INTERNATIONAL JOURNAL OF ROBUST AND NONLINEAR CONTROL
Int. J. Robust Nonlinear Control (in press)
Published online in Wiley InterScience (www.interscience.wiley.com). DOI: 10.1002/rnc.1206

Analysis of response of second-order sliding mode controllers to external inputs in frequency domain

Igor Boiko[1,*,†], Ivan Castellanos[2,3] and Leonid Fridman[2]

[1] *University of Calgary, 2500 University Dr. N.W., Calgary, Alta., Canada T2N 1N4*
[2] *National Autonomous University of Mexico, UNAM 04510, México, D.F., Mexico*
[3] *Autonomous University of Ciudad Juárez., Henry Dunant 4016, Ciudad Juárez, Mexico*

SUMMARY

Response of systems with the second-order sliding-mode (SOSM) control algorithms to an external harmonic excitation is analysed in the frequency domain. The analysis is done *via* application of the concept of the equivalent gain of a nonlinear function. The equivalent gain is used as a model of propagation of averaged values of signals through nonlinear elements of the system. Two SOSM algorithms: twisting and super-twisting are analysed. The results are presented in the format of the conventional amplitude and phase frequency response. Copyright © 2007 John Wiley & Sons, Ltd.

Received 28 April 2006; Revised 31 October 2006; Accepted 12 March 2007

KEY WORDS: high-order sliding mode; frequency methods; input–output analysis

1. INTRODUCTION

Chattering is usually an undesirable effect and displays system performance deterioration in comparison with ideal sliding [1, 2]. However, this possibly is not the only manifestation of the system performance deterioration caused by the presence of parasitic dynamics. In first-order sliding mode (SM) control systems performance deterioration was discovered not only in the autonomous mode but also with respect to external input tracking [3].

The second-order sliding-mode (SOSM) algorithms were originally designed as a tool for the chattering attenuation [4, 5]. However, in [6, 7] was shown that the system controlled by a SOSM algorithm exhibits chattering in the presence of parasitic dynamics. Since every real system

*Correspondence to: Igor Boiko, University of Calgary, 2500 University Dr. N.W., Calgary, Alta., Canada T2N 1N4.
† E-mail: i.boiko@ieee.org

Contract/grant sponsor: CONACyT (Consejo Nacional de Ciencia y Tecnología); contract/grant number: 43807-Y
Contract/grant sponsor: PAPIIT (Programa de Apoyo a Proyectos de Investigación e Innovación Tecnológica), UNAM; contract/grant number: 107006-2

always possesses parasitic dynamics of some kind the chattering phenomenon is an unavoidable feature of every real SOSM control system [6, 7].

Therefore, analysis of the closed-loop performance of systems governed by SOSM with respect to external input tracking would be of some interest too. In the autonomous mode a SOSM system exhibits fast periodic motions or chattering [6, 7]. If a relatively slow input is applied to such a system the system would exhibit the mode that would be a combination of chattering and external input effect (propagation), so that the output of the SOSM controller would be a pulse-width-modulated signal, and the output of the system would be a combination (sum) of the somewhat distorted (lagged and attenuated) input and the chattering component [8]. The aim of this paper is to assess those distortions of the input signal propagation. In particular, we aim to assess SOSM control system response to harmonic signals of different frequencies and amplitudes.

Analysis of input–output properties of the SOSM system can be done *via* the use of the so-called linearization due to the 'chatter smoothing effect' [8, 9]. With this approach, for analysis of input–output properties of the system with respect to external inputs, the nonlinear functions can be replaced with equivalent gains that describe the propagation of averaged values through those nonlinear functions [3, 8, 9]. The present paper proposes this kind of analysis. The two most popular SOSM algorithms: twisting and super-twisting are analysed with the use of the approach proposed below.

2. TWISTING ALGORITHM AND ITS DESCRIBING FUNCTION ANALYSIS

The twisting algorithm [4, 5] is one of the simplest and most popular algorithms among the SOSM algorithms. There are two ways of using the twisting algorithm [4, 5]: to apply it to the principal dynamics of relative degree two, or to apply it to the principal dynamics of relative degree one and introduce an integrator in series with the plant (twisting-as-a-filter). For the principal dynamics of relative degree two, it can be formulated as follows. Let the plant (or the plant plus actuator) be given by the following differential equations which would comprise principal and parasitic dynamics:

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y = Cx, \quad \sigma = f - y \tag{1}$$

where A, B and C are matrices of respective dimensions, $x \in \mathbb{R}^n, u \in \mathbb{R}^1, f$ is the input, y is the output of the plant, σ is the error signal. The control u of the twisting algorithm is given as follows [4, 5]:

$$u(t) = c_1 \cdot \operatorname{sign}(\sigma) + c_2 \cdot \operatorname{sign}(\dot{\sigma}) \tag{2}$$

where c_1 and c_2 are positive values, $c_1 > c_2 > 0$. Figure 1 shows the system given by Equations (1) and (2), where the reference signal $f = f_0 \equiv 0$. We assume that the plant is asymptotically stable, and is a low-pass filter. We shall also use the plant description in the form of the transfer function $W(s)$, which can be obtained from formulas (1) as follows: $W(s) = C(Is - A)^{-1}B$.

