Technical Notes and Correspondence

Cheap Suboptimal Control of an Integral Sliding Mode for Uncertain Systems With Delays

Valery Y. Glizer, Leonid M. Fridman, and Vladimir Turetsky

***Abstract*—A cheap control problem for a system with state delays and matched uncertainties is studied. First, such a cheap control problem is considered for the *nominal* linear system associated with the original one. A suboptimal state-feedback composite control for this problem is constructed using a singular perturbation technique. Employing this composite control in the nominal system generates the *suboptimal nominal trajectory*. Secondly, based on the composite control, an integral sliding mode controller is designed for the original system, providing its robust motion along the suboptimal nominal trajectory. An illustrative numerical example is presented.**

***Index Terms*—Cheap control, singular perturbation, sliding mode control, time delay system.**

I. Introduction and Problem Formulation

Dynamics of real life controlled systems often contains unmeasured terms (uncertainties). Assuming the uncertainty to be zero, one can construct an optimal control (with respect to a prechosen cost functional) for the resulting *(nominal)* system. However, the employment of this *nominal* control in the original system generates the trajectory depending on a realization of the uncertainty. Thus, the system motion subject to this control is sensitive to the uncertainty. The *sliding mode* method is one of the most simple and effective tools of designing a controller providing an insensitivity of a desired system motion with respect to the matched uncertainties (see, e.g., [1]–[3] and references therein). This motion takes place in a *sliding manifold*, and it is called a *sliding mode*. Due to the insensitivity of the sliding mode to the uncertainties, an optimal sliding mode can be realized with respect to a given cost functional. The extension of the sliding mode approach is the *integral sliding mode* one (see, e.g., [4]). A controller, constructed by this approach, uses not only the information on the current value of the state variable but also the full history of the system motion. The latter (in contrast with a conventional sliding mode controller) allows an integral sliding mode controller to track a *nominal trajectory* from the very beginning of the control process, ensuring the insensitivity of the system motion to the matched uncertainties.

In this paper, we consider the following controlled uncertain system:

$$
\begin{aligned}
dz(t)/dt = Az(t)+Hz(t-h)+\int_{-h}^{0} G(\tau)z(t+\tau)d\tau \\
+[B+C(z(t),z(t-h),t)]u(t) \\
+w(z(t),z(t-h),t), \qquad z(\tau)=\varphi(\tau), \qquad \tau\in[-h,0]
\end{aligned}
\tag{1}
$$

where $t \in (0,T]$; $z(t) \in E^n$, $u(t) \in E^r$, $n \geq r$ (u is a control); $h \geq 0$ is a given constant time delay; $A, H, G(\tau)$, and B are given t-invariant matrices of corresponding dimensions; $\operatorname{rank} B = r$; $G(\tau)$ is piecewise continuous for $\tau \in [-h,0]$; $C(z(t),z(t-h),t)$ and $w(z(t),z(t-h),t)$ are an unmeasured control coefficient uncertainty and an unmeasured disturbance; $T > 0$ is a given control process duration; and $\varphi(\tau)$ is a given vector-function continuous for $\tau \in [-h,0]$.

The following is assumed.

A1) The uncertainty $C(z(t),z(t-h),t)$ and the disturbance $w(z(t),z(t-h),t)$ satisfy the following matching conditions: there exist an $r\times r$-matrix $U(z,\zeta,t)$ and an r-vector $\gamma(z,\zeta,t)$ such that

$$C(z,\zeta,t) = BU(z,\zeta,t), \quad \|U(z,\zeta,t)\| \leq \delta < 1 \tag{2}$$

$$w(z,\zeta,t) = B\gamma(z,\zeta,t), \quad \|\gamma(z,\zeta,t)\| \leq f(z,\zeta,t) \tag{3}$$

where $(z,\zeta,t) \in E^n \times E^n \times [0,T]$; δ and $f(z,\zeta,t)$ are a known positive constant and a known positive continuous function, respectively; and $\|\cdot\|$ denotes the Euclidean norm of either a matrix or a vector.

The performance index of the control process for (1) is

$$J(u) \triangleq \int_0^T [z'(t)Dz(t) + \varepsilon^2 u'(t)Mu(t)]dt \rightarrow \min_{u(t)} \tag{4}$$

where the prime denotes the transposition; D is symmetric positive semidefinite and M is symmetric positive definite matrices of corresponding dimensions; and $\varepsilon > 0$ is a small parameter ($\varepsilon \ll 1$), meaning that the control in (4) is "cheap" with respect to the state.

The cheap control problem has considerable importance in many topics of control theory and its applications (see, e.g., [5]–[7]). The smallness of the control cost yields the singular perturbation in the Hamilton–Jacobi–Bellman equation and in the Hamilton boundary-value problem, associated with the original problem by control optimality conditions. The cheap control problem for systems without delays was extensively studied (see, e.g., [6] and [8]–[10] and references therein). However, in spite of the considerable importance of studying delayed dynamics controlled systems (see, e.g., [11]–[13]), the cheap control problem for such systems was considered only in a few works [14], [15]. Also, only a few works study an application of the integral sliding mode approach to uncertain systems with delays (see [16] and [17], where systems with matched disturbances and control delays in the dynamics are considered).

In the sequel, we assume the following.

A2) $\det(B'DB) \neq 0$.

Transform the state and the control in (1) and (4) as follows:

$$
\begin{aligned}
z(t) &= LZ(t), \quad Z(t) = \operatorname{col}(x(t),y(t)) \\
u(t) &= \varepsilon^{-1}v(t)
\end{aligned}
\tag{5}
$$

where $x(t) \in E^{n-r}, y(t) \in E^r; L = (L_1, B), L_1 = B_c - B(B'DB)^{-1}B'DB_c$; and B_c is a complement matrix to B.

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V. Y. Glizer and V. Turetsky are with the Faculty of Aerospace Engineering, Technion—Israel Institute of Technology, Haifa 32000, Israel (e-mail: valery@techunix.technion.ac.il; turetsky@aerodyne.technion.ac.il).

L. M. Fridman is with the Faculty of Engineering, National Autonomous University of Mexico, CP 045510 Mexico City, Mexico (e-mail: lfridman@servidor.unam.mx).

