood of any point $x \in \Omega$

(i) The system in (1) is assumed to have a vector relative degree $r = \{r_1, r_2, \ldots, r_m\}$, i.e.

$$\begin{aligned} &L_{g_j} L_f^k h_i(x) = 0 \quad \forall j = 1, \ldots, m \quad \forall k < r_i - 1 \quad \forall i = 1, \ldots, m \\ &L_{g_j} L_f^{r_i-1} h_i(x) \neq 0 \quad \text{for at least one } 1 \leqslant j \leqslant m \end{aligned} \tag{2}$$

(ii) The $m \times m$ matrix

$$E(x) = \begin{bmatrix} L_{g_1}(L_f^{r_1-1}h_1) & L_{g_2}(L_f^{r_1-1}h_1) & \cdots & L_{g_m}(L_f^{r_1-1}h_1) \\ L_{g_1}(L_f^{r_2-1}h_2) & L_{g_2}(L_f^{r_2-1}h_2) & \cdots & L_{g_m}(L_f^{r_2-1}h_2) \\ \vdots & \vdots & \vdots & \vdots \\ L_{g_1}(L_f^{r_m-1}h_m) & L_{g_2}(L_f^{r_m-1}h_m) & \cdots & L_{g_m}(L_f^{r_m-1}h_m) \end{bmatrix} \tag{3}$$

is nonsingular;

(iii) The distribution $\Gamma = \text{span}\{g_1, g_2, \ldots, g_m\}$ is involutive.

A well-known property of systems of the form in (1) which satisfy assumptions (i) and (ii) is summarized in the following lemma [15].

Lemma
Suppose that assumptions (i) and (ii) are valid for the system (1). Then the row vectors

$$\begin{gathered} \mathrm{d}h_1(x), \mathrm{d}L_f h_1(x), \ldots, \mathrm{d}L_f^{r_1-1} h_1(x) \\ \mathrm{d}h_2(x), \mathrm{d}L_f h_2(x), \ldots, \mathrm{d}L_f^{r_2-1} h_2(x) \\ \vdots \\ \mathrm{d}h_m(x), \mathrm{d}L_f h_m(x), \ldots, \mathrm{d}L_f^{r_m-1} h_m(x) \end{gathered} \tag{4}$$

are linearly independent. □

The lemma conditions are also interpreted in [16] as the notion of local weak observability.

3. PROBLEM FORMULATION AND MAIN RESULT

The problem considered in this paper is to design an asymptotic observer that generates the estimates $\hat{x}(t), \hat{\varphi}(t)$ for the state $x(t)$, and the disturbance $\varphi(t)$ of the system (1)–(3) given the measurements $y = h(x)$, i.e.

$$\lim_{t \to \infty} ||\hat{x}(t) - x(t)|| = 0 \tag{5}$$

$$\lim_{t \to \infty} ||\hat{\varphi}(t) - \varphi(t)|| = 0 \tag{6}$$

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3.1. Coordinate transformation

The system given by (1)–(3) with an involutive distribution $\Gamma = \mathrm{span}\{g_1, g_2, \dots, g_m\}$ and total relative degree $r = \sum_{i=1}^{m} r_i < n$ can be presented in a new basis that is introduced as follows:

$$\{\xi^{\mathrm{T}}, \eta^{\mathrm{T}}\}^{\mathrm{T}} : \xi^i = \begin{pmatrix} \xi_1^i \\ \xi_2^i \\ \vdots \\ \xi_{r_i}^i \end{pmatrix} = \begin{pmatrix} \phi_1^i(x) \\ \phi_2^i(x) \\ \vdots \\ \phi_{r_i}^i(x) \end{pmatrix} = \begin{pmatrix} h_i(x) \\ L_f h_i(x) \\ \vdots \\ L_f^{r_i - 1} h_i(x) \end{pmatrix} \in \Re^{r_i} \quad \forall i = 1, \dots, m$$

$$\xi = \begin{pmatrix} \xi^1 \\ \xi^2 \\ \vdots \\ \xi^m \end{pmatrix}, \quad \eta = \begin{pmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_{n-r} \end{pmatrix} = \begin{pmatrix} \phi_{r+1}(x) \\ \phi_{r+2}(x) \\ \vdots \\ \phi_n(x) \end{pmatrix} \tag{7}$$