Assume that in the autonomous mode ($f \equiv 0$) a periodic motion occurs in the system with the twisting algorithm and find the parameters of this periodic motion. The system can be analysed with the use of the Describing Function (DF) method. We assume that the output of the plant displays harmonic oscillation. Find the DF N of the twisting algorithm as the first harmonic of

DOI: 10.1002/rnc

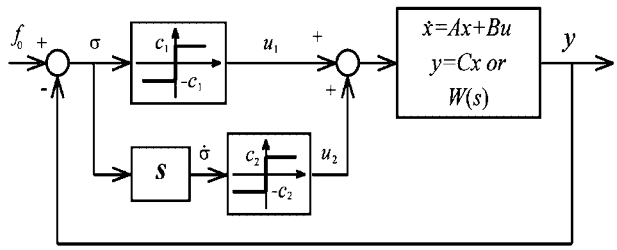

Figure 1. Diagram of twisting algorithm.

the periodic control signal divided by the amplitude of $\sigma(t)$ (or of $y(t)$) [9]:

$$N = \frac{\omega}{\pi a_1}\int_0^{2\pi/\omega} u(t)\sin\omega t\,\mathrm{d}t + \mathrm{j}\frac{\omega}{\pi a_1}\int_0^{2\pi/\omega} u(t)\cos\omega t\,\mathrm{d}t \tag{3}$$

where a_1 is the amplitude of the input to the nonlinearity (of $y(t)$ in our case) and $\omega=\Omega$ is the frequency of $y(t)$. However, the twisting algorithm can be analysed as the parallel connection of two ideal relays where the input to the first relay is the sliding variable and the input to the second relay is the derivative of the sliding variable. The DFs for those nonlinearities are known [9]. For the first relay, the DF is: $N_1 = 4c_1/\pi a_1$, and for the second relay it is: $N_2 = 4c_2/\pi a_2$, where a_2 is the amplitude of $\mathrm{d}\sigma/\mathrm{d}t$ (or of $\mathrm{d}y/\mathrm{d}t$). Also, take into account the relationship between y and $\mathrm{d}y/\mathrm{d}t$ in the Laplace domain, which gives the relationship between the amplitudes a_1 and a_2: $a_2 = a_1\Omega$, where Ω is the frequency of the oscillation. Using the notation of the twisting algorithm, we can rewrite this equation as follows:

$$N = N_1 + sN_2 = \frac{4c_1}{\pi a_1} + \mathrm{j}\Omega\frac{4c_2}{\pi a_2} = \frac{4}{\pi a_1}(c_1 + \mathrm{j}c_2) \tag{4}$$

where $s = \mathrm{j}\Omega$. Let us note that the DF of the twisting algorithm depends on the amplitude value only. This suggests the technique of finding the parameters of the limit cycle—*via* the solution of the harmonic balance equation [9] for Ω and a_1:

$$W(\mathrm{j}\Omega)N(a_1,\Omega) = -1 \tag{5}$$

where $W(\mathrm{j}\omega), \omega \in [0;\infty)$ is the complex frequency response (Nyquist plot) of the plant. Using the notation of the twisting algorithms, this equation can be rewritten as follows:

$$W(\mathrm{j}\Omega) = -\frac{1}{N(a_1,\Omega)} \tag{6}$$

where the function at the right-hand side is given by

$$-\frac{1}{N(a_1,\Omega)} = -\frac{1}{N(a_1)} = \pi a_1\frac{-c_1 + \mathrm{j}c_2}{4(c_1^2 + c_2^2)}$$

The negative reciprocal of the DF is a function of the amplitude only and does not depend on the frequency of the oscillation. Equation (5) is equivalent to the condition of the complex frequency response characteristic of the open-loop system intersecting the real axis in the point $(-1, \mathrm{j}0)$. The graphical illustration of the technique of solving equation (5) is given in Figure 2. The function $-1/N$ is a straight line the slope of which depends on c_2/c_1 ratio. This line is