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Due to (5), (1) and (4) become

$$\begin{aligned} dx(t)/dt = {} & F_1x(t) + F_2y(t) + K_1x(t-h) \\ & + K_2y(t-h) + \int_{-h}^{0}[N_1(\tau)x(t+\tau) \\ & + N_2(\tau)y(t+\tau)]d\tau,\ t\in(0,T] \end{aligned} \tag{6}$$

$$\begin{aligned} \varepsilon dy(t)/dt = {} & \varepsilon\Big\{F_3x(t) + F_4y(t) + K_3x(t-h) \\ & + K_4y(t-h) + \int_{-h}^{0}[N_3(\tau)x(t+\tau) \\ & + N_4(\tau)y(t+\tau)]d\tau\Big\} \\ & + (I_r + V(Z(t),Z(t-h),t))v(t) \\ & + \varepsilon\Gamma(Z(t),Z(t-h),t), \quad t\in(0,T] \end{aligned} \tag{7}$$

$$x(\tau) = \psi_x(\tau), \quad y(\tau) = \psi_y(\tau), \quad \tau\in[-h,0] \tag{8}$$

$$\begin{aligned} \mathcal{J}(v) \triangleq {} & \int_0^T [x'(t)D_xx(t) + y'(t)D_yy(t) \\ & + v'(t)Mv(t)]dt \to \min_{v(t)} \end{aligned} \tag{9}$$

where I_k is the k-dimensional identity matrix;

$$\mathrm{col}(\psi_x(\tau),\psi_y(\tau)) = L^{-1}\varphi(\tau) \triangleq \psi_Z(\tau);$$

and

$$\begin{aligned} \begin{pmatrix} F_1 & F_2 \\ F_3 & F_4 \end{pmatrix} &= L^{-1}AL \triangleq F \\ \begin{pmatrix} K_1 & K_2 \\ K_3 & K_4 \end{pmatrix} &= L^{-1}HL \triangleq K \\ \begin{pmatrix} N_1(\tau) & N_2(\tau) \\ N_3(\tau) & N_4(\tau) \end{pmatrix} &= L^{-1}G(\tau)L \triangleq N(\tau) \\ V(Z(t),Z(t-h),t) &= U(LZ(t),LZ(t-h),t) \\ \Gamma(Z(t),Z(t-h),t) &= \gamma(LZ(t),LZ(t-h),t) \\ D_x = L_1'DL_1, &\quad D_y = B'DB. \end{aligned}$$

In the sequel, we deal with this new control problem.

Note that dynamics (6) and (7) in this problem is singularly perturbed [9]. Due to (2) and (3), for all $(Z,\eta,t)\in E^n\times E^n\times[0,T]$

$$\begin{aligned} &\|V(Z,\eta,t)\| \le \delta < 1 \\ &\|\Gamma(Z,\eta,t)\| \le g(Z,\eta,t) \triangleq f(LZ,L\eta,t). \end{aligned} \tag{10}$$

The objective of this paper is to design a *robust controller* transferring system (6)–(8) to a preconstructed "manifold" in zero time, keeping it there until the end of the control process and providing the systems motion on this "manifold" to be suboptimal with respect to (9) for all sufficiently small $\varepsilon > 0$. This objective is achieved by two steps. First, the *nominal optimal control problem* (NOCP), associated with (6)–(9), is treated. The NOCP is obtained from (6)–(9) by setting there $V(\cdot)\equiv 0, \Gamma(\cdot)\equiv 0$, i.e., it consists of the *nominal system* [(6)–(8) with $V(\cdot)\equiv 0$ and $\Gamma(\cdot)\equiv 0$] and performance index (9). For the NOCP, a suboptimal state-feedback composite control is constructed by its asymptotic decomposition into two lower dimension ε-free subproblems (the slow and fast ones) and employing optimal feedback controls of the latter. Secondly, based on this control and using the integral sliding mode approach, a required robust controller for system (6)–(8) is designed.

To our best knowledge, the combination of the integral sliding mode approach and cheap control approach to design a controller, providing a robust and suboptimal motion of an uncertain system on some preconstructed "manifold," never has been considered in the literature. Such a combination allows one to design a controller gathering the advantages of both approaches.

II. Asymptotic Decomposition of NOCP

A. Slow Subproblem

Setting formally $\varepsilon = 0$ in the NOCP, and then redenoting in the resulting problem the variables x, y, v, and $\mathcal{J}$ by x_s, y_s, v_s, and $\mathcal{J}_s$, respectively, one obtains after some rearrangement

$$\begin{aligned} dx_s(t)/dt = {} & F_1x_s(t) + K_1x_s(t-h) \\ & + \int_{-h}^{0}N_1(\tau)x_s(t+\tau)d\tau \\ & + F_2y_s(t) + K_2y_s(t-h) \\ & + \int_{-h}^{0}N_2(\tau)y_s(t+\tau)d\tau, \qquad t\in(0,T] \end{aligned} \tag{11}$$

$$\begin{aligned} x_s(\tau) &= \psi_x(\tau), \qquad \tau\in[-h,0] \\ y_s(\tau) &= \psi_y(\tau), \qquad \tau\in[-h,0) \end{aligned} \tag{12}$$

$$v_s(t) = 0, \qquad t\in[0,T] \tag{13}$$

$$\mathcal{J}_s = \int_0^T [x_s'(t)D_xx_s(t) + y_s'(t)D_yy_s(t)]dt \to \min. \tag{14}$$

Since $y_s(t), t\in[0,T]$ does not satisfy any equation, the minimization of (14) can be fulfilled by a proper choice of $y_s(t), t\in[0,T]$, i.e., $y_s(t)$ can be considered as a control in (11), (12), and (14). This optimal control problem is called the reduced optimal control problem (ROCP). The ROCP has the delay not only in the state variable but also in the control variable. The ROCP, along with (13), constitutes the slow subproblem associated with the NOCP.

Based on results of [18], one directly obtains the following lemma.

