INT. J. CONTROL, 2003, VOL. 76, NO. 8, 770–780

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Stabilization of amplitude of oscillations via relay delay control

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Time delay does not allow realizing an ideal sliding mode, but implies oscillations in the state space. It is shown that relay delay controllers allow us to achieve stabilization for amplitude of oscillations suppressing uncertainties in a time delay even in the case when the time delay is variable. Sufficient conditions for a relay delay stabilization are found. The obtained results are illustrated in the example of the relay delay stabilization for the inverted pendulum.

1. Introduction

Relay control systems are widely used due to the following main reasons:

- relay controllers suppress bounded uncertainties (see Utkin 1992);
- there are such control systems where only sign of variables is observable (see Choi and Hedrick 1996, Li and Yurkovich 1999).

Time delay that usually take place in relay and sliding mode control systems must be taken into account for system analysis and design (see for example Utkin *et al.* 1999). On the other hand, time delay does not allow to design the sliding mode control in the state space. Moreover, Fridman *et al.* (2002) have shown that even in the simplest one-dimensional delayed relay control system only oscillatory solutions can occur. This is why the main directions in relay delayed control are as follows.

1.1. *The research of time delay compensation*

Pade approximation of delay reducing the relay delay output tracking problem to the sliding mode control for nonminimum phase system was suggested by Shtessel *et al.* (2002). Roh and Oh (1999) designed the sliding mode control in the space of predictor variables (see Gouaisbaut *et al.* 1999, Richard *et al.* 2001). This approach allowed us to solve the eigenvalues assignment problem without any restriction on time delay and spectral properties of the open loop system. But Sing (2001) and Fridman *et al.* (2001) remarked that sliding mode control design in the predictor variable space:

- cannot compensate even the matching uncertainties;
- in the simplest case, when the dimensions of the state space and the control vector are the same, sliding mode design in the predictor variable space suppresses the uncertainties in the predictor variable space but cannot guarantee full compensation of the uncertainties in the state variable space.

Robustness properties of Smith predictors with respect to uncertainties in the time delay was studied by Palmor (1980) and Furutani and Araki (1998). The conditions of robustness of Smith predictors with respect to uncertainties in the time delay are formulated by Furutani and Araki (1998) in terms of the stability margins.

1.2. *Control of amplitudes of oscillations*

P.I. delayed relay control algorithm for amplitude of oscillations control applied to a one-dimensional system with delay in the input was suggested by Akian *et al.* (1997).

Fridman *et al.* (2002) have shown that any solution of the equation

$$\dot{x}(t) = \alpha x - p\,\mathrm{sign}[x(t-h)]$$

with the initial conditions

$$|\varphi(0)| < p\frac{2-\mathrm{e}^{\alpha h}}{\alpha\,\mathrm{e}^{\alpha h}} \tag{1}$$

for all $t \in [T_0, \infty), T_0 > 0$, is located in the *domain of stabilization*

$$|x| < p(\mathrm{e}^{\alpha h} - 1)/\alpha \tag{2}$$

under the *stabilization condition*

$$0 < \alpha h < \ln 2 \tag{3}$$

It is important to remark that

- the condition (3) is a sufficient and necessary condition for the relay delayed stabilization (see Fridman *et al.* 2002);
- the size of the *domain of stabilization* is proportional to the control gain.

Fridman *et al.* (2002) proposed the following algorithm for controlling the motion amplitudes: since after finite

Received 12 July 2002. Accepted 2 November 2002.
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International Journal of Control ISSN 0020–7179 print/ISSN 1366–5820 online © 2003 Taylor & Francis Ltd
http://www.tandf.co.uk/journals
DOI: 10.1080/0020717031000099083

time all solutions coincide with the periodic solution, one can extrapolate the next zero for the periodic solution, and reduce the control gain near to the periodic solution zero. This algorithm needs the knowledge of the sign of the state variable with delay only but requires *stabilization condition* (3) to hold. This algorithm is valid for any constant delay satisfying condition (3) and does not depend on the delay value. *Stabilization condition* (3) and algorithm for stabilization was generalized by Shustin *et al.* (2003) to the case of *second-order* relay delay systems.

A relay delay controller proposed in this paper needs information only about amplitude of oscillations with delay. Stabilization properties of the proposed algorithm are based on estimations of oscillations amplitude. Relay control algorithms present with two main advantages:

(a) Robustness with respect to uncertainties in the time delay.

Proposed relay delay controller suppresses uncertainty in the time delay in following sense: once being designed for the upper bound of uncertainty in the time delay for a given system, this controller ensures the stabilization of this system for any values of the time delay less than the upper bound, even in the case when the delay is variable.

(b) Design of relay delay controllers for the multi-input multi-output (MIMO) case.

The proposed relay delay algorithm does not use the periodicity of solutions. It allows us to generalize the stabilization condition (3) and to design the relay delay controller for *unstable* MIMO systems.

The paper is organized as follows. The properties of the relay delay controller for the simplest scalar case are introduced in § 2. In § 3 a modification of the control algorithm is suggested for systems having two unstable complex conjugate roots. In § 4 a notion of ε stabilization specifying the properties of the relay delay controllers for the general case is introduced. The algorithm for ε stabilization for single-input single-output (SISO) systems with unstable eigenvalue is suggested in § 5. In § 6 this algorithm is generalized for MIMO systems having real unstable eigenvalues. This algorithm is generalized for arbitrary controllable MIMO system. In § 8 the proposed algorithm is used for the relay delay control of inverted pendulum.

2. Scalar case

Consider the problem of the oscillations stabilization for the scalar unstable system

$$\dot{x} = \lambda x + f(x,t) + u$$

with the help of the relay delayed control $u = -p\,\mathrm{sign}[x(t-h(t))]$, where $h(t)$ is a continuous bounded time delay function satisfying the inequality $0 < h(t) \le h_0$ for all $t \ge 0$, and $0 \le \lambda h_0 < L = \ln 2$. Then the equation describing the behaviour of the control system has the form

$$\dot{x} = \lambda x + f(x,t) - p\,\mathrm{sign}[x(t-h(t))] \qquad (4)$$

with the initial conditions

$$x(t) = \varphi(t), \qquad \varphi \in \mathbf{C}[-h_0, 0] \qquad (5)$$

Fridman *et al.* (2002) have shown that in such a system there exists a countable set of periodic solutions and all other solutions to (4) and (5) after a finite time will coincide with one of the periodic solutions. This means that the stabilization in the usual sense cannot be achieved. Let us describe a special type of stabilization taking place in the relay delayed systems.

Choose and fix $\varepsilon > 0$.

Problem statement: Try to find $\delta > 0, p > 0$ such that for all

$$\varphi(t)\colon |\varphi(0)| < \delta$$

the solution $x(t)$ of the Cauchy problem (4) and (5) satisfies the inequality $|x(t)| < \varepsilon$, $(0 < t < \infty)$.

When the solution of this problem exists we will say the system (4) is ε-stable and we have achieved ε-stabilization of system (4).

Algorithm of ε-stabilization:

(1) Choose and fix $\varepsilon > 0$.