DOI: 10.1002/rnc

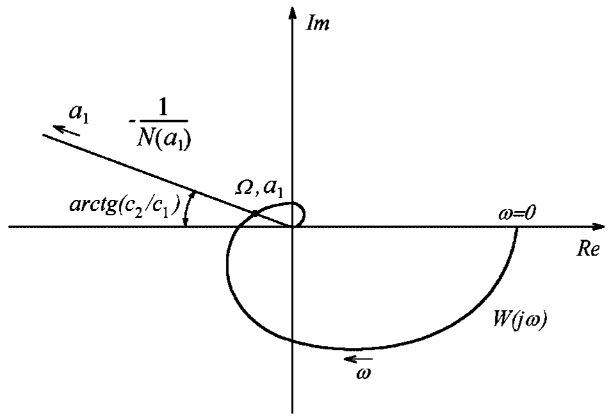

Figure 2. Finding a periodic solution in a system with the twisting algorithm.

located in the second quadrant of the complex plane. The point of intersection of this function and of the Nyquist plot $W(j\omega)$ provides the solution of the periodic problem. This point gives the frequency of the oscillation and the amplitude a_1. Therefore, if the transfer function of the plant (or plant plus actuator) has relative degree higher than *two*, a periodic motion occurs in such a system. For this reason, if an actuator of first or higher order is added to the plant of relative degree *two* driven by the twisting controller a periodic motion occurs in the system. The conditions of the existence of a periodic solution in a system with the twisting controller can be derived from analysis of Figure 2 Obviously, every system with a plant of relative degree *three* and higher would have a point of intersection with the negative reciprocal of the DF of the twisting algorithm and, therefore, a periodic solution would exist. Another modification of the twisting algorithm is its application to a plant with relative degree one with the introduction of the integrator. This is usually referred to as the 'twisting-as-a-filter' algorithm. The above reasoning is applicable in this case too. The introduction of the integrator in series with the plant makes the relative degree of this part of the system equal to two. As a result, any actuator introduced in the loop increases the overall relative degree to at least three. In this case, there always exists a point of intersection of the Nyquist plot of the serial connection of the actuator, the plant and the integrator and of the negative reciprocal of the DF of the twisting algorithm Figure 2. Thus, if an actuator of first or higher order is added to the plant with relative degree one, a periodic motion occurs in the system with the twisting-as-a-filter-algorithm.

3. PROPAGATION OF SLOW INPUT THROUGH THE SYSTEM WITH TWISTING ALGORITHM

Let us describe propagation of averaged signals through the nonlinear functions using the concept of the equivalent gain [8, 9] Assume that the input to the first relay is an asymmetric harmonic signal

$$\sigma(t) = \sigma_0 + a_1 \cdot \sin(\Omega t) \tag{7}$$

Define the equivalent gain as the derivative of the averaged output of the nonlinearity with respect to the averaged input noting that the averaged input is σ_0. For the first relay,

DOI: 10.1002/rnc

the equivalent gain will be [8, 9]:

$$k_{n1} = \left.\frac{\partial u_1}{\partial \sigma_0}\right|_{\sigma_0=0} = \frac{2 \cdot c_1}{\pi \cdot a_1} \tag{8}$$

For the second relay, we also assume an asymmetric harmonic input. The equivalent gain can be obtained as follows:

$$k_{n2} = \left.\frac{\partial u_2}{\partial \dot{\sigma}_0}\right|_{\sigma_0=0} = \frac{2 \cdot c_2}{\pi \cdot a_1 \Omega} \tag{9}$$

With those gains available, we can build now the linearized model of averaged motions in the system with the twisting algorithm (Figure 3). The transfer function of the closed-loop system is as follows:

$$W_{f_0-y}(s) = \frac{(k_{n1} + sk_{n2})C(sI - A)^{-1}B}{1 + (k_{n1} + sk_{n2})C(sI - A)^{-1}B} \tag{10}$$

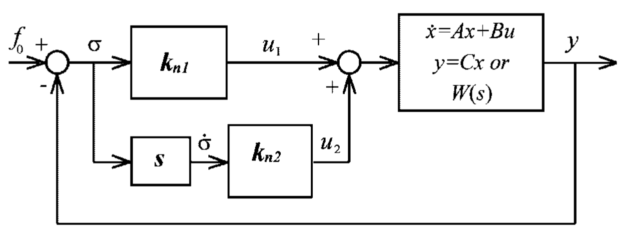

Figure 3. Block diagram of the system with the linearized twisting algorithm.

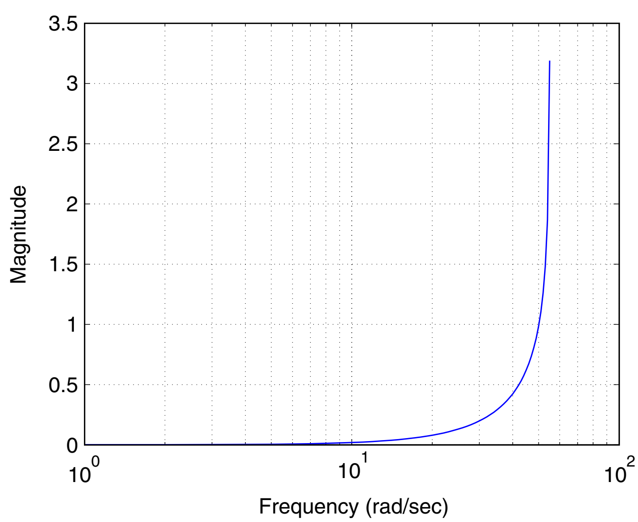

Figure 4. Closed-loop frequency response for the case of plant of relative degree two $W_p(s) = 1/(s^2 + s + 1)$ and actuator of relative degree one $W_a(s) = 1/(0.01s + 1)$, controlled by twisting algorithm.