It is well known (see Proposition 5.1.2 on p. 222 of [15]) that if assumption (i) is satisfied then it is always possible to find $n - r$ functions $\phi_{r+1}(x), \dots, \phi_n(x)$ such that the mapping

$$\Phi(x) = \mathrm{col}\{\phi_1^1(x), \dots, \phi_{r_1}^1(x), \dots, \phi_1^m(x), \dots, \phi_{r_m}^m(x), \phi_{r+1}(x), \dots, \phi_n(x)\} \in \Re^n \tag{8}$$

is a local diffeomorphism in a neighbourhood of any point $x \in \bar{\Omega} \subset \Omega \subset \Re^n$, which means

$$x = \Phi^{-1}(\xi, \eta) \tag{9}$$

Furthermore, for a system given by (1)–(3) with an involutive distribution $\Gamma = \mathrm{span}\{g_1, g_2, \dots, g_m\}$ i.e assumption (iii) it is always possible to identify the functions $\phi_{r+1}(x), \dots, \phi_n(x)$ in such a way that

$$L_{g_j} \phi_i(x) = 0 \quad \forall i = r+1, \dots, n \ \forall j = 1, \dots, m \tag{10}$$

in a neighbourhood of any point $x \in \bar{\Omega} \subset \Omega \subset \Re^n$.

Taking into account Equations (7) and (8), the system given by (1)–(3) with an involutive distribution $\Gamma = \mathrm{span}\{g_1, g_2, \dots, g_m\}$ and a total relative degree $r = \sum_{i=1}^{m} r_i < n$ can be written in the form

$$\dot{\xi}^i = \Lambda_i \xi^i + \psi^i(\xi, \eta) + \lambda^i(\xi, \eta, \varphi(t)) \quad \forall i = 1, \dots, m \tag{11}$$

$$\dot{\eta} = q(\xi, \eta) \tag{12}$$

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where

$$\Lambda_i = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \in \Re^{r_i \times r_i}, \quad \psi^i(\xi,\eta) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ L_f^{r_i} h_i(x) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ L_f^{r_i} h_i(\Phi^{-1}(\xi,\eta)) \end{pmatrix}$$

$$\lambda^i(\xi,\eta,\varphi(t)) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \sum_{j=1}^m L_{g_j} L_f^{r_i-1} h_i(x)\varphi_j(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \sum_{j=1}^m L_{g_j} L_f^{r_i-1} h_i(\Phi^{-1}(\xi,\eta))\varphi_j(t) \end{pmatrix} \quad \forall i = 1,\ldots,m$$

Remark

In this paper, it has been assumed that the total relative degree $r = \sum_{i=1}^m r_i < n$. The developments, however, are also applicable to the case when $r = n$ with minor modifications. In this situation, there will be no internal dynamics and all the results will be finite time in nature.

3.2. Internal dynamics

It is assumed that for some norm-bounded $\xi = \bar{\xi}(t) : ||\bar{\xi}(t)|| \leqslant L_\xi$, there exists an unique and norm-bounded solution of the equations of the internal dynamics (12) $\eta = \bar{\eta}(t) : ||\bar{\eta}(t)|| \leqslant L_\eta$.

This norm-bounded solution of the internal dynamics (12) is assumed to be locally asymptotically stable: this means that, first of all, it is stable in a Lyapunov sense and, second, there exists an $\varepsilon > 0$ such that $\forall \bar{\eta}(t_0)$ satisfying $||\eta(t_0) - \bar{\eta}(t_0)|| < \varepsilon \Rightarrow \lim_{t\to\infty} ||\eta(t) - \bar{\eta}(t)|| = 0$. Such an assumption guarantees that there exists a domain $\Theta : ||\eta(t_0)|| \in L_{\eta(t_0)}$ so that a solution $\eta = \eta(t, t_0), \eta(t_0) \in \Theta$, asymptotically converges to a solution $\eta = \bar{\eta}(t, t_0)$ with some unknown initial condition $\bar{\eta}(t_0) \in \Theta$ and forced by $\xi = \bar{\xi}(t)$, i.e. $\lim_{t\to\infty} ||\eta(t, t_0) - \bar{\eta}(t)|| = 0$.

Remark

Of course, not all systems satisfy this assumption (in the same way not all systems have stable zero dynamics for instance). In addition, for general nonlinear systems, this requirement may be difficult to check.

3.3. Higher-order sliding-mode observer

Definition 1

System (1)–(3) is said to be *locally detectable*, if

- total relative degree is $r = \sum_{i=1}^m r_i < n$;

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- the distribution $\Gamma = \text{span}\{g_1, g_2, \ldots, g_m\}$ is involutive;
- the internal dynamics (12) are locally asymptotically stable.

