Mini–Max Integral Sliding-Mode Control for Multimodel Linear Uncertain Systems

A. Poznyak, L. Fridman, and F. J. Bejarano

***Abstract*—An original linear time-varying system with matched and unmatched disturbances and uncertainties is replaced by a finite set of dynamic models such that each one describes a particular uncertain case including exact realizations of possible dynamic equations as well as external unmatched bounded disturbances. Such a tradeoff between an original uncertain linear time varying dynamic system and a corresponding higher order multimodel system containing only matched uncertainties leads to a linear multi-model system with known unmatched bounded disturbances and unknown matched disturbances as well. Each model from a given finite set is characterized by a quadratic performance index. The developed minimax integral sliding mode control strategy gives an optimal minimax linear quadratic (LQ)-control with additional integral sliding mode term. The design of this controller is reduced to a solution of an equivalent mini–max LQ problem that corresponds to the weighted performance indices with weights from a finite dimensional simplex. The additional integral sliding mode controller part completely dismisses the influence of matched uncertainties from the initial time instant. Two numerical examples illustrate this study.**

***Index Terms*—Optimal control, sliding-mode control.**

I. Introduction

Sliding-mode control is a powerful nonlinear control technique that has been intensively developed during the last 35 years [8], [13]. The sliding mode controller drives the system state to a "custom-built" sliding (switching) surface and constrains the state to this surface thereafter. A system motion in a sliding surface, named *sliding mode,* is robust with respect to disturbances and uncertainties matched by a control but sensitive to unmatched ones. The sliding mode design approach usually consits of two steps [7],[11],[13]. First, the switching surface is designed such that the system motion in sliding mode satisfies design specifications. Second, a control function is designed making the switching surface attractive to the system state.

However, this control design strategy has three main disadvantages [13].

- The classical sliding-mode controllers are robust in the case of matched disturbances only.
- The designed controller ensures the optimality only after the entrance point into the sliding mode.
- The trajectory of the designed solution is not robust even with respect to the matched disturbances on a time interval preceding the sliding motion.

Reference [12] proposes a new sliding-mode design concept, namely integral sliding mode (ISM) *without reaching phase.* The order of the motion equation in ISM is equal to the order of the original system, rather than reduced by the dimension of the control input. As a result, robustness of the trajectory for a system driven by a smooth control law can be guaranteed throughout an entire response of the system starting from the initial time instance. However, ISM preserves only the trajectories driven by smooth controllers and needs the knowledge of the controller's derivative. The uniform formulation of the ISM design principle is developed in this note without the use of any information about the derivative of the control law needed for the robustification.

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A. Antecedents

As the antecedents reference we would like to single out the following lines of investigations.

- In [6], the ISM are used for the robustification of the optimal control problem for system with matched and unmatched uncertainties. It was shown that ISM allows to reduce the solution of the initial uncertain optimal problem to the solution of an optimal problem with unmatched uncertainties only.
- The problem of minimax sliding-mode design with optimal reaching phase solved in [11] for multimodel systems.
- In [2], the sliding mode approach was used for robust control design together with H_∞.
- In some papers, special switching surfaces *(dynamic sliding manifolds)* was introduced to robustify the solutions of different control problems: [14] (frequency shaping problem), [9] (precision control of a magnetic suspension actuator).
- The sliding-mode control design was suggested in [3] for systems with both matched and unmatched uncertainties using output information only.
- The classical sliding-mode approach was used in [11] for minimax control for multimodel linear uncertain systems.

B. Motivation

1) In the presence of unmatched uncertainties, the classical sliding-mode control [13] cannot be formulated, since it may successfully compensate only uncertainties or disturbances of "matched type."
2) In turn, an optimal control requires a complete knowledge of system dynamic equations. Therefore, in the situation when there is any unmeasured (even "matched-type") uncertainties another design concept must be developed.
3) The implementation of the integral sliding-mode approach is expected to be able to eliminate the influence of matched uncertainties in the right-hand side of the dynamic equation starting from the initial time and, after that, when we will have only unmatched uncertainties, but with completely known scenarios, the "worst-case" optimization procedure may be applied.
4) The corresponding optimization problem is usually treated as a *minimax control* dealing with different classes of partially known models [4], [10], and [5]. The minimax control problem can be formulated in such a way that the operation of the maximization is taken over a set of uncertainty and the operation of the minimization is taken over control strategies within a given resource set (usually a convex compact). In view of this concept, the original system model is replaced (approximated) by a finite set of dynamic models such that each model describes a particular uncertain case including exact realizations of possible dynamic equations as well as external bounded disturbances. An example of such situation could be the reusable launch vehicle attitude control dealing with a dynamic model which contains an uncertain matrix of inertia (various payloads in a cargo bay) and is affected by unknown bounded disturbances such as wind gusts (usually modeled by table look up data corresponding to different launch sites and months of a year). The design of the mini–max sliding mode controller that optimizes the worst flight

scenarios will reduce the risk of loss of a vehicle and a loss of a crew. Others examples can be found in [11].