DOI: 10.1002/rnc

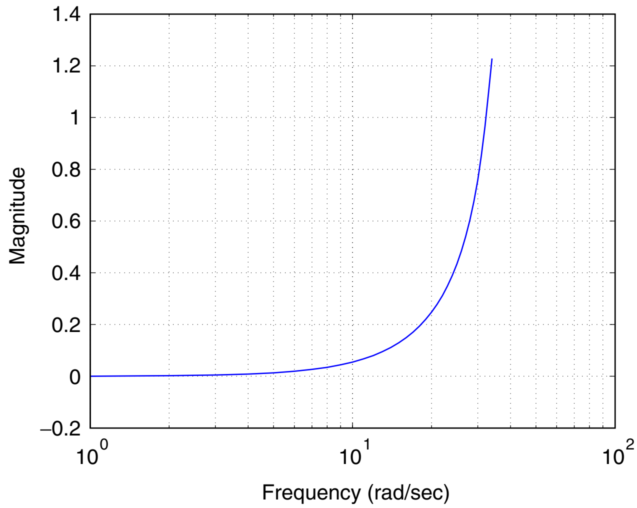

Figure 5. Closed-loop frequency response for the case of plant of relative degree two $W_p(s) = 1/(s^2 + s + 1)$ and actuator of relative degree two $W_a(s) = 1/(0.0001s^2 + 0.01s + 1)$, controlled by twisting algorithm.

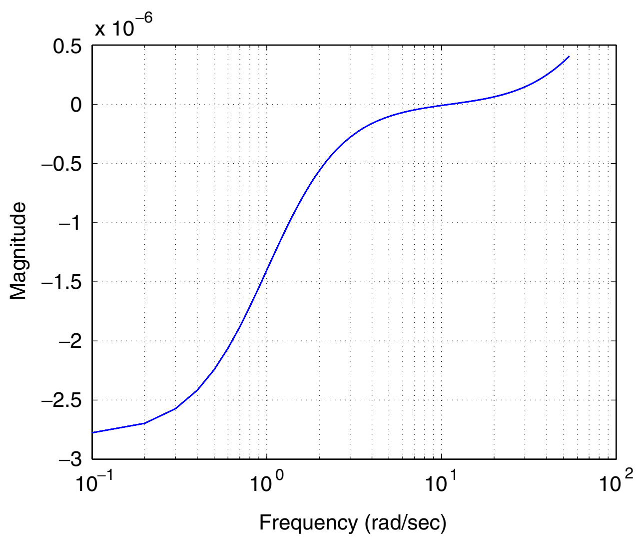

Figure 6. Closed-loop frequency response for the case of plant of relative degree one $W_p(s) = (s + 1)/(s^2 + s + 1)$ and actuator of relative degree one $W_a(s) = 1/(0.01s + 1)$, controlled by twisting-as-a-filter algorithm.

From the formula of the transfer function, all characteristics of the closed-loop frequency response can be easily obtained. Examples of a few magnitude frequency responses are given in Figures 4–7. The amplitudes of the relays are selected as $c_1 = 0.8$ and $c_2 = 0.6$. Analysis of those characteristics shows that up to a certain frequency the system driven by the twisting algorithm provides a good closed-loop performance (quality of external signal tracking). However, at

DOI: 10.1002/rnc

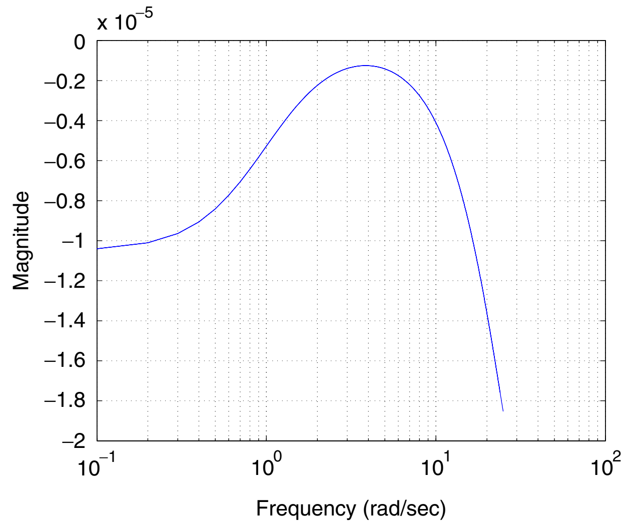

Figure 7. Closed-loop frequency response for the case of plant of relative degree one $W_p(s) = (s+1)/(s^2+s+1)$ and actuator of relative degree two $W_a(s) = 1/(0.0001s^2 + 0.01s + 1)$, controlled by twisting-as-a-filter algorithm.

higher frequencies the quality deteriorates, and the higher the frequency the lower the quality of tracking.

