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# Nonlocal stabilization via delayed relay control rejecting uncertainty in a time delay

L. Fridman[1,*,†], V. Strygin[2] and A. Polyakov[2]

[1] *National Autonomous University of Mexico, DEP-FI, UNAM, Edificio "A", Ciudad Universitaria, A. P. 70-256, C.P.04510, Mexico, DF, Mexico*
[2] *Department of Applied Mathematics, Voronezh State University, Universitetskaja pl.1, Voronezh 394693, Russia*

SUMMARY

Sufficient conditions for a robust relay delayed non-local stabilization of linear systems are found, which relate the upper bound of an uncertainty in a time delay and the maximum of the real part of system spectrum. Algorithm of delayed relay control gain adaptation for non-local stabilization is suggested. The proposed algorithm suppresses bounded uncertainties in the time delay: once this relay delayed control law for the upper bound of uncertainty in the time delay for given system is designed, we ensure non-local stabilization for all values of the time delay less than the upper bound even in the case of a variable delay.


KEY WORDS: input delay; relay control; semiglobal stability; uncertain systems; robust control

## 1. INTRODUCTION

Relay control systems are widely used due to the following main reasons:

- relay controllers suppress bounded uncertainties (see Reference [1]);
- there are control systems in which only sign of variables is observable (References [2–4]).

Time delay that usually takes place in relay and sliding mode control systems must be taken into account for system analysis and design (see for example Reference [1]). On the other hand, time delay does not allow to design the sliding mode control in the space of state variables. Moreover, in References [5, 6], it was shown that even in the simplest one-dimensional delayed relay control system only oscillatory solutions can occur. That is why the main directions in relay delayed control are:

(a) *The research of time delay compensation*: Pade approximation of delay that reduces the relay delay output tracking problem to the sliding mode control for non-minimum phase system was suggested in Reference [7]. Roh and Oh [8] designed the sliding mode control in the space of predictor variables (see also [9]). This approach allowed to solve eigenvalues assignment

*Correspondence to L. Fridman, National Autonomous University of Mexico, DEP-FI, UNAM, Edificio 'A', Ciudad Universitaria, A.P. 70-256, C.P. 04510, Mexico, D.F
† E-mail: lfridman@verona.fi-p.unam.mx

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problem without any restriction on time delay and spectral properties of open-loop system. However in References [10, 11], it was remarked that sliding mode control design in the space of predictor variables

- cannot compensate even the matching uncertainties;
- in the simplest case of square systems, if the dimensions of state space and control vector are the same, sliding mode design in the space of predictor variables can remove the uncertainties in the space of predictor variables but cannot guarantee compensation of uncertainties in the space of state variables.

Robustness properties of the Smith predictors with respect to uncertainties in time delay were studied in a series of papers (see for example References [12, 13]). The conditions of robustness of Smith predictors with respect to the uncertainty in the time delay are formulated by Furutani and Araki [13] in terms of stability margins.

(b) *Control of amplitudes of oscillations*: P.I. control algorithms for amplitude control for one-dimensional relay system with delay in the input was suggested in Reference [14].

Fridman *et al.* [5] have shown that any solution to the following equation:

$$\dot{x}(t) = \alpha x - p\,\mathrm{sign}[x(t-h)] \tag{1}$$

with the initial conditions

$$x(t) = \varphi(t), \quad t \in [-h, 0], \qquad |\varphi(0)| < p\frac{2 - e^{\alpha h}}{\alpha e^{\alpha h}} = r_0 \tag{2}$$

for all $t \in [T_0, \infty)$, $T_0 > 0$ is bounded:

$$|x| < p(e^{\alpha h} - 1)/\alpha = r_\infty \tag{3}$$

whenever stabilization condition

$$0 < \alpha h < \ln 2 \tag{4}$$

holds. It is important to remark that

- Condition (4) is sufficient and necessary condition relay delayed stabilization;
- The size of the domain of stabilization is proportional to the control gain $p$.

In References [5, 6] the following algorithm for controlling the motion amplitudes was developed: since after finite time all solutions coincide with the periodic solution, one can extrapolate the next zero for the periodic solution, and reduce the control gain near to zero of the periodic solution. This algorithm needs only the knowledge of the sign of state variable with delay but requires stabilization condition (4). This algorithm is valid for any constant delay satisfying condition (4) and does not depend on the value of delay. Stabilization condition (4) and algorithm for stabilization were generalized by Shustin [15] for linear second-order relay delay systems.

Strygin *et al.* [16] have generalized stabilization condition (4) for MIMO systems and proposed a delayed relay control algorithm, allowing to reach local stabilization of oscillations amplitudes for controllable systems.

In this paper, the algorithm of delayed relay switching of the control gain is suggested using the knowledge of solutions amplitudes at the delayed time moment. It allows to achieve non-local stabilization of amplitudes of the oscillations rejecting uncertainty in the time delay.

 

## 2. PROBLEM FORMULATION

Consider a linear system with delayed control and uncertainties of the form

$$\frac{dx}{dt} = Ax + Bu(x(t - h(t))) + f(x, t) \tag{5}$$

where $x \in R^n$ is the vector of state space, $A, B$ are real matrices, $h(t)$ $(0 < h(t) \leqslant h_0$ for all $t \in [0, \infty))$ is the continuous function describing an uncertainty in the time delay, $u \in R^m$ is the relay control vector bounded in every bounded domain $\|x\| \leqslant D$, $x \in R^n$, the function $f(x, t)$ is continuous on $t$, smooth on $x$ and corresponds to the presence of an uncertainty in the model of the plant.

Let us denote as $x(t)$ the solution to system (5) with the initial condition:

$$x(t) = \varphi(t),^{1} \quad (-h_0 \leqslant t \leqslant 0), \ \varphi(t) \in \mathbf{C}[-h_0, 0] \tag{6}$$

The existence and uniqueness for the solutions to the Cauchy problem (5) and (6) is proved in Reference [17]. We will consider the problem of nonlocal stabilization for the system (5). As it was shown in [5], the amplitude of oscillations in relay delay systems is proportional to the relay control gain. That is why to achieve nonlocal stabilization for system (5) we need to use sufficiently big initial relay control gain in order to stabilize the solutions of (5) with sufficiently big initial conditions. Consequently, the desired relay control law should depend on the size of the initial domain. On the other hand, due to the oscillatory properties of relay delay systems solutions [5], one can conclude that it is impossible to achieve asymptotic stability for the solutions to system (5) via relay controllers with a finite number of switches. That is why it is impossible to ensure semiglobal stabilization in the sense of Isidory [18] via relay delayed control law with finite number of switching in control gain. In the paper the following modification of the semiglobal stability notion is used.

*Definition 1*

The zero solution to system (5) is said to be practically semiglobal stabilizable, if for any $R > \varepsilon > 0$ there are $\delta > 0$, $D = D(R, \varepsilon) > 0$, relay delay control $u(t - h(t))$ and time moment $T > 0$, such that the inequalities $\|\varphi(0)\| < R$ and $\|f(t, x)\| < \delta\varepsilon \ \forall x : \|x\| < D$ imply the inequality

$$\sup_{t \in [T, \infty)} \|x(t)\| < \varepsilon$$

In this paper, relay delayed control law and non-local stability conditions are proposed ensuring practical semiglobal stabilization for the zero solution to system (5). The proposed control law requires:

- amplitude of solutions at delayed time moment;
- upper bound of the time delay;
- upper bound of initial conditions;
- size of the desired neighbourhood of the zero solution to system (5).

[1] The values of relay delayed control for $t \in [0, h(0)]$ will be defined below through the value of the initial function $\varphi(t), t \in [-h_0, 0]$ but for the state variable $x$ only the restriction at the initial point $x(0) = \varphi(0)$ is necessary.

 

The paper is organized as follows. The properties of relay delay control for the simplest scalar case are introduced in Section 3. In Section 4 a modification of relay control is suggested for controlled system having two unstable complex conjugate roots. In Section 5 the design procedure for relay delayed algorithm is described. In Section 6 proposed algorithm is compared with the linear control algorithm. All proofs are given in the appendix.

## 3. SCALAR CASE

To explain the basics of proposed algorithm, let us return first to Equation (1) and conditions (2) and (4). Let us find conditions ensuring that the magnitude of steady oscillations $r_\infty$ is less than the magnitude of the initial conditions $r_0$:

$$r_\infty/r_0 < \frac{(e^{\alpha h}-1)/\alpha}{(2-e^{\alpha h})/(\alpha e^{\alpha h})} < 1$$

and

$$e^{2\alpha h} - e^{\alpha h} < 2 - e^{\alpha h}, \quad \alpha h < \tfrac{1}{2}\ln 2$$

Consider the scalar control system with the continuous uncertain time delay $h(t), 0 < h(t) < h_0$ and smooth bounded uncertainty $f(x,t)$

$$\dot{x} = \alpha x + u(x(t-h(t))) + f(x,t) \tag{7}$$

$$x(t) = \phi(t), \quad t \in [-h_0, 0] \tag{8}$$

Suppose that

$$\alpha h_0 < L = \frac{1}{2}\ln 2 \tag{9}$$

Now, the idea of the desired control algorithm is the following:

- Consider the amplitude of stabilization domain for each value of relay delayed control gain as the amplitude of the initial conditions for the next step;
- by decreasing the control gain, enter into a smaller neighbourhood of zero.

