Technical Notes and Correspondence

Frequency Domain Input–Output Analysis of Sliding-Mode Observers

Igor Boiko and Leonid Fridman

***Abstract*—The sliding-mode (SM) observer dynamics are analyzed in the frequency domain. It is shown that the SM observer is essentially a relay feedback-feedforward system. It is feedback with respect to the measured variable of the system being observed, and the feedforward with respect to the control applied to the system being observed. Having noted this, the input-output properties of observer dynamics are analyzed with the use of the locus of a perturbed relay system (LPRS) method. The performance of the observer is assessed via the use of the frequency response techniques. An example of performance analysis of the observer dynamics in the frequency domain is given.**

***Index Terms*—Frequency domain, nonlinear observers, sliding modes (SMs).**

I. Introduction

The idea of using a dynamical system to obtain estimates of the system states from measurable system variables was first proposed by Luenberger [1]. This dynamical system received the name *observer*. In that approach, the observer dynamical system is driven by the control and by the difference between the output of the observer and the output of the plant. This difference should ideally become zero, which would indicate that the state estimates from the observer have become equal to the states of the plant. Sliding modes (SM) can be used for the purpose of observation if generated in the observer dynamical system, and the system is designed in such a way that the difference between the output of the observer and the output of the plant becomes the *sliding variable*. The control should be designed to provide the existence of SM in the observer dynamical system. SM observers were analyzed in a number of publications (see, for example, respective chapters of textbooks [2] and [3] and recent tutorials [4] and [5]). Only an ideal SM in the observer dynamical system was analyzed in those works (as well as in the publications referenced above). For that reason, in a steady mode, the observation is done with zero error. The nonzero observation error only appears in the transients while the process is converging. After the process has converged (this time may be infinite though, and for that reason some observers are called *asymptotic observers*) the observation error becomes zero and ideal observation occurs. Another feature of the traditional analysis is that SM observers were always analyzed as *stabilization systems* but not as *servo systems*.

However, no ideal SM exists in any real application. Parasitic dynamics always exist along with the principal dynamics. Even if we consider an observer, where the control is realized by means of a digital processor the processor itself would introduce a delay into the observer loop. In addition to that, the sensors that measure the variables of the plant would also contribute to the error of observation, despite the fact that they are outside the observer loop, due to mismatch between the observer model that does not include the parasitic dynamics and the plant that includes this dynamics. It was shown in [8] that due to the parasitic dynamics presence not only chattering occurs but the averaged motions in the SM system differ from those of the reduced-order system model. Naturally, we can expect the same effect in the SM observers too.

Considering the fact that a SM observer can be analyzed as a servo system, this analysis would involve solving the *input–output problem*. The input–output problem in discontinuous control was considered in [7], [8]. The locus of a perturbed relay system (LPRS) method was proposed as a tool for input–output analysis of system with discontinuous (relay) control. This method is going to be used below for input–output analysis of SM observers.

Manuscript received October 11, 2005; revised April 19, 2006 and July 3, 2006. Recommended by Associate Editor S. Celiskovsky.

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Digital Object Identifier 10.1109/TAC.2006.884995

II. Problem Formulation

Consider an n-dimensional version of the observer proposed by Utkin [2]. Let the linear plant, the states of which are supposed to be observed, be the nth-order dynamical system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u \tag{1}$$

$$y = \mathbf{C}\mathbf{x} \tag{2}$$

where $\mathbf{x} \in R^n$ is the state vector, $y \in R^1$ is the measurable system output, $\mathbf{A} \in R^{n\times n}$, $\mathbf{B} \in R^{n\times 1}$, $\mathbf{C} \in R^{1\times n}$ are matrices. The pair $(\mathbf{C}, \mathbf{A})$ is assumed to be observable.

The SM observer can be designed in the same form as the original system (1), (2) with addition of an *output injection* being an observer correction input that depends on the error between the output of the observer and the output of the plant (system to be observed)

$$\dot{\widehat{\mathbf{x}}} = \mathbf{A}\widehat{\mathbf{x}} + \mathbf{B}u + \mathbf{L}\,\mathrm{sign}(y - \widehat{y}) \tag{3}$$

$$\widehat{y} = \mathbf{C}\widehat{\mathbf{x}} \tag{4}$$

where $\widehat{\mathbf{x}} \in R^n$ is an estimate of the system state vector, $\widehat{y} \in R^1$ is an estimate of the system output, and $\mathbf{L} \in R^{n\times 1}$ is a gain matrix.