Lemma 1: Under assumption A2), the optimal control of the ROCP exists, is unique, and has the feedback form

$$\begin{aligned} & y_s[x_s(t),x_{sh}(t),y_{sh}(t),t] \\ & = -D_y^{-1}\Big\{[F_2'P_s(t) + Q_{s2}'(t,0)]x_s(t) \\ & \quad + \int_{-h}^{0}[F_2'Q_{s1}(t,\tau) + R_{s1}'(t,\tau,0)]x_s(t+\tau)d\tau \\ & \quad + \int_{-h}^{0}[F_2'Q_{s2}(t,\tau) + R_{s2}(t,0,\tau)]y_s(t+\tau)d\tau\Big\} \end{aligned} \tag{15}$$

where $x_{sh}(t) = \{x_s(t+\tau)\,\forall\tau\in[-h,0)\}, y_{sh}(t) = \{y_s(t+\tau)\forall\tau\in[-h,0)\}$; the matrices $P_s(t), Q_{si}(t,\tau), R_{si}(t,\tau,\rho), (i=1,2)$, along with an additional matrix $R_{s0}(t,\tau,\rho)$, form the unique solution of the set of Riccati-type functional-differential equations in the domain $\mathcal{T} = \{(t,\tau,\rho): t\in[0,T], \tau\in[-h,0], \rho\in[-h,0]\}$

$$\begin{aligned} & dP_s(t)/dt \\ & = -P_s(t)F_1 - F_1'P_s(t) - Q_{s1}(t,0) \\ & \quad - Q_{s1}'(t,0) - D_x + [P_s(t)F_2 + Q_{s2}(t,0)]D_y^{-1} \\ & \quad \times [P_s(t)F_2 + Q_{s2}(t,0)]', P_s(T) = 0 \end{aligned} \tag{16}$$

$$\begin{aligned} & (\partial/\partial t - \partial/\partial\tau)Q_{s1}(t,\tau) \\ & = -F_1'Q_{s1}(t,\tau) - P_s(t)N_1(\tau) - R_{s0}(t,0,\tau) \\ & \quad + [P_s(t)F_2 + Q_{s2}(t,0)]D_y^{-1}[F_2'Q_{s1}(t,\tau) \\ & \quad + R_{s1}'(t,\tau,0)], \quad Q_{s1}(T,\tau) \\ & = 0, \quad Q_{s1}(t,-h) = P_s(t)K_1 \end{aligned} \tag{17}$$

$$\begin{aligned}
&(\partial/\partial t - \partial/\partial\tau)Q_{s2}(t,\tau)\\
&\quad = -F_1'Q_{s2}(t,\tau) - P_s(t)N_2(\tau) - R_{s1}(t,0,\tau)\\
&\qquad + [P_s(t)F_2 + Q_{s2}(t,0)]D_y^{-1}[F_2'Q_{s2}(t,\tau)\\
&\qquad + R_{s2}(t,0,\tau)], \quad Q_{s2}(T,\tau)\\
&\quad = 0,\ Q_{s2}(t,-h) = P_s(t)K_2 \qquad (18)\\
&(\partial/\partial t - \partial/\partial\tau - \partial/\partial\rho)R_{s0}(t,\tau,\rho)\\
&\quad = -N_1'(\tau)Q_{s1}(t,\rho) - Q_{s1}'(t,\tau)N_1(\rho)\\
&\qquad + [Q_{s1}'(t,\tau)F_2 + R_{s1}(t,\tau,0)]D_y^{-1}\\
&\qquad \times [F_2'Q_{s1}(t,\rho) + R_{s1}'(t,\rho,0)], \quad R_{s0}(T,\tau,\rho) = 0 \qquad (19)\\
&(\partial/\partial t - \partial/\partial\tau - \partial/\partial\rho)R_{s1}(t,\tau,\rho)\\
&\quad = -N_1'(\tau)Q_{s2}(t,\rho) - Q_{s1}'(t,\tau)N_2(\rho)\\
&\qquad + [Q_{s1}'(t,\tau)F_2 + R_{s1}(t,\tau,0)]D_y^{-1}\\
&\qquad \times [F_2'Q_{s2}(t,\rho) + R_{s2}(t,0,\rho)], \quad R_{s1}(T,\tau,\rho) = 0 \qquad (20)\\
&(\partial/\partial t - \partial/\partial\tau - \partial/\partial\rho)R_{s2}(t,\tau,\rho)\\
&\quad = -N_2'(\tau)Q_{s2}(t,\rho) - Q_{s2}'(t,\tau)N_2(\rho)\\
&\qquad + [Q_{s2}'(t,\tau)F_2 + R_{s2}(t,\tau,0)]D_y^{-1}\\
&\qquad \times [F_2'Q_{s2}(t,\rho) + R_{s2}(t,0,\rho)], \quad R_{s2}(T,\tau,\rho) = 0 \qquad (21)\\
&R_{s0}(t,-h,\tau)\\
&\quad = K_1'Q_{s1}(t,\tau), \quad R_{s0}(t,\tau,-h) = Q_{s1}'(t,\tau)K_1\\
&R_{s1}(t,-h,\tau)\\
&\quad = K_1'Q_{s2}(t,\tau), \quad R_{s1}(t,\tau,-h) = Q_{s1}'(t,\tau)K_2\\
&R_{s2}(t,-h,\tau)\\
&\quad = K_2'Q_{s2}(t,\tau), \quad R_{s2}(t,\tau,-h) = Q_{s2}'(t,\tau)K_2. \qquad (22)
\end{aligned}$$

Remark 1: Although (16)–(22) look like an unsimple problem, it is simpler (less dimensional) than the set of Riccati-type matrix functional-differential equations associated with the NOCP by the control optimality conditions (see, e.g., [18]). Taking into account the symmetry of the respective unknown matrices in both sets, one obtains that in the latter set the number of the unknown scalar functions, depending on t, (t,τ) and (t,τ,ρ), is $n(n+1)/2$, n^2, and $n(n+1)/2$, respectively, while in (16)–(22), the number of such functions is $(n-r)(n-r+1)/2$, $n(n-r)$, and $n(n+1)/2$, respectively. Moreover, the set of Riccati-type equations, associated with the NOCP, depends on ε, while (16)–(22) is ε-free, which allows one to solve it once and then to use its solution in designing the NOCP suboptimal composite control for all sufficiently small $\varepsilon > 0$.