The derivatives $\xi^i_j(t)\ \forall i = 1, \ldots, m\ \forall j = 1, \ldots, r_i$ of the measured outputs $y_i = h_i(x)$ can be estimated in finite time by the higher-order sliding-mode differentiator [14]. This can be written in the form

$$\begin{aligned}
\dot{z}^i_0 &= v^i_0 \\
v^i_0 &= -\lambda^i_0 |z^i_0 - y_i(t)|^{(r_i/(r_i+1))} \operatorname{sign}(z^i_0 - y_i(t)) + z^i_1 \\
\dot{z}^i_1 &= v^i_1 \\
v^i_1 &= -\lambda^i_1 |z^i_1 - v^i_0|^{((r_i-1)/r_i)} \operatorname{sign}(z^i_1 - v^i_0) + z^i_2 \\
&\vdots \\
\dot{z}^i_{r_i-1} &= v^i_{r_i-1} \\
v^i_{r_i-1} &= -\lambda^i_{r_i-1} |z^i_{r_i-1} - v^i_{r_i-2}|^{(1/2)} \operatorname{sign}(z^i_{r_i-1} - v^i_{r_i-2}) + z^i_{r_i} \\
\dot{z}^i_{r_i} &= -\lambda^i_{r_i} \operatorname{sign}(z^i_{r_i} - v^i_{r_i-1})
\end{aligned} \tag{13}$$

for $i = 1, \ldots, m$. By construction,

$$\begin{gathered}
\hat{\xi}^1_1 = \hat{\phi}^1_1(x) = z^1_0, \ldots, \hat{\xi}^1_{r_1} = \hat{\phi}^1_{r_1}(x) = z^1_{r_1-1}, \quad \hat{\dot{\xi}}^1_{r_1} = \hat{\dot{\phi}}^1_{r_1}(x) = z^1_{r_1} \\
\vdots \\
\hat{\xi}^m_1 = \hat{\phi}^m_1(x) = z^m_0, \ldots, \hat{\xi}^m_{r_m} = \hat{\phi}^m_{r_m}(x) = z^m_{r_m-1}, \quad \hat{\dot{\xi}}^m_{r_1} = \hat{\dot{\phi}}^m_{r_m}(x) = z^1_{r_m}
\end{gathered} \tag{14}$$

Therefore, the following exact estimates are available in finite time:

$$\hat{\xi}^i = \begin{pmatrix} \hat{\xi}^i_1 \\ \hat{\xi}^i_2 \\ \vdots \\ \hat{\xi}^i_{r_i} \end{pmatrix} = \begin{pmatrix} \hat{\phi}^i_1(x) \\ \hat{\phi}^i_2(x) \\ \vdots \\ \hat{\phi}^i_{r_i}(x) \end{pmatrix} \in \Re^{r_i} \quad \forall i = 1, \ldots, m \quad \hat{\xi} = \begin{pmatrix} \hat{\xi}^1 \\ \hat{\xi}^2 \\ \vdots \\ \hat{\xi}^m \end{pmatrix} \in \Re^r \tag{15}$$

Next, integrating Equation (12), with $\hat{\xi}$ replacing ξ

$$\dot{\hat{\eta}} = q(\hat{\xi}, \hat{\eta}) \tag{16}$$

and with some initial condition $\hat{\eta}(t_0) \in \Theta$ from the stability domain Θ of the internal dynamics (12), a solution $\hat{\eta}(t)$ is obtained. This solution $\hat{\eta}(t)$ converges asymptotically to an unknown

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(unobservable) solution $\eta(t)$ that passes through an unknown initial condition $\eta(t_0)$. In other words, the asymptotic estimate $\hat{\eta}(t)$ of $\eta(t)$ can be obtained locally:

$$\hat{\eta} = \begin{pmatrix} \hat{\eta}_1 \\ \hat{\eta}_2 \\ \vdots \\ \hat{\eta}_{n-r} \end{pmatrix} = \begin{pmatrix} \hat{\phi}_{r+1}(x) \\ \hat{\phi}_{r+2}(x) \\ \vdots \\ \hat{\phi}_n(x) \end{pmatrix} \tag{17}$$

Finally, the asymptotic estimate for the mapping (8) is identified as

$$\Phi(\hat{x}) = \mathrm{col}\{\hat{\phi}_1^1(\hat{x}), \ldots, \hat{\phi}_{r_1}^1(\hat{x}), \ldots, \hat{\phi}_1^m(\hat{x}), \ldots, \hat{\phi}_{r_m}^m(\hat{x}), \hat{\phi}_{r+1}(\hat{x}), \ldots, \hat{\phi}_n(\hat{x})\} \in \Re^n \tag{18}$$

The asymptotic estimate $\hat{x}$ of the state vector x can be easily identified *via* Equations (9) and (18) as

$$\hat{x} = \Phi^{-1}(\hat{\xi}, \hat{\eta}) \tag{19}$$

It is worth noting that the operation (19) is also local and can be performed, for example, by inverting a Jacobian of the map (18) that is nonsingular in a vicinity of some point x.