(2) Suppose that $f(x,t)$ is an uncertainty and we can find $K > 0$ such that $|f(x,t)| < K\varepsilon$ for all $|x| \le \varepsilon$.

(3) Let

$$h_0 < \min\left\{\frac{L}{\lambda + K}, \frac{1}{\lambda}\ln\frac{2\lambda + 4K}{\lambda + 4K}\right\}$$

(4) Let $\delta = \varepsilon((2 - e^{\lambda h_0})/2e^{\lambda h_0})$, $p = (\lambda + K)\varepsilon$.

The ε-stabilization of the trivial solution to the system (4) for this choice of parameters is proved in § A.1 of the Appendix.

3. Two-dimensional system with unstable complex conjugate eigenvalues

Consider the case, when the control system is of second order and has unstable complex eigenvalues. In this case the system is of the form

$$\left.\begin{aligned} \dot{x} &= \alpha x - \beta y + u_1 + f_1(x,y,t) \\ \dot{y} &= \alpha y + \beta x + u_2 + f_2(x,y,t) \end{aligned}\right\} \qquad (6)$$

where $x(t), y(t) \in R$, α, β are real numbers, u_1, u_2 are the real controls, $f_1(x,y,t)$, $f_2(x,y,t)$ are uncertainties, $\beta \ne 0, \alpha > 0$.

Let

$$u_1 = -p_1\,\mathrm{sign}[x(t-h(t))\cos(\beta h_0) - y(t-h(t))\sin(\beta h_0)]$$

and

$$u_2 = -p_2\,\mathrm{sign}[x(t-h(t))\sin(\beta h_0) + y(t-h(t))\cos(\beta h_0)]$$

Now the behaviour of system (6) is described by Cauchy problem

$$\left.\begin{aligned}\dot{x} &= \alpha x - \beta y - p_1\,\mathrm{sign}[x(t-h(t))\cos(\beta h_0)\\ &\quad - y(t-h(t))\sin(\beta h_0)] + f_1(x,y,t)\\ \dot{y} &= \alpha y + \beta x - p_2\,\mathrm{sign}[x(t-h(t))\sin(\beta h_0)\\ &\quad + y(t-h(t))\cos(\beta h_0)] + f_2(x,y,t)\end{aligned}\right\} \qquad (7)$$

$$x(t) = x_0(t), \qquad y(t) = y_0(t), \qquad -h_0 \le t \le 0. \qquad (8)$$

Our goal is to find the parameters p_1 and p_2, such that from the inequality $\sqrt{x_0^2(t) + y_0^2(t)} \le \varepsilon/2$ it follows that

$$\sqrt{x^2(t) + y^2(t)} \le \varepsilon \qquad (0 \le t \le \infty) \qquad (9)$$

Algorithm of ε-stabilization:

(1) Choose and fix $\varepsilon > 0$.

(2) Suppose that there exists $K > 0$ such that

$$\sqrt{f_1^2(x,y,t) + f_2^2(x,y,t)} < K\varepsilon$$

for all x, y: $\sqrt{x^2 + y^2} < \varepsilon$.

(3) Suppose that $h_0 < \pi/4(2K+\beta)$. Let us denote as

$$M = \max_{t\in[0,\pi/4-(2K+\beta)h_0]} \tfrac{1}{2}t\cos(t + (2K+\beta)h_0 + \pi/4)$$

Assume that $h_0 < M/(\alpha + K)$.

(4) Choose $\delta = \varepsilon/2$ and $p_1 = p_2 = (\sqrt{2}/4)\alpha'\varepsilon$, where $\alpha' \in (0, (\pi/4h_0) - 2K - \beta)$: $\alpha + K < \alpha'\cos(\alpha' h_0 + (2K+\beta)h_0 + \pi/4)/2$.

The ε-stabilization of the trivial solution to the system (6) for this choice of parameters δ, p_1 and p_2 is proved in § A.2 of the Appendix.

Remark 1: Let $\Phi(\psi)(-\infty < \psi < \infty)$ be the 2π-periodic piece-wise constant function determined in the interval $[0, 2\pi)$ as

$$\Phi(\psi) = \begin{cases} e^{i\pi/4} & \text{for } 0 \le \psi < \pi/2\\ e^{i3\pi/4} & \text{for } \pi/2 \le \psi < \pi\\ e^{i5\pi/4} & \text{for } \pi \le \psi < 3\pi/2\\ e^{i7\pi/4} & \text{for } 3\pi/2 \le \psi < 2\pi \end{cases}$$

Suppose that $z = x + iy$, where x, y are real numbers. Then equation (6) may be rewritten in the form

$$\dot{z} = (\alpha + i\beta)z - \frac{\alpha'\varepsilon}{2}\Psi(\beta + \arg(z(t-h(t)))) + f(z,t) \qquad (10)$$

where

$$f(z,t) = f_1\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}, t\right) + if_2\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}, t\right)$$

Here we have used

$$\cos(\arg z(t-h(t))) = x(t-h(t))[x^2(t-h(t)) + y^2(t-h(t))]^{-1/2}$$

$$\sin(\arg z(t-h(t))) = y(t-h(t))[x^2(t-h(t)) + y^2(t-h(t))]^{-1/2}$$

$$\exp^{i\pi/4} = 2^{-1/2} + i2^{-1/2}, \ldots$$

4. Problem statement

Consider the system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = Ax + Bu(t-h(t)) + f(x,t) \qquad (11)$$

where $x \in R^n, u \in R^m$, A, B are real matrices, $h(t), 0 < h(t) \le h_0$ is a continuous function describing uncertainties in the time delay, $u \in R^m$ is the relay control vector, and $f(x,t)$ is continuous on t and smooth on x corresponding to the presence of an uncertainty in the model of the plant. Suppose that the system (11) consists of an input or output time delay and the matrix A has characteristic roots with positive real part.

In this paper we will find the relay controller of the form

$$u(t-h(t)) = F(\mathrm{sign}\,S_1(x(t-h(t)), \ldots, \mathrm{sign}\,S_k(x(t-h(t)))),$$

$$S = (S_1, S_2, \ldots, S_k)^{\mathrm{T}}$$

and the pair (S, F) belongs to the class of smooth functions Q transforming $S: R^n \to R^k, F: R^k \to R^m$. Let us denote as $x(t)$ the solution to the system (11) with initial conditions

$$x(t) = \varphi(t), \qquad (-h_0 \le t \le 0)$$

Definition 1: The zero solution to the system (11) is said to be ε-stabilizable, if for any $\varepsilon > 0$ there exist $\delta > 0$ and the relay delay control $u(t-h(t))$ such that from the inequality $\|\varphi(0)\| < \delta$ it follows that

$$\sup_{t\in[0,\infty]} \|x(t)\| < \varepsilon$$

Remark 2: It is necessary to note that S and F could not depend on $\varepsilon > 0$.

5. ε-Stabilization of SISO systems

Consider a SISO system of the form

$$\dot{x} = Ax + bu + f(x,t) \tag{12}$$

where $b = (b_1, b_2, \dots, b_n)^{\mathrm{T}}$, u is a scalar control and $f(x,t)$ is uncertainty.