Remark 2: There are two main approaches to approximate solution of (16)–(22). The first one is based on a finite-difference approximation of the derivatives with respect to τ and ρ (see, e.g., [19] and references therein). The second one is based on an iterative solution with an approximate linearizing the problem at each iteration (see, e.g., [20] and references therein). In the case $K_2 = 0$, $N_2(\cdot) \equiv 0$, i.e., when the slow subsystem does not contain the delayed fast state variable, the ROCP does not contain the delayed control. The latter circumstance allows one to simplify (16)–(22) considerably. In this case, $Q_{s2}(\cdot) \equiv 0$, $R_{s1}(\cdot) \equiv 0$, $R_{s2}(\cdot) \equiv 0$; (16)–(22) becomes the set of three (16), (17), and (19) with the respective boundary conditions for $R_{s0}(t,\tau,\rho)$ from (22), while (15) becomes $y_s[x_s(t), x_{sh}(t), t] = -D_y^{-1}F_2'[P_s(t)x_s(t) + \int_{-h}^0 Q_{s1}(t,\tau)x_s(t+\tau)d\tau]$. Further simplification is obtained in the undelayed case ($h = 0$), where (16)–(22) becomes (16) with $Q_{si}(t,0) \equiv 0, (i = 1,2)$ and (15) becomes $y_s[x_s(t), t] = -D_y^{-1}F_2'P_s(t)x_s(t)$.

Remark 3: An important feature of the ROCP optimal control is that, at any current time-instant t, this control depends on the state and control values distributed over the interval $[t-h, t)$, i.e., on the distributed state and control delays. It was shown in [21] that a digital computer implementation of a control law, depending on a distributed control delay, is a very unsimple task. Fortunately, the ROCP optimal control is only an intermediate result needed for obtaining a composite suboptimal control in the NOCP. This composite control is independent of a distributed control delay (see Section III), although it does depend on a distributed state delay. The latter circumstance does not create problems for its digital computer implementation (see Section V).

B. Fast Subproblem

Using the fast subsystem of the NOCP, corresponding to (7), and cost functional (9), the fast subproblem is obtained in the way.

1) The state $x(\cdot)$ is removed from the fast subsystem of the NOCP and (9).
2) The transformation of variables $t = \varepsilon\xi$, $y(\varepsilon\xi) = y_f(\xi)$, $v(\varepsilon\xi) = v_f(\xi)$, $\mathcal{J}(v(\varepsilon\xi)) = \varepsilon\mathcal{J}_f(v_f(\xi))$ is made in the resulting equations, where ξ, y_f, v_f, and $\mathcal{J}_f$ are new independent variable, state, control, and cost functional.

Thus, one has the problem

$$\begin{aligned}
dy_f(\xi)/d\xi &= \varepsilon[F_4 y_f(\xi) + K_4 y_f(\xi - h/\varepsilon)\\
&\quad + \int_{-h}^0 N_4(\tau)y_f(\xi + \tau/\varepsilon)d\tau] + v_f(\xi),\\
&\qquad \xi \in (0, T/\varepsilon]\\
\mathcal{J}_f(v_f) &= \int_0^{T/\varepsilon}[y_f'(\xi)D_y y_f(\xi) + v_f'(\xi)Mv_f(\xi)]d\xi \to \min_{v_f(\xi)}.
\end{aligned} \tag{23}$$

Neglecting in (23) the terms multiplied by ε, and replacing there T/ε by $+\infty$, yields the fast subproblem associated with the NOCP

$$\begin{aligned}
dy_f(\xi)/d\xi &= v_f(\xi), \qquad \xi > 0\\
\mathcal{J}_f(v_f) &= \int_0^{+\infty}[y_f'(\xi)D_y y_f(\xi)\\
&\quad + v_f'(\xi)Mv_f(\xi)]d\xi \to \min_{v_f(\xi)}.
\end{aligned} \tag{24}$$

Using [22], one directly has the following lemma.

Lemma 2: Under assumption A2), the fast subproblem (24) with a given $y_f(0)$ has the unique state-feedback optimal control

$$v_f[y_f(\xi)] = -M^{-1}P_f y_f(\xi) \tag{25}$$

where P_f is the unique positive definite solution of the algebraic Riccati equation $P_f M^{-1} P_f - D_y = 0$. The optimal trajectory $y_f(\xi)$ of this problem satisfies the inequality $\|y_f(\xi)\| \le a\exp(-\beta\xi)\|y_f(0)\|, \xi \ge 0$. Here and in the sequel, a and β denote some positive constants independent of ε.

III. Composite Suboptimal Control of NOCP

The algorithm of the formal designing the composite control consists of two stages. At the first stage, the following auxiliary control function v_a is constructed:

$$v_a[Z(t), Z_h(t), t] = v_s(t) + v_f[y(t) - y_s[x(t), x_h(t), \bar{y}_h(t), t]] \tag{26}$$

where $v_s(t)$ and $v_f[\cdot]$ are given by (13) and (25); $Z_h(t) \triangleq \{Z(t+\tau)\ \forall\tau \in [-h,0)\}$; $\bar{y}_h(t) \triangleq y_h(t) - y_{fh}(t/\varepsilon), y_{fh}(t/\varepsilon) \triangleq \{y_f[(t+\tau)/\varepsilon]\ \forall\tau \in [-h,0]\}, y_f(\xi) = 0\ \forall\xi < 0$.

Calculating (26) yields, after some rearrangement

$$\begin{aligned} v_a&[Z(t), Z_h(t), t] \\ &= -M^{-1}P_f \\ &\times \Bigg\{y(t) + D_y^{-1}\Bigg[(F_2'P_s(t) \\ &\quad + Q_{s2}'(t,0))x(t) + \int_{-h}^{0}(F_2'Q_{s1}(t,\tau) \\ &\quad + R_{s1}'(t,\tau,0))x(t+\tau)d\tau + \int_{-h}^{0}(F_2'Q_{s2}(t,\tau) \\ &\quad + R_{s2}(t,0,\tau))(y(t+\tau) - y_f[(t+\tau)/\varepsilon])d\tau\Bigg]\Bigg\}. \end{aligned} \tag{27}$$

It is seen that (27) depends not only on $x(\cdot)$ and $y(\cdot)$ but also on $y_f(\cdot)$. At the second stage, we eliminate $y_f(\cdot)$ from $v_a(\cdot)$. Using Lemma 2 yields the estimate for all sufficiently small $\varepsilon > 0$ and $t \in [0,T]$: $\|\int_{-h}^{0}(F_2'Q_{s2}(t,\tau)+R_{s2}(t,0,\tau))y_f[(t+\tau)/\varepsilon]d\tau\| \le a\varepsilon$. Due to the latter, the term in (27), depending on $y_f(\cdot)$, vanishes as $\varepsilon \to +0$. Neglecting this term converts (27) to the composite state-feedback control of the NOCP

$$\begin{aligned} v_c&[Z(t), Z_h(t), t] \\ &= -M^{-1}P_f \\ &\times \Bigg\{y(t) + D_y^{-1}\Bigg[(F_2'P_s(t) \\ &\quad + Q_{s2}'(t,0))x(t) + \int_{-h}^{0}(F_2'Q_{s1}(t,\tau) \\ &\quad + R_{s1}'(t,\tau,0))x(t+\tau)d\tau + \int_{-h}^{0}(F_2'Q_{s2}(t,\tau) \\ &\quad + R_{s2}(t,0,\tau))y(t+\tau)d\tau\Bigg]\Bigg\}. \end{aligned} \tag{28}$$

Let $\mathcal{J}_\varepsilon^*$ be the optimal value of the cost functional in the NOCP and $\mathcal{J}_\varepsilon(v_c)$ be the value of this cost functional obtained by employing $v(t) = v_c[Z(t), Z_h(t), t]$ in the NOCP.