Let us choose $0 < \varepsilon < R < +\infty$, and define the constants $\alpha', \delta^{\max}, \gamma, N$ as follows:

(1) the interval $I_{\alpha h_0} = (\alpha e^{\alpha h_0}/(2-e^{\alpha h_0}), \alpha/(e^{\alpha h_0}-1))$ is not empty due to (9) and we can select

$$\alpha' \in I_{\alpha h_0} \tag{10}$$

(2) $\delta^{\max} = \alpha(\alpha'(2-e^{\alpha h_0})/\alpha e^{\alpha h_0} - 1)$.
(3) $\gamma = (1 + (\alpha' + \delta^{\max})/\alpha)e^{\alpha h_0} - (\alpha' + \delta^{\max})/\alpha$.
(4) $\log_3(R/\varepsilon) \leqslant N < \log_3(R/\varepsilon) + 1$.

Suppose that there exists such $\delta > 0$ that $\delta < \delta^{\max}$ and $\alpha' \pm \delta \in I_{\alpha h_0}$, and

$$|f(x,t)| < \delta\varepsilon \quad \text{for all } x : |x| < 3^N \varepsilon\gamma \tag{11}$$

 

Consider the system of neighbourhoods of zero: $U_k = \{x\colon \|x\| < v_k\}, v_k = 3^{k-1}\varepsilon\gamma,\ k = 1, \dots, N,$ where the indicator function

$$H_{v_k}(|x|) = \begin{cases} 1 & \text{for } |x| > v_k \\ 0 & \text{for } |x| \leqslant v_k \end{cases} \tag{12}$$

whose zero value indicates that $x \in U_k$. In this case the desired control law takes the form

$$u(t-h(t)) = u_{R\varepsilon}(x(t-h(t))) = -\alpha'\varepsilon\left(1 + 2\sum_{n=1}^{N} 3^{n-1} H_{v_n}(|x(t-h(t))|)\right)\mathrm{sign}[x(t-h(t))] \tag{13}$$

*Remark 1*

The radius of neighbourhoods $U_k$ and the control gain increase thrice in such a step. This factor $r$ can be chosen, for example, as the solution of some optimal problem. However, if $r > 3$, then we need to make condition (9) stronger. On the other hand, for $r < 3$ the number of relay control elements grows up.

This control law has the following properties:

$1^\circ$. The set of control values is $M = \{\pm\alpha'\varepsilon, \pm 3\alpha'\varepsilon, \dots, \pm 3^k\alpha'\varepsilon, \dots, \pm 3^N\alpha'\varepsilon\}$.
$2^\circ$. If $v_k < |x(t-h(t))| < v_{k+1}$, then, $|u(x(t-h(t)))| = 3^k\alpha'\varepsilon$.
$3^\circ$. If $|x(t-h(t))| < v_{k+1}$, then, $|u(x(t-h(t)))| \leqslant 3^k\alpha'\varepsilon\,(1 \leqslant k \leqslant N-1)$.
$4^\circ$. $|u(x(t-h(t)))| \leqslant 3^N\alpha'\varepsilon\,(\forall t \geqslant 0)$.
$5^\circ$. $v_k < 3^k\varepsilon\ \forall k \in \{1, 2, \dots, N+1\}$.
$6^\circ$. $v_{k+1} - 3^k\varepsilon(\alpha' - \delta)/\alpha = 3^k\varepsilon(\gamma - (\alpha' - \delta)/\alpha) < 0\ \forall k \in \{0, 1, 2, \dots, N\}$. It is easy to see that

$$\begin{aligned}\gamma - \frac{\alpha' - \delta}{\alpha} &= \left(1 + \frac{\alpha'}{\alpha}\right)e^{\alpha h_0} + \frac{e^{\alpha h_0}}{\alpha}\delta^{\max} - 2\frac{\alpha'}{\alpha} + \frac{\delta - \delta^{\max}}{\alpha} \\ &< \left(1 + \frac{\alpha'}{\alpha}\right)e^{\alpha h_0} + \frac{e^{\alpha h_0}}{\alpha}\delta^{\max} - 2\frac{\alpha'}{\alpha} = 0\end{aligned}$$

*Theorem 1*

For any initial condition $\phi(t) : |\phi(0)| \leqslant R$, there exists such time moment $T > 0$, that for any $t > T$ we have $|x(t)| < \varepsilon$.

## 4. STABILIZATION OF SECOND ORDER SYSTEM WITH UNSTABLE COMPLEX CONJUGATE EIGENVALUES

Consider the system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} u_1(x(t-h(t)), y(t-h(t))) \\ u_2(x(t-h(t)), y(t-h(t))) \end{pmatrix} + \begin{pmatrix} f_1(x, y, t) \\ f_2(x, y, t) \end{pmatrix} \tag{14}$$

where $(x(t), y(t))^{\mathrm{T}} \in R^2$ is the state vector, $\alpha > 0$ and $\beta > 0$, $(u_1, u_2)^{\mathrm{T}} \in R^2$ is the control vector, $f_1(x, t), f_2(x, t)$ are uncertainties and $h(t)\ 0 < h(t) \leqslant h_0$, for all $t \in [0, \infty)$.

 

Choose $0<\varepsilon<R$ and suppose that $0<\beta h_0<\pi/4$ and $\alpha h_0<M$, where

$$M = \max_{t\in[0,\pi/4-\beta h_0]} \frac{t}{6\sqrt{2}}\cos\left(t+\frac{\pi}{4}+\beta h_0\right)$$

Define the constants $\alpha', \delta^{\max}, N$ as follows:

(1) $\alpha' \in (0, \pi/4h_0 - \beta) : \alpha < \alpha' \cos(\alpha' h_0 + \pi/4 + \beta h_0)/\, 6\sqrt{2}$.
(2) $\delta^{\max} = 1/2h_0(\arccos 6\sqrt{2}\alpha/\alpha' - (\alpha' + \beta)h_0 - \pi/4)$.
(3) $\log_3(R/\varepsilon) \leqslant N < \log_3(R/\varepsilon) + 1$.

Assume that there exists such constant $\delta$ that $\delta<\delta^{\max}$ and $\alpha+\delta<\alpha'/6\sqrt{2}\cos(\alpha' h_0+\pi/4+\beta h_0+2\delta h_0)$ ensuring the estimation of the uncertainty:

$$\sqrt{f_1^2(x,t)+f_2^2(x,t)}<\delta\varepsilon \quad \text{for } \forall x,y : \sqrt{x^2+y^2}<3^N\varepsilon\sqrt{2}$$

Now the desired control vector takes the form

$$\begin{pmatrix} u_1(x(t-h(t)), y(t-h(t))) \\ u_2(x(t-h(t)), y(t-h(t))) \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \cos\beta h_0 & -\sin\beta h_0 \\ \sin\beta h_0 & \cos\beta h_0 \end{pmatrix}\begin{pmatrix} u_{R\varepsilon}(x(t-h(t))) \\ u_{R\varepsilon}(y(t-h(t))) \end{pmatrix} \tag{15}$$

where $u_{R\varepsilon}$ has been already considered in the previous section

$$u_{R\varepsilon}(\cdot) = -\alpha'\varepsilon\left(1+2\sum_{i=1}^{N} 3^{i-1} H_{v_i}(|\cdot|)\right)\operatorname{sign}[\cdot]$$

where $v_k = 3^{k-1}\varepsilon$.

*Theorem 2*
For any initial condition $\phi(t)$: $\sqrt{\phi_1^2(0)+\phi_2^2(0)}<R$, there exists time moment $T>0$: $\sqrt{x^2(t)+y^2(t)}<\varepsilon$, for all $t>T$.

*Remark 2*
Suppose that $x+\mathrm{i}y=\rho(t)\mathrm{e}^{\mathrm{i}\varphi(t)}$ and $f_1(x,y,t)+\mathrm{i}f_2(x,y,t)=q(t)\mathrm{e}^{\mathrm{i}\psi(t)}$, where $x,y$ are real numbers. Then, Equation (14) under control (15) may be written in the form

$$\dot{\rho} = \alpha\rho(t) - \frac{\alpha' r(t)}{2}\varepsilon\cos(k(t)+\beta h_0-\varphi(t)) + q(t)\cos(\psi(t)-\varphi(t)) \tag{16}$$

$$\dot{\varphi}(t) = \beta - \frac{\alpha' r(t)}{2\rho(t)}\sin(k(t)+\beta h_0-\varphi(t)) + \frac{q(t)}{\rho(t)}\sin(\psi(t)-\varphi(t)) \tag{17}$$

where

$$r(t) = \sqrt{3^{2n}+3^{2m}} \quad \text{if } v_n<|y(t-h(t))|\leqslant v_{n+1}, v_m<|x(t-h(t))|\leqslant v_{m+1} \tag{18}$$

$$k(t) = \begin{cases} \operatorname{arctg}(\frac{3^n}{3^m}) & \text{if } v_n<y(t-h(t))\leqslant v_{n+1}, v_m<x(t-h(t))\leqslant v_{m+1} \\ -\operatorname{arctg}(\frac{3^n}{3^m}) & \text{if } -v_{n+1}\leqslant y(t-h(t))<-v_n, v_m<x(t-h(t))\leqslant v_{m+1} \\ \pi-\operatorname{arctg}(\frac{3^n}{3^m}) & \text{if } v_n<y(t-h(t))\leqslant v_{n+1}, -v_{m+1}\leqslant x(t-h(t))<-v_m \\ -\pi+\operatorname{arctg}(\frac{3^n}{3^m}) & \text{if } -v_{n+1}\leqslant y(t-h(t))<-v_n, -v_{m+1}\leqslant x(t-h(t))<-v_m \end{cases} \tag{19}$$

 

*Remark 3*

(1°). $v_k \leqslant \varepsilon r(t) \leqslant \sqrt{2}v_k$ if $\sqrt{2}v_k < \rho(t-h(t)) < v_{k+1}$.