Suppose that

(1) $f(x,t)$ is the matching uncertainty and $f(x,t) = bg(x,t)$,
(2) the pair $\{A, b\}$ is controllable, and consequently the vectors $\{b, Ab, \dots, A^{n-1}b\}$ are linearly independent.

Denote by $\varphi_A(\lambda) = \lambda^n + \alpha_1\lambda^{n-1} + \dots + \alpha_n$ the characteristic polynomial of the matrix A. Let us introduce the controllability basis into R^n as

$$\begin{aligned} e_1 &= A^{n-1}b + \alpha_1 A^{n-2}b + \dots + \alpha_{n-1}b \\ e_2 &= A^{n-2}b + \alpha_1 A^{n-3}b + \dots + \alpha_{n-2}b \\ &\vdots \\ e_{n-1} &= Ab + \alpha_1 b \\ e_n &= b \end{aligned}$$

System (12) in this basis takes the form

$$\left.\begin{aligned} \dot{y}_1 &= y_2 \\ \dot{y}_2 &= y_3 \\ &\vdots \\ \dot{y}_n &= -\alpha_n y_1 - \alpha_{n-1}y_2 - \dots - \alpha_1 y_n \\ &\quad + u(t - h(t)) + g(x,t) \end{aligned}\right\} \tag{13}$$

and it is possible to rewrite the systems (12) and (13) in the form of the nth order equation

$$y_1^{(n)} + \alpha_1 y_1^{(n-1)} + \dots + \alpha_n y_1 = u + g(x,t) \tag{14}$$

Assume that (**i**) the characteristic equation of the matrix A: $\varphi_A(\lambda) = 0$ has only one positive root $\lambda_1 \in (0, \ln 2)$, and the other roots of this equation have negative real part.

In such a case the polynomial $\varphi_A(\lambda)$ becomes

$$\varphi_A(\lambda) = (\lambda - \lambda_1)\psi(\lambda),$$

where the polynomial

$$\psi(\lambda) = \lambda^{n-1} + \beta_{n-1}\lambda^{n-2} + \dots + \beta_1$$

has only roots with negative real part. Then for equation (14) one has the differential equation

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} - \lambda_1\right)\psi\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)y_1 = u + g(x,t)$$

Suppose that

$$\begin{aligned} z(t) &= \psi\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)y_1 \\ &= y_1^{(n-1)}(t) + \beta_{n-1}y_1^{(n-2)}(t) + \dots + \beta_1 y_1(t) \end{aligned}$$

For $z(t)$ we will have the scalar equation

$$\frac{\mathrm{d}}{\mathrm{d}t}z = \lambda_1 z + u + g(x,t), \quad \lambda_1 \in (0, \ln 2)$$

From §2 it follows that the trivial solution of the equation $\dot{z} = \lambda_1 z$ is ε-stabilizable with control

$$u = -p\,\mathrm{sign}\, z(t - h(t))$$

The corresponding control law for equation (14) has the form

$$u = -p\,\mathrm{sign}\{Y\}$$

where $Y = y_1^{(n-1)}(t - h(t)) + \beta_{n-1}y_1^{(n-2)}(t - h(t)) + \dots + \beta_1 y_1(t - h(t))$. Returning to the state variables $y_1, y_2, \dots, y_n$ we obtain

$$\begin{aligned} \dot{y}_1 &= y_2 \\ \dot{y}_2 &= y_3 \\ &\vdots \\ \dot{y}_n &= -\alpha_n y_1 - \alpha_{n-1}y_2 - \dots - \alpha_1 y_n + g(x,t) \\ &\quad - p\,\mathrm{sign}\{Y_\beta\} \end{aligned}$$

where

$$\begin{aligned} Y_\beta &= \beta_1 y_1(t - h(t)) + \beta_2 y_2(t - h(t)) \\ &\quad + \dots + \beta_{n-1}y_{n-1}(t - h(t)) + y_n(t - h(t)) \end{aligned}$$

Let us denote as P the matrix used to transform the basis $\{h_j\} = (0, \dots, 1, \dots, 0)$ into the basis $\{e_1, \dots, e_n\}$. Then for the initial variables $x = P^{-1}y$ one has

$$\dot{x} = Ax - pb\,\mathrm{sign}\{\gamma^{\mathrm{T}}x(t - h(t))\} + f(x,t)$$

where $\gamma = P^*\beta$, $\beta^{\mathrm{T}} = (\beta_1, \beta_2, \dots, \beta_{n-1}, 1)$.

Theorem 1: *The zero solution of system* (12) *under assumption* (**i**) *is* ε *stabilizable with the control law*

$$u = -p\,\mathrm{sign}(\gamma, x(t - h(t)))$$

The algorithm for ε-stabilization will be defined in the next section for one more general case.

6. ε-stabilization of MIMO systems in real case

Consider the initial system (11) in the general case

$$u \in R^m, \quad 1 < m < n, \quad B = (b_1, b_2, \dots, b_m), \quad (b_j \in R^n)$$

Suppose that:

(1) $f(x,t) = Bg(x,t)$ and

$$g(x,t) = (g_1(x,t), g_2(x,t), \dots, g_m(x,t))^{\mathrm{T}}$$

(2) the pair $\{A, B\}$ is controllable,

(3) for every $j = 1, 2, \ldots, m$ the vectors $b_j, Ab_j, \ldots, A^{n_j-1}b_j$, are linearly independent, and vectors $A^{n_j}b_j$ are linear combination of the vectors

$$b_j, Ab_j, \ldots, A^{n_j-1}b_j$$

Then the space

$$E_j = \text{Span}\{b_j, Ab_j, \ldots, A^{n_j-1}b_j\}$$

is the invariant space for the matrix A, and the pair $\{A, b_j\}$ is controllable into the E_j. Suppose that $E_i \cap E_j = \varnothing$ $(i \neq j)$, and

$$R^n = E_1 \oplus E_2 \oplus \cdots \oplus E_m$$

Let us denote by $\varphi_j(\lambda) = \lambda^{n_j} + \alpha_{1j}\lambda^{n_j-1} + \cdots + \alpha_{nj}$ the characteristic polynomial of matrix $A_j = A|_{E_j}$. Suppose that for A_j and φ_j assumption (**i**) is true. This means that in each E_j we can choose the canonical basis in the form

$$e_{1j} = A^{n_j-1}b_j + \alpha_{1j}A^{n_j-2}b_j + \cdots + \alpha_{n_j j}b_j$$
$$\vdots$$
$$e_{n_j j} = b_j$$

Then for the pair $\{A_j, b_j\}$ into the E_j one has

$$A_j = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -\alpha_{n_j j} & -\alpha_{n_j-1j} & -\alpha_{n_j-2j} & -\alpha_{n_j-3j} & \cdots & -\alpha_{1j} \end{pmatrix}$$