Theorem 1: Under assumption A2), there exists a number $\varepsilon_* > 0$ such that the following inequality is satisfied for all $\varepsilon \in (0, \varepsilon_*]$: $0 < \mathcal{J}_\varepsilon(v_c) - \mathcal{J}_\varepsilon^* \le a\varepsilon^2(\|\psi_Z(\tau)\|_C)^2$, where $\|\cdot\|_C$ denotes the uniform norm in the space $C[b,c;E^k]$ of k-dimensional vector functions, defined and continuous on a closed interval $[b,c]$.

Proof: The theorem is a direct consequence of [15, Theorem 5.1].

IV. Robust Control of Original System (6)–(8)

In this section, a robust controller for (6)–(8) is designed using the integral sliding mode approach. For any given $\varepsilon > 0$, consider the following integral sliding mode "manifold" subject to (8):

$$\begin{aligned} \sigma&[Z(\cdot), Z_h(\cdot)](t) \\ &\triangleq y(t) - \psi_y(0) \\ &- \int_0^t \Bigg\{F_3x(s) + F_4y(s) \\ &\quad + K_3x(s-h) + K_4y(s-h) \\ &\quad + \int_{-h}^{0}[N_3(\tau)x(s+\tau) + N_4(\tau)y(s+\tau)]d\tau \\ &\quad + \varepsilon^{-1}v_c[Z(s), Z_h(s), s]\Bigg\}ds = 0, \quad t \in [0,T]. \end{aligned} \tag{29}$$

Let $Z(t)$, $t \in [0,T]$ be a solution of (6)–(8) for some given $v(t)$.

Theorem 2: Let assumption A2) and inequalities (10) be satisfied. Let $d\sigma[Z(\cdot), Z_h(\cdot)](t)/dt = 0$ for some $t \in (0,T]$. Then, for this t and any given $\varepsilon > 0$, $Z(t)$ satisfies the nominal system with $v(t) = v_c[Z(t), Z_h(t), t]$.

Proof: By using (29), the equality $d\sigma[Z(\cdot), Z_h(\cdot)](t)/dt = 0$ yields

$$\begin{aligned} dy(t)/dt &- F_3x(t) - F_4y(t) - K_3x(t-h) \\ &- K_4y(t-h) - \int_{-h}^{0}[N_3(\tau)x(t+\tau) \\ &+ N_4(\tau)y(t+\tau)]d\tau - \varepsilon^{-1}v_c[Z(t), Z_h(t), t] = 0. \end{aligned} \tag{30}$$

Substituting (7) into (30), one obtains after some rearrangement the algebraic equation with respect to $v(t)$. This equation, due to the first inequality in (10), has the unique solution

$$\begin{aligned} v(t) = [&I_r + V(Z(t), Z(t-h), t)]^{-1} \\ &\times\{v_c[Z(t), Z_h(t), t] \quad - \varepsilon\Gamma(Z(t), Z(t-h), t)\}. \end{aligned} \tag{31}$$

Substituting (31) into (7) directly yields the statement of the theorem.

Corollary 1: Let assumption A2) and (10) be satisfied. Let (29) be satisfied on some interval $(t_1, t_2] \subseteq (0,T]$. Then, for all $t \in (t_1, t_2]$ and any given $\varepsilon > 0$, $Z(t)$ satisfies the nominal system with $v(t) = v_c[Z(t), Z_h(t), t]$.

Proof: Since (29) is satisfied for all $t \in (t_1, t_2]$, $d\sigma[Z(\cdot), Z_h(\cdot)](t)/dt = 0$ for these t. Now, the corollary is a direct consequence of Theorem 2.

Let us extend the approach, suggested in [23] for the case of undelayed systems, to the case of system (6)–(8). Namely, let us consider the following integral sliding mode controller:

$$\begin{aligned} v &= v_{\mathrm{ism}}^{so}[Z(t), Z_h(t), t] \\ &= v_{vs}[Z(t), Z_h(t), t] + v_c[Z(t), Z_h(t), t] \end{aligned} \tag{32}$$

where the variable structure part v_{vs} has the form shown in (33) at the bottom of the page, and, for any fixed $t \in [0,T]$, $\chi[Z(t), Z_h(t), t]$ is a given functional defined on the direct product $E^n \times C[-h,0;E^n]$ and satisfying the inequality

$$\begin{aligned} \chi[Z(t), Z_h(t), t] \ge \{&\delta\|v_c[Z(t), Z_h(t), t]\| + \varepsilon g(Z(t) \\ &Z(t-h), t) + \alpha\}/(1-\delta) \end{aligned} \tag{34}$$

$\alpha > 0$ is a given constant. In the sequel, $v_{\mathrm{ism}}^{so}[Z(t), Z_h(t), t]$ is called the *suboptimal integral sliding mode controller.*

$$v_{vs}[Z(t), Z_h(t), t] = \begin{cases} -\{\chi[Z(t), Z_h(t), t]/\|\sigma[Z(\cdot), Z_h(\cdot)](t)\|\}\sigma[Z(\cdot), Z_h(\cdot)](t), & \text{if } \sigma[Z(\cdot), Z_h(\cdot)](t) \ne 0, \\ 0, & \text{if } \sigma[Z(\cdot), Z_h(\cdot)](t) = 0 \end{cases} \tag{33}$$

Theorem 3: Let assumption A2) and (10) be satisfied. Then, for any given $\varepsilon > 0$, the trajectory $Z = Z(t)$ of (6)–(8) subject to control (32) satisfies (29) for all $t \in [0, T]$.