(2°). From the definition of $\varphi(t)$ it follows that $\varphi(t) \in (-\infty, +\infty)$. This means that $\varphi(t) = \varphi_z \times (t) + 2\pi z,\ -\pi < \varphi_z(t) < \pi$, where $z \in Z$. On the other hand, $-\pi < k(t) < \pi$ and consequently $|k \times (t) - \varphi(t-h(t))| \leqslant \pi/4 + 2\pi|z|,\ z \in Z$.

## 5. CONTROL ALGORITHM FOR MIMO CASE

### 5.1. *Structure of projectors*

Assume that the spectrum $\sigma(A)$ of the matrix $A$ has the following structure $\sigma(A) = \sigma_+ \cup \sigma_-$, where $\sigma_+$ and $\sigma_-$ are the sets of matrix $A$ eigenvalues with the positive and negative real part, respectively. Then the state space $E = R^n$ could be represented in the form of a direct sum $E = E_+ \oplus E_-$, where $E_+$ and $E_-$ are the invariant subspaces with respect to $A$. Consider two projectors $P$ and $Q$, transforming $P: R^n \to E_+, Q: R^n \to E_-$. Suppose that

(1) $\dim E_+ = \mathrm{rank}(PB)$

(2) $\sigma_+ = \{\lambda_i\}_{i=1}^{l} \bigcup \{\alpha_j \pm i\beta_j\}_{j=1}^{v},\ \lambda_i h_0 < L, L = \frac{1}{2}\ln 2$

$$0 < \beta_i < \frac{\pi}{4h_0},\quad \alpha_i h_0 < M_j,\quad M_j = \max_{t \in [0, \pi/4 - \beta_j h_0]} \frac{t}{6\sqrt{2}} \cos\left(t + \frac{\pi}{4} + \beta_i h_0\right) \tag{20}$$

and all the eigenvalues from $\sigma_+$ are simple.

Assume that the spectrum $\sigma_+$ of matrix $A$ satisfies condition (1) and the following representations hold

$$Ah_i = \lambda_i h_i,\quad (i = \overline{1, l})$$

and

$$Ah_{l+2j-1} = \alpha_j h_{l+2j-1} - \beta_j h_{l+2j}$$

$$Ah_{l+2j} = \beta_j h_{l+2j-1} + \alpha_j h_{l+2j},\ (j = \overline{1, v})$$

Consider the conjugate matrix $A^*$ and suppose that $f_1, f_2, \ldots, f_m,\ m = 2v + l$ are the following eigenvectors:

$$A^* f_i = \lambda_i f_i,\quad (i = \overline{1, l})$$

and

$$A^* f_{l+2j-1} = \alpha_j f_{l+2j-1} + \beta_j f_{l+2j},\quad A^* f_{l+2j} = -\beta_j f_{l+2j-1} + \alpha_j f_{l+2j}\quad (j = \overline{1, v})$$

Then, the above projectors can be written in the following form:

$$Px = \sum_{i=1}^{m} (x, g_i) h_i,\qquad Qx = x - Px,\quad g_i = \frac{f_i}{\|f_i\|},\ (i = \overline{1, l})$$

$$g_{l+2j-1} = c_{11}^{j} f_{l+2j-1} + c_{12}^{j} f_{l+2j}$$

$$g_{l+2j} = c_{21}^{j} f_{l+2j-1} + c_{22}^{j} f_{l+2j},\quad (j = \overline{1, v})$$

 

where

$$c_{11}^{j} = \frac{\|f_{l+2j}\|^2}{\|f_{l+2j-1}\|^2 \cdot \|f_{l+2j}\|^2 - (f_{l+2j-1}, f_{l+2j})^2}$$

$$c_{12}^{j} = c_{21}^{j} = \frac{(f_{l+2j-1}, f_{l+2j})}{\|f_{l+2j-1}\|^2 \cdot \|f_{l+2j}\|^2 - (f_{l+2j-1}, f_{l+2j})^2}$$

$$c_{22}^{j} = \frac{\|f_{l+2j-1}\|^2}{\|f_{l+2j-1}\|^2 \cdot \|f_{l+2j}\|^2 - (f_{l+2j-1}, f_{l+2j})^2}$$

Suppose that $e_j = (\alpha_{j1}, \alpha_{j2}, \ldots, \alpha_{jn})^{\mathrm{T}}$ is some basis in $R^n$ space such that $e_j = h_j$ $(j = \overline{1, m})$. Let us introduce the new basis into $R^n$ in the form

$$\bar{e}_i = h_i, \quad (i = \overline{1, m}), \qquad \bar{e}_{m+j} = e_{m+j} - Pe_{m+j}, \quad (j = \overline{1, n-m})$$

The matrix of transition from the canonical basis to the new basis $\{\bar{e}_i\}$ has the form

$$G = (\bar{e}_1, \ldots, \bar{e}_n) = \begin{pmatrix} \alpha_{11} & \alpha_{21} & \cdots & \alpha_{m1} & \alpha_{m+1\,1} - \sum_{i=1}^{m} \alpha_{i1}(e_{m+1}, g_i) & \cdots \\ \alpha_{12} & \alpha_{22} & \cdots & \alpha_{m2} & \alpha_{m+1\,2} - \sum_{i=1}^{m} \alpha_{i2}(e_{m+1}, g_i) & \cdots \\ \vdots & \vdots & \vdots & \vdots & & \\ \alpha_{1n} & \alpha_{2n} & \cdots & \alpha_{mn} & \alpha_{m+1\,n} - \sum_{i=1}^{m} \alpha_{in}(e_{m+1}, g_i) & \cdots \\ & & & & & \end{pmatrix}$$

and

$$J = G^{-1}AG = \begin{pmatrix} A^{+} & 0 \\ 0 & A^{-} \\ & \end{pmatrix}$$

In this case, the following representation holds

$$G^{-1}B = \begin{pmatrix} B^{+} \\ B^{-} \end{pmatrix}$$

where $B^{+} = m \times m$, $B^{-} = n - m \times m$ and $rank(B^{+}) = m$.

Denoting $z = (z_1, z_2)^{\mathrm{T}} = G^{-1}x$, one can rewrite system (5) as follows:

$$\begin{aligned} \dot{z}_1 &= A^{+}z_1 + B^{+}u + g_1(z, t) \\ \dot{z}_2 &= A^{-}z_2 + B^{-}u + g_2(z, t) \end{aligned} \tag{21}$$

where $g(z,t) = (g_1(z,t), g_2(z,t))^{\mathrm{T}} = G^{-1}f(t, Gz)$,

$$A^{+} = \begin{pmatrix} \lambda_1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \lambda_l & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \cdots & 0 & \alpha_1 & -\beta_1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \beta_1 & \alpha_1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & \alpha_v & -\beta_v \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & \beta_v & \alpha_v \end{pmatrix}$$

### 5.2. *Control design*

Algorithm of practical semiglobal stabilization

(1) Fix $R > 0$ and $\varepsilon > 0$
(2) Choose $\alpha_i{}' > 0$ and $\delta_i^{\max} > 0$ such that

$$\alpha_i{}' \in I_{\lambda_i h_0}, \quad \delta_i^{\max} = \lambda_i\left(\alpha_i{}' \frac{2 - \mathrm{e}^{\lambda_i}}{\lambda_i \mathrm{e}^{\lambda_i}} - 1\right), \quad (i = 1, \ldots, l)$$

where

$$I_{\lambda_i h_0} = \left(\frac{\lambda_i \mathrm{e}^{\lambda_i h_0}}{2 - \mathrm{e}^{\lambda_i h_0}}, \frac{\lambda_i}{\mathrm{e}^{\lambda_i h_0} - 1}\right)$$

(3) Define $\alpha_{l+j}{}' > 0$ and $\delta_{l+j}^{\max} > 0$ such that

$$\alpha_{l+j}{}' \in \left[0, \frac{\pi}{4h_0} - \beta_j\right]: \ \alpha_j < \frac{\alpha_{l+j}{}'}{6\sqrt{2}} \cos\left(\frac{\pi}{4} + \alpha_{l+j}{}' h_0 + \beta_{l+j} h_0\right)$$

$$\delta_j^{\max} = \frac{1}{2h_0}\left(\arccos \frac{6\sqrt{2}\alpha_j}{\alpha_j{}'} - (\alpha_j{}' + \beta_j)h_0 - \frac{\pi}{4}\right) \quad j = 1, \ldots, v$$

(4) Matrix $A^{-}$ is stable. Consequently, there exists $\mu > 0$ and $C > 0$ such that $\|\mathrm{e}^{QAt}\| \leqslant C\mathrm{e}^{-\mu t}$. Put $\Delta^{\max} = \max \delta_i^{\max}$, $R_0 = \|G^{-1}\|R$ and $\varepsilon_0 \in (0, \varepsilon V)$, where

$$V = \min\left\{\frac{1}{2\|G\|}, \frac{\mu}{4\|G\|C(\|\bar{\alpha}'\| \cdot \|B^{-}[B^{+}]^{-1}\| + \Delta^{\max})}\right\}$$

$\bar{\alpha}' = (\alpha_1{}', \ldots, \alpha_l{}', \alpha_{l+1}{}', \alpha_{l+1}{}', \ldots, \alpha_{l+v}{}', \alpha_{l+v}{}')^{\mathrm{T}}$.