$b'_j = (0, 0, \ldots, 0, 1)^{\mathrm{T}}$. The matrix A in such case has the block diagonal form $A = \text{diag}\{A_1, A_2, \ldots, A_m\}$, and $B = (b'_1, b'_2, \ldots, b'_m)^{\mathrm{T}}$. The system (11) has the following block form

$$\dot{y}_{1j} = y_{2j}, \quad \dot{y}_{2j} = y_{3j}, \ldots, \dot{y}_{n_j-1j} = y_{n_j j}$$
$$\dot{y}_{n_j j} = -\alpha_{n_j j}y_{1j} - \cdots - \alpha_{1j}y_{n_j j} + u_j + g_j(x, t)$$

Taking into account that λ_j is a root of the polynomial $\varphi_j(\lambda)$, one can suppose that

$$\varphi_j(\lambda) = (\lambda - \lambda_j)[\lambda^{n_j-1} + \beta_{j1}\lambda^{n_j-2} + \cdots + \beta_{jn-1}] \quad (15)$$

$$\alpha_{j1} = \beta_{j1} - \lambda_j, \qquad \alpha_{j2} = \beta_{j2} - \lambda_j\beta_{j1}, \ldots \quad (16)$$

$$\alpha_{jn_j-1} = \beta_{jn_j-1} - \lambda_j\beta_{jn_j-2}, \qquad \alpha_{jn_j} = -\lambda_j\beta_{jn_j-1}$$

Substituting (16) into (15), we obtain

$$\dot{y}_{1j} = y_{2j}, \qquad \dot{y}_{2j} = y_{3j}, \ldots, \dot{y}_{n_j-1j} = y_{n_j j} \quad (17)$$
$$\dot{y}_{nj} = \lambda_j[\beta_{jn_j-1}y_{1j} + \beta_{jn_j-2}y_{2j} + \cdots + y_{n_j j}]$$
$$- \{\beta_{jn_j-2}y_{2j} + \cdots + \beta_{j1}y_{n_j j}\} + u_j + g_j(x, t)$$

Multiplying the first equation of the system (17) by β_{jn_j-1}, the second equation by $\beta_{jn_j-2}, \ldots$ and the $(n_j - 1)$th equation by β_{j1}, and adding the result, to the last one, we obtain

$$\dot{z}_j = \lambda_j z_j + u_j + g_j(x, t)$$

where $z_j = \beta_{jn_j-1}y_{1j} + \beta_{jn_j-2}y_{2j} + \cdots + y_{n_j j}$.

Now it is possible to rewrite the system (17) in the form

$$\dot{\bar{y}} = A_0^j\bar{y} + \bar{b}z_j(t), \qquad \dot{z}_j = \lambda_j z_j + u_j + g_j(x, t)$$

$$A_0^j = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & & & & \\ 0 & 0 & 0 & \cdots & 1 \\ -\beta_{jn_j-1} & \cdots & \cdots & \cdots & -\beta_{j1} \end{pmatrix}$$

where $\bar{b} = (0, 0, \ldots, 1)^{\mathrm{T}}$, $u_j = -p_j \operatorname{sign} z_j(t-1)$. The matrix A_0 is stable.

Returning to initial system (15), we will have

$$\dot{x} = Ax + B\begin{pmatrix} -p_1 \operatorname{sign}(P^*\beta_1, x(t - h(t))) \\ -p_2 \operatorname{sign}(P^*\beta_2, x(t - h(t))) \\ \cdots \\ -p_m \operatorname{sign}(P^*\beta_m, x(t - h(t))) \end{pmatrix} + f(x, t)$$

$$\beta_1 = \begin{pmatrix} \beta_{11} \\ \beta_{12} \\ \vdots \\ \beta_{1n_1-1} \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \qquad \beta_2 = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \beta_{21} \\ \beta_{22} \\ \vdots \\ \beta_{2n_2-1} \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots$$

where $\beta_{j\sigma}$ are the coefficients of $\varphi_j(\lambda)$ and P is the matrix used to transform the basis $\{h_j\} = (0, \ldots, 1, \ldots, 0)$ into the basis $\{e_{ij}\}$.

Theorem 2: *Suppose that the matrix A has the simple eigenvalues with positive*

$$\lambda_1, \lambda_2, \ldots, \lambda_l \in (0, L), \qquad L = \ln(2)$$

The remaining eigenvalues of matrix A have negative real part.

Then the zero solution of system (11) *is* ε*-stabilizable*.

Algorithm of ε-stabilization:

(1) Choose and fix $\varepsilon > 0$.

(2) Let there exist $K > 0$: $\|g(x, t)\| < K\varepsilon$ for all $\|x\| \leq \varepsilon$.

(3) Let C_i, γ_i be such that

$$\|e^{A_0^i t}\| < C_i e^{-\gamma_i t}$$

(4) Assume that

$$\varepsilon_i < \frac{\varepsilon}{\|P^{-1}\| m} \min\left\{\frac{1}{2}, \frac{\gamma_i}{3C_i}\right\}, \qquad K_i = K\varepsilon/\varepsilon_i$$

(5) Suppose that

$$h_0 < \min\left\{\frac{L}{\lambda_i + K_i}, \frac{1}{\lambda_i} \ln \frac{2\lambda_i + 4K_i}{\lambda_i + 4K_i}\right\}$$

(6) Let

$$\delta < \varepsilon_i \frac{2 - e^{\lambda_i h_0}}{2\|P\| e^{\lambda_i h_0}} \quad \text{and} \quad p = (\lambda_i + K_i)\varepsilon_i$$

7. ε-Stabilization of MIMO system in general case

Assume that the spectrum $\sigma(A)$ of the matrix A consists of two parts

$$\sigma(A) = \sigma_+ \cup \sigma_-$$

where σ_+ and σ_- are the sets of eigenvalues of the matrix A with positive and negative real part respectively. Then the state space $E = R^n$ could be represented in the form $E = E_+ \oplus E_-$, where E_+ and E_- are the invariant subspaces with respect to A. Consider two projectors P and Q, transforming

$$P: R^n \to E_+, \qquad Q: R^n \to E_-$$

Denoting $y = Px$, $z = Qx$ one can rewrite system (11) in the form

$$\left.\begin{aligned} \dot{y} &= A^+ y + B^+ u + f_1(x,t) \\ \dot{z} &= A^- z + B^- u + f_2(x,t) \end{aligned}\right\} \tag{18}$$

where $y \in E_+$, $z \in E_-$, $A^+ = PA$, $A^- = QA$, $B^+ = PB = (b_1^+, b_2^+, \dots, b_m^+)$, $B^- = QB$, $f_1(x,t) = Pf(x,t)$, $f_2(x,t) = Qf(x,t)$.