Proof: Consider the Lyapunov–Krasovskii functional

$$W[Z(\cdot), Z_h(\cdot)](t) = (1/2)\sigma'[Z(\cdot), Z_h(\cdot)](t)\sigma[Z(\cdot), Z_h(\cdot)](t). \qquad (35)$$

Differentiating $W[Z(\cdot), Z_h(\cdot)](t)$ with respect to t along the trajectory $Z(\cdot)$ of (6)–(8), (32) yields, after some rearrangement

$$\begin{aligned} dW[Z(\cdot), Z_h(\cdot)](t)/dt &= \sigma'[Z(\cdot), Z_h(\cdot)](t) d\sigma[Z(\cdot), Z_h(\cdot)](t)/dt \\ &= \sigma'[Z(\cdot), Z_h(\cdot)](t) \{\varepsilon^{-1}[I_r \\ &\quad + V(Z(t), Z(t-h), t)]v_{\mathrm{ism}}^{so}[Z(t), Z_h(t), t] \\ &\quad + \Gamma(Z(t), Z(t-h), t) \\ &\quad - \varepsilon^{-1} v_c[Z(t), Z_h(t), t]\}, \qquad t \in [0, T]. \end{aligned} \qquad (36)$$

Let $\sigma[Z(\cdot), Z_h(\cdot)](t) \neq 0$ at some $t \in [0, T]$. Using (32), (33), and (36), one has

$$\begin{aligned} dW[Z(\cdot), Z_h(\cdot)](t)/dt &= -\chi[Z(t), Z_h(t), t]\|\sigma[Z(\cdot), Z_h(\cdot)](t)\|/\varepsilon \\ &\quad + \sigma'[Z(\cdot), Z_h(\cdot)](t)\{\varepsilon^{-1} V(Z(t), Z(t-h), t) \\ &\quad \times v_{\mathrm{ism}}^{so}[Z(t), Z_h(t), t] + \Gamma(Z(t), Z(t-h), t)\} \end{aligned} \qquad (37)$$

yielding, by using (10) and (34)

$$dW[Z(\cdot), Z_h(\cdot)](t)/dt \leq -\varepsilon^{-1}\alpha\|\sigma[Z(\cdot), Z_h(\cdot)](t)\| \quad \sigma[Z(\cdot), Z_h(\cdot)](t) \neq 0. \qquad (38)$$

Due to (8), (29), and (35), $W[Z(\cdot), Z_h(\cdot)](t) = 0$ for $t = 0$. The latter, along with (35) and (38), implies that $W[Z(\cdot), Z_h(\cdot)](t) \equiv 0 \ \forall t \in [0, T]$. Hence, $\sigma[Z(\cdot), Z_h(\cdot)](t) \equiv 0 \ \forall t \in [0, T]$, which completes the proof of the theorem.

The following proposition presents the case where, for all sufficiently small $\varepsilon > 0$, the functional $\chi[Z(t), Z_h(t), t]$ can be chosen subject to a simpler inequality than (34).

Corollary 2: Let assumption A2) and (10) be satisfied. Let a constant $\mu > 0$ exist such that $g(Z, \eta, t) \leq \mu \ \forall (Z, \eta, t) \in E^n \times E^n \times [0, T]$. Let, for some constant $\alpha > 0$, the functional $\chi[Z(t), Z_h(t), t]$ satisfy the inequality $\chi[Z(t), Z_h(t), t] \geq \{\delta\|v_c[Z(t), Z_h(t), t]\| + \alpha\}/(1-\delta)$. Then, for any $\varepsilon \in (0, \alpha/\mu)$, the trajectory $Z = Z(t)$ of (6)–(8) subject to control (32) satisfies (29) for all $t \in [0, T]$.

Proof: Considering Lyapunov–Krasovskii functional (35) and using the same arguments as in the proof of Theorem 3, one obtains $dW[Z(\cdot), Z_h(\cdot)](t)/dt \leq -\varepsilon^{-1}(\alpha - \varepsilon g(Z(t), Z_h(t), t))\|\sigma[Z(\cdot), Z_h(\cdot)]\|, \sigma[Z(\cdot), Z_h(\cdot)](t) \neq 0$.

This inequality, along with $g(Z, \eta, t) \leq \mu$, directly yields the statement of the corollary.

Remark 4: Two important features of the integral sliding mode controller (32) should be emphasized. First, the variable structure part (33) of (32) is not derived as a solution of an optimal control problem. Therefore, cost functional (9) is not used directly in the design of (33). This cost functional is used directly only for design of the composite state-feedback control (28) of the NOCP, the second term in (32). Due to Theorems 2 and 3 and Corollaries 1 and 2, the first term in (32) is responsible for keeping the original uncertain system on the "manifold" (29), where its motion coincides with the motion of the nominal system subject to (28). According to Theorem 1, the second term in (32) makes the motion of the original system on (29) be suboptimal with respect to (9). Due to such a structure of controller (32), cost functional (9) calculated along the trajectory of the original system, generated by (32), depends, in general, on the uncertainties, while the trajectory is insensitive to the uncertainties. The value of cost functional (9), in which only the second term of (32) is taken for the calculation, is robust with respect to the uncertainties. Moreover, if the uncertainties vanish, the full value of cost functional (9) coincides with $\mathcal{J}_\varepsilon(v_c)$. The second feature is that (32) transfers the original system from its initial position to (29) in zero time, i.e., the system motion, generated by this control, is a sliding mode from $t = 0$ to $t = T$.

V. EXAMPLE

Consider a particular case of (1) and (4) with $n = 2, r = 1, h = 0.5$, $T = 2$, and

$$A = \begin{pmatrix} 3 & -2 \\ 2 & -3 \end{pmatrix}, \quad H = \begin{pmatrix} -1.6 & 5.4 \\ -0.4 & 2.6 \end{pmatrix}$$
$$G(\tau) \equiv 0, B = \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \quad D = \begin{pmatrix} 0.24 & -0.56 \\ -0.56 & 1.64 \end{pmatrix}, \quad M = 1$$