4. SUPER-TWISTING ALGORITHM AND ITS DF ANALYSIS

4.1. *DF of super-twisting algorithm*

The super-twisting algorithm is one of the popular algorithms among the SOSM algorithms. It is used for the principal dynamics of relative degree *one*. The control $u(t)$ for the super-twisting algorithm is given as a sum of two components [5]

$$u(t) = u_1(t) + u_2(t) \tag{11}$$

$$\dot{u}_1 = -\gamma\,\mathrm{sign}(\sigma)$$

$$u_2 = \begin{cases} -\lambda|s_0|^\rho\,\mathrm{sign}(\sigma) & \text{if } |\sigma| > s_0 \\ -\lambda|\sigma|^\rho\,\mathrm{sign}(\sigma) & \text{if } |\sigma| \leqslant s_0 \end{cases}$$

where α, ρ and s_0 are design parameters, $0.5 \leqslant \rho < 1$. The system under analysis can be represented in the form of the block diagram as in Figure 8. For an arbitrary value of the power ρ in (11), the formula of the DF of such nonlinear function can be given as follows:

$$N_2 = \frac{2\lambda a_y^{\rho-1}}{\pi}\int_0^\pi (\sin\psi)^{\rho+1}\,\mathrm{d}\psi = \frac{2\lambda a_y^{\rho-1}}{\sqrt{\pi}}\frac{\Gamma(\rho/2+1)}{\Gamma(\rho/2+1.5)} \tag{12}$$

DOI: 10.1002/rnc

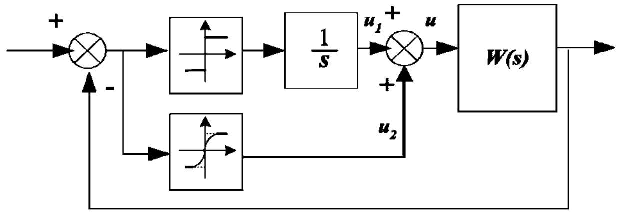

Figure 8. Block diagram of the system with the super-twisting algorithm.

where a_y is the amplitude of variable σ (of y), $a_y \leqslant s_0$ (that is considered the most important range of the amplitude values for the analysis of the steady state) and Γ is the gamma-function [10]. With the square root nonlinearity ($\rho = 0.5$), the DF formula can be derived as

$$\begin{aligned} N_2 &= \frac{2\lambda}{\pi a_y} \int_0^{\pi} \sqrt{a_y \sin \psi} \sin \psi \, \mathrm{d}\psi \\ &= \frac{2\lambda}{\sqrt{\pi a_y}} \frac{\Gamma(1.25)}{\Gamma(1.75)} \approx \frac{1.1128\lambda}{\sqrt{a_y}} \end{aligned} \tag{13}$$

The DF of the first component of the super-twisting algorithm can be written as follows:

$$N_1 = \frac{4\gamma}{\pi a_y} \frac{1}{\mathrm{j}\omega}$$

which is a result of the cascade connection of the ideal relay having the DF equal to $4\gamma/\pi a_y$ [7] and the integrator with the transfer function $1/s$ (for the harmonic signal the Laplace variable s can be replaced with $\mathrm{j}\omega$). Taking into account both control components, we can rewrite the DF of the super-twisting algorithms as follows:

$$N = N_1 + N_2 = \frac{4\gamma}{\pi a_y} \frac{1}{\mathrm{j}\omega} + \frac{1.1128\lambda}{\sqrt{a_y}} \tag{14}$$

Let us note that the DF of the super-twisting algorithm depends on both: the amplitude and the frequency values. The parameters can be found *via* the solution of the harmonic balance equation (5), where the DF N is given by (14). The function at the negative reciprocal of the DF can be represented by the following formula:

$$\begin{aligned} -\frac{1}{N(a_y, \omega)} &= \frac{1}{1.1128 \dfrac{\lambda}{\sqrt{a_y}} + \dfrac{4\gamma}{\pi a_y} \dfrac{1}{\mathrm{j}\omega}} \\ &= -\frac{0.8986 \dfrac{\sqrt{a_y}}{\lambda} + \mathrm{j}1.1329 \dfrac{\gamma}{\lambda^2} \dfrac{1}{\omega}}{1 + 1.3092 \dfrac{\gamma^2}{\lambda^2} \dfrac{1}{a_y \omega^2}} \end{aligned} \tag{15}$$