5) So, the suggested idea is to modify both approaches (integral sliding-mode and minimax optimization) in order to bring together these advantages and ensure the successful control design in this complex situation.

C. Basic Assumptions and Restrictions

Since the original system model is uncertain, in this work

- we consider *a finite set of dynamic models* (as an approximation of a convex compact) such that each model describes exactly a particular unmatched uncertainty; the presence of matched bounded uncertainties is admitted;
- each model from a finite set is supposed to be given by a system of *linear time-varying* ODE with matched uncertainties may be a nonlinear nature;
- the performance of each model is characterized by a *LQ-criterion with a finite horizon*;
- the same control action is assumed to be applied to all models simultaneously and designed based on *an integral joint sliding function* as well as on *the mini–max LQ-criterion (fuzzy or mixing)*.

D. Main Contribution

This study demonstrates that

- the designed control, including the integral sliding-mode component, provides the best dynamics for the worst transient response to a disturbance input from a finite (*a priori* known) set of unmatched uncertainties and disturbances in the presence of the bounded matched part which is shown to be compensated by the integral sliding-mode control part from the start-point of the process, that is, from $t=0$;
- the LQ problem formulation leads to the design of the mini–max controller in *a linear weighted format with respect to system state*;
- the corresponding optimal weighting coefficients are computed based on a *Riccati equation parametrized by a vector,* defined on a finite dimensional simplex.

II. Problem Statement

Let us consider a controlled linear uncertain system

$$\dot{x}(t) = A(t)x(t) + B(t)u(x,t) + \zeta(t), \qquad x(0) = x^0 \tag{1}$$

where $x(t) \in R^n$ is the state vector at time $t \in [0,T]$, $u(x,t) \in R^m$ is a control action, ζ is an external disturbance (or uncertainty), $A(t), B(t)$ are assumed to be partially continuous. We will assume the following.

1) The matrix $B(t)$ is known, it has a full-rank for all $t \geq 0$ and its pseudoinverse matrix B^+ is bounded

$$\begin{aligned} &\text{rank}\, B(t) = m \\ &\|B^+(t)\| \leq b^+ \\ &B^+(t) := [B^\intercal(t)B(t)]^{-1}B^\intercal(t) \\ &b^+ = \text{const} > 0 \end{aligned}$$

and the matrix $A(t)$ may take a finite number of fixed and a priory known functions, that is, $A(t) \in \{A^1(t), A^2(t), \ldots, A^N(t)\}$ where N is a finite number of *possible dynamic scenarios,* here $A^\alpha(t)$ ($\alpha \in \{1,\ldots,N\}$) is supposed to be bounded, that is

$$\sup_{t\geq 0} \sup_{\alpha=\overline{1,N}} \|A^\alpha(t)\| \leq a^+, \qquad a^+ = \text{const} > 0. \tag{2}$$

2) An external disturbances ζ are represented in the following manner:

$$\zeta(t) = g(x,t) + \xi(t), \qquad t \in [0,T] \tag{3}$$

where $g(\cdot)$ is unmeasured smooth uncertainty representing the perturbations which satisfies the so-called "*standard matching condition*," that is, $g \in span\, B$, or, in other words, $g(x,t) \in \Omega$ where

$$\begin{aligned} \Omega := \{g(x,t) : g(x,t) = B\gamma(x,t) \\ \|\gamma(x,t)\| \leq q\|x\| + p, \qquad q,p > 0\} \end{aligned} \tag{4}$$

and $\xi(t)$ is a disturbance taking the finite number of alternative functions, that is, $\xi(t) \in \Xi =: \{\xi^1(t), \ldots, \xi^N(t)\}$ where $\xi^\alpha(t)$ ($\alpha \in \{1,\ldots,N\}$) are known (smooth enough) bounded functions such that $\|\xi(t)\| \leq \xi^+$ for all $t \in [0,T]$.

So, for each concrete realization of possible scenarios we obtain the following dynamics:

$$\begin{aligned} &\dot{x}^\alpha(t) = A^\alpha(t)x^\alpha(t) + B(t)u(x,t) + g(x^\alpha,t) + \xi^\alpha(t) \\ &x^\alpha(0) = x^0. \end{aligned} \tag{5}$$

Our goal is to design a control law which allows us to eliminate completely the matched part of uncertainties $g(x,t)$ and after that, using the rest of the control possibilities, to minimize a given performance index corresponding a worst possible scenario of the system dynamics.