Denote the sliding variable as follows:

$$\sigma = y - \widehat{y}. \tag{5}$$

The elements of $\mathbf{L}$ must be such that the reachability condition of the SM and stability of the reduced SM dynamics should be insured. It is shown in [2] and [3] that the matrix $\mathbf{L}$ can be selected to provide the convergence of the sliding variable σ to zero in finite time and asymptotic convergence of the estimation error for the system variables. We assume that conditions of existence of SM in the observer dynamics are satisfied. The subject of analysis of the present note is the effect of the parasitic dynamics that are inevitably present in the dynamics of a practical realization of an observer. Let us consider the observer to be a dynamical system that has two inputs and one output. One of those two inputs $y(t)$ must be followed (tracked) by the observer output $\widehat{y}(t)$

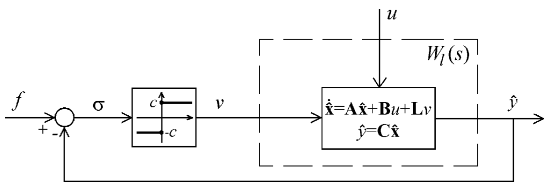

Fig. 1. Relay servo system representation of the SM observer.

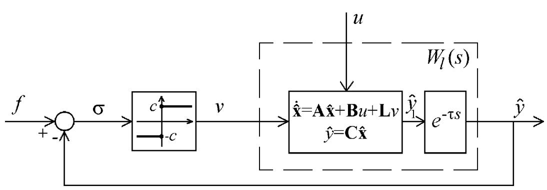

Fig. 2. Relay servo system representation of the SM observer with parasitic dynamics.

as precisely as possible. The other input $u(t)$ can be treated as a feedforward. Therefore, it makes sense to consider the observer as a *relay servo system* (Fig. 1).

With this representation, we are now able to apply the LPRS approach [7], [8] to analysis of the SM observer performance. Yet, it is worth noting that a SM observer is designed the way that theoretically in the system Fig. 1 an ideal SM occurs. As a result, in a steady state, variables $\widehat{y}(t)$ and $y(t)$ are always equal, and the state estimates $\widehat{\mathbf{x}}(t)$ are equal to the plant states $\mathbf{x}(t)$. We need to determine the nature the parasitic dynamics to obtain a result different from the ideal sliding and observation accuracy different from the ideal one. Considering a digital realization of the SM observer one can note that the parasitic dynamics of the SM observer comes from the delay in the output $\widehat{y}(t)$ calculation [9]. Let us show now that in a relay feedback system, the digital realization of the observer equations is manifested as a delay. Consider only calculations that are done within the observer loop. Suppose that the computing of the observer output begins from the calculation of the sliding variable $\sigma(t)$, after that the discontinuous control $v(t)$ is calculated, and so on up to the observer output $\widehat{y}(t)$. All those calculations are done over one execution period τ of the controller. Therefore, at the calculation of the sliding variable, the value of the observer output calculated on the previous execution period is used. This is equivalent to introducing the time delay in the observer loop (Fig. 2). It should be noted that the execution period for computing $\widehat{y}(t)$ and the sampling rate for measuring $y(t)$ are two different notions. However, for the purpose of our analysis we can assume that the values of those two variables are equal. This is a simplified approach. In reality, there may be different combinations of sampling rate and execution period. The results would also depend on the algorithm of the numeric integration.

Let us proceed from the following assumption. Since in the case of absence of the parasitic dynamics the SM is realized ideally, the digital implementation would exhibit chattering with the period equal to two execution periods of the algorithm (see [10], for example). On the other hand the system with a time delay (Fig. 2) would exhibit the same kind of motion. Obviously, by varying the time delay the frequency of chattering in the second case can be tuned to be equal to the frequency of chattering in the first case. If we assume that the solution of system (3), (4) is obtained exactly at the digital realization of the SM observer then the considered two SM observer dynamics: the digital realization and the time delay realization would be equivalent. In other words, knowing the execution period we can find a certain equivalent time delay that the dynamics of those two representations of the SM observer would be equivalent. A similar approach to analysis of discrete-time SM systems was proposed in [11]. Therefore, the use of time delay in the model represents a certain generic approach to evaluation of the SM observer performance.