(5) Define $v_i^n = \gamma_i 3^{n-1}\varepsilon_0$, where $\gamma_i = (1 + (\alpha_i{}' + \delta_i^{\max})/\lambda_i)\mathrm{e}^{\lambda_i h_0} - (\alpha_i{}' + \delta_i^{\max})/\lambda_i$, $i = 1, \ldots, l$ and $v_{l+j}^n = 3^{n-1}\varepsilon_0$, $(j = 1, \ldots, 2v)$.

(6) $\log_3(R_0/\varepsilon_0) \leqslant N < \log_3(R_0/\varepsilon_0) + 1$.

Suppose that

$$\|f(x,t)\| < \frac{\delta_k \varepsilon_0}{\|G^{-1}\|} \quad \forall x : \|x\| < D \tag{22}$$

where $\delta_k < \delta_k^{\max}$, $(k = 1, \ldots, l + v)$ and $\alpha_i{}' \pm \delta_i \in I_{\lambda_i h_0}$, $(i = 1, .., l)$, $\alpha_j + \delta_{l+j} < \alpha_{l+j}{}'/6\sqrt{2}\cos(\alpha_{l+j}{}' h_0 + \pi/4 + \beta_j h_0 + 2\delta_{l+j} h_0)$, $(j = 1, \ldots, v)$ and $D = (7/4 + 3/2C)3^N \varepsilon$.

Design the delayed switching surface $\sigma_i(y_i(t - h(t)))$ in the form

$$\sigma_i(y_i(t - h(t))) = -\bar{\alpha}_i{}' \varepsilon_i \left( 1 + 2 \sum_{n=1}^{N} 3^{n-1} H_{v_i^n}(|y_i(t - h(t))|) \right) \operatorname{sign}[y_i(t - h(t))]$$

where $i = 1, 2, \ldots, m = l + 2v$.

Define the switching surface vector such as

$$\sigma(z_1(t - h(t))) = (\sigma_1(z_{11}(t - h(t))), \sigma_2(z_{12}(t - h(t))), \ldots, \sigma_m(z_{1m}(t - h(t))))^{\mathrm{T}}$$

where $z = (z_1, z_2)^{\mathrm{T}} = G^{-1}x$. Then, the desired control vector in the form

$$u(x(t - h(t))) = [B^+]^{-1} \begin{pmatrix} I^l & 0 & \cdots & 0 \\ 0 & S_1 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & \cdots & 0 & S_v \end{pmatrix} \sigma(G^{-1}x(t - h(t)) \tag{23}$$

where $I^l$ is the identity $l \times l$ matrix and

$$S_j = \frac{1}{2} \begin{pmatrix} \cos \beta_j & -\sin \beta_j \\ \sin \beta_j & \cos \beta_j \end{pmatrix}, \quad j = 1, \ldots, v$$

*Theorem 3*

If conditions (1) and (2) hold, then system (5) is practically semiglobally stabilizable.

*Remark 4*

Condition (1) is more restrictive than the usual conditions of stabilization. To satisfy (1) it is necessary that the vector control has the same dimension as the number of unstable poles in the open-loop system. On the other hand condition (1) needs sometimes less than controllability of the pair $(A, b)$.

Consider two simple examples:

$$\dot{x} = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u \tag{24}$$

and

$$\dot{x} = \begin{pmatrix} -1 & 0 \\ 1 & 2 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u \tag{25}$$

It is easy to see that system (24) is controllable, since $\operatorname{rank}[Ab, b] = 2$. However, the stability condition $\dim E_+ = \dim(Pb)$ is not satisfied, since $\dim E_+ = \dim R^2 = 2 \neq \operatorname{rank}(Pb) = 1$. At the same time, system (25) is uncontrolled, but condition (1) is true, since $\dim(E_+) = 1 = \operatorname{rank}(Pb)$. Moreover, it is easy to see that for SISO systems, when the matrix $A$ has only one simple unstable root, condition (1) is equivalent to the controlability of the pair $\{A, b\}$.

## 6. NUMERICAL EXAMPLE

Consider the following unstable system:

$$\dot{x} = \begin{pmatrix} -7.1869 & -2.0400 & -6.4796 & -4.2994 \\ 3.8908 & 1.1200 & 3.4189 & 2.2182 \\ 0.3914 & 0.0066 & 0.2752 & 0.1754 \\ 1.2945 & 0.5000 & 1.7170 & 1.1516 \end{pmatrix} x + \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \\ 4 & 3 & 2 \end{pmatrix} u(x(t-h(t))) + f(x,t) \tag{26}$$

where $h(t) = 0.5 + 0.5\sin(30t)$, $(0 \leqslant h(t) \leqslant h_0 = 1)$,

$$f(x(t),t) = (0.0130\sin(t), -0.0063\sin(t), -0.0602\sin(t), 0.0557\sin(t))^{\mathrm{T}} \tag{27}$$

The open-loop system has the following eigenvalues $\lambda_1 = 0.34, \lambda_{2,3} = 0.01 \pm 0.1i, \lambda_4 = -5$. We will consider solution to system (26) with the initial condition

$$\phi(t) = (\cos(t), \cos(2t), \sin(3t), \sin(4t))^{\mathrm{T}} \tag{28}$$

### 6.1. Relay control design

Let us design the relay delayed control law for system (26). Put $R = 2.5$, $\varepsilon = 0.1$. Let us choose

$$\alpha_1{}' = 0.83 \in \left(\frac{\lambda_1 \mathrm{e}^{\lambda_1}}{2-\mathrm{e}^{\lambda_1}}, \frac{\lambda_1}{\mathrm{e}^{\lambda_1}-1}\right)$$

$$\alpha_2{}' = \alpha_3{}' = 0.34 \in \left[0, \frac{\pi}{4h_0} - 0.1\right] : 0.01 < \frac{\alpha_2{}'}{6\sqrt{2}}\cos\left(\frac{\pi}{4} + \alpha_2{}' + 0.1\right)$$

$$v_1^k = 3^{k-1}\varepsilon_0\gamma, \quad v_2^k = v_3^k = 3^{k-1}\varepsilon_0$$

where $\varepsilon_0 = 0.01 \in (0, \varepsilon V_i)$ (see (4) in Section 5.2), $\gamma = 2.4 = (1 + (\alpha_1{}' + \delta_1^{\max})/\lambda_1)\mathrm{e}^{\lambda_1} - (\alpha_1{}' + \delta_1^{\max})/\lambda_1$ and $\delta^{\max} = \lambda_1(\alpha_1{}'(2 - \mathrm{e}^{\lambda_1}/\lambda_1\mathrm{e}^{\lambda_1}) - 1)$. In this case the desired relay delay control can be designed as follows:

$$u(t-h(t)) = \begin{pmatrix} 1.4930 & 1.5137 & -0.4382 \\ -4.0657 & -3.8731 & 0.9320 \\ 3.4226 & 2.8463 & -0.5022 \end{pmatrix} \begin{pmatrix} \sigma_1(y_1(t-h(t))) \\ \sigma_2(y_2(t-h(t))) \\ \sigma_3(y_3(t-h(t))) \end{pmatrix} \tag{29}$$

where $\sigma_i(y_i(t-h(t)) = -0.01\alpha_i{}'(1 + 2\sum_{k=1}^{6} 3^{k-1} H_{v_i^k}(|y_i(t-h(t))|))\mathrm{sign}[y_i(t-h(t))]$,

$$y(t-h(t)) = \begin{pmatrix} 0.3852 & 0 & 2.3115 & 1.5410 \\ -0.1877 & 0.0816 & -2.5958 & -0.3673 \\ -1,7713 & -2.7264 & -1.7632 & -1.1346 \\ 1.6388 & 0.4682 & 1.4047 & -0.9364 \end{pmatrix} x(t-h(t))$$

### 6.2. Linear control

The pair $\{A, b_1\}$ is controllable. From Ackermann's formula it follows

$$u_1 = [0, 0, 0, 1][b_1, Ab_1, A^2b_1, A^3b_1]^{-1}(A - l_1I^4)(A - l_2I^4)(A - l_3I^4)(A - l_4I^4)x(t - h(t))$$

For $l_1 = -0.7,\ l_2 = -0.8,\ l_3 = -0.9,\ l_4 = -1$ the linear control vector has the form

$$u(t - h(t)) = \begin{pmatrix} 0.4 & 0.7316 & 0.5423 & -0.2492 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} x(t - h(t)) \tag{30}$$

Figure 1 shows the $x_1(t)$ co-ordinate of the solution to system (26), driven by a linear controller, leaving the $\varepsilon = 0.1$ vicinity of zero but at the same time the corresponding solution to system (26), driven by relay delay controller, oscillating inside the desired neighbourhood of zero. Moreover, the simulation results show that if we increase the parameters $l_i$ the stability neighbourhood will grow. On the other hand, decreasing the parameters $l_i$ results in system unstability. The simulation results show that the linear control algorithm does not allow to achieve the desired $\varepsilon$-neighbourhood of zero. Increasing the parameters $l_i$ implies increasing of the stability neighborhood. Decreasing the parameters $l_i$ implies system instability.