The function $-1/N(a_y, \omega)$ is of two variables: the amplitude and the frequency. It can be depicted as a number of plots representing the amplitude dependence, with each of those plots

DOI: 10.1002/rnc

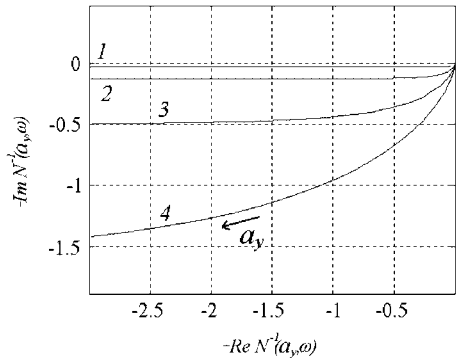

Figure 9. Plots of the function $-1/N$.

corresponding to a certain frequency. The frequency range of interest lies below the frequency corresponding to the intersection of the Nyquist plot and the real axis. The plots of function $-1/N(a_y,\omega)$ are depicted in Figure 9. The plots 1–4 correspond to four different frequencies, with the following relationship: $\omega_1 > \omega_2 > \omega_3 > \omega_4$. Each of those plots represents the dependence of the DF on the amplitude value. Function $-N^{-1}(a_y,\omega)$ (where $\omega = \text{const}$) has an asymptote at $a_y \to \infty$, which is the horizontal line $-\mathrm{j}1.1329(\gamma/\lambda^2)1/\omega$. Also, it is easy to show that

$$\lim_{\substack{a_y \to 0 \\ \omega=\text{const}}} \arg(-N^{-1}(a_y,\omega)) = -\pi/2$$

5. PROPAGATION OF SLOW INPUT THROUGH THE SYSTEM WITH SUPER-TWISTING ALGORITHM

Let us describe propagation of averaged signals through the nonlinear functions of the super-twisting algorithm in the same way as through the twisting algorithm—using the concept of the equivalent gain [8, 9] (Figure 10). Assume that the input to the controller is an asymmetric harmonic signal:

$$\sigma(t) = \sigma_0 + a_y \cdot \sin(\omega t) \tag{16}$$

The equivalent gain k_n for the first nonlinear function is

$$k_{n1} = \frac{2\gamma}{\pi a_y} \tag{17}$$

where γ is the amplitude output of the relay. k_{n2} can be derived in accordance with its definition as follows. The averaged output of the second nonlinearity is

$$v_{20} = \frac{1}{T}\int_0^T |\sigma(t)|^\rho \, \mathrm{d}t \tag{18}$$

If $\rho = 0.5$ (that is considered a typical value) then

$$v_{20} = \frac{\lambda}{T}\int_0^T |(\sigma_0 + a_y \cdot \sin(\omega t)|^{1/2} \, \mathrm{d}t \tag{19}$$

Taking derivative of (19) results in the following formula:

$$k_{n2} = \left.\frac{\partial v_{20}}{\partial \sigma_0}\right|_{\sigma_0=0} = \lambda \frac{\Gamma(\frac{1}{4})}{2\sqrt{a_y}\,\Gamma(\frac{3}{4})} \approx \lambda \frac{0.8346}{\sqrt{a_y}} \tag{20}$$

With both equivalent gains available, we can easily build the linearized model of the dynamical system. The nonlinear functions of the super-twisting controller need to be replaced with the respective equivalent gains for that purpose, and the subsequent analysis can be done as of a linear system. The transfer function of the system input–outputs is defined as follows:

$$W_{f_0-y}(s) = \frac{(k_{n1}/s + k_{n2})C(sI-A)^{-1}B}{1 + (k_{n1}/s + k_{n2})C(sI-A)^{-1}B} \tag{21}$$

The frequency response of the super-twisting controller for $\rho = 0.5$, $\gamma = 0.8$ and $\lambda = 0.6$ is shown in Figures 11 and 12.

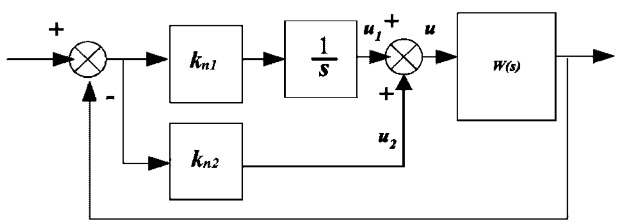

Figure 10. Linearized model of the super-twisting algorithm.