Combining the last equations in the ith subsystem in (11) in a new system, we obtained:

$$\dot{\xi}^i_{r_i} = L_f^{r_i} h_i(\Phi^{-1}(\xi, \eta)) + \sum_{j=1}^{m} L_{g_j} L_f^{r_i - 1} h_i(\Phi^{-1}(\xi, \eta))\varphi_j(t) \quad \forall i = 1, \ldots, m \tag{20}$$

Since the finite-time exact estimates $\hat{\dot{\xi}}^i_{r_1}$ of $\dot{\xi}^i_{r_1}$ $\forall i = 1, \ldots, m$ are available *via* the higher-order sliding-mode differentiator (13), (14), and using the estimates $\hat{\xi}, \hat{\eta}$ for ξ, η in (20), the asymptotic estimate $\hat{\varphi}(t)$ of the disturbance $\varphi(t)$ in (1) can be identified

$$\hat{\varphi}(t) = E^{-1}(\Phi^{-1}(\hat{\xi}, \hat{\eta})) \left[\begin{pmatrix} \hat{\dot{\xi}}^1_{r_1} \\ \hat{\dot{\xi}}^2_{r_2} \\ \vdots \\ \hat{\dot{\xi}}^m_{r_m} \end{pmatrix} - \begin{pmatrix} L_f^{r_1} h_1(\Phi^{-1}(\hat{\xi}, \hat{\eta})) \\ L_f^{r_2} h_2(\Phi^{-1}(\hat{\xi}, \hat{\eta})) \\ \vdots \\ L_f^{r_m} h_m(\Phi^{-1}(\hat{\xi}, \hat{\eta})) \end{pmatrix} \right] \tag{21}$$

Based on the developments in this section, the following theorems are true.

Theorem 1

If system (1)–(3) is locally detectable in the sense of Definition 1, the higher-order sliding-mode observer (13), (14), (19), (21) asymptotically estimates the state x and the disturbance $\varphi(t)$ in the system, and hence the goals of the observer design (5) and (6) are met.

When the total relative degree of the system is $r = n$, all the states are estimated in finite time.

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Theorem 2
Suppose that system (1)–(3) is locally detectable in the sense of Definition 1 and the measured outputs are corrupted with noise which is a Lebesgue-measurable function of time with maximal magnitude ε. Then the higher-order sliding-mode observer (13), (14), (19), (21) ensures a state observation error accuracy of the order of $\varepsilon^{2/(\bar{r}+1)}, \bar{r} = \max r_i, i = 1, \ldots, m$.

Theorem 3
Suppose that the outputs of system (1)–(3) are measured at discrete sampling times with a sufficiently small sampling interval δ. Then the higher-order sliding-mode observer (13), (14), (19), (21), after some transient, ensures a state observation error accuracy of the order of δ^2.

Remark
When the total the relative degree of the system $r = n$, all the states are estimated in finite time.

4. EXAMPLES

Example 1
Consider a satellite system which is modelled as in [17] as

$$\dot{\rho} = v$$

$$\dot{v} = \rho\omega^2 - \frac{k_{\mathrm{g}} M}{\rho^2} + d$$

$$\dot{\omega} = -\frac{2v\omega}{\rho} - \frac{\theta\omega}{m}$$

where ρ is the distance between the satellite and the Earth centre, v is the radial speed of the satellite with respect to the Earth, ω is the angular velocity of the satellite around the Earth, m and M are the mass of the satellite and the Earth, respectively, k_{g} represents the universal gravity coefficient, and θ is the damping coefficient. The quantity d which affects the radial velocity equation is assumed to be a disturbance which is to be reconstructed/estimated. Let $x := \mathrm{col}(x_1, x_2, x_3) := (\rho, v, \omega)$. The satellite system can be rewritten as follows:

$$\dot{x} = \begin{pmatrix} x_2 \\ x_1 x_3^2 - \dfrac{k_1}{x_1^2} \\ -\dfrac{2x_2 x_3}{x_1} - k_2 x_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \mathrm{d}(t) \tag{22}$$

$$y = x_1 \tag{23}$$

where y is the system output, $k_1 = k_{\mathrm{g}} M$ and $k_2 = \theta/m$.