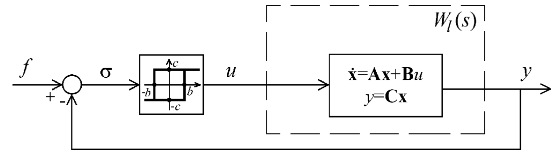

Fig. 3. Relay servo system.

With this representation, we can analyze the observer performance in terms of the response of the relay servo system (Fig. 2) to two inputs: u and y. This is a complex task, which, however, can be fulfilled via application of the LPRS method that is designed for input–output analysis of relay systems.

III. The Concepts of the Locus of a Perturbed Relay System (LPRS) Approach

In [7], [8] the LPRS was introduced as a method of analysis and design of relay servo systems having a linear plant (Fig. 3). Let us call the part of the relay servo system that is given by the linear differential equations the *linear part*. With respect to the SM observer, the linear part will be the one given by (3) and (4).

The LPRS was defined as a complex function $J(\omega)$ in the frequency domain as follows:

$$J(\omega) = -\frac{1}{2}\lim_{f_0 \to 0}\frac{\sigma_0}{u_0} + j\frac{\pi}{4c}\lim_{f_0 \to 0} y(t)|_{t=0} \tag{6}$$

where $t = 0$ is the time of the switch of the relay from "$-c$" to "$+c$", ω is the frequency of the periodic motion. The frequency in (6) is the frequency of the self-excited oscillations varied by changing the hysteresis $2b$ while all other parameters of the system are considered constant; σ_0, u_0 and $y(t)|_{t=0}$ can, therefore, be considered functions of ω, and ω a function of the hysteresis $2b$. The limit in the imaginary part of (6) is the value of $y(t)$ at the time of the switch in the symmetric oscillations. Thus, $J(\omega)$ is defined as a characteristic of the response of the linear part to the unequally spaced pulse input $u(t)$ subject to $f_0 \to 0$ as the frequency ω is varied.

A few techniques of the LPRS computing—for different types of plant description—were proposed. If the plant is represented by (1), (2) the LPRS is given by the following formula:

$$\begin{aligned} J(\omega) = {} & -0.5\mathbf{C}\left[\mathbf{A}^{-1} + \frac{2\pi}{\omega}(\mathbf{I} - e^{(2\pi/\omega)\mathbf{A}})^{-1}e^{(\pi/\omega)\mathbf{A}}\right]\mathbf{B} \\ & + j\frac{\pi}{4}\mathbf{C}(\mathbf{I} + e^{(\pi/\omega)\mathbf{A}})^{-1}(\mathbf{I} - e^{(\pi/\omega)\mathbf{A}})\mathbf{A}^{-1}\mathbf{B} \end{aligned} \tag{7}$$

where matrix $\mathbf{A}$ is assumed to be invertible (for noninvertible $\mathbf{A}$, another formula of the LPRS is obtained in [8]). Another useful formula of $J(\omega)$ was derived for the case of the linear part given by a transfer function $W_l(s)$ [7], [8]. The formula has the form of infinite series, which is convenient if the plant is given by the transfer function. This formula can be used for linear parts containing a time delay

$$J(\omega) = \sum_{k=1}^{\infty}(-1)^{k+1}\mathrm{Re}\, W_l(k\omega) + j\sum_{k=1}^{\infty}\frac{\mathrm{Im} W_l[(2k-1)\omega]}{2k-1} \tag{8}$$

where $W_l(s) = \mathbf{C}(\mathbf{I}s - \mathbf{A})^{-1}\mathbf{B}$ or considered a given function.

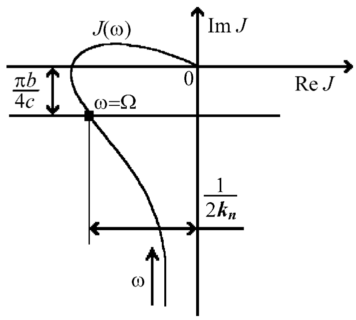

Fig. 4. LPRS and oscillations analysis.