$\varphi(\tau) = \mathrm{col}(16\tau + 7,\ 9\tau + 3)$. The matrix $C(\cdot)$ and the vector $w(\cdot)$ in (1) satisfy (2) and (3) with $\delta = 0.5$, $f(\cdot) \equiv 1$. Based on these data, one obtains the matrix L in (5)

$$L = \begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix}$$

yielding (6)–(9) as

$$\begin{aligned} dx(t)/dt &= x(t) + 2y(t) - x(t-0.5) \\ x(\tau) &= \tau + 1, \tau \in [-0.5, 0] \\ \varepsilon dy(t)/dt &= \varepsilon[2x(t) - y(t) + x(t-0.5) + 2y(t-0,5)] \\ &\quad + [1 + V(Z(t), Z(t-0.5), t)]v(t) \\ &\quad + \varepsilon\Gamma(Z(t), Z(t-0.5), t); \quad y(\tau) = 4\tau + 1, \\ &\qquad \tau \in [-0.5, 0] \end{aligned} \qquad (39)$$

$$\mathcal{J}(v) \triangleq \int_0^2 [x^2(t) + 2y^2(t) + v^2(t)]dt \to \min_{v(t)}. \qquad (40)$$

In (39), $t \in (0, 2]$; $Z(\cdot) = \mathrm{col}(x(\cdot), y(\cdot))$; and $V(\cdot)$ and $\Gamma(\cdot)$ satisfy (10) with $\delta = 0.5$ and $g(Z, \eta, t) \equiv 1$. Due to (11), (12), and (14), the ROCP, associated with (39) and (40), has the form

$$\begin{aligned} dx_s(t)/dt &= x_s(t) - x_s(t-0.5) + 2y_s(t), \quad t \in (0, 2] \\ x_s(\tau) &= \tau + 1,\ \tau \in [-0.5, 0] \\ \mathcal{J}_s(y_s) &\triangleq \int_0^2 \left[x_s^2(t) + 2y_s^2(t)\right] dt \to \min_{y_s(t)}. \end{aligned} \qquad (41)$$

By Lemma 1 and Remark 2, the optimal feedback control of (41) is

$$y_s[x_s(t), x_{sh}(t), t] = -\left[P_s(t)x_s(t) + \int_{-0.5}^0 Q_{s1}(t, \tau)x_s(t+\tau)d\tau\right] \qquad (42)$$

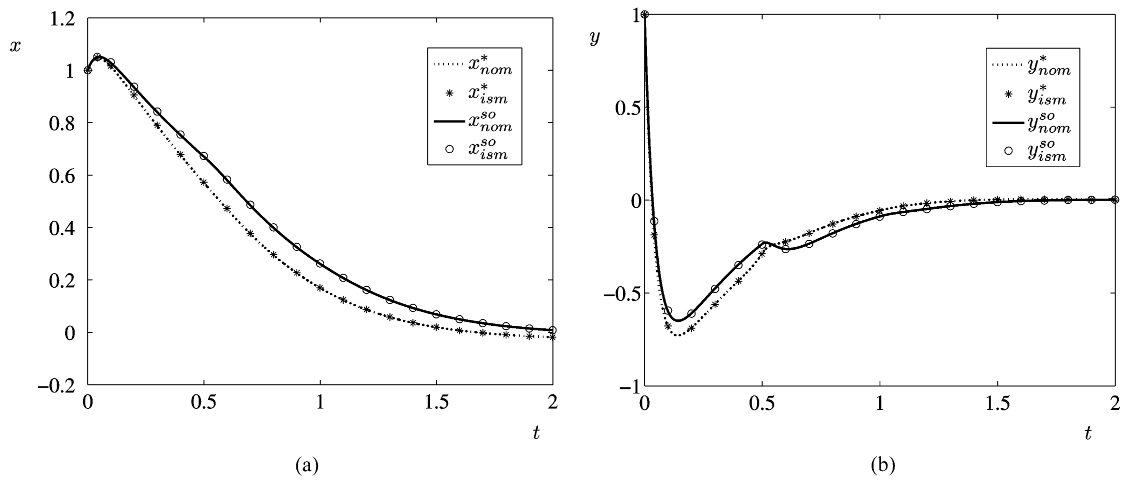

Fig. 1. Integral sliding modes and nominal trajectories.

where $P_s(t)$ and $Q_{s1}(t,\tau)$ are obtained from the unique solution of the set of equations

$$\begin{aligned} &dP_s(t)/dt \\ &\quad = -2P_s(t) - 2Q_{s1}(t,0) - 1 + 2P_s^2(t), \qquad P_s(2) = 0 \\ &(\partial/\partial t - \partial/\partial\tau)Q_{s1}(t,\tau) \\ &\quad = -Q_{s1}(t,\tau)[1-2P_s(t)] - R_{s0}(t,0,\tau), \qquad Q_{s1}(2,\tau) = 0 \\ &(\partial/\partial t - \partial/\partial\tau - \partial/\partial\rho)R_{s0}(t,\tau,\rho) \\ &\quad = 2Q_{s1}(t,\tau)Q_{s1}(t,\rho), \qquad R_{s0}(2,\tau,\rho) = 0 \\ &Q_{s1}(t,-0.5) \\ &\quad = -P_s(t), \qquad R_{s0}(t,-0.5,\tau) = R_{s0}(t,\tau,-0.5) \\ &\quad = -Q_{s1}(t,\tau). \end{aligned} \tag{43}$$

In this example, (43) is solved numerically by using the first-order finite-difference approximation of the derivatives with respect to τ and ρ, and then by applying the Euler method to the obtained set of ordinary differential equations.

Due to (24), the fast subproblem, associated with (39) and (40), has the form

$$\begin{aligned} &dy_f(\xi)/d\xi = v_f(\xi) \\ &\mathcal{J}_f(v_f) \triangleq \int_0^{+\infty} \left[2y_f^2(\xi) + v_f^2(\xi)\right] d\xi \to \min_{v_f(\xi)}. \end{aligned} \tag{44}$$

By Lemma 2, the optimal feedback control of (44) is $v_f[y_f(\xi)] = -\sqrt{2}y_f(\xi)$. Now, using the latter, along with (42) and results of Section III, one obtains the composite suboptimal control for the NOCP, associated with (39) and (40)

$$\begin{aligned} v_c[Z(t),Z_h(t),t] = -\sqrt{2}\Bigg\{&P_s(t)x(t) + y(t) \\ &+ \int_{-0.5}^{0} Q_{s1}(t,\tau)x(t+\tau)d\tau\Bigg\}. \end{aligned} \tag{45}$$