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By direct computation, it follows that

$$L_g h(x) = 0, \quad L_g L_f h(x) = 1$$

and thus the system (22)–(23) has global relative degree 2. Choose the coordinate transformation as $T: \xi_1 = x_1, \xi_2 = x_2, \eta = x_1^2 x_3$. Note that for $x_1 \neq 0$, this transformation is invertible and an analytic expression for the inverse can be obtained as $x_1 = \xi_1$, $x_2 = \xi_2$, $x_3 = \eta/\xi_1^2$, since $x_1 = \rho$ is the distance of the satellite from the centre of the Earth $x_1 \neq 0$. It follows that in the new coordinate system $\mathrm{col}(\xi_1, \xi_2, \eta)$, system (22)–(23) can be described by

$$\dot{\xi}_1 = \xi_2$$

$$\dot{\xi}_2 = \frac{\eta^2}{\xi_1^3} - \frac{k_1}{\xi_1^2} + d$$

$$\dot{\eta} = -k_2 \eta$$

Clearly, the system internal dynamics are asymptotically stable since $k_2 > 0$. Therefore, system (21)–(22) is locally asymptotically observable. From (13) and (14), the higher-order sliding-mode differentiator is described by

$$\begin{aligned}
\dot{z}_0^1 &= v_0^1 \\
v_0^1 &= -\lambda_0^1 |z_0^1 - y|^{2/3} \operatorname{sign}(z_0^1 - y) + z_1^1 \\
\dot{z}_1^1 &= v_1^1 \\
v_1^1 &= -\lambda_1^1 |z_1^1 - v_0^1|^{1/2} \operatorname{sign}(z_1^1 - v_0^1) + z_2^1 \\
\dot{z}_2^1 &= -\lambda_2^1 \operatorname{sign}(z_2^1 - v_1^1)
\end{aligned}$$

Define $\hat{\xi}_1 = z_0^1$, $\hat{\xi}_2 = z_1^1$, $\dot{\hat{\xi}}_2 = z_2^1$. Then,

$$\hat{\xi} := \begin{bmatrix} \hat{\xi}_1 \\ \hat{\xi}_2 \end{bmatrix} \in \Re^2$$

is an estimate of ξ and the estimate for η can be obtained from the equation $\dot{\hat{\eta}} = -k_2 \hat{\eta}$. Therefore, the estimate of the disturbance $d(t)$ is available online, and from (21),

$$\hat{d}(t) = \dot{\hat{\xi}}_2 - \frac{\hat{\eta}^2}{\hat{\xi}_1^3} + \frac{k_1}{\hat{\xi}_1^2}$$

is a reconstruction for the disturbance $d(t)$. As in [17], the parameters have been chosen as follows: $m = 10, M = 5.98 \times 10^{24}, k_g = 6.67 \times 10^{-11}$ and $\theta = 2.5 \times 10^{-5}$. For simulation purposes, choose $d(t) = \mathrm{e}^{-0.002t} \sin(0.02t)$. The differentiator gains λ_i^j are chosen as $\lambda_0^1 = 2$ and $\lambda_1^1 = \lambda_2^1 = 1$. In the following simulation, the initial values $x_0 = (10^7, 0, 6.3156 \times 10^{-4})$ for the plant states in the original coordinates whilst for the observer $z_0 = (1.001 \times 10^7, 0, 1)$ and $\hat{\eta}_0 = 6.3156 \times 10^{-4}$ (in the transformed coordinate system). Figures 1 and 2 show that the states and the disturbance signal $d(t)$ can be reconstructed faithfully.

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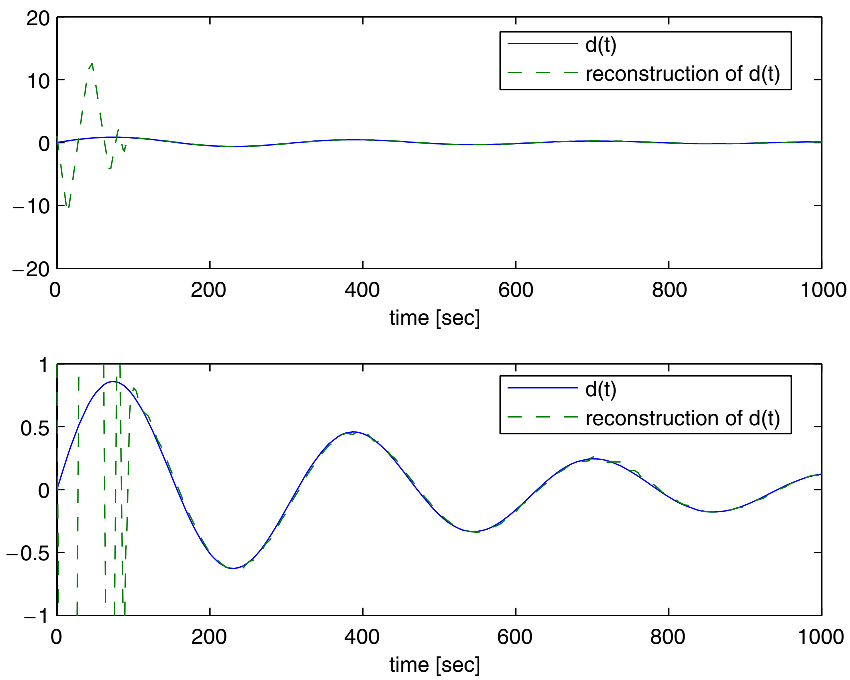

Figure 1. The disturbance signal $d(t)$ and its reconstruction signal $\hat{d}(t)$.