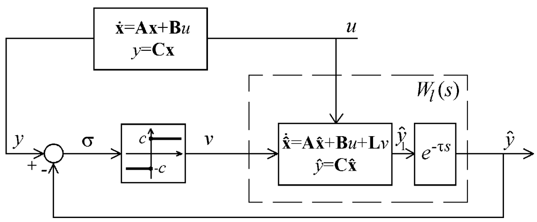

Fig. 5. Plant and observer model.

With the plant model available, the LPRS can be computed at various frequencies and the LPRS plot can be drawn on the complex plane (an example of the LPRS is given in Fig. 4). What is important is that the LPRS is a characteristic of the relay feedback system and can be computed from the plant model. Once the LPRS is computed, the frequency Ω of the symmetric periodic solution can be determined from the following equation:

$$Im\ J(\Omega) = -\frac{\pi b}{4c} \tag{9}$$

which corresponds to finding the point of intersection of the LPRS and the horizontal line that lies below the real axis at $-\pi b/(4c)$ (Fig. 4), and the *equivalent gain* [7], [8] of the relay (the gain of the relay with respect to the averaged motions) can be determined as

$$k_n = \left.\frac{\partial u_0}{\partial \sigma_0}\right|_{\sigma_0=0} = -\frac{1}{2ReJ(\Omega)} \tag{10}$$

which corresponds to the distance between the intersection point and the imaginary axis. Both formulas: (9) and (10) directly follow from the definition (6).

With the formulas of the LPRS available, input-output analysis of the relay feedback system (Fig. 3) can be done in the same manner as with the use of the describing function method [12] (however, involvement of the *filtering hypothesis* is not needed any longer). The relay function can be replaced with the equivalent gain (10) and the input–output properties of the relay system can be analyzed as the properties of the resulting linearized system.

IV. SM Observer Performance Analysis and Characteristics

With the representation of the SM observer as a relay servo system, we can formulate some performance measures of the observer. In our analysis, we have to consider some properties of this servo system typical of the observers only. The main feature of the observer input-output

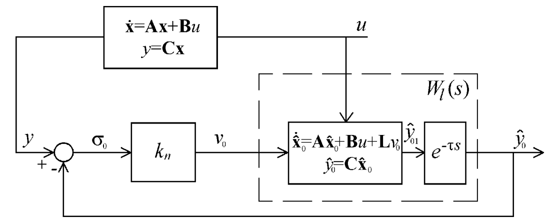

Fig. 6. Linearized model of plant and observer.

dynamics is that there are two different inputs to the system that are not independent (Fig. 5).

From the function of the observer it follows that, apart from the initial transient time, the values of $y(t)$ and of $\widehat{y}(t)$ are very close. The variable $y(t)$ is a result of the propagation of input $u(t)$ through the dynamics of the plant. On the other hand, if we assume that the model of the observer ideally matches the model of the plant we can conclude that by setting $v(t) \equiv 0$ we can obtain the result: $\widehat{y}(t) = y(t)$, which essentially means that the discontinuous control $v(t)$ is needed to compensate for the mismatch between the dynamics of the plant and the dynamics of the observer model. Since the values of $y(t)$ and of $\widehat{y}(t)$ are close, with the value of $\widehat{y}(t)$ alternately becoming slightly above and slightly below $y(t)$, the control $v(t)$ is almost equally spaced, and $u(t)$ plays the role of the "equivalent control."

At first, consider possible methodology of analysis of the observer dynamical system given in Fig. 2. From the LPRS approach, we know that the averaged forced motions in the system Fig. 2 can be analyzed via the use of the equivalent gain of the relay concept and the linearized model, which can be obtained from the original model by replacing the relay function with the equivalent gain. The linear part of the system for the LPRS analysis is the dynamics of the observer model and the parasitic dynamics (time delay) marked in the diagram Fig. 2 with the dashed line. This model can also be represented by the transfer function $W_l(s)$ from control $v(t)$ to the output $\widehat{y}(t)$.