Based on (45), the integral sliding mode manifold (29) becomes in this example as

$$\begin{aligned} &\sigma[Z(\cdot),Z_h(\cdot)](t) \\ &\quad \triangleq y(t) - 1 - \int_0^t \{2x(s) - y(s) \\ &\quad + x(s-0.5) + 2y(s-0.5) \\ &\quad + \varepsilon^{-1}v_c[Z(s),Z_h(s),s]\}ds = 0, \qquad t\in[0,2]. \end{aligned} \tag{46}$$

TABLE I
VALUES OF J^{so}_{nom}, J^*_{nom}, J^{so}_{ism}, AND J^*_{ism}

ε	0.1	0.08	0.06	0.04	0.02
J^{so}_{nom}	1.2353	1.0839	0.9535	0.8387	0.7351
J^*_{nom}	1.1884	1.0605	0.9433	0.8352	0.7345
J^{so}_{ism}	1.2457	1.0935	0.9626	0.8472	0.7433
J^*_{ism}	1.1996	1.0708	0.9529	0.8441	0.7427

The suboptimal integral sliding mode controller $v^{so}_{\mathrm{ism}}[y(t),y_h(t),t]$ is obtained by using (32) and (33) with $v_c[Z(t),Z_h(t),t]$ and $\sigma[Z(\cdot),Z_h(\cdot)](t)$ given by (45) and (46). Moreover, using Corollary 2 and that $\delta = 0.5$, one can take $\chi[Z(t),Z_h(t),t] = |v_c[Z(t),Z_h(t),t]| + 0.22$, implying that the controller $v^{so}_{\mathrm{ism}}[Z(t),Z_h(t),t]$ is robust at least for $\varepsilon \in (0,0.11)$. This controller is compared to the integral sliding mode controller $v^*_{\mathrm{ism}}[Z(t),Z(t)_h,t]$, constructed by using (29), (32), and (33) where the composite control $v_c[Z(t),Z_h(t),t]$ is replaced by the optimal state-feedback control $v^*[Z(t),Z_h(t),t]$ of the NOCP. The set of Riccati-type functional-differential equations, associated with this control, is solved similarly to (43). The functional $\chi[\cdot]$ in the expression for $v^*_{\mathrm{ism}}[\cdot]$ has the same form as in the expression for $v^{so}_{\mathrm{ism}}[\cdot]$ with replacing $v_c[\cdot]$ by $v^*[\cdot]$.

Employing $v^{so}_{\mathrm{ism}}[\cdot]$ and $v^*_{\mathrm{ism}}[\cdot]$ in the original uncertain system (39) yields the integral sliding modes $Z^{so}_{\mathrm{ism}}(t) = \mathrm{col}(x^{so}_{\mathrm{ism}}(t), y^{so}_{\mathrm{ism}}(t))$ and $Z^*_{\mathrm{ism}}(t) = \mathrm{col}(x^*_{\mathrm{ism}}(t), y^*_{\mathrm{ism}}(t))$, respectively. These integral sliding modes are evaluated by cost functional (40). Let J^{so}_{ism} and J^*_{ism} be the respective values of this cost functional. Also, let J^*_{nom} be the optimal value of the cost functional in the NOCP and J^{so}_{nom} be the value of the same cost functional obtained by employing the composite control in the nominal system. In Table I, all these values are presented as functions of ε. The calculations for (39) are carried out with $V[Z(t),Z(t-0.5),t] \equiv -0.5$, $\Gamma[Z(t),Z(t-0.5),t] \equiv 1$. The numerical solution of the set of Riccati-type equations, associated with $v^*[Z(t),Z_h(t),t]$, is obtained for the numbers of mesh points of 150 in τ and ρ, and of 2400 in t. These numbers provide the accuracy of the respective value of the cost functional in the NOCP to be within 0.25% for all considered values of ε. The numerical solution of (43) is obtained for the same numbers of mesh points. It is seen from Table I that $0 < J^{so}_{\mathrm{nom}} - J^*_{\mathrm{nom}} < 4.69\varepsilon^2$ for $\varepsilon \in (0,0.1]$. The initial function [see (39)] is $\psi_Z(\tau) = \mathrm{col}(\tau+1, 4\tau+1)$, $\tau \in [-0.5, 0]$ and $\|\psi_Z(\tau)\|_C = \sqrt{2}$. Thus, one directly obtains $0 < J^{so}_{\mathrm{nom}} - J^{*}_{\mathrm{nom}} < 2.35\varepsilon^2(\|\psi_Z(\tau)\|_C)^2$ for $\varepsilon \in (0, 0.1]$, which accords Theorem 1. Moreover, one has from Table I that $0 < J^{so}_{\mathrm{ism}} - J^{*}_{\mathrm{ism}} < 2.31\varepsilon^2(\|\psi_Z(\tau)\|_C)^2$ for $\varepsilon \in (0, 0.1]$. Comparison of this inequality to the previous one for J^{so}_{nom} and J^{*}_{nom} shows that, in spite of the presence of the uncertainties in the original system, the suboptimal integral sliding mode controller keeps the same accuracy of the corresponding value of the cost functional as is obtained for the composite control in the NOCP.

In Fig. 1(a), the first components of the integral sliding modes $x^{so}_{\mathrm{ism}}(t)$ and $x^{*}_{\mathrm{ism}}(t)$, as well as the first components of the corresponding nominal trajectories $x^{so}_{\mathrm{nom}}(t)$ and $x^{*}_{\mathrm{nom}}(t)$, are depicted for $\varepsilon = 0.06$. In Fig. 1(b), the respective second components are shown. It is seen that the integral sliding modes practically coincide with the respective nominal trajectories. Moreover, the integral sliding mode $Z^{so}_{\mathrm{ism}}(t) = \mathrm{col}(x^{so}_{\mathrm{ism}}(t), y^{so}_{\mathrm{ism}}(t))$ is close to $Z^{*}_{\mathrm{ism}}(t) = \mathrm{col}(x^{*}_{\mathrm{ism}}(t), y^{*}_{\mathrm{ism}}(t))$.

VI. Conclusion

The controlled system with matched uncertainties and point-wise and distributed state delays was considered. The control cost in the performance index for this system is small with respect to the state cost. Using the singular perturbation technique and the integral sliding mode approach, an integral sliding mode "manifold" and a respective robust controller were designed for this system. It was shown that this controller transfers the system from its initial position to the integral sliding mode "manifold" in zero time, keeps this system on the "manifold" until the end of the control process, and provides the system motion on the "manifold" (the sliding mode) to be suboptimal with respect to the cheap control performance index.

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