### 6.3. System with nonlinear uncertainty

Consider system (26) with the nonlinear uncertainty

$$f_1(x(t), t) = \left(0.0130\left(\frac{x_1(t)}{5}\right)^3, -0.0063\left(\frac{x_1(t)}{5}\right)^3,\ -0.0602\left(\frac{x_1(t)}{5}\right)^3,\ 0.0557\left(\frac{x_1(t)}{5}\right)^3\right)^{\mathrm{T}} \tag{31}$$

The designed relay delayed control ensures a practically semiglobal stabilization of system (26). System (26) with uncertainties (31) under linear control is unstable (see Figure 2).

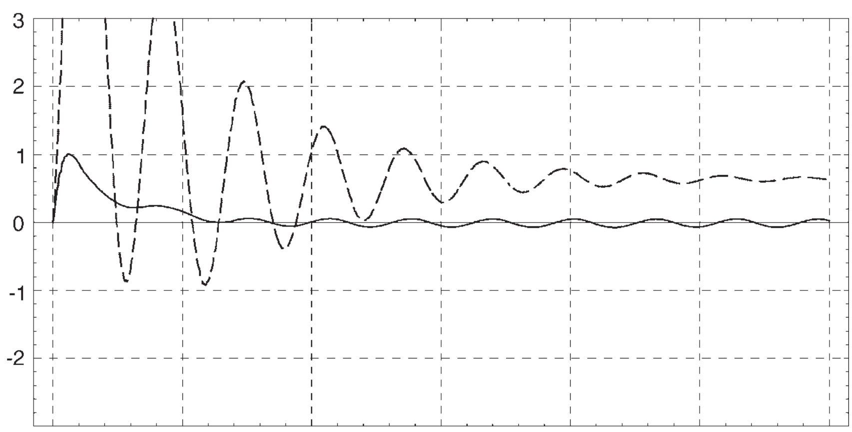


Figure 1. System state $x_4(t)$ (dotted and continuous lines describe the solutions for linear controller and relay delayed control law, respectively).

 

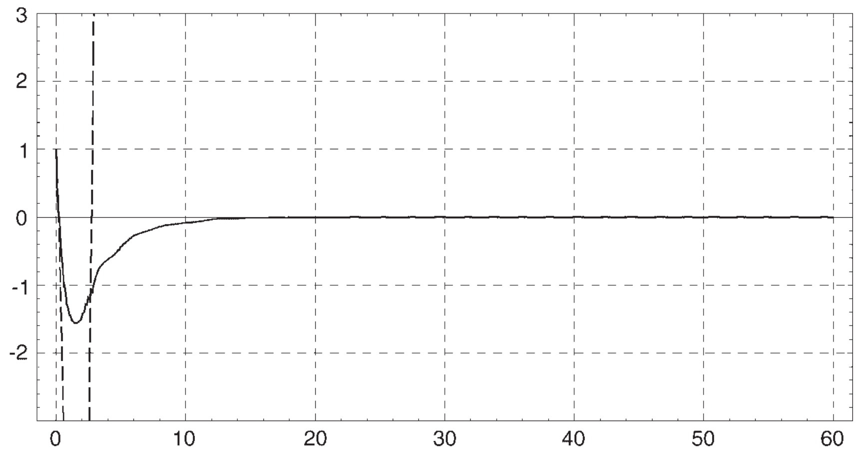


Figure 2. $x_1(t)$ co-ordinate of system (26) solution with nonlinear uncertainty (31) (dotted and continuous lines describe $x_1(t)$ for linear controller and relay delayed control law, respectively).

## 7. CONCLUSION

An algorithm of delayed relay control gain adaptation for the practical semiglobal stabilization is suggested that requires delayed information about amplitude of oscillations and upper bound of the time delay. The proposed algorithm rejects bounded uncertainties in time delay: once we have designed the control law for the upper bound of the uncertainty in the time delay for a given system, we can ensure the practical semiglobal stabilization of zero solution for any values of the time delay less than the upper bound, even in the case where the delay is variable.

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## APPENDIX A

### *A.1. Proofs for the scalar case*

(1) *Staying in the neighbourhood*

*Lemma A1*

If for $k \in \{0, 1, \ldots, N-1\}$ there exists $T \geqslant 0$ such that $|x(t)| < v_{k+1}$ for all $t \in [T - h_0, T]$ and $|x(T)| \leqslant 3^k \varepsilon$, then $|x(t)| \leqslant v_{k+1}$ for all $t \geqslant T$.

*Proof*

Suppose by contradiction that there exists $T' \geqslant T$ such that $|x(T')| > v_{k+1}$. Then, from the condition $|x(T)| \leqslant 3^k \varepsilon$ it follows that there exists such $t^* > T : |x(t^*)| = 3^k \varepsilon$ and $|x(t)| > 3^k \varepsilon$, for all $t \in (t^*, T']$, and moreover there exists $T^* > t^* : |x(T^*)| = v_{k+1}$ and $|x(t)| < v_{k+1}, \forall t \in [t^*, T^*)$. Now we can suppose that $x(T') > v_{k+1}$. Then $x(t^*) = 3^k \varepsilon$ and $x(T^*) = v_{k+1}$.

 

Let us show that $T^* - t^* \geqslant h_0$ estimates the upper bound of $x(t)$ for $t \in [t^*, T^*]$. Taking into account that $|x(t)| < v_{k+1}$ for $t \in [T - h_0, T]$ and $|x(t)| < v_{k+1}$ for $t \in [T, T^*]$, one has $|u(t - h \times (t))| < \alpha' 3^k \varepsilon$ for $t \in [T, T^*]$. Then

$$\dot{x} \leqslant \alpha x + (\alpha' + \delta) 3^k \varepsilon$$

$$x(t^*) = 3^k \varepsilon$$

and

$$x(t) \leqslant 3^k \varepsilon \left( \left( 1 + \frac{\alpha' + \delta}{\alpha} \right) e^{\alpha(t - t^*)} - \frac{\alpha' + \delta}{\alpha} \right)$$

For $t = T^*$ one has

$$\begin{aligned} x(T^*) = v_{k+1} &= 3^k \varepsilon \left( \left( 1 + \frac{\alpha' + \delta^{\max}}{\alpha} \right) e^{\alpha h} - \frac{\alpha' + \delta^{\max}}{\alpha} \right) \\ &\leqslant 3^k \varepsilon \left( \left( 1 + \frac{\alpha' + \delta}{\alpha} \right) e^{\alpha(T^* - t^*)} - \frac{\alpha' + \delta}{\alpha} \right) \end{aligned}$$

which yields $T^* - t^* \geqslant h_0$. Let us note that in this case from $x(t) > 3^k \varepsilon > v_k$ for $t \in [t^*, T']$ one has $\operatorname{sign}[x(t - h(t))] = 1$ and $H_{v_n}(|x(t - h(t))|) = 1$ for $n = \overline{1, k}$, $t \in [t^* + h_0, T']$, which means

$$u(t) \geqslant -3^k \alpha' \varepsilon, \quad \forall t \in [t^* + h_0, T'] \tag{A1}$$

Now

$$\dot{x}(T^*) \leqslant \alpha v_{k+1} - 3^k (\alpha' - \delta) \varepsilon < 0 \tag{A2}$$

This means that at $t = T^*$ the function $x(t)$ is decreasing on $[T^*, T']$, and $x(T') < v_{k+1}$. This is a contradiction in the initial assumption. □

*Corollary A1*

*Proof*
If $|x(0)| \leqslant R$, then

$$|x(t)| \leqslant v_{N+1} = 3^N \varepsilon \gamma \quad \forall t \geqslant 0$$

It is obvious, that condition $|x(t)| \leqslant v_{k+1}$ for $t \in [T - h_0, T]$ and $k \in \{1, \dots, N-1\}$ is equivalent to $|u(t)| \leqslant \alpha' 3^k \varepsilon$ $\forall t \in [T, T + h_0]$. Lemma A1 is true even for $k = N$. Taking into account that $N \geqslant \log_3 R/\varepsilon$, one has $|x(0)| \leqslant R \leqslant 3^N \varepsilon$. □

(2) *Existence of the next zero*

*Lemma A2*
If $|x(0)| \leqslant R$, then for all $t \geqslant 0$ there exists $T \geqslant t : x(T) = 0$.