Methodology of input–output analysis of the dynamics given in Fig. 5 is presented in [7], [8]. However, the observer analysis has its specifics due to unknown value of the equivalent time delay. As a result, analysis can be done via: a) identification of the equivalent time delay of the continuous-time model of the observer—via matching the frequencies of chattering, where the frequency of chattering of the continuous-time system should be computed from (9), where $b = 0$, with the LPRS computed per (8) (or using other techniques given in [8]); b) computing the LPRS for the linear part of the relay system (corresponding to $W_l(s)$); c) computing the equivalent gain value using formula (10); and d) replacing the relay with the equivalent gain and carrying out analysis of the linearized observer dynamics. The linearized plant-observer dynamics can be represented as in Fig. 6. In Fig. 6, subscript "0" is used to indicate the averaged on the period of chattering variables.

We have thus obtained a linear model of the plant and the observer, from which various characteristics of the observer accuracy can be derived. Those characteristics are different from the frequency and the amplitude of chattering that can be obtained from the relay feedback representation of the observer (Fig. 2). If, for example, we follow the conventional approach to servo systems analysis we can formulate a dynamical accuracy criterion as a frequency response of the error signal $\sigma(t)$ to the harmonic excitation $u(t)$ of variable frequency. This characteristic can be presented as a magnitude and a phase responses:

$$M = 20\lg|W_{u-\sigma}(j\omega)| \tag{11}$$

$$\varphi = \arg W_{u-\sigma}(j\omega) \tag{12}$$

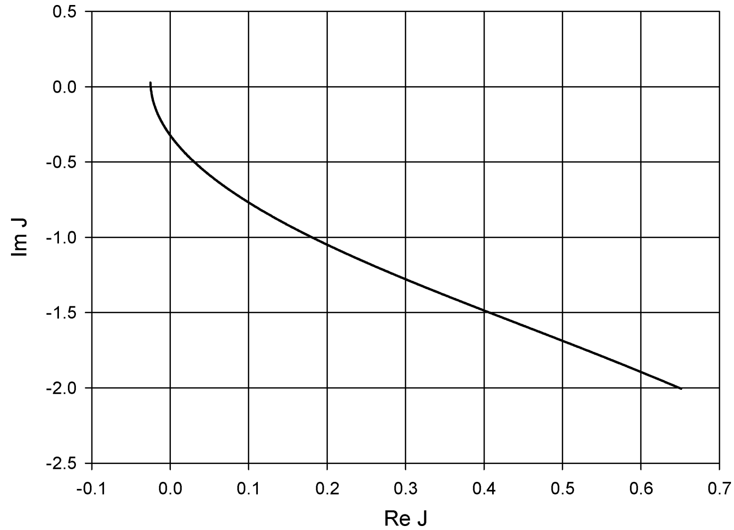

Fig. 7. LPRS of the SM observer linear part.

where M is the magnitude response, φ is the phase response, $W_{u-\sigma}(s)$ is the transfer function from $u(t)$ to $\sigma(t)$ that is given by

$$W_{u-\sigma}(s) = \mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}\frac{1-e^{-\tau s}}{1+k_n\mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{L}e^{-\tau s}}. \quad (13)$$

Such characteristics as the bandwidth, resonant frequencies and others can be easily obtained from the frequency response of the linearized plant-observer model too.

V. Example of SM Observer Performance Analysis

Consider an example of performance analysis of two-dimensional version of the SM observer. Let the plant be the second-order system (1), (2) with $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & -3 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\mathbf{C} = [1 \quad 1]$. And the observer dynamics be given by (3), (4) with $\mathbf{L} = [1 \quad 4]^T$. Assume that the input to the system is a harmonic oscillation of variable frequency: $u(t) = \sin(\omega t)$. Let the equivalent delay be $\tau = 0.01$ s. The corresponding execution period that would cause such delay will be determined below. Write an expression for the transfer function of the linear part in the relay servo dynamics

$$\begin{aligned} W_l(s) &= \mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{L}e^{-\tau s} = \frac{5s+6}{s^2+3s+1}e^{-0.01s} \\ &= \left(\frac{\frac{25+3\sqrt{5}}{10}}{s+\frac{3+\sqrt{5}}{2}} + \frac{\frac{25-3\sqrt{5}}{10}}{s+\frac{3-\sqrt{5}}{2}}\right)e^{-0.01s} \\ &= \frac{1.2112}{0.3819s+1}e^{-0.01s} + \frac{4.7888}{2.6180s+1}e^{-0.01s}. \end{aligned} \quad (14)$$