Example 2
Consider the fifth-order nonlinear system

$$\dot{x} = \underbrace{\begin{bmatrix} -2x_1 - x_2 \\ x_1 \\ -x_3^3 - 2x_3 - x_4 \\ x_3 \\ (x_2 - 4)\dfrac{2x_5 + \sin x_5}{2 + \cos x_5} \end{bmatrix}}_{f(x)} + \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 + (2x_5 + \sin(x_5))^2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}}_{G(x):=[g_1(x), g_2(x)]} \underbrace{\begin{bmatrix} \varphi_1(t) \\ \varphi_2(t) \end{bmatrix}}_{\varphi(t)} \tag{24}$$

$$\begin{aligned} y_1 &= h_1(x) = x_2 \\ y_2 &= h_2(x) = x_4 \end{aligned} \tag{25}$$

in the domain $\Omega = \{(x_1, x_2, x_3, x_4, x_5) | \, |x_2| < 3.5, x_1, x_3, x_4, x_5 \in \Re\}$, where $x \in \Re^5$ is the system state, $y = \text{col}(y_1, y_2)$ is the system output, and $\varphi(t) = [\varphi_1(t)\varphi_2(t)]^T$ is the system input which will be reconstructed.

By direct computation, it follows that

$$L_{g_1}h_1 = L_{g_2}h_2 = 0$$

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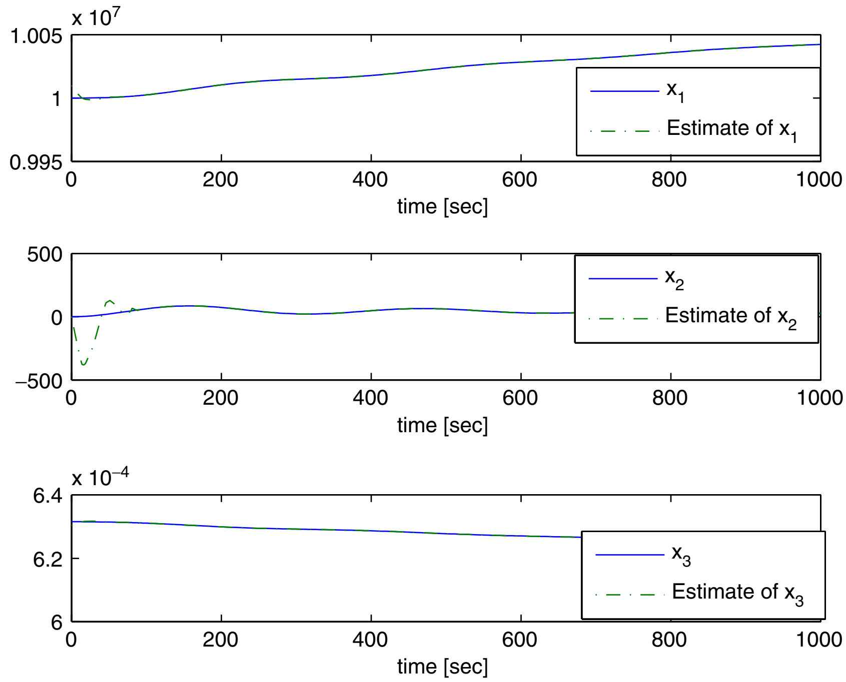

Figure 2. The response of system states and their estimates.