Suppose that $\operatorname{rank} B^+ = \dim E^+ = k$. This means that the vectors $\{b_j^+\}(j = \overline{1,k})$ are linearly independent, and that following representation holds

$$Ah_i = \lambda_i h_i, \qquad h_i = s_{1i} b_1^+ + s_{2i} b_2^+ + \dots + s_{ki} b_k^+, \qquad i = \overline{1,l}$$

$$A^+ h_{l+2j-1} = \alpha_j h_{l+2j-1} - \beta_j h_{l+2j},$$

$$A^+ h_{l+2j} = \beta_j h_{l+2j-1} - \alpha_j h_{l+2j}$$

$$h_{l+2j-1} = s_{1l+2j-1} b_1^+ + s_{2l+2j-1} b_2^+ + \dots + s_{kl+2j-1} b_k^+$$

$$h_{l+2j} = s_{1l+2j} b_1^+ + s_{2l+2j} b_2^+ + \dots + s_{kl+2j} b_k^+$$

$$j = \overline{1,\nu}$$

where $l + 2\nu = k$. Let us design the control u as

$$u = \begin{pmatrix} S_0 & 0 \\ 0 & 0 \end{pmatrix} \sigma$$

where the matrix S_0 consisting of $\{s_{ij}\}$ coefficients that represents the eigenvectors of the matrix A^+ in the basis $b_1^+, \dots, b_k^+$. The function $\sigma = (\sigma_1(y(t - h(t))), \sigma_2(y(t - h(t))), \dots, \sigma_k(y(t - h(t))), 0, \dots, 0)^{\mathrm{T}}$ will be defined below. Denote by $B^+ S_0 = (h_1, h_2, \dots, h_k, 0, \dots, 0) = (T, 0)$. Substituting the variables $v = T^{-1} y$ into (18), we will have

$$\left.\begin{aligned} \dot{v} &= D^+ v + (I^k \ 0)\sigma + T^{-1} f_1(x,t) \\ \dot{z} &= A^- z + B^- u + f_2(x,t) \end{aligned}\right\} \tag{19}$$

where

$$D^+ = \begin{pmatrix} \lambda_1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \lambda_l & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \cdots & 0 & \alpha_1 & -\beta_1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \beta_1 & \alpha_1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & \alpha_\nu & -\beta_\nu \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & \beta_\nu & \alpha_\nu \end{pmatrix}$$

Let us design $\sigma(v(t - h(t)))$ in the form

$$\sigma_i(v(t - h(t))) = -p_i \operatorname{sign}(v_i(t - h(t))), \qquad i = 1, 2, \dots, l \tag{20}$$

$$\sigma_{l+2j-1}(v(t - h(t))) = -p_{l+2j-1} \operatorname{sign}(v_{l+2j-1}(t - h(t)) \cos\beta_j - v_{l+2j}(t - h(t)) \sin\beta_j) \tag{21}$$

$$\sigma_{l+2j}(v(t - h(t))) = -p_{l+2j} \operatorname{sign}(v_{l+2j-1}(t - h(t)) \sin\beta_j + v_{l+2j}(t - h(t)) \cos\beta_j) \tag{22}$$

$$j = 1, 2, \dots, \nu$$

Now one can conclude that the system (19) is ε-stabilizable and finally we will have

$$\dot{x} = Ax + B\begin{pmatrix} S_0 & 0 \\ 0 & 0 \end{pmatrix} \sigma(Px(t - h(t))) + f(x,t) \tag{23}$$

Remark 3: Let us write the projectors as

$$Px = \sum_{i=1}^{k} (x, g_i) h_i$$

$$Qx = x - Px$$

where $h_1, h_2, .., h_k$ are the eigenvectors of the matrix A and g_i are found in Appendix A.3.

Let us have the basis

$$e_j = (\alpha_{j1}, \alpha_{j2}, \dots, \alpha_{jn})^{\mathrm{T}}, \quad j = 1, \dots, n$$

in R^n. Let us introduce the new basis into R^n as

$$\bar{e}_i = e_i, \quad (i = \overline{1,k})$$
$$\bar{e}_{k+j} = Qe_{k+j}, \quad (j = \overline{1,n-k})$$

The matrix transforming the old basis $\{h_i\}$ into the new basis $\{e_i\}$ has the form

$$G = \begin{pmatrix} \alpha_{11} & \alpha_{21} & \cdots & \alpha_{k1} & \alpha_{k+1\,1} - \sum_{i=1}^{k} \alpha_{i1}(e_{k+1}, g_i) & \cdots \\ \alpha_{12} & \alpha_{22} & \cdots & \alpha_{k2} & \alpha_{k+1\,2} - \sum_{i=1}^{k} \alpha_{i2}(e_{k+1}, g_i) & \cdots \\ \cdots & \cdots & \cdots & \cdots & & \\ \alpha_{1n} & \alpha_{2n} & \cdots & \alpha_{kn} & \alpha_{k+1\,n} - \sum_{i=1}^{k} \alpha_{in}(e_{k+1}, g_i) & \cdots \end{pmatrix}$$

and

$$J = G^{-1}AG = \begin{pmatrix} D^+ & 0 \\ 0 & A^- \end{pmatrix}$$

In this case the following representation holds

$$\begin{pmatrix} PB \\ QB \end{pmatrix} = \begin{pmatrix} B^+ \\ B^- \end{pmatrix} = \begin{pmatrix} B_0^+ & B_1^+ \\ B_0^- & B_1^- \end{pmatrix}$$

where $B_0^+ - k \times k$, $B_1^+ - k \times m - k$, $B_0^- - n - k \times k$, $B_1^- - n - k \times m - k$ and $\det(B_0^+) \neq 0$. Then system (23) can be rewritten

$$\dot{x} = Ax + B\begin{pmatrix} [B_0^+]^{-1} & 0 \\ 0 & 0 \end{pmatrix}\sigma(Px(t - h(t))) + f(x,t) \tag{24}$$

Algorithm of ε-stabilization:

(1) Choose and fix $\varepsilon > 0$.

(2) Let there exists $K > 0$: $\|f(x,t)\| < K\varepsilon$ for all x: $\|x\| < \varepsilon$.

(3) Let

$$\varepsilon_i < \frac{\varepsilon}{2(l+2\nu)\|h_i\|}, \qquad K_i = \frac{K\|g_i\| \cdot \|h_i\|}{2(l+2\nu)}$$

(4) Assume that

$$h_0 < \min\left\{\frac{L}{\lambda_i + K_i}, \frac{1}{\lambda_i}\ln\frac{2\lambda_i + 4K_i}{\lambda_i + 4K_i}\right\}, \quad i = \overline{1,l}$$
$$h_0 < \frac{\pi}{4(2K_{l+2j} + \beta_j)}, \quad j = \overline{1,\nu}$$

Moreover suppose that for

$$M_j = \max_{0 \le t \le \pi/4 - (2K_{l+2j} + \beta_j)h_0} \left(\tfrac{1}{2}t\cos(t + (2K_{l+2j} + \beta_j)h_0 + \pi/4)\right)$$

the inequality $h_0 < M_j/(\alpha + K_{l+2j})$, $j = \overline{1,\nu}$ holds.

(5) Let

$$\delta < \frac{\varepsilon_i(2 - e^{\lambda_i h_0})}{2\|g_i\|e^{\lambda_i h_0}}$$

and $p_i = (\lambda_i + K_i)\varepsilon_i$ for $i = \overline{1,k}$.