Now, compute the LPRS corresponding to the transfer function (14) using the formula for the first-order plus dead time dynamics [8] with the transfer function $W(s) = Ke^{-\tau s}/(Ts+1)$ and the additivity property of the LPRS [8, Th. 1]

$$J(\omega) = \frac{K}{2}(1-\alpha e^{\gamma}\, cosech\, \alpha) + j\frac{\pi}{4}K\left(\frac{2e^{-\alpha}e^{\gamma}}{1+e^{-\alpha}} - 1\right)$$

where $\alpha = \pi/T\omega$ and $\gamma = \tau/T$. The LPRS of the observer dynamics is presented in Fig. 7. Calculate the frequency of chattering as per formula (9) assuming zero hysteresis ($b = 0$): $\Omega = 158.48\ \mathrm{s}^{-1}$. This would correspond to the execution period equal to $\pi/\Omega = 0.0198$ s, which illustrates the correspondence between the execution period and the equivalent delay. Using formula (10) calculate the equivalent gain of the relay: $k_n = 20.08$. Now, we can compute the observation error for various frequencies of the input signal as the magnitude of the transfer function (13). The plot providing the output error versus the frequency of the input signal is given in Fig. 8. From this plot, we can also derive some other characteristics of the analyzed SM observer. For example, the resonant frequency of the observer characteristic is about 138 s^{-1}, and the observer bandwidth measured at the level of 1% observation error is about 76 s^{-1}, maximum observation error is about 2.85%. At low frequencies, observation error grows linearly with the logarithm of the input signal frequency. It should be noted that the frequency-domain input–output analysis in a SM system makes sense only for the frequencies below the frequency of chattering. For that reason, the observation error is computed only in this frequency range.

The observation error can also be computed for x_1 and x_2 using the transfer functions from the system input to the difference of the variable $x_{1,2}$ and its estimate $\hat{x}_{1,2}$

$$\begin{aligned} W_{u-x1}(s) &= \mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}\frac{1-e^{-\tau s}}{1+k_n\mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{L}e^{-\tau s}}\mathbf{C}_1(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{L} \end{aligned} \quad (15)$$

where $\mathbf{C}_1 = [1 \quad 0]$.

$$\begin{aligned} W_{u-x2}(s) &= \mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}\frac{1-e^{-\tau s}}{1+k_n\mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{L}e^{-\tau s}}\mathbf{C}_2(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{L} \end{aligned} \quad (16)$$

where $\mathbf{C}_2 = [0 \quad 1]$. The error of observation for variables x_1 and x_2 versus the frequency of the input signal is depicted in Fig. 9. The computer simulations qualitatively match very well the results obtained

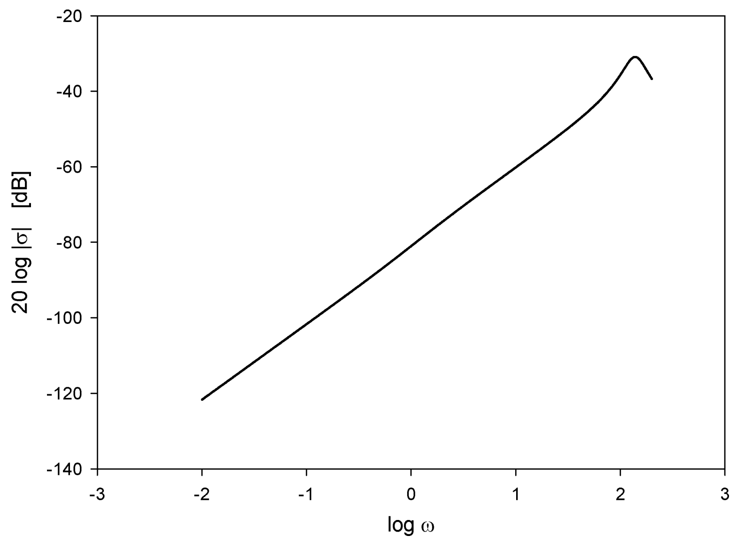

Fig. 8. Output error magnitude versus frequency.