III. Control Design Challenge

Now, the control design problem can be formulated as follows: *design the control $u = u(x,t)$ in the form*

$$\left.\begin{aligned} u(x,t) = u_0(x,t) + u_1(x,t) \\ u_1(x,t) = u_{1\text{corr}} + u_{1\text{comp}} \end{aligned}\right\} \tag{6}$$

where $u_1(x,t)$ is a term named the ISM control part, $u_{1\text{comp}}$ is responsible for the exact compensation of the unmeasured matched part of uncertainties $g(x,t)$ for a finite minimal possible compensation time t_{comp}, $u_{1\text{corr}}$ is a correction term for the linear part of ISM equations, and $u_0(x,t)$ is a control function minimizing the worst possible scenario in the sense of some *LQ*-index over a finite horizon $T \geq t_{\text{comp}}$, that is

$$\min_{u_0 \in R^m} \max_{\alpha=\overline{1,N}} h^\alpha \tag{7}$$

$$\begin{aligned} h^\alpha := \frac{1}{2}(x^\alpha(T), Lx^\alpha(T)) + \frac{1}{2} \\ \times \int_{t=t_{\text{comp}}}^{T} [(x^\alpha(t), Qx^\alpha(t)) \\ +(u_0(t) + u_{1\text{corr}}(x,t), R(u_0(t) \\ +u_{1\text{corr}}(x,t)))]\, dt \end{aligned}$$

$$Q = Q^\top \geq 0, \qquad L = L^\top \geq 0 \quad R = R^\top > 0. \tag{8}$$

Below will be shown that $t_{\text{comp}} = 0$.

IV. Design Principles

Substitution of the control law (6) and (3) into (1) yields

$$\begin{aligned}\dot{x}(t) &= A(t)x(t) + B(t)u_0(x,t) \\ &\quad + B(t)u_1(x,t) + g(x,t) + \xi(t) \\ x(0) &= x^0.\end{aligned} \tag{9}$$

Define the auxiliary "*sliding*" function $s(x,t) \in R^m$ as

$$s(x,t) = z(x,t) + s_0(x,t) \tag{10}$$

where $s_0(x,t)$ and $z(x,t)$ are also some auxiliary variables which will be defined below. Then, it follows:

$$\begin{aligned}\dot{s}(x,t) &= \dot{z}(x,t) + G(x,t)\left[A(t)x + B(t)u_0(x,t)\right. \\ &\quad \left.+B(t)\gamma(x,t) + B(t)u_1(x,t) + \xi(t)\right] + \frac{\partial s_0(x,t)}{\partial t}\end{aligned} \tag{11}$$

with $G(x,t) = \partial s_0(x,t)/\partial x$. This means that integral sliding mode design needs a continuously differentiable function s_0. Select the auxiliary variable z as the solution to the differential equation

$$\begin{aligned}\dot{z}(x(t),t) &= -G(x(t),t)[B(t)u_0(x(t))] - \frac{\partial s_0(x(t),t)}{\partial t} \\ z((x(0),0)) &= -s_0(x(0),0).\end{aligned} \tag{12}$$

Here, we should emphasize that we only know that $A(t) \in \left\{A^1(t), A^2(t), \ldots, A^N(t)\right\}$, but we do not know which of these matrices is the matrix of our realization. That is why, in difference with [12], we do not include the matrix $A(t)$ in the function $\dot{z}$. Then the equation for $s(x,t)$ becomes as follows:

$$\begin{aligned}\dot{s}(x(t),t) &= G(x(t),t) \\ &\quad \times [B(t)\gamma(x(t),t) + B(t)u_1(x(t),t) + \xi(t)] \\ &\quad + G(x(t),t)A(t)x(t) \\ s(x(0),0) &= 0.\end{aligned} \tag{13}$$

In order to realize a *sliding mode dynamics*, let us design the relay control in form

$$\begin{aligned}u_1(x,t) &= -M(x)\mathrm{Sign}[s(t)] \\ M(x) &= \bar{q}\|x(t)\| + \bar{p} + \rho = \text{scalar} \\ \rho &> b^+\xi^+\end{aligned} \tag{14}$$

$\bar{p} \geq p$, $\mathrm{Sign}[s(t)] = [\mathrm{sign}[s_1(t)], \mathrm{sign}[s_2(t)], \ldots, \mathrm{sign}[s_m(t)]]^T$, $\bar{q} \geq q + b^- a^+$ (a^+ is a positive constant), that implies

$$\begin{aligned}\dot{s}(x(t),t) &= G(x(t),t) \\ &\quad \times [B(t)(\gamma(x,t) - M(x)\mathrm{Sign}[s(t)]) + \xi(t)] \\ &\quad + G(x(t),t)A(t)x(t).\end{aligned}$$