*Proof*
Suppose in contradiction that there exists $t^*$ such that for all $t \geqslant 0$ $x(t) \neq 0$. Consider the case when $x(t) > 0$. Then, for $t > t^* + h_0$ we will have $\operatorname{sign}[x(t - h(t))] = 1$. Equation (7) takes the form

$$\dot{x} = \alpha x - \alpha' \varepsilon \left( 1 + 2 \sum_{n=1}^{N} 3^{n-1} H_{v_n}(|x(t - h(t))|) \right) + f(x, t)$$

 

Now

$$\dot{x} \leqslant \alpha x - (\alpha' - \delta)\varepsilon \tag{A3}$$

Let us show that $x(t) > v_1$ for all $t \geqslant t^* + h_0$. Suppose that it is not true and $\exists t^1 \geqslant t^* + h_0 : x \times (t^1) \leqslant v_1$. Then inequality (A3) yields

$$x(t) \leqslant \left( v_1 - \frac{(\alpha' - \delta)\varepsilon}{\alpha} \right) \mathrm{e}^{\alpha(t - t^1)} + \frac{(\alpha' - \delta)\varepsilon}{\alpha} = v(t)$$

Taking into account that the first coefficient before the exponent in the last equation is negative, one can conclude that $v(t)$ is a decreasing function and there exists $\overline{t^0} : v(\overline{t^0}) = 0$, then $x(\overline{t^0}) \leqslant 0$, which contradicts with condition $x(t) > 0$. This means that $x(t) > v_1$. Analogously, we can prove that $x(t) > v_2$, etc. Finally, we will have the inequality $x(t) > v_{N+1}$, which contradicts Corollary A1. □

(3) *Reduction of the amplitude of oscillations*

*Lemma A3*
If $|x(t)| \leqslant v_{k+1}$, then for all $t \geqslant T$, there exists $T_1$, such that for all $t \geqslant T_1$
$|x(t)| \leqslant v_k$

*Proof*
(1) *Consider the case*

$$\alpha' + \delta \leqslant \frac{1}{3} \frac{\alpha}{\mathrm{e}^{\alpha h_0} - 1}$$

Then, from condition (3) and 5° it follows that $\dot{x} \leqslant \alpha x + (\alpha' + \delta)\varepsilon 3^k$. Suppose that $x \times (t^*) = 0$, $t^* > T + h_0$ for all $t \geqslant T + h_0$. Then

$$x(t) \leqslant \frac{(\alpha' + \delta)\varepsilon 3^k}{\alpha} \mathrm{e}^{\alpha(t - t^*)} - \frac{(\alpha' + \delta)\varepsilon 3^k}{\alpha}$$

At $t = t^* + h_0$ we will have

$$\begin{aligned} x(t^* + h_0) &\leqslant \frac{3^k(\alpha' + \delta)\varepsilon}{\alpha} \mathrm{e}^{\alpha h_0} - \frac{3^k(\alpha' + \delta)\varepsilon}{\alpha} \\ &= 3^k \varepsilon (\alpha' + \delta) \frac{\mathrm{e}^{\alpha h_0} - 1}{\alpha} \leqslant 3^k \varepsilon \frac{1}{3} \frac{\alpha}{\mathrm{e}^{\alpha h_0} - 1} \frac{\mathrm{e}^{\alpha h_0} - 1}{\alpha} = 3^{k-1}\varepsilon \end{aligned}$$

For $t \in [t^*, t^* + h_0]$ one has $|x(t)| \leqslant 3^{k-1}\varepsilon \leqslant v_k$ and $x(t^* + h_0) \leqslant 3^{k-1}\varepsilon$.

From Lemma A1 it follows that for all $t \geqslant t^*$ $x(t) \leqslant v_k$. Analogously, we can have $x(t) \geqslant -v_k$.

(2) *Consider the case*

$$\alpha' + \delta > \frac{1}{3} \frac{\alpha}{\mathrm{e}^{\alpha h_0} - 1}$$

Let $t = t^*$ is the zero of the solution $x(t)$ such that

$$t^* \geqslant T + 2h_0 - \frac{1}{\alpha} \ln\left( 1 + \frac{\alpha}{3(\alpha' + \delta)} \right) \tag{A4}$$

Then for $t \geqslant t^*$ one has

$$x(t) \leqslant \frac{(\alpha' + \delta)\varepsilon 3^k}{\alpha} \mathrm{e}^{\alpha(t - t^*)} - \frac{(\alpha' + \delta)\varepsilon 3^k}{\alpha} = v(t)$$

 

Consequently, the function is increasing on $v(t)$ on $[t^*, t^1]$ for some $t^1 \geqslant t^*$: and $v(t^1) = 3^{k-1}\varepsilon$. From the definition of function $v(t)$ it follows:

$$\frac{(\alpha' + \delta)\varepsilon 3^k}{\alpha} e^{\alpha(t^1 - t^*)} - \frac{(\alpha' + \delta)\varepsilon 3^k}{\alpha} = 3^{k-1}\varepsilon$$

$$\frac{\alpha' + \delta}{\alpha} e^{\alpha(t^1 - t^*)} - \frac{\alpha' + \delta}{\alpha} = \frac{1}{3}$$

and

$$t^1 - t^* = \frac{1}{\alpha} \ln\left(1 + \frac{\alpha}{3(\alpha' + \delta)}\right) \tag{A5}$$

Consider now two cases

($\alpha$) $|x(t)| \leqslant v_k$ for $t \in [T + h_0, t^*]$, then taking into account (A4) and (A5), one has $t^1 - T \geqslant 2h_0$, which means that for $t \in [T + h_0, t^1]$, $|x(t)| < v_k$, $|x(t^1)| \leqslant 3^{k-1}\varepsilon$, and one can conclude that Lemma A3 follows from Lemma A1.

($\beta$) Suppose that there exists $\bar{t} \in [T, t^*] : x(\bar{t}) > v_k$. Then from the continuity of $x(t)$ it follows that $\exists t^2 \geqslant T : x(t^2) = v_k$ and $x(t) \leqslant v_k$, for all $t \in [t^2, t^*]$.

Consequently, $|x(t)| < v_k$, $\forall t \in [t^2, t^1]$. Let us estimate the lower band of $x(t)$ for $t \in [t^2, t^*]$. Then the differential inequality $\dot{x} \geqslant \alpha x - (\alpha' + \delta)\varepsilon 3^k$ with the initial conditions $x(t^2) = v_k$, implies that

$$x(t) \geqslant \left(v_k - \frac{(\alpha' + \delta)\varepsilon 3^k}{\alpha}\right) e^{\alpha(t - t^2)} + \frac{(\alpha' + \delta)\varepsilon 3^k}{\alpha}$$

Let us rewrite this inequality at $t = t^*$ in the form

$$0 \geqslant \left(v_k - \frac{(\alpha' + \delta)\varepsilon 3^k}{\alpha}\right) e^{\alpha(t^* - t^2)} + \frac{(\alpha' + \delta)\varepsilon 3^k}{\alpha}$$

Then

$$0 \geqslant (\varepsilon 3^{k-1}((\alpha + \alpha' + \delta)e^{\alpha h_0} - \alpha' - \delta) - (\alpha' + \delta)\varepsilon 3^k)e^{\alpha(t^* - t^2)} + (\alpha' + \delta)\varepsilon 3^k$$

and

$$0 \geqslant ((\alpha + \alpha' + \delta)e^{\alpha h_0} - 4(\alpha' + \delta))e^{\alpha(t^* - t^2)} + 3(\alpha' + \delta)$$

$$(2(\alpha' + \delta) + (2 - e^{\alpha h_0})(\alpha' + \delta) - \alpha e^{\alpha h_0})e^{\alpha(t^* - t^2)} \geqslant 3(\alpha' + \delta)$$

Now from (10) it follows that $(2 - e^{\alpha h_0})(\alpha' + \delta) - \alpha e^{\alpha h_0} > 0$ and

$$t^* - t^2 \geqslant \frac{1}{\alpha} \ln \frac{3(\alpha' + \delta)}{(4 - e^{\alpha h_0})(\alpha' + \delta) - \alpha e^{\alpha h_0}}$$

Taking into account the last inequality, one has

$$\begin{aligned} t^1 - t^2 &= t^1 - t^* + t^* - t^2 \\ &\geqslant \frac{1}{\alpha} \ln\left(1 + \frac{\alpha}{3(\alpha' + \delta)}\right) + \frac{1}{\alpha} \ln \frac{3(\alpha' + \delta)}{(4 - e^{\alpha})(\alpha' + \delta) - \alpha e^{\alpha}} \\ &= \frac{1}{\alpha} \ln \frac{3(\alpha' + \delta) + \alpha}{(4 - e^{\alpha})(\alpha' + \delta) - \alpha e^{\alpha}} \end{aligned}$$

 

It is easy to show that

$$\frac{1}{\alpha}\ln\frac{3(\alpha'+\delta)+\alpha}{(4-\mathrm{e}^{\alpha h_0})(\alpha'+\delta)-\alpha\mathrm{e}^{\alpha h_0}}\geqslant h_0$$

Then $t^1-t^2\geqslant h_0$.