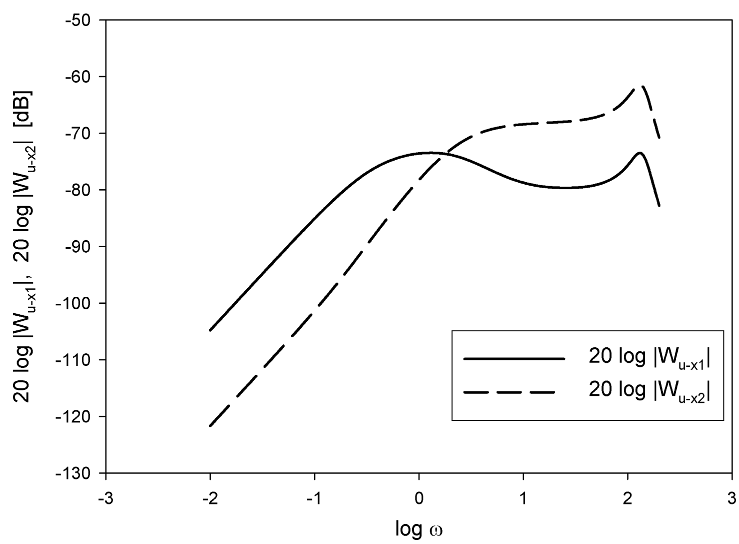

Fig. 9. Observation error of x_1 and x_2 versus frequency.

analytically. However, the numeric results may differ from the analytical results—depending on the used method of integration, step value, execution order of the model, etc.

VI. Conclusion

It was shown that the SM observer is essentially a relay feedback-feedforward system and can be analyzed as such. Having noted this, it is proposed above that the performance of the SM observer should be estimated in terms of the characteristics of the frequency response of this relay feedback-feedforward system. Respective analysis should involve replacement of the original digital implementation of the SM observer with a certain SM system having a time delay. The latter can be analyzed with the use of the LPRS method, which involves computing the frequency-domain characteristic called the LPRS, solving the equation for the frequency of chattering, computing the value of the equivalent gain of the relay for the averaged values of the variables, building the linearized model of the averaged motions via the replacement of the relay characteristic with the equivalent gain. Analysis of the linearized system can be done with the use of applicable methods. It is found that the dynamical performance of the SM observer is not ideal. Because of the existence of parasitic dynamics in the observer loop (time delay due to discrete implementation of the algorithm) there always exists a nonzero observation error – even after the initial transient. This observation error depends on a number of factors (including the execution period of the algorithm), and can be numerically evaluated with the use of the approach presented above. The provided example illustrates the proposed approach in detail and proves that no ideal observation occurs in an implementation of the SM observers.

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A Block Triangular Form for Nonlinear Observer Design

Yebin Wang and Alan F. Lynch

***Abstract*—This note reconsiders existence conditions for an established block triangular form (BTF) for a broad class of multi-output nonlinear systems. Systems transformable into BTF coordinates allow for decentralized observer design performed subsystem-at-a-time effectively treating upper subsystem states as measurements. Sufficient conditions are given to ensure that the output is linear in BTF coordinates. An example illustrates the BTF construction and a typical observer design.**

***Index Terms*—Differential geometry, nonlinear systems, observers.**

I. Introduction

We consider uncontrolled multiple-output systems

$$\dot{\zeta} = f(\zeta) \quad y = h(\zeta) \tag{1}$$

with C^∞ vector fields $f : \mathbb{R}^n \to \mathbb{R}^n$, and C^∞ output functions $h : \mathbb{R}^n \to \mathbb{R}^p$. Exact error linearization (EEL) is an established observer design method based on an observer form (OF) which leads to linear time-invariant (LTI) error dynamics when expressed in OF coordinates. After the original single-output work on EEL in [1] and [2] many extensions have been proposed [3]–[10]. OF-based methods may be difficult to apply in practice due to restrictive existence conditions, and this has led to approaches which relax these conditions. For example, researchers have considered removing the linearity constraint on the output [11] or the use of high gain [12]. The OF and related special forms, such as that considered in this note, define coordinates in which the system has a special structure which facilitates observer design. This may lead to a more complex design applicable to a restricted system class. However, the benefits of exploiting a particular system form, even if only locally defined, are established and include simple observer structures and reduced observer gains which make the design less susceptible to measurement noise and model uncertainty.