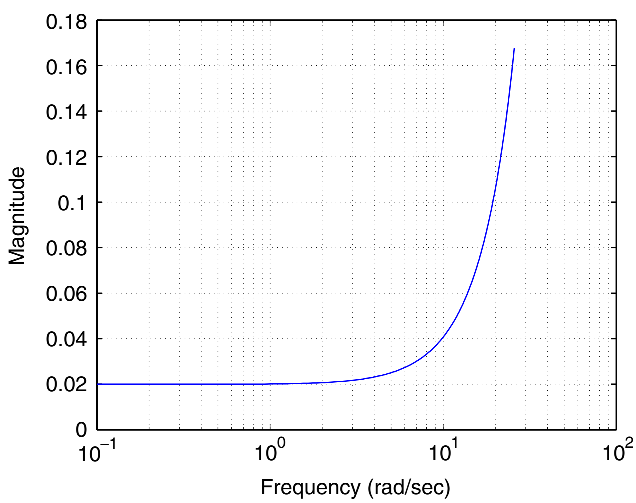

Figure 11. Closed-loop frequency response for the case of plant of relative degree one $W_p(s) = (s+1)/(s^2+s+1)$ and actuator of relative degree one $W_a(s) = 1/(0.01s+1)$, controlled by super-twisting algorithm.

DOI: 10.1002/rnc

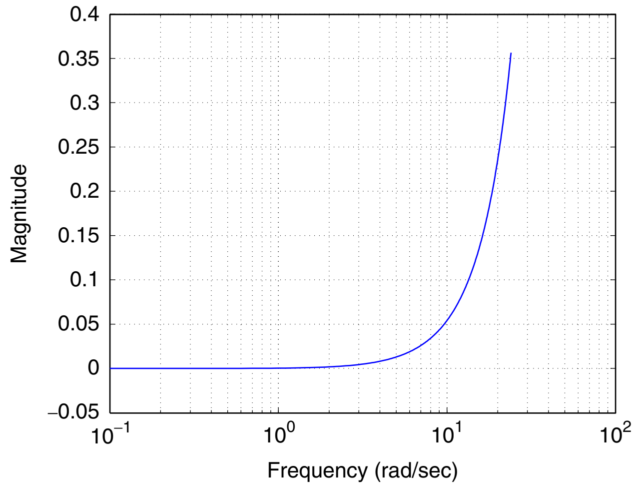

Figure 12. Closed-loop frequency response for the case of plant of relative degree one $W_p(s) = (s+1)/(s^2+s+1)$ and actuator of relative degree two $W_a(s) = 1/(0.0001s^2 + 0.01s + 1)$, controlled by super-twisting algorithm.

6. FREQUENCY-DOMAIN ANALYSIS OF DIFFERENTIATOR BASED ON SUPER-TWISTING ALGORITHM

Consider now a practical real-time differentiation problem. Let input signal $f(t)$ be a measurable locally bounded continuously differentiable function defined on $[0, \infty)$. Let us apply the super-twisting algorithm [5, 11] to keep $f(t) - y = 0$. The averaged control $u_0(t)$ would be the derivative of $f(t)$ if the plant is chosen to be the integrator. The algorithm itself is given by expression (13), the principal dynamics will be the integrator, and the parasitic dynamics will be denoted by the transfer function $W(s)$ [12]. Therefore, the plant dynamics (that would include the principal and parasitic dynamics) are given by the following transfer function: $W(s)/s$ (Figure 13). Compute the frequency response of the differentiator for $\gamma = 1.5$ and $\lambda = 1.1$. As per Equation (21), the transfer function of the differentiator (affected by parasitic dynamics $W(s)$) can be given as follows:

$$W_{f(t)-y}(s) = \frac{(k_{n1} + sk_{n2})sW(s)}{s^2 + (k_{n1} + sk_{n2})W(s)} \tag{22}$$

The frequency response of the differentiator (with the account of parasitic dynamics) is presented in Figure 14. One can see that at lower frequencies the effect of parasitic dynamics is small and the frequency response of the differentiator coincides with the one of the ideal differentiator. At higher frequencies, the effect of the parasitic dynamics is revealed as the downfall character of the magnitude response, which indicates that the differentiator has a limited bandwidth. The results of simulations of the super-twisting algorithm based differentiator [11] are presented in Table II. Those results closely match to the analytical results presented in Table I.

DOI: 10.1002/rnc

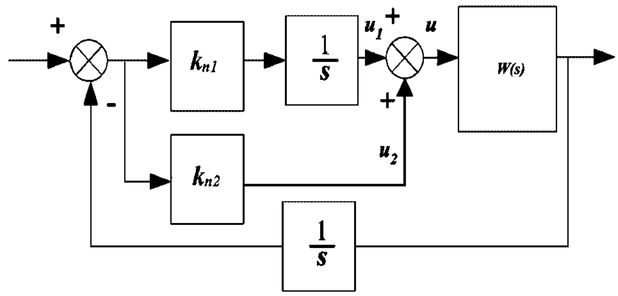

Figure 13. Linearized model of the differentiator ($W(s) = 1$ for ideal case).