(4) Proof of Theorem 1 $\square$

(1) Let us show that there exists such time moment $t=T_1$, that for all $t\geqslant T_1$ $|x(t)|<v_1$. Following, Corollary A1 one has $|x(t)|\leqslant v_{N+1}$ for $t\geqslant 0$. Lemma A3 yields that there exists such time moment that $t=t^1$ that $|x(t)|\leqslant v_N$ $\forall t\geqslant t^1$. Analogously for $N$th step for all $t\geqslant t^n$ one has

$$|x(t)|\leqslant v_1 \tag{A6}$$

(2) Inequality (A2) holds only if for $t\geqslant t^n+h_0$ one has $|u(t)|\leqslant\alpha'\varepsilon$. Moreover, from Lemma A2 it follows that $\exists T\geqslant t^n+h_0 : x(T)=0$. Let us show that $|x(t)|\leqslant\varepsilon$, $\forall t\geqslant T$. Suppose in contradiction that, if $\exists\bar{t}\geqslant T : x(\bar{t})>\varepsilon$, then $\exists T^*\geqslant T : x(T^*)=0$, and $x(t)>0\ \forall t\in(T^*,\bar{t}]$. Let us find the upper bound of $x(t)$ for $t\in[T^*,\bar{t}]$. Then the inequality

$$\dot{x}\leqslant\alpha x+(\alpha'+\delta)\varepsilon, x(T^*)=0$$

implies that

$$x(t)\leqslant\frac{(\alpha'+\delta)\varepsilon}{\alpha}\mathrm{e}^{\alpha(t-T^*)}-\frac{(\alpha'+\delta)\varepsilon}{\alpha}$$

and for the time moment $T^*+h_0$ the last inequality takes the form

$$x(T^*+h_0)\leqslant\frac{(\alpha'+\delta)\varepsilon}{\alpha}\mathrm{e}^{\alpha h_0}-\frac{(\alpha'+\delta)\varepsilon}{\alpha}=\varepsilon(\alpha'+\delta)\frac{\mathrm{e}^{\alpha h_0}-1}{\alpha}\leqslant\varepsilon$$

This means that there exists the time moment $T^*+h_0$

$$\dot{x}=\alpha x-(\alpha'-\delta)\varepsilon$$

but $x(T^*+h_0)\leqslant\varepsilon\leqslant v_1$. Consequently, from (6°) it follows:

$$\dot{x}(T^*+h_0)\leqslant\alpha v_1-(\alpha'-\delta)\varepsilon<0$$

This means that for $t\geqslant T^*+h_0$ the solution $x(t)$ will decrease until the next switching moment. Now one can conclude that at some time moment $x(t_*)=0$. This equality contradicts with condition $x(t)>0\ \forall t\in(T^*,\bar{t}]$.

*A.2. Proofs for the second-order system with unstable complex eigenvalues*

(1) *Staying in the neighbourhood*

*Lemma A4*

If there exists such time moment $T>0$ that $\rho(t)\leqslant v_k$ for all $t\in[T-h_0,T]$, then $\rho(t)<v_k$ for all $t>T$.

*Proof*

Let us suppose in contradiction that there exists $T'>T : \rho(T')=v_k$. Then there are two cases

(1) $\sqrt{2}v_{k-1}<\rho(t)\leqslant v_k$ for all $t\in[T,T']$. Let us denote $t^*=T-h_0$. It is easy to see that $T'-t^*>h_0$.

(2) There exists $t^*>T : \rho(t^*)=\sqrt{2}v_{k-1}$ and $\sqrt{2}v_{k-1}<\rho(t)\leqslant v_k$ for all $t\in(t^*,T']$.

Let us show that in this case the inequality $T' - t^* > h_0$ is correct.

Since $\varepsilon r(t) \leqslant \sqrt{2} v_{k-1}$ for $\rho(t - h(t)) : \sqrt{2} v_{k-1} < \rho(t - h(t)) \leqslant v_k$ (see (1°), Remark 2) then for $t > t^*$ we have

$$\dot{\rho} \leqslant \alpha\rho + \frac{\alpha' v_{k-1}}{\sqrt{2}} + \delta\varepsilon < \alpha\rho + \frac{(\alpha' + 2\delta) v_{k-1}}{\sqrt{2}}$$

$$\rho(t^*) = \sqrt{2} v_{k-1}$$

Then

$$\rho(T') = v_k \leqslant \left( \sqrt{2} v_{k-1} + \frac{(\alpha' + 2\delta) v_{k-1}}{\sqrt{2}\alpha} \right) e^{\alpha(T' - t^*)} - \frac{(\alpha' + 2\delta) v_{k-1}}{\sqrt{2}\alpha}$$

Hence, we have $T' - t^* > h_0$.

To study the term

$$\cos[k(t) + \beta h_0 - \varphi(t)]$$

on the right-hand side of (16) we should integrate Equation (17) on the interval $[t - h \times (t), t], t \in [t^* + h_0, T']$. It is easy to see that

$$\varphi(t) = \varphi(t - h(t)) + \beta h(t) - \alpha' \xi(t) + \chi(t) \tag{A7}$$

where

$$\xi(t) = \frac{1}{2} \int_{t-h(t)}^{t} \frac{r(\tau)}{\rho(\tau)} \sin[k(\tau) + \beta - \varphi(\tau)] \, d\tau \tag{A8}$$

$$\chi(t) = \int_{t-h(t)}^{t} q(\tau) \rho^{-1}(\tau) \sin[\psi(\tau) - \varphi(\tau)] \, d\tau \tag{A9}$$

Obviously,

$$|\xi(t)| < \frac{1}{2} h(t) < h_0 \quad \text{and} \quad |\chi(t)| < 2\delta h_0$$

Now substituting (A7) into (16) we have

$$\rho' = \alpha\rho - \frac{\varepsilon r(t)}{2} \alpha' \cos[k(t) - \varphi(t - h(t)) + \beta(h_0 - h(t)) + \alpha' \xi(t) + \chi(t)] + q(t) \cos(\psi(t) - \varphi(t)) \tag{A10}$$

From (2°) we obtain

$$\cos[k(t) - \varphi(t - h(t)) + \beta(h_0 - h(t)) + \alpha' \xi(t) + \chi(t)] \geqslant \cos(\alpha' h_0 + 2\delta h_0 + \beta h_0 + \pi/4) \tag{A11}$$

Hence, for $t \in [t^* + h_0, T']$

$$\rho' \leqslant \alpha\rho - \frac{\alpha' v_{k-1}}{2} \cos(\alpha' h_0 + 2\delta h_0 + \beta h_0 + \pi/4) + \delta\varepsilon$$
$$\leqslant v_k \left[ \alpha + \delta - \frac{\alpha'}{6} \cos(\alpha' h_0 + 2\delta h_0 + \beta h_0 + \pi/4) \right] < 0$$

That is why equality $\rho(T') = v_k$ cannot be achieved.

 

(2) *Transition into the smaller neighbourhood*

*Lemma A5*
If $\rho(t) < v_k$ for all $t > T$ and there exists a time moment $t^* > T + h_0$ such that $\rho(t^*) = v_{k-1}/\sqrt{2}$, then, $\rho(t) < v_{k-1}$ for all $t > t^*$.

*Proof*
Suppose in contradiction that there exists $T' > t^* : \rho(T') = v_{k-1}$. In this case

$$\dot{\rho} < \alpha\rho + \frac{(\alpha' + 2\delta)v_{k-1}}{\sqrt{2}}$$

$$\rho(t^*) = v_{k-1}/\sqrt{2}$$

Then

$$v_k = \rho(T') \leqslant \left( v_{k-1}/\sqrt{2} + \frac{(\alpha' + 2\delta)v_{k-1}}{\sqrt{2}} \right) e^{\alpha(T' - t^*)} - \frac{(\alpha' + 2\delta)v_{k-1}}{\sqrt{2}}$$

Hence

$$T' - t^* > \frac{1}{\alpha} \ln \frac{\sqrt{2}\alpha + \alpha' + 2\delta}{\alpha + \alpha' + 2\delta}$$

If $\rho(t) < v_{k-1}$ for all $t \in [T, t^*]$, then $T' - T > h_0$. From Lemma A4 it follows that $\rho(t) < v_{k-1}$ for all $t > T + h_0$. This means that there exists a time moment $T_0 > T$ such that $\rho(T_0) = v_{k-1}$ and $\rho(t) < v_{k-1}$ for all $t \in (T_0, t^*]$.

$$\dot{\rho} \geqslant \alpha\rho - \frac{(\alpha' + 2\delta)v_{k-1}}{\sqrt{2}}, \quad \rho(T_0) = v_{k-1}$$

This means that

$$v_{k-1}/\sqrt{2} = \rho(t^*) \geqslant \left( v_{k-1} - \frac{(\alpha' + 2\delta)v_{k-1}}{\sqrt{2}\alpha} \right) e^{\alpha(T' - t^*)} + \frac{(\alpha' + 2\delta)v_{k-1}}{\sqrt{2}\alpha}$$

Consequently,

$$t^* - T_0 > \frac{1}{\alpha} \ln \frac{\alpha' + 2\delta - \alpha}{\alpha' + 2\delta - \sqrt{2}\alpha}$$

Finally,

$$T' - T_0 = T' - t^* + t^* - T_0 > \frac{1}{\alpha} \ln \frac{\sqrt{2}\alpha + \alpha' + 2\delta}{\alpha + \alpha' + 2\delta} \cdot \frac{\alpha' + 2\delta - \alpha}{\alpha' + 2\delta - \sqrt{2}\alpha} > h_0$$

Let us prove the last inequality

$$\sqrt{2}\alpha^2 + \alpha(\alpha' + 2\delta)(\sqrt{2} - 1)\frac{e^{\alpha h_0} + 1}{e^{\alpha h_0} - 1} - (\alpha + 2\delta)^2 > 0$$

$$\sqrt{2}\alpha^2 + (\alpha' + 2\delta)\left( \alpha(\sqrt{2} - 1)\frac{e^{\alpha h_0} + 1}{e^{\alpha h_0} - 1} - (\alpha + 2\delta) \right) > 0$$

 

Now

$$\alpha(\sqrt{2}-1)\frac{e^{\alpha h_0}+1}{e^{\alpha h_0}-1} > \alpha + 2\delta$$

It is easy to see that

$$\mu(\alpha) = \alpha h_0 \frac{e^{\alpha h_0}+1}{e^{\alpha h_0}-1}$$

is an increasing function of $\alpha$ and consequently $\min_{\alpha\in[0,M/h_0]} \mu(\alpha) = 2$. Then

$$\alpha h_0(\sqrt{2}-1)\frac{e^{\alpha h_0}+1}{e^{\alpha h_0}-1} \geqslant 2(\sqrt{2}-1) > \pi/4 > h_0(\alpha' + 2\delta) \qquad \square$$

(3) *Reduction of the amplitudes of oscillations*

*Lemma A6*

If $\rho(t) < v_k$ for all $t > T$ then there exists a time moment $T' > T + h_0$, such that $\rho(t) < v_{k-1}$ for all $t > T'$.