(6) Let $\delta < \varepsilon_{l+2j}/2$ and

$$p_{l+2j-1} = p_{l+2j} = \frac{\alpha_j'\sqrt{2}}{4}\varepsilon_{l+2j}$$

$$\alpha_j' \in \left(0, \frac{\pi}{4h_0} - 2K_{l+2j} - \beta_j\right) : \alpha_j + K_{l+2j}$$
$$< \alpha_j'\cos(\alpha_j' h_0 + (2K_{l+2j} + \beta)h_0 + \pi/4)/2, \quad j = \overline{1,\nu}$$

(7) Let $C, \gamma > 0$ be such that both

$$\|e^{A^- t}\| \le Ce^{-\gamma t}$$

and

$$\gamma > 4\frac{C}{\varepsilon}(\|B_0^-| \cdot \|[B_0^+]^{-1}\| \cdot \|\bar{p}\| + \|G\|K)\max_{j=\overline{k+1,n}}\|e_j\|$$

$\bar{p} = (p_1, p_2, \ldots, p_k)^{\mathrm{T}}$ hold.

8. Relay delay control inverted pendulum

Consider the problem of an inverted pendulum stabilization via relay delayed control. The model of the pendulum has the form

$$\ddot{\theta} + k\dot{\theta} - p\sin(\theta) = u(t - h(t)) + f(\theta, \dot{\theta}, t) \tag{25}$$

where θ is an inclination angle, k is a friction coefficient, $p = g/l$, where l is a length of pendulum, $h(t)$ is time delay, $f(\theta, \dot{\theta}, t)$ is an uncertainty. Linearizing (25) we will have

$$\ddot{\theta} + k\dot{\theta} - p\theta = u(t - h(t)) + g(\theta, \dot{\theta}t)$$

The characteristic equation has two real roots of opposite signs

$$\lambda_1 = \tfrac{1}{2}(\sqrt{k^2 + 4p} - k)$$
$$\lambda_2 = \tfrac{1}{2}(-\sqrt{k^2 + 4p} - k)$$

Assume that $0 < \lambda_1 h_0 < \ln 2$. In this case the equation (25) can be rewritten in the form

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} - \lambda_1\right)\left(\frac{\mathrm{d}}{\mathrm{d}t} - \lambda_2\right)\theta = u(t - h(t)) + g(\theta, \dot{\theta}, t)$$

Denoting $z = \dot{\theta} - \lambda_2\theta$ we will have

$$\dot{z} = \lambda_1 z + u(t - h(t)) + g(\theta, t)$$

Let us design the controller in the form

$$u = -q\,\mathrm{sign}[z(t - h(t)]$$

Returning to the original system (25), we will have

$$\ddot{\theta} + k\dot{\theta} - p\sin(\theta) = -q\,\mathrm{sign}[\dot{\theta}(t - h(t)) - \lambda_2\theta(t - h(t))] + f(\theta, \dot{\theta}, t)$$

Consider the case when

$$\ddot{\theta} + 2.9\dot{\theta} - 0.3\sin(\theta) = u(t - h(t)) + 0.003\sin(t) \tag{26}$$

$$u = -q\,\mathrm{sign}(\dot{\theta}(t - h(t)) + 3\theta(t - h(t))$$
$$\theta(t) = 0.01\sin(t) \tag{27}$$

$$\dot{\theta}(t) = 0.01\cos(t) \quad \text{for } t \in [-h_0, 0] \tag{28}$$

If the upper bound of the time delay is $h_0 = 5$, then for the system (26)–(28) conditions of the Theorem 1 hold. The relay delayed control law ensuring ε of the inverted pendulum (26)–(28) takes the form

$$u = -1.3\varepsilon \operatorname{sign}(\dot{\theta}(t - h(t)) + 3\theta(t - h(t))) \tag{29}$$

Figures 1–3 are illustrating the results of implementation of delayed relay control law (29) in the system (26)–(28) for constant delay $h(t) \equiv 5$ and $\varepsilon = 0.1$. Figures 4–6 show the behaviour of the pendulum (26)–(28) for $h(t) = 3 + 2\sin(70t)$. This confirms the main property of delayed relay controller: once being designed for the upper bound of uncertainty in the time delay $h(t) \equiv 5$ for a given system, this controller ensures the stabiliz-

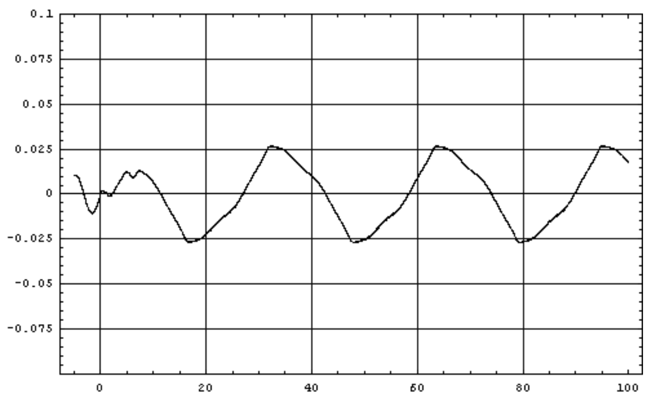

Figure 1. Inclination angle $\theta\,(h(t) \equiv 5)$.

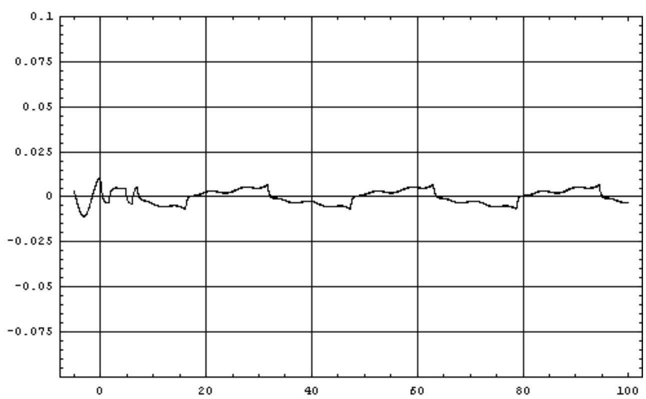

Figure 2. Angular speed $\dot{\theta}\,(h(t) \equiv 5)$.

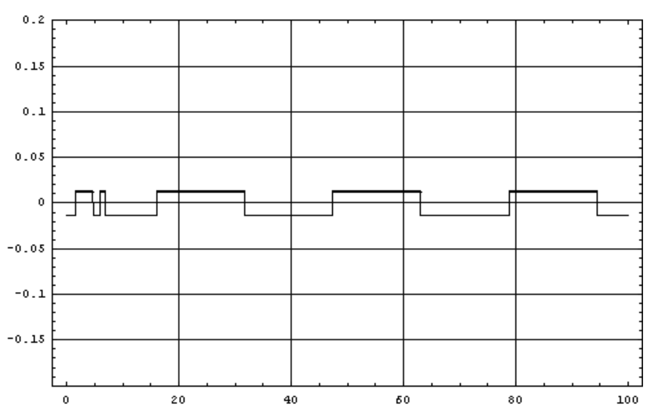

Figure 3. Relay control law $u\,(h(t) \equiv 5)$.