In [13] and [12], to design the integral sliding-mode controllers the auxiliary function $s_0(x,t) = u_0(x,t)$ is used. However, such a choice of auxiliary function requires a smoothness of $u_0(x,t)$. On the other hand, the minimax controllers do not have the continuos derivative ([4], [10] and [5]). So, modifying the integral sliding mode procedure, we select $s_0(x,t) := [B(t)]^+ x$. Consequently $G(x(t),t) = [B(t)]^+$ and, for the Lyapunov function $V(s) = 1/2\|s\|^2$, in view of (4) and (2), and using the inequality $\sum_{i=1}^m |s_i| \geq \|s\|$, it follows:

$$\begin{aligned}\frac{d}{dt}V &= (s, \dot{s}) \\ &= (s, [B(t)]^+ B(t)(\gamma(x,t) - M(x)\mathrm{Sign}[s(t)]) \\ &\quad + [B(t)]^+\xi(t)) + (s, [B(t)]^+ A(t)x(t)) \\ &\leq -\|s\|(M(x) - \|\gamma(x,t)\| - \|B^+(t)\|\xi^+ \\ &\quad - \|[B(t)]^+\| \cdot \|A(t)\| \cdot \|x(t)\|) \\ &\leq -\|s\|[(\bar{q} - q - b^+a^+)\|x(t)\| \\ &\quad + (\bar{p} - p) + \rho - b^+\xi^+] \\ &\leq -\|s\|[\rho - b^+\xi^+] \leq 0.\end{aligned}$$

So, in view of (12), we derive

$$V(s(x(t),t)) \leq V(s(x(0),0)) = \frac{1}{2}\|s(x(0),0)\|^2 = 0$$

that implies for all $t \geq 0$ the following identities:

$$\begin{aligned}s(t) &= 0 \\ \dot{s}(t) &= 0.\end{aligned} \tag{15}$$

It means that **the integral sliding mode control (14) completely compensates the effect of the matched uncertainty g from the beginning of the process.** The relations (15) and (13) leads to the following representations:

$$\begin{aligned}&B(t)[\gamma(x,t) + u_{1eq}(x,t)] \\ &\quad + B(t)B^+(t)(\xi(t) + A(t)x(t)) = 0 \\ &u_{1eq}(x,t) = u_{1\mathrm{corr}} + u_{1\mathrm{comp}} \\ &u_{1\mathrm{comp}} = -\gamma(x,t) - B^+(t)\xi(t) \\ &u_{1\mathrm{corr}} = -B^+(t)A(t)x(t)\end{aligned}$$

and, hence

$$\begin{aligned}\dot{x}(t) &= A(t)x(t) + B(t)(u_0(x) - B^+(t)A(t)x(t)) \\ &\quad + [I - B(t)B^+(t)]\xi(t), \qquad \xi(t) \in \Xi.\end{aligned} \tag{16}$$

Remark 1: $\xi_{eq} := \left[I - BB^+\right]\xi \in \ker B^+$. This means that vector ξ_{eq} is a projection of the vector ξ on the space $\ker \mathbf{B}^+$.

V. Optimal Control Design

Returning to the multimodel case when $A(t)$ may take one of possible scenarios $A^\alpha(t)(\alpha = \overline{1,N})$, one can conclude that the multimodel system dynamics into the ISM takes the form

$$\begin{aligned}\dot{x}^\alpha(t) &= A^\alpha(t)x(t) + B(t)(u_0(x) - B^+(t)A^\alpha(t)x(t)) \\ &\quad + [I - B(t)B^+(t)]\xi^\alpha(t), \xi^\alpha(t) \in \Xi\end{aligned} \tag{17}$$

and LQ-index (8) becomes

$$\begin{aligned}h^\alpha &:= \frac{1}{2}(x^\alpha(T), Lx^\alpha(T)) + \frac{1}{2} \\ &\quad \times \int_{t=t_{\mathrm{comp}}}^{T} [(x^\alpha(t), Qx^\alpha(t)) \\ &\quad + [u_0(t) - (B^+(t)A^\alpha(t)x^\alpha(t)) \\ &\quad R(u_0(t) - B^+(t)A^\alpha(t)x^\alpha(t))]]dt.\end{aligned} \tag{18}$$

The next and last step is to apply the mini–max LQ control [4], [10] to the plant (17) and obtain the control $u_0(x)$ which together with u_1(14) solves the minimax problem for (18). It is necessary to remark here that unlike to [4], [10] in (17) only unmatched uncertainties can occur.

Now, with the extended system $\dot{\mathbf{x}}(t) = \mathbf{A}_{eq}(t)\mathbf{x}(t)+\mathbf{B}(t)u_0(\mathbf{x})+\mathbf{d}$ and according to [4], [10], this control is as follows:

$$u_0 = -R^{-1}\mathbf{B}^{\mathsf{T}}[\mathbf{P}_\lambda \mathbf{x} + \mathbf{p}_\lambda] + \mathbf{B}^{+}\mathbf{A}\mathbf{\Lambda}\mathbf{x} \tag{19}$$

where the matrix $\mathbf{P}_\lambda = \mathbf{P}_\lambda^T \in R^{nN\times nN}$ is the solution of the parametrized differential matrix Riccati (20), shown at the bottom of the page, and the shifting vector $\mathbf{p}_\lambda$ satisfies (21), as shown at the bottom of the page. Here