and

$$\begin{bmatrix} L_{g_1}L_f h_1 & L_{g_2}L_f h_1 \\ L_{g_1}L_f h_2 & L_{g_2}L_f h_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1+(2x_5+\sin x_5)^2 \end{bmatrix}$$

which is nonsingular in $\Re^5$. Therefore, system (24)–(25) has relative degree $\{2,2\}$. Further, $\Phi = \text{span}\{g_1, g_2\}$ is an involutive distribution and thus Assumptions (i)–(iii) are satisfied. This implies that system (24)–(25) is weakly observable in the domain Ω. Then, under the coordinate transformation: $\xi_1^1 = x_2$, $\xi_2^1 = x_1$, $\xi_1^2 = x_4$, $\xi_2^2 = x_3$, $\eta = 2x_5 + \sin x_5$, the system (22)–(23) in the new coordinate system can be described by

$$\dot{\xi}_1^1 = \xi_2^1$$

$$\dot{\xi}_2^1 = -2\xi_2^1 - \xi_1^1 + \varphi_1(t)$$

$$\dot{\xi}_1^2 = \xi_2^2$$

$$\dot{\xi}_2^2 = -(\xi_2^2)^3 - \xi_1^2 - 2\xi_2^2 + (1+\eta^2)\varphi_2(t)$$

$$\dot{\eta} = (\xi_1^1 - 4)\eta$$

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It is clear that it is not possible to obtain an analytic inverse for the inverse transformation because the fifth coordinate x_5 cannot be expressed analytically as a function of η since $\eta = 2x_5 + \sin x_5$.

Clearly, the system internal dynamics are asymptotically stable in the domain Ω. Therefore, system (24)–(25) is locally asymptotically observable. From (13) and (14), the high-order sliding-mode differentiator (13) is described by

$$\begin{aligned}
\dot{z}_0^1 &= v_0^1\\
v_0^1 &= -\lambda_0^1 |z_0^1 - y_1|^{2/3}\,\mathrm{sign}\,(z_0^1 - y_1) + z_1^1\\
\dot{z}_1^1 &= v_1^1\\
v_1^1 &= -\lambda_1^1 |z_1^1 - v_0^1|^{1/2}\,\mathrm{sign}\,(z_1^1 - v_0^1) + z_2^1\\
\dot{z}_2^1 &= -\lambda_2^1\,\mathrm{sign}\,(z_2^1 - v_1^1)\\
\dot{z}_0^2 &= v_0^2\\
v_0^2 &= -\lambda_0^2 |z_0^2 - y_2|^{2/3}\,\mathrm{sign}\,(z_0^2 - y_2) + z_1^2\\
\dot{z}_1^2 &= v_1^2\\
v_1^2 &= -\lambda_1^2 |z_1^2 - v_0^2|^{1/2}\,\mathrm{sign}\,(z_1^2 - v_0^2) + z_2^2\\
\dot{z}_2^2 &= -\lambda_2^2\,\mathrm{sign}\,(z_2^2 - v_1^2)
\end{aligned}$$

Define $\hat{\xi}_1^1 = z_0^1$, $\hat{\xi}_2^1 = z_1^1$, $\dot{\hat{\xi}}_2^1 = z_2^1$, $\hat{\xi}_1^2 = z_0^2$, $\hat{\xi}_2^2 = z_1^2$, $\dot{\hat{\xi}}_2^2 = z_2^2$. Then,

$$\hat{\xi} := \begin{bmatrix} \hat{\xi}_1 \\ \hat{\xi}_2 \end{bmatrix} := \begin{bmatrix} \hat{\xi}_1^1 \\ \hat{\xi}_2^1 \\ \hat{\xi}_1^2 \\ \hat{\xi}_2^2 \end{bmatrix} \in \Re^4$$

is an estimate of ξ and the estimate for η can be obtained from equation

$$\dot{\hat{\eta}} = (\hat{\xi}_1^1 - 4)\hat{\eta}$$

Therefore,

$$\hat{\varphi}(t) = \begin{bmatrix} \dot{\hat{\xi}}_2^1 + \hat{\xi}_1^1 + 2\hat{\xi}_2^1 \\ (\dot{\hat{\xi}}_2^2 + (\hat{\xi}_2^2)^2 + \hat{\xi}_1^2 + 2\hat{\xi}_2^2)/(1 + \hat{\eta}^2) \end{bmatrix}$$

is available online and from (19) it is a reconstruction for the input $\varphi(t)$. For simulation purposes, choose $\varphi_1(t) = \sin(0.5t)$ and $\varphi_2(t) = 0.5\sin(0.5t) + 0.5\cos t$. From [16], λ_i^j can be chosen as $\lambda_0^1 = \lambda_0^2 = 3$, $\lambda_1^1 = \lambda_1^2 = 1.5$, and $\lambda_2^1 = \lambda_2^2 = 1.1$. The simulation with the initial values $x_0 = (0, 0.1, 0, -0.2, 0.2)$, $z_0 = (0, 0, 0, -0.2, 0)$, and $\hat{\eta}_0 = 0.5$ are shown in the following figures. Figure 3 shows the states and the estimates in the original coordinate system. Here only asymptotic convergence is achieved. To obtain the values of $\hat{x}$ in terms of $\hat{\xi}$ and $\hat{\eta}$, it has been necessary to embed in the simulation an iteration scheme to extract $\hat{x}_5$ from

