B. Brief Discussion

The results presented clearly show that the disturbance model and the observer used for the augmented system affect significantly the closed-loop performance of MPC in the presence of plant/model mismatch and unmeasured disturbances. In Fig. 3 we can see that MPC 1 and MPC 2, based on disturbance models and observers designed with the proposed method, reject the unmeasured disturbances much more efficiently than MPC 0, even during the second phase of the rejection ($t \geq 50$) when the actual disturbance is consistent with the disturbance model used by MPC 0. Most likely the reason for the worse behavior of MPC 0 is that it is based on a fixed-structure observer (steady-state Kalman filter), while the design of MPC 1 and MPC 2 exploits its freedom in choosing the observer gain. Indeed, the existence result of [9] proves that an observer gain exists for MPC 0 that provides the same behavior of MPC 1 or MPC 2.

Notice that this performance improvement is associated to a "better" (not a "larger") input usage. In fact, this occurs because the output prediction errors go to zero much more quickly for MPC 1 and MPC 2 than for MPC 0. We can also see, as expected [5], that MPC 3 is not able to guarantee offset-free control in the two controlled variables because it uses a disturbance model with two integrating disturbances only. Fig. 4 shows that efficient disturbance rejection and low sensitivity to output noise is achieved by the proposed method. It is also interesting to notice the effect of the tuning parameter α on the closed-loop performance: the lower α, the more effective the estimator in rejecting disturbances (but also the more sensitive to output and process noise). Hence, by simply varying this single tuning parameter, one can trade off between effectiveness in obtaining offset-free control and low sensitivity to noise.

VI. Conclusion

In this note, we proposed a novel method to design a disturbance model and its associated observer for offset-free model predictive control. This objective was achieved by synthesizing a "dynamic" observer for the nominal system and by showing that, when offset-free control is required, this is equivalent to choosing an integrating disturbance model and a static observer gain for the augmented system. The dynamic observer was designed by solving an appropriate $\mathcal{H}_\infty$ control problem, aimed at minimizing the effect of external unmeasured disturbances (and plant/model mismatch) on the output prediction. There is a single scalar parameter to choose in the proposed design method, which trades off between the aggressiveness in the rejection of disturbances and the resiliency to output and process noise. A simple application example showed the effectiveness of the proposed method and the benefits that can be achieved with respect to other more common choices of disturbance models and observers.

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Performance Analysis of Second-Order Sliding-Mode Control Systems With Fast Actuators

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***Abstract*—The robustness of the second-order sliding-mode control (2-SMC) algorithm known as the "Generalized Sub-optimal" algorithm is analyzed with respect to the cascade introduction of a linear fast actuator. It is shown that if the actuator dynamics are sufficiently fast then the system trajectories converge to an invariant set that includes the second-order sliding domain $s = \dot{s} = 0$. It is also shown that the size of the invariant set has quadratic dependence with respect to the actuator "small parameter" μ for the sliding variable s, and linear dependence for its derivative $\dot{s}$. This means that in the steady state the system trajectories converge to an invariant domain described by the following conditions: $|s| \leq O(\mu^2)$ and $|\dot{s}| \leq O(\mu)$. A simulation example is given to confirm the proposed analysis.**

***Index Terms*—Fast actuators, nonlinear systems, sliding-mode control (SMC), uncertain systems, variable structure systems.**

I. Introduction

The second-order sliding-mode control (2-SMC) approach was considered an interesting topic by many researchers and practitioners over the last decade (see [16], [4], [21], [19], [9], [20] and references therein). Some attractive features of 2-SMC, as compared to the classical first order sliding mode control (1-SMC), are commonly recognized: higher accuracy of motions [16], chattering reduction [21], finite-time convergence for systems with relative degree two [3],

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[15], the capability of utilizing the Coulomb friction in positioning systems [19].

Such an interesting performance can be obtained by means of a discontinuous control switching at theoretically infinite frequency. In practice, digital implementation [2] and switching delay [15] cause a detriment of the control accuracy which depends on the sampling frequency and the switching delay respectively. These results are similar to those well established for classical 1-SMC [23], [24].