$$\mathbf{A} := \begin{bmatrix} A^1 & 0\cdot\cdot & 0 \\ & \cdot & \\ \cdot & & \cdot \\ & \cdot & \\ 0 & \cdot\cdot 0 & A^N \end{bmatrix}$$

$$\mathbf{A}_{eq} := \begin{bmatrix} A^1_{eq} & 0\cdot\cdot & 0 \\ & \cdot & \\ \cdot & & \cdot \\ & \cdot & \\ 0 & \cdot\cdot 0 & A^N_{eq} \end{bmatrix}$$

$$A^\alpha_{eq} = [I - BB^{+}]A^\alpha$$

$$\mathbf{Q} := \begin{bmatrix} Q^1 & 0\cdot\cdot & 0 \\ & \cdot & \\ \cdot & & \cdot \\ & \cdot & \\ 0 & \cdot\cdot 0 & Q^N \end{bmatrix}$$

$$\mathbf{L} := \begin{bmatrix} L & 0\cdot\cdot & 0 \\ & \cdot & \\ 0 & & 0 \\ & \cdot & \\ 0 & \cdot\cdot 0 & L \end{bmatrix}$$

$$\mathbf{\Lambda} := \begin{bmatrix} \lambda_1 I_{n\times n} & 0\cdot\cdot & 0 \\ & \cdot & \\ 0 & & 0 \\ & \cdot & \\ 0 & \cdot\cdot 0 & \lambda_N I_{n\times n} \end{bmatrix}$$

$$Q^\alpha = Q + [B^{+}(t)A^\alpha(t)]^\top R B^{+}(t)A^\alpha(t) \tag{22}$$

and

$$\mathbf{B}^\top := [\,B(t)^{1\top} \quad \ldots \quad B(t)^{N\top}\,] \in R^{r\times nN};$$
$$\mathbf{B}^{+} := [\,B^{+}(t) \quad \ldots \quad B^{+}(t)\,]$$
$$\mathbf{d}^\top := \left[(\xi^1_{eq})^\top, \ldots, (\xi^N_{eq})^\top\right] \in R^{1\times nN};$$
$$\xi^\alpha_{eq} = [\,I - B(t) \quad B^{+}(t)\,]\,\xi^\alpha.$$

The matrix $\mathbf{\Lambda} = \mathbf{\Lambda}(\lambda^*)$ is defined by (22) with the weight vector $\lambda = \lambda^*$ solving the following finite-dimensional optimization problem:

$$\begin{aligned}
\lambda^* &= \arg\min_{\lambda\in S^N} J(\lambda) \\
J(\lambda) &:= \max_{\alpha=\overline{1,N}} h^\alpha \\
&= \frac{1}{2}\mathbf{x}^\top(0)\mathbf{P}_\lambda(0)\mathbf{x}(0) + \mathbf{x}^\top(0)\mathbf{p}_\lambda(0) \\
&\quad + \frac{1}{2}\max_{i=\overline{1,N}}\Bigg[\int_0^T \Big[x^{i\top}(t)Q^i x^i(t) \\
&\qquad + 2x^{i\top}(t)(B^{+}A^i)^\top \\
&\qquad \Big(\mathbf{B}^\top[\mathbf{P}_\lambda\mathbf{x} + \mathbf{p}_\lambda] - R\mathbf{B}^{+}\mathbf{A}\mathbf{\Lambda}\mathbf{x}\Big)\Big]\,dt \\
&\qquad + x^{i\top}(T)Lx^i(T)\Bigg] \\
&\quad - \frac{1}{2}\sum_{i=1}^N \lambda_i\Bigg[\int_0^T \Big[x^{i\top}(t)Q^i x^i(t) \\
&\qquad + 2x^{i\top}(t)(B^{+}A^i)^\top(\mathbf{B}^\top[\mathbf{P}_\lambda\mathbf{x} + \mathbf{p}_\lambda] \\
&\qquad - R\mathbf{B}^{+}\mathbf{A}\mathbf{\Lambda}\mathbf{x})\Big]\,\mathbf{dt} + x^{i\top}(T)Lx^i(T)\Bigg] \\
&\quad + \frac{1}{2}\int_{t=0}^T \mathbf{p}_\lambda^\top[2\mathbf{d} - \mathbf{B}R^{-1}\mathbf{B}^\top\mathbf{p}_\lambda]dt \\
S^N &= \left\{\boldsymbol{\lambda}\in\Re^N : \lambda_\alpha \ge 0 \quad \sum\nolimits_{\alpha=1}^N \lambda_\alpha = 1\right\}.
\end{aligned} \tag{23}$$

This means that, u_0 is a linear combination of a feedback part (proportional to x) and a shifting vector $\mathbf{p}_\lambda$ which is indeed an open loop control part.