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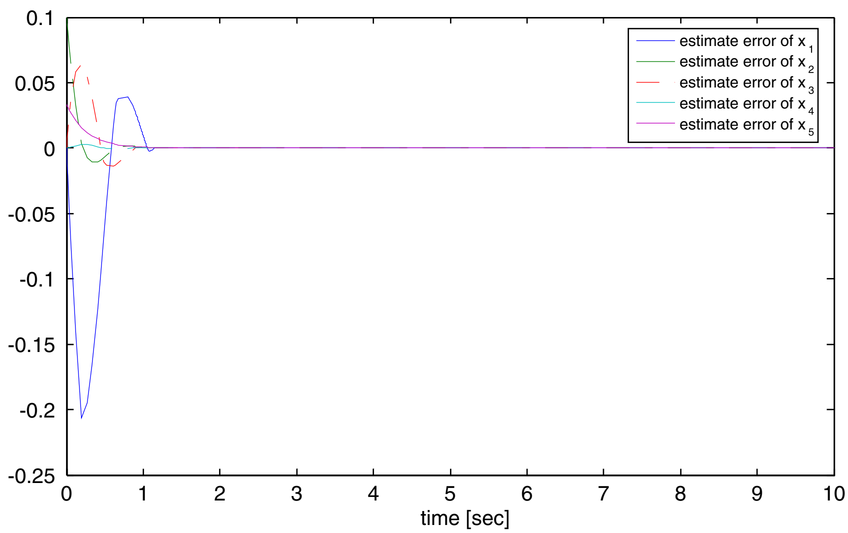

Figure 3. The original state variables $x(t)$ and their estimates $\hat{x}(t)$.

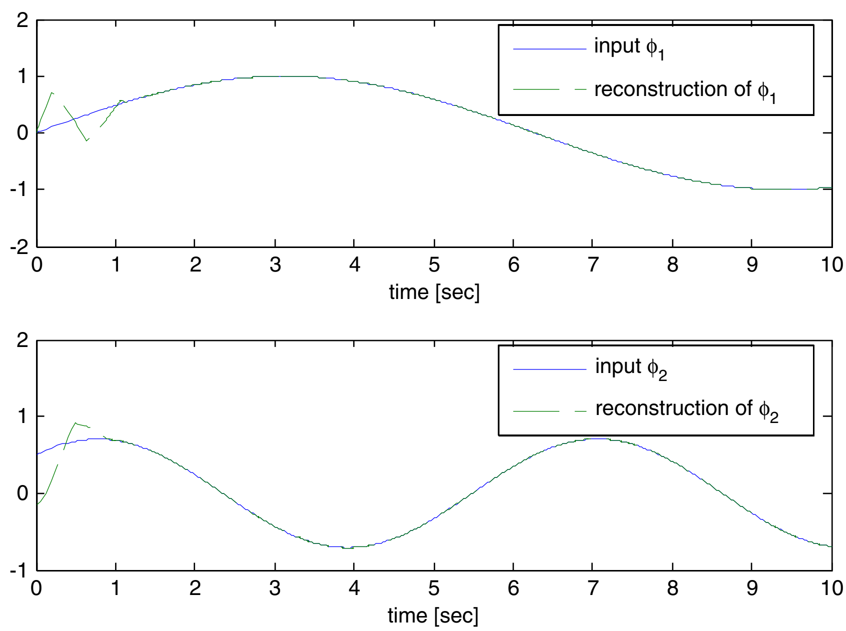

Figure 4. The inputs of system (22)–(23) and their reconstruction signal.

$\hat{\eta} = 2\hat{x}_5 + \sin \hat{x}_5$ given $\hat{\eta}$. Figure 4 shows the estimate of the unknown inputs. From the simulation, it is observed that the proposed strategy can reconstruct the input faithfully after approximately 1.1 s.

5. CONCLUSIONS

In this paper, an exact observer scheme for nonlinear locally detectable systems with unknown inputs has been proposed based on higher-order sliding-mode concepts. The approach is applicable for a class of nonlinear systems with unknown inputs, which enter affinely. The systematic design approach consists of two steps: first the transformation of the system to the Brunovsky canonical form; and second the application of higher-order sliding-mode differentiators for each coordinate of the output vector error.

The proposed scheme ensures exact finite state estimation for the observable variables and asymptotic exact estimation of the unobservable variables for the case when the system has stable internal dynamics. When the total the relative degree of the system $r = n$, all the states are estimated in finite time. In addition to estimating the states, the unknown inputs can also be identified asymptotically.

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