This note presents a block triangular form (BTF) which relates to EEL in that it also identifies coordinates in which the system has a special structure. The BTF is more broadly applicable than the OF and decomposes a p-output system (1) into p subsystems with the dynamics of the ith subsystem depending on the state of the ith subsystem itself, the states of the "upper" jth subsystem, $j < i$, and the system output. In general we do not constrain the dependence of the output in the BTF coordinates. The BTF first appeared in [6] as an intermediate step to computing a so-called block triangular observer normal form and was further investigated in [13]. The primary reason for transforming a system into BTF is that it permits lower dimensional observer designs to be performed subsystem-at-a-time while effectively treating "upper" subsystem states as known measurements [6]. Existence conditions for the BTF are weaker than those of the special forms used in EEL [4], [5] and can, therefore, be applied to a broader class of systems. As in [6], we reconsider the application of the results in [14] to establish modified BTF existence conditions. It is important to note the similarity of the existence conditions and their proof which are presented here with those given in [6]. The present work's contribution is to point out the oversight in [6]; the upper subsystem variables cannot be treated as parameters.

II. System Forms

In order to state the existence conditions for the BTF we introduce an observable form which exists if we make an observability assumption.

Definition 2.1: System (1) is locally observable at $\zeta^0 \in \mathbb{R}^n$ with indexes $\lambda_i > 0, 1 \le i \le p$, if

1) $\sum_{i=1}^{p} \lambda_i = n$;
2) after suitable reordering of the h_i's, $\dim(\mathrm{span}\{L_f^{j-1}\mathrm{d}h_i(\zeta^0), 1 \le j \le \lambda_i, 1 \le i \le p\}) = n$.

The indexes λ_i in Definition 2.1 are not uniquely defined [6]. As shown in Section IV, nonuniqueness of λ_i can add a degree of freedom when checking BTF existence conditions. That is, for a particular choice of indexes λ_i the system may be transformable to BTF and for another choice it is not.

A. Observable Form

When (1) is observable according to Definition 2.1 we can define new coordinates $x_j^i = L_f^{j-1} h_i, 1 \le j \le \lambda_i, 1 \le i \le p$ to locally transform (1) into an observable form whose ith subsystem Σ^i is

$$\Sigma^i : \begin{cases} \dot{x}^i = f^i\left(x^{\langle i \rangle}, y_{[i+1,p]}^{\langle \alpha^i \rangle}\right) = \begin{bmatrix} x_2^i \\ x_3^i \\ \vdots \\ \varphi^i\left(x^{\langle i \rangle}, y_{[i+1,p]}^{\langle \alpha^i \rangle}\right) \end{bmatrix} \\ y_i = x_1^i \end{cases} \tag{2}$$

where $x^{\langle i \rangle} = ((x^1)^T, \ldots, (x^i)^T)^T \in \mathbb{R}^{\nu_i}$ with $\nu_i = \sum_{k=1}^{i} \lambda_k$, $x^i = (x_1^i, \ldots, x_{\lambda_i}^i)^T$, $y_{[i+1,p]}^{\langle \alpha^i \rangle} = ((y_{i+1}^{\langle \alpha_{i+1}^i \rangle})^T, \ldots, (y_p^{\langle \alpha_p^i \rangle})^T)^T$, $y_j^{\langle \alpha_j^i \rangle} = (y_j, \dot{y}_j, \ldots, y_j^{(\alpha_j^i)})^T$, and the nonnegative indexes $\alpha_j^i, 1 \le i < j \le p$ denote the highest order of time derivative of y_j appearing in $\varphi^i =$

Manuscript received June 27, 2005; revised December 5, 2005 and June 30, 2006. Recommended by Associate Editor M.-Q. Xiao. This work was supported by the Natural Sciences and Engineering Council of Canada (NSERC) under Research Grant 249681-02.

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Digital Object Identifier 10.1109/TAC.2006.884996