So, we can summarize the designed control algorithm as follows.

Step 1) For a fixed control u_0, we construct the "nominal" systems **(17)** and the corresponding *LQ*-index **(18)**.

Step 2) Construct the control u_0 using the extended system **(22)**.

Step 3) Design the *ISM* law u_1 in the form **(14)**, compensating the matched part of the uncertainties from the beginning of the process completely.

Step 4) Apply the control $u = u_0 + u_1$ to the closed-loop system **(1)**.

$$\begin{cases} \dot{\mathbf{P}}_\lambda + \mathbf{P}_\lambda(\mathbf{A}_{eq} + \mathbf{B}\mathbf{B}^{+}\mathbf{A}\mathbf{\Lambda}) + (\mathbf{A}_{\mathbf{eq}} + \mathbf{B}\mathbf{B}^{+}\mathbf{A}\mathbf{\Lambda})^{\mathsf{T}}\mathbf{P}_\lambda - \mathbf{P}_\lambda\mathbf{B}R^{-1}\mathbf{B}^{\mathsf{T}}P_\lambda \\ +\mathbf{\Lambda}(\mathbf{Q}_{eq} - (\mathbf{B}^{+}\mathbf{A})^\top R\mathbf{B}^{+}\mathbf{A}\mathbf{\Lambda}) = 0; \qquad \mathbf{P}_\lambda(T) = \mathbf{\Lambda}\mathbf{L} \end{cases} \tag{20}$$

$$\begin{cases} \dot{\mathbf{p}}_\lambda + (\mathbf{A}_{eq} + \mathbf{B}\mathbf{B}^{+}\mathbf{A}\mathbf{\Lambda})^{\mathsf{T}}\mathbf{p}_\lambda - \mathbf{P}_\lambda\mathbf{B}R^{-1}\mathbf{B}^{\mathsf{T}}\mathbf{p}_\lambda + \mathbf{P}_\lambda\mathbf{d} = \mathbf{0} \\ \mathbf{p}_\lambda(T) = 0 \end{cases}. \tag{21}$$

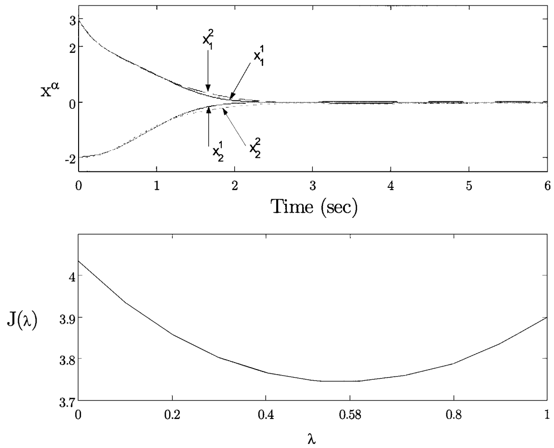

Fig. 1. Trajectories of the states variables for thesystem (24) and Performance index J.

A. Examples

Example 1: Let us consider two possible scenarios $(N = 2)$ with

$$\begin{aligned}
A^1 &= \begin{bmatrix} -0.2\cdot t & 2\cdot t \\ -0.3\cdot t & -1.5\cdot t \end{bmatrix}\\
A^2 &= \begin{bmatrix} -0.25\cdot t & 2.3\cdot t \\ -0.27\cdot t & -1.7\cdot t \end{bmatrix}\\
B^\top &= [2 \quad t]\\
g^\top &= [1.2\sin(4\pi t) \quad 0.6t(\sin 4\pi t)]\\
(\xi^1)^\top &= [0.2\cdot\sin(\pi\cdot t) \quad 0.25]\\
(\xi^2)^\top &= [0.5 \quad 0.3\cdot\sin(\pi\cdot t)].
\end{aligned} \tag{24}$$

Selecting $R = 1$, $Q = 1$, $L = I$, $T = 6$, we obtain (see Fig. 1) $\lambda_1^* = 0.58,\ \lambda_2^* = 0.42$ and $J(\lambda^*) = 3.744$. The corresponding state variable dynamics is depicted at Fig. 1 and the control law is in Fig. 2.

Example 2: Consider the case of three possible scenarios $(N = 3)$ where

$$\begin{aligned}
A^1 &= \begin{bmatrix} -1 & 2 \\ 0 & -0.5 \end{bmatrix}\\
A^2 &= \begin{bmatrix} -0.5 & 2.2 \\ 0 & -0.7 \end{bmatrix}\\
A^3 &= \begin{bmatrix} -1.3 & 1.5 \\ 0 & -0.8 \end{bmatrix}\\
B^\top &= [2 \quad 2]\\
g^\top &= [0.8\cdot x_1 \quad 0.8\cdot x_1]\\
(\xi^1)^\top &= [0.62\cdot\sin(2\cdot\pi\cdot t) \quad 0.13]\\
(\xi^2)^\top &= [0.2 \quad 0.7]\\
(\xi^3)^\top &= [0.55 \quad 0.15].
\end{aligned} \tag{25}$$

Selecting $R = 1$, $Q = I$, $L = I$, $T = 6$ we obtain the optimal weights $\lambda_1^* = 0\ \ \lambda_2^* = 0,\ \lambda_3^* = 1$ and the functional $J(\lambda^*) = 4.365$. The corresponding state variables dynamics is shown at Fig. 3 and control law is in Fig. 4.