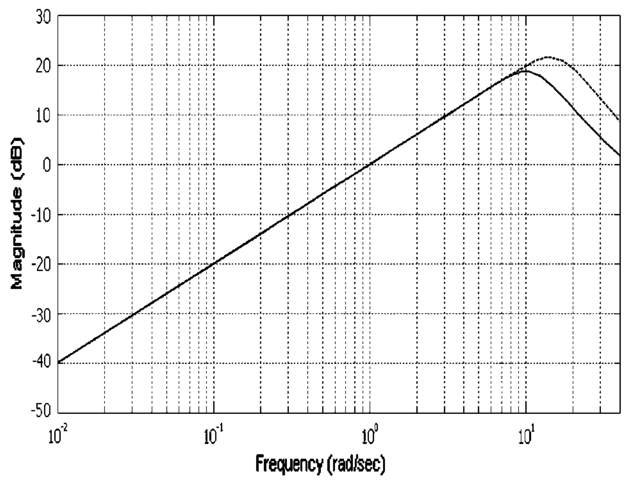

Figure 14. Frequency response of the differentiator with parasitic dynamics $W(s) = 1/(0.0001s^2 + 0.01s + 1)$ (lower line) and $W(s) = 1/(0.01s + 1)$ (upper line).

Table I. Results of DF analysis of differentiator based on super-twisting algorithm.

	DF analysis	
$W(s)$	$1/(0.01s+1)$	$1/(0.0001s^2+0.01s+1)$
Amplitude a_y	2.43×10^{-4}	7.75×10^{-4}
Frequency Ω (rad/s)	88.58	66.31
Equivalent gain k_{n1}	3929.8	1232.2
Equivalent gain k_{n2}	58.89	32.98

DOI: 10.1002/rnc

Table II. Results of simulations of differentiator based on super-twisting algorithm.

	Simulations	
$W(s)$	$1/(0.01s+1)$	$1/(0.0001s^2+0.01s+1)$
Amplitude a_y	2.45×10^{-4}	7.84×10^{-4}
Frequency Ω (rad/s)	87.69	65.80
Equivalent gain k_{n1}	3897.7	1218.0
Equivalent gain k_{n2}	58.65	32.79

7. CONCLUSIONS

The frequency-domain analysis of the SOSM control system response to external signals is analysed with the use of the concepts of linearization *via* 'chatter smoothing' and equivalent gain. It is shown that the systems exhibit closed-loop performance deterioration in the presence of parasitic dynamics. This performance deterioration is frequency dependent. At lower frequencies, the deterioration is smaller than at higher frequencies. Thus, the methodology of performance analysis of SOSM affected by parasitic dynamics is developed based on commonly used techniques of the transfer functions and frequency response.

REFERENCES

1. Tsypkin YZ. *Relay Control Systems*. Cambridge Press: U.K., 1984.
2. Utkin VI. *Sliding Modes in Optimization and Control Problems*. Springer: New York, 1992.
3. Boiko I. Analysis of sliding modes in the frequency domain. *International Journal of Control* 2005; **78**(13):969–981.
4. Emelyanov SV, Korovin SK, Levantovsky LV. Second order sliding modes in controlling uncertain systems. *Soviet Physics Doklady* 1986; **24**(4):63–68.
5. Levant A. Sliding order and sliding accuracy in sliding mode control. *International Journal of Control* 1993; **58**(6):1247–1263.
6. Boiko I, Fridman L, Castellanos MI. Analysis of second order sliding mode algorithms in the frequency domain. *IEEE Transactions on Automatic Control* 2004; **49**(6):946–950.
7. Boiko I, Fridman L. Analysis of chattering in continuous sliding-mode controllers. *IEEE Transactions on Automatic Control* 2005; **50**(9):1442–1446.
8. Gelb A, Vander Velde WE. *Multiple-Input Describing Functions and Nonlinear System Design*. McGraw-Hill: New York, 1968.
9. Atherton DP. *Nonlinear Control Engineering—Describing Function Analysis and Design*. Van Nostrand Company Limited: Workingham, Berks, U.K., 1975.
10. Burlington RS. *Handbook of Mathematical Tables and Formulas*. McGraw-Hill: New York, 1973.
11. Levant A. Robust exact differentiation via sliding mode technique. *Automatica* 1998; **34**:379–384.
12. Shtessel YB, Krupp DR, Shkolnikov IA. 2-sliding mode control for nonlinear plants with parametric and dynamic uncertainties. *AIAA Guidance, Navigation and Control Conference*, Denver, CO, 2000; 1–9.

DOI: 10.1002/rnc