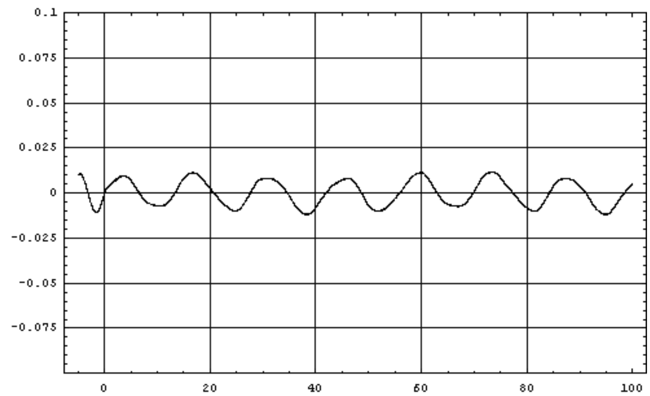

Figure 4. Inclination angle $\theta\,(h(t) = 3 + 2\sin(70t))$.

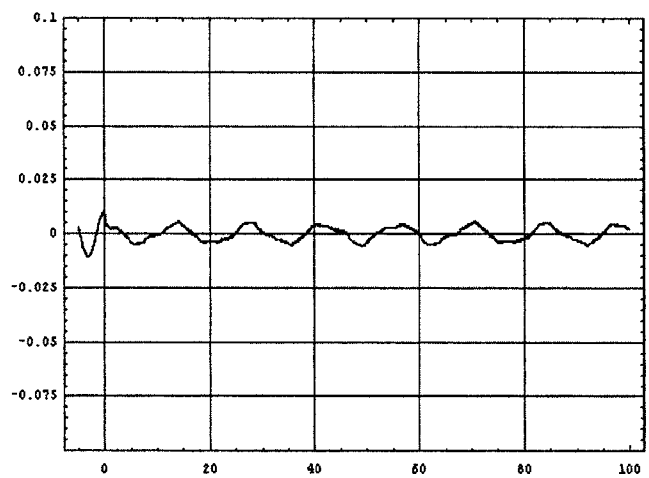

Figure 5. Angular speed $\dot{\theta}\,(h(t) = 3 + 2\sin(70t))$.

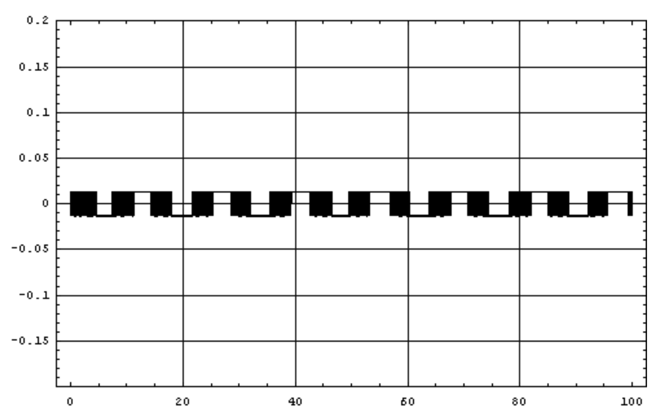

Figure 6. Relay control law $u\,(h(t) = 3 + 2\sin(70t))$.

ation of this system for the variable delay $h(t) = 3 + 2\sin(70t)$.

9. Conclusions

In this paper the possibilities of stabilization for unstable control system via relay delay control are discussed. The concept of ε stabilization characterizing specific features of relay delay systems is formulated. A sufficient condition for such a kind of relay delayed stabilization are found relating the upper bound of uncertainty in the time delay and the maximum of the real part of the system spectrum. The algorithm of the relay delay control allowing to achieve ε stabilization is suggested. Obtained results are illustrated on the example of the relay delay stabilization for the inverted pendulum.

This allows us to conclude the following:

1. Time delay does not allow us to realize the ideal sliding mode, but implies oscillations in the space of state variables. Nevertheless, relay delay controllers allow us to achieve stabilization of the amplitude of oscillations, and suppress uncertainties in the time delay in the following sense: once being designed for the upper bound of an uncertainty in the time delay for a given system this relay delayed control law ensures stabilization for any values of the time delay less than the upper bound even in the case when the delay is variable.
2. Proposed algorithm allows to achieve only *local* stabilization and should be extended in order to achieve non-local or semiglobal stabilization.

Appendix

A.1. *Proof of ε-stabilization for the scalar case*

Let us show that choosing p and h_0 we ensuring the ε-stabilization for system (4). Let us suppose by contradiction that it is not true. Then there exists such $T > 0$ that $|x(T)| = \varepsilon$, but $|x(t)| < \varepsilon$ for all $t \in [0, T)$. In this case there exists a time moment $t^* \in (0, T)$ such that $|x(t^*)| = \delta$ and $\delta < |x(t)| < \varepsilon$ for all $t \in (t^*, T)$.

Let $x(t^*) = \delta$ and $x(T) = \varepsilon$. Let us show that $T - t^* > h_0$

$$\dot{x} \leq \lambda x + f(x, t) + p \leq \lambda x + p + K\varepsilon$$
$$x(t^*) = \delta$$

Then

$$x(T) = \varepsilon \leq \left(\delta + \frac{p + K\varepsilon}{\lambda}\varepsilon\right) e^{\lambda(T - t^*)} - \frac{p + K\varepsilon}{\lambda}\varepsilon$$

Hence, the inequality

$$e^{\lambda(T - t^*)} > \frac{\lambda + p\varepsilon + K}{\lambda(\delta/\varepsilon) + p\varepsilon + K} > e^{\lambda h_0}$$

implies the inequality $T - t^* > h_0$. Now for all $t \in [t^* + h_0, T]$ we have $\text{sign}[x(t - h(t))] = 1$ and

$$\dot{x}(t) \leq \lambda x - p + f(x, t) < (\lambda + K)\varepsilon - p = 0$$

Hence, $x(t)$ is decreasing function in the interval $[t^* + h_0, T]$. This means that equality $x(T) = \varepsilon$ will never achieved.

Another case could be proved analogously.