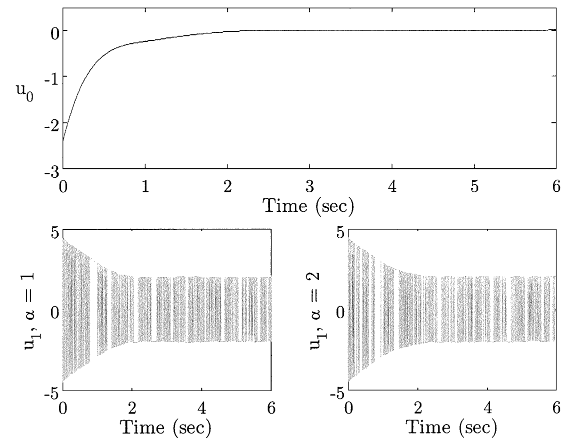

Fig. 2. Controls u_0 and u_1 for $\alpha = 1 \quad \alpha = 2$.

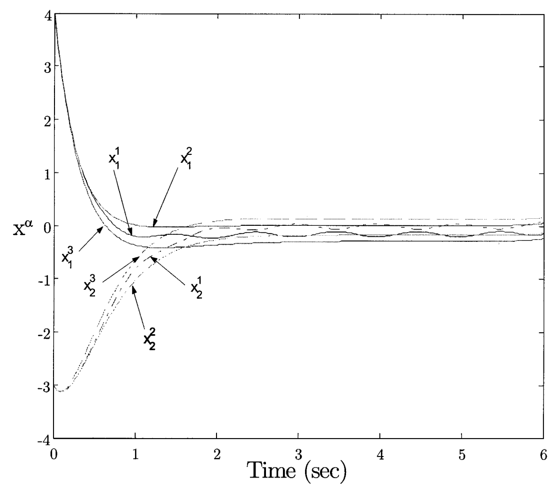

Fig. 3. Trajectories of the states variables for (25).

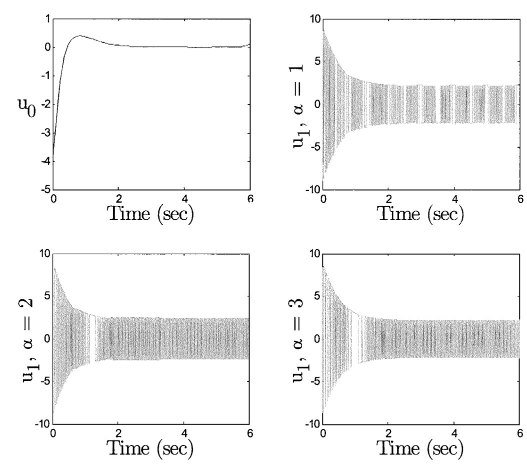

Fig. 4. Controls u_0 and u_1 for $\alpha = 1, \alpha = 2$, and $\alpha = 3$.

VI. Conclusions

The problem of **robust** optimal control design is considered for a linear multimodel system with bounded disturbances and uncertainties. With this aim the methods of integral sliding mode control and minimax robust optimal control are modified. The suggested designed control, includes terms corresponding to an integral sliding-mode component as well as a minimax optimization part. The integral sliding-mode component

- compensates of the matching part of the uncertainty **beginning from the start-point of the process,** that is, from $t = 0$;
- allows to make minimax control design for unmatched uncertainties only.

So, *the minimax optimization control provides now the best dynamics for the worst transient response to a disturbance input from a finite (a priori known) set of unmatched uncertainties.*

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REFERENCES

[1] J. Ackermann, D. Kaesbaver, W. Sienel, and R. Steinhauser, *Robust Control, Systems With Uncertain Physical Parameters*. London, U.K.: Springer-Verlag, 1994.

[2] L. Aguilar, L. Orlov, and L. Acho, “Nonlinear H-infinity control of nonsmooth time-varying systems with application to friction mechanical manipulators,” *Automatica*, vol. 39, no. 9, pp. 1531–1542, 2003.

[3] S. K. Bag, S. K. Spurgeon, and C. Edwards, “Output feedback sliding mode design for linear uncertain systems,” *Proc. Inst. Elect. Eng. Control Theory Applicat.*, vol. 144, pp. 209–216, 1997.