*Proof*

Suppose in contradiction that for any $t > T + h_0$ there exists $t^* > t : \rho(t^*) \geqslant v_{k-1}$.

Let us show that there exists $t^* > T + h_0$: $\rho(t^*) = v_{k-1}/\sqrt{2}$. Suppose in contradiction that: $v_{k-1}/\sqrt{2} < \rho(t) \leqslant v_k$ for all $t > T + h_0$. In this case

$$\varphi(t) = \varphi(t - h(t)) + \beta h(t) - \alpha'\xi(t) + \chi(t), \quad (\forall t > T + h_0)$$

where

$$|\xi(t)| < h_0 \quad \text{and} \quad |\chi(t)| < 2\delta h_0$$

Then

$$\dot{\rho} \leqslant \alpha\rho - \frac{\alpha' v_{k-2}}{2}\cos(\alpha' h_0 + 2\delta h_0 + \beta h_0 + \pi/4) + \delta v_{k-1}$$

It is easy to see that if $\rho(t) \leqslant \sqrt{2}v_{k-1}$, then $\dot{\rho} < 0$ and there exists $t^* > T + h_0$: $\rho(t^*) = v_{k-1}/\sqrt{2}$. So we have $\rho(t) \geqslant \sqrt{2}v_{k-1}$ for all $t > T + h$. However, in this case $\varepsilon r(t) > v_{k-1}/\sqrt{2}$ and

$$\dot{\rho} < (\alpha + \delta)v_k - \frac{\alpha' v_{k-1}}{2}\cos(\alpha' h_0 + 2\delta h_0 + \beta h_0 + \pi/4) < 0$$

This means that there exists $t^* > T + h_0$ such that $\rho(t^*) = v_{k-1}/\sqrt{2}$. $\square$

## *A.3. Proof of Theorem 2*

(1) Let us show that there exists $\rho(t) < \sqrt{2}v_{N+1}$ for all $t > 0$.

It is easy to see that $\varepsilon r(t) \leqslant \sqrt{2}v_N$ for any $\rho(t)$ (see Remark 2).

$$\dot{\rho} \leqslant \alpha\rho + \frac{(\alpha' + 2\delta)v_N}{\sqrt{2}}$$

$$\rho(0) = v_{N+1}$$

Then

$$\rho(h) < \left(v_{N+1} + \frac{(\alpha' + \delta)v_N}{\sqrt{2}\alpha}\right)e^{\alpha h} - \frac{(\alpha' + \delta)v_N}{\sqrt{2}\alpha}$$

$$= v_{N+1}\left(e^{\alpha h_0} + \frac{\alpha' + \delta}{3\sqrt{2}}\frac{e^{\alpha h_0} - 1}{\alpha}\right) < \sqrt{2}v_{N+1}$$

(2) Let us show that there exists a time moment $T_1 > 0$ such that $\rho(t) < v_{N+1}$ for all $t > T_1$. Suppose the opposite. In this case, $\rho(t) > v_{N+1}/\sqrt{2}$ for all $t > 0$. Otherwise, we can find time moment $T_1$ (see Lemma A5).

Let us integrate Equation (17) over the interval $[t - h(t), t]$ for $t > h_0$

$$\varphi(t) = \varphi(t - h(t)) + \beta h(t) - \alpha'\xi(t) + \chi(t)$$

where

$$|\chi(t)| < h_0, \quad \text{and} \quad |\xi(t)| < 2\delta h_0$$

Then

$$\dot{\rho} < (\alpha + \delta)\sqrt{2}v_{N+1} - \frac{\alpha' v_N}{2}\cos(\pi/4 + \alpha' h_0 + 2\delta h_0 + \beta h_0) < 0$$

(3) Now from Lemma A6 we have $\rho(t) < v_N$ etc. On the $N$th step we will have $\rho(t) < v_1 = \varepsilon$.

*A.4. Practical semiglobal stabilization of system via designed control*

Suppose that the control vector $u(t - h(t))$ was designed in accordance with the algorithm described in Section 5.

Since $f(t, x)$ satisfies Condition (22) and $|z_i(0)| < R_i$ then from Theorems 1 and 2 one can conclude that there exists $T_i > 0$ such that $|z_{1i}(t)| < \varepsilon_0$, $(i = \overline{1, m})$.

Let us denote $T^{\max} = \max T_i$. Then according to the control property (3°) we have $|\sigma_i(y_i \times (t - h(t)))| \leqslant \alpha_i{}'\varepsilon_0$ for any $t > T^{\max} + h_0$. In Section 5.2 it was proposed that

$$\|e^{QAt}\| \leqslant Ce^{-\mu t}$$

Consequently, for all $t > T^{\max} + h_0$ from (21) we have

$$\|z_2(t)\| \leqslant Ce^{-\mu(t - T^{\max} - h_0)}\|z_2(T^{\max} + h_0)\| + C\frac{M}{\mu}(1 - e^{-\mu(t - T^{\max} - h_0)}) \quad \text{(A12)}$$

where $M$ is a positive constant. Then, there exists time moment $T' > T^{\max} + h_0$ such that

$$\|z_2(t)\| < 2C\frac{M}{\mu}$$

It is easy to see that

$$M \leqslant \varepsilon_0\,(\|\bar{\alpha}'\| \cdot \|B^-[B^+]^{-1}\| + \Delta^{\max})$$

where $\bar{\alpha}' = (\alpha_1', \ldots, \alpha_l{}', \alpha_{l+1}{}', \alpha_{l+1}{}', \ldots, \alpha_{l+v}{}', \alpha_{l+v}{}')^{\mathrm{T}}$, and $\Delta^{\max} = \max \delta_i^{\max}$.

Since the parameter $\varepsilon_0$ has been chosen such that

$$\varepsilon_0 < \frac{\varepsilon\mu}{4C\|G\|(\|\alpha'\| \cdot \|B^-\| \cdot \|[B_0^+]^{-1} + \Delta^{\max})}$$

 

we have

$$M < \frac{\mu\varepsilon}{4C\|G\|}$$

Then, for all $t > T'$

$$\|x(t)\| = \|Gz(t)\| \leqslant \|G\|(\|z_1(t)\| + \|z_2(t)\|)$$
$$< \|G\|\left(\varepsilon_0 + 2C\frac{M}{\mu}\right) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon$$

Let us show that

$$x(t) \in U_{\max}, \quad \forall t > 0$$

where $U_{\max} = \{x \in R^n : \|x\| < D = (7/4 + 3/2C)\varepsilon 3^N\}$. From Corollary 1 and Theorem 2 (see Section A.3) we have

$$|z_{1\,i}(t)| < 3^N \varepsilon_0 \gamma_i < 3^{N+1}\varepsilon_0 \quad (i = 1, \ldots, l)$$

and

$$|z_{1\,l+k}| < 3^N \varepsilon_0 \sqrt{2} < 3^{N+1}\varepsilon_0 \quad (k = 1, \ldots, 2v)$$

Taking into account (A12) one has

$$\|z_2(t)\| \leqslant C\mathrm{e}^{-\mu t}\|z_2(0)\| + C\frac{M_{\max}}{\mu}(1 - \mathrm{e}^{-\mu t})$$

where

$$M_{\max} \leqslant 3^N \varepsilon_0 (\|\bar{\alpha}'\| \cdot \|B^-[B^+]^{-1}\| + \Delta^{\max}) < 3^N \frac{\varepsilon\mu}{4\|G\|C}, \quad t > 0$$

Then

$$\|x(t)\| \leqslant \|G\|(\|z_1(t)\| + \|z_2(t)\|) \leqslant \|G\|(3^{N+1}\varepsilon_0 + CR_0 + CM_{\max}/\mu)$$
$$< \|G\|\left(3^{N+1}\frac{\varepsilon}{2\|G\|} + C3^{N+1}\varepsilon_0 + \frac{1}{4\|G\|}\varepsilon 3^N\right)$$
$$= 3^N\varepsilon(3/2 + 3/2C + 1/4) < (7/4 + 3/2C)3^N\varepsilon = D$$

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