A.2. *Proof of ε-stabilization for the complex case*

Proof: Let us introduce the polar coordinates in the equation (10) by formula $z = \rho(t) \exp i\varphi(t)$. Then

$$\begin{aligned} \rho' e^{i\varphi(t)} + i\varphi'(t)\rho(t) e^{i\varphi(t)} &= (\alpha + i\beta)\rho(t) e^{i\varphi(t)} - \tfrac{1}{2}\alpha' \Phi(\beta h_0 + \varphi(t - h(t))) \\ &\quad + f(z, t) \end{aligned} \tag{30}$$

where

$$\begin{aligned} \Phi(\beta h_0 + \varphi(t - h(t))) &= \frac{\sqrt{2}}{2}(\text{sign}[\cos(\varphi(t - h(t)) + \beta h_0)] \\ &\quad + i\,\text{sign}[\sin(\varphi(t - h(t)) + \beta h_0)]) \end{aligned}$$

The function Φ has only four values: $e^{i\pi/4}$, $e^{i3\pi/4}$, $e^{i5\pi/4}$, $e^{i7\pi/4}$. Moreover, the value of $\Phi(\beta h_0 + \varphi(t - h(t)))$ depends from $\beta h_0 + \varphi(t - h(t))$ only. Denote

$$\Phi(\beta h_0 + \varphi(t - h(t))) = e^{ik(t)} \tag{31}$$

where

$$-\pi/4 + 2\pi l \leq k(t) - \varphi(t - h(t)) - \beta h_0 \leq \pi/4 + 2\pi l \tag{32}$$

l is the integer number. Substituting (30) into (7) we get

$$\rho' + i\varphi'\rho = (\alpha + i\beta)\rho - \tfrac{1}{2}\alpha' e^{i(k(t) - \varphi(t))} + q(t) e^{i(\psi(t) - \varphi(t))} \tag{33}$$

where $f(z, t) = q(t) e^{i\psi(t)}$.

In other words we have

$$\begin{aligned} \rho' &= \alpha\rho - \tfrac{1}{2}\varepsilon\alpha' \cos(k(t) - \varphi(t)) \\ &\quad + q(t)\cos(\psi(t) - \varphi(t)) \end{aligned} \tag{34}$$

$$\begin{aligned} \varphi' &= \beta - \tfrac{1}{2}\alpha'\rho^{-1} \sin(k(t) - \varphi(t)) \\ &\quad + q(t)\rho^{-1}\sin(\psi(t) - \varphi(t)) \end{aligned} \tag{35}$$

Let us show that this choice of initial conditions ensure the ε-stabilization for the system (6). Suppose that it is not true. Then there exists the smallest positive root $t = T$ of the equation $\rho(T) = \varepsilon$. The inequality

$$|z(t)| = r(t) < \varepsilon/2 \quad (-h \leq t \leq 0)$$

implies existence such the last time moment $t = t^* > 0$ that $\rho(t^*) = \varepsilon/2$ and hence, we have $\rho(t) > \varepsilon/2\,(t^* < t \le T)$ or

$$\tfrac{1}{2}\rho^{-1}(t) < 1 \quad (t^* < t \le T) \tag{36}$$

Further

$$\rho' \le \alpha\rho + \alpha'\varepsilon/2, \quad \rho(t^*) = \varepsilon/2 \quad (t^* < t \le T) \tag{37}$$

Hence

$$\rho(T) = \varepsilon \le \frac{\varepsilon}{2}(1 + \alpha'\alpha^{-1})\,\mathrm{e}^{\alpha(T-t^*)} - \frac{1}{2}\alpha'\alpha^{-1}\varepsilon, \ \ (t^* \le t \le T)$$

This implies

$$T^* - t^* > \frac{1}{\alpha}\ln\left(\frac{2\alpha + \alpha'}{\alpha + \alpha'}\right) \tag{38}$$

Since $\alpha' < \alpha(2 - \mathrm{e}^{\alpha h_0})(\mathrm{e}^{\alpha h_0} - 1)^{-1}$, it is easy to show that $T^* - t^* > h_0$.

To study the term $\cos[k(t) - \varphi(t)]$ in the right hand side of (35) we should integrate the equation (35) on the interval $[t - h(t), t], t \in [t^* + h_0, T]$. It is easy to see that

$$\varphi(t) = \varphi(t - h(t)) + \beta h(t) - \alpha'\xi(t) + \chi(t) \tag{39}$$

where

$$\xi(t) = \frac{\varepsilon}{2}\int_{t-h(t)}^{t} \rho^{-1}(\tau)\sin[k(\tau) - \varphi(\tau)]\,\mathrm{d}\tau \tag{40}$$

$$\chi(t) = \int_{t-h(t)}^{t} q(\tau)\rho^{-1}(\tau)\sin[\psi(\tau) - \varphi(\tau)]\,\mathrm{d}\tau \tag{41}$$

Obviously

$$|\xi(t)| \le h_0 \quad \text{and} \quad |\chi(t)| < 2Kh_0 \tag{42}$$

Now substituting (39) into (34) we have

$$\rho' = \alpha\rho - \frac{\varepsilon}{2}\alpha'\cos[k(t) - \varphi(t - h(t)) - \beta h(t) + \alpha'\xi(t) + \chi(t)] + q(t)\cos(\psi(t) - \varphi(t)) \tag{43}$$

From (32) and (42) we obtain

$$\cos[k(t) - \varphi(t - h(t)) - \beta h_0 + \beta(h_0 - h(t)) + \alpha'\xi(t) + \chi(t)] \ge \cos((\alpha' + 2K + \beta)h_0 + \pi/4) \tag{44}$$

Hence, for $t \in [t^* + h_0, T]$

$$\varrho' \le \alpha\rho - \alpha'\frac{\varepsilon}{2}\cos((\alpha' + 2K + \beta)h_0 + \pi/4) + K\varepsilon \le \varepsilon\Big[\alpha + K - \frac{\alpha}{2}\cos((\alpha' + 2K + \beta)h_0 + \pi/4)\Big] < 0$$

That is why equality $\rho(T) = \varepsilon$ cannot be achieved for $t = T$ because $t = T$ is the smallest positive root of this equation. □

A.3. *Structure of projectors*

Consider the conjugate matrix A^* and suppose that $f_1, f_2, \ldots, f_k$ are the eigenvectors of A^*

$$A^* f_i = \lambda_i f_i, \quad (i = \overline{1, l})$$

and

$$A^* f_{l+2j-1} = \alpha_j f_{l+2j-1} + \beta_j f_{l+2j}$$
$$A^* f_{l+2j} = -\beta_j f_{l+2j-1} + \alpha_j f_{l+2j}$$

Let us introduce the vectors g_i

$$g_i = \frac{f_i}{\|f_i\|}, \quad (i = \overline{1, l})$$

$$g_{l+2j-1} = c_{11}^j f_{l+2j-1} + c_{12}^j f_{l+2j}$$
$$g_{l+2j} = c_{21}^j f_{l+2j-1} + c_{22}^j f_{l+2j}$$
$$(j = \overline{1, \nu})$$

where

$$c_{11}^j = \frac{\|f_{l+2j}\|^2}{\|f_{l+2j-1}\|^2\|f_{l+2j}\|^2 - (f_{l+2j-1}, f_{l+2j})^2}$$

$$c_{12}^j = c_{21}^j = \frac{(f_{l+2j-1}, f_{l+2j})}{\|f_{l+2j-1}\|^2\|f_{l+2j}\|^2 - (f_{l+2j-1}, f_{l+2j})^2}$$

$$c_{22}^j = \frac{\|f_{l+2j-1}\|^2}{\|f_{l+2j-1}\|^2 \cdot \|f_{l+2j}\|^2 - (f_{l+2j-1}, f_{l+2j})^2}$$

Now it is easy to show that the projector P could be rewritten in the form

$$Px = \sum_{i=1} k(x, g_i)h_i$$

where $h_1, h_2, \ldots, h_k$ are eigenvectors of the matrix A.

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