[4] V. G. Boltyanskii and A. S. Poznyak, “Robust maximum principle in minimax control,” *Int. J. Control*, vol. 72, pp. 305–314, 1999.

[5] V. G. Boltyanski and A. S. Poznyak, “Robust maximum principle for a measured space as uncertainty set,” in *Dynamic Systems & Applications*. Atlanta, GA: Dynamic Publishers, 2002, vol. 11, pp. 277–292.

[6] W. J. Cao and J.-X. Xu, “Nonlinear integral-type sliding surface for both matched and unmatched uncertain systems,” in *Proc. Amer. Control Conf.*, vol. 6, 2001, pp. 4369–4374.

[7] L. Fridman, “Sliding mode control for systems with fast actuators: Singularly perturbed,” in *Variable Structure Systems: Toward the 21st Century*, X. Yu and J.-X. Xu, Eds. London, U.K.: Springer-Verlag, 2002, vol. 274, Lecture Notes in Control and Information Science, pp. 391–415.

[8] A. Levant, “Universal SISO sliding-mode controllers with finite-time convergence,” *IEEE Trans. Automat. Contr.*, vol. 46, pp. 1447–1451, Sept. 2001.

[9] S. Mittal and C.-H. Menq, “Precision motion control of a magnetic suspension actuator using a robust nonlinear compensation scheme,” *IEEE/ASME Trans. Mechatron.*, vol. 2, pp. 268–280, Dec. 1997.

[10] A. Poznyak, T. Duncan, B. Pasik-Duncan, and V. Boltyansky, “Robust maximum principle for minimax linear quadratic problem,” *Int. J. Control*, vol. 75, no. 15, pp. 1170–1177, 2002.

[11] A. Poznyak and Y. Shtessel, “Mini–max sliding mode control with minimal time reaching phase,” in *Proc. Amer. Control Conf.*, Denver, CO, 2003, pp. 174–179.

[12] V. Utkin and J. Shi, “Integral sliding mode in systems operating under uncertainty conditions,” in *Proc. 35th IEEE Conf. Decision Control*, Kobe, Japan, 1996, pp. 4591–4596.

[13] V. Utkin, J. Guldner, and J. Shi, *Sliding Modes in Electromechanical Systems*. London, U.K.: Taylor and Francis, 1999.

[14] K. D. Young and U. Ozguner, “Frequency shaping compensator designor sliding mode,” *Int. J. Control*, vol. 57, no. 5, pp. 1005–1019, 1993.

Distributed Multirate Interacting Multiple Model Fusion (DMRIMMF) With Application to Out-of-Sequence GMTI Data

Lang Hong, Shan Cong, and Devert Wicker

***Abstract*—This note develops a distributed approach for fusing ground moving target indicator data with out-of-sequence (OOS) measurements. A multirate interacting multiple model (MRIMM) fusion algorithm is developed for effectively fusing multirate information. The multirate approach provides an excellent framework for efficient information retrodiction and forward update. A multirate interacting multiple model filter is employed locally to track a target with or without maneuvering behavior. The combination of global MRIMM fusion and local MRIMM tracking proves to be powerful for tracking and fusing maneuvering and nonmaneuvering targets in an environment of OOS measurement reporting.**

***Index Terms*—Multirate processing, out-of-sequence (OOS) measurements, target tracking, track fusion.**

I. INTRODUCTION

Ground moving target indicator (GMTI) radar has demonstrated its powerful surveillance/ reconnaissance capability in many military and law enforcement operations. Its ability to timely provide detailed information throughout the theater is critical to real-time command and control in a battlefield. In the near future, a surveillance operation may include a network of GMTI platforms. As there are thousands of objects in a surveillance region, an enormous tracking effort is needed to correlate radar measurements into target trajectories. Meanwhile, under a combat condition, the measurements from multiple platforms will not be synchronized, because each individual platform has its own scan rate and the communication network cannot guarantee to deliver measurements on time. Asynchronized measurements inevitably make the order of measurements uncertain, which creates an out-of-sequence (OOS) reporting phenomenon. Tracking with OOS measurements poses a challenge to GMTI tracker design. Although the delay of each OOS measurement may be only a fraction of a scan to a few scans, this challenge cannot simply be ignored, because this phenomenon is due to the nature of networking and is expected to occur frequently.

Although the OOS reporting phenomenon exists widely in the real world applications, only limited research papers have been published [1], [14] among which Bar-Shalom made a significant contribution in formalizing the problem and initial development [1]. His work focused on an optimality study of filtering OOS measurements with delay time less than one scan, which can be considered as a generalization of traditional smoothing algorithms. Based on our recently developed multirate techniques [7]–[9], we have developed a general multirate filtering algorithm for arbitrary OOS measurements [10] and resolved three key issues: 1) an efficient processing structure for information retrodiction, 2) an efficient memory structure for storing historical information, and 3) an efficient computational structure for tracking nonmaneuvering and maneuvering targets.

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