In implementation, the actuator bandwidth is often much larger than that of the closed-loop system, and for this reason its dynamics is usually neglected in the controller design. In some cases, nevertheless, unmodeled, even fast, dynamics may become critical as they increase the relative degree. It is well known [1] that in the case of linear plant and actuator, the relay feedback control systems may have a stable equilibrium point only if the overall relative degree of the combined actuator and plant dynamics is one or two, otherwise the system exhibits chattering.

For smooth control systems, the influence of actuators on the system behavior could be analyzed within the framework of classical singular perturbation theory (see, for example, [14]) based on spectrum separation. Those methods could not be used for the chattering analysis in sliding mode control systems ([23, Ch. 5]). In [10]–[12] special techniques for the analysis of singularly perturbed relay control systems were developed, allowing to estimate the amplitude and frequency of chattering and to correct the sliding mode equations in terms of fast actuators and sensors time constants. Generally speaking, in [23], and [10]–[12] it was shown that the accuracy of 1-SMC with fast actuators is proportional to the "small parameter" μ of the actuator's dynamics ($|s| \leq O(\mu)$, s being the sliding variable). These results were confirmed in [5] via recently developed "locus of a perturbed relay system" (LPRS) method allowing to analyze the robustness of the relay control systems in the frequency domain.

The chattering analysis and adjustment of the 2-SMC algorithm in the frequency domain was performed in [6], [7], in which the cascade connection of a linear plant and a linear actuator was analyzed. Necessary conditions for the existence of periodic motions, and a methodology for computing the parameters of those motions, were obtained via, both, the describing function analysis and the "modified Tsypkin locus".

In this note, we analyze the effect of introducing a fast actuator into a nonlinear system driven by the Generalized Suboptimal (G-SO) algorithm [4]. We consider the plant being *nonlinear* dynamics, and the linear actuator dynamics being a *singular perturbation* to the original system, and estimate the accuracy of G-SO algorithm in terms of the fast actuator time constant μ. We prove that the trajectory of the control system, with the G-SO algorithm and a sufficiently fast linear actuator, enters into an invariant μ-vicinity of the second-order sliding domain $s = \dot{s} = 0$ after a finite-time transient. Such vicinity is characterized by $|s| \leq O(\mu^2)$ and $|\dot{s}| \leq O(\mu)$. This result justifies successful practical implementations of 2-SMC presented in the literature [4], [20] .

This note is organized as follows. Section II presents the problem statement and Section III contains the main result. A simulation example illustrating the proposed results is suggested in Section IV. Section V concludes this note by summarizing the results of this research and presenting some final remarks.

II. Problem Formulation

Let us consider the following class of nonlinear uncertain single-input dynamics in regular form [23]

$$\dot{\mathbf{x}} = \mathbf{h}(\mathbf{x}, \boldsymbol{\xi}) \tag{1}$$

$$\dot{\boldsymbol{\xi}} = \mathbf{f}(\mathbf{x}, \boldsymbol{\xi}) + \mathbf{g}(\mathbf{x}, \boldsymbol{\xi}) z_1(t)$$

$$\equiv \begin{bmatrix} \xi_2 \\ f(\mathbf{x}, \boldsymbol{\xi}) \end{bmatrix} + \begin{bmatrix} 0 \\ g(\mathbf{x}, \boldsymbol{\xi}) \end{bmatrix} z_1(t) \tag{2}$$

where $\mathbf{x} \in R^m$ is the internal dynamics state-vector, $\boldsymbol{\xi} = [\xi_1, \xi_2] \equiv [s, \dot{s}] \in R^2$ is the vector collecting the sliding output $s \in R$ and its derivative, $z_1 \in R$ is the plant control input, $t \in [t_0, \omega)$ is the time variable and $\omega \geq t_0$ is the maximal existence time interval of the system trajectories. Let $\mathbf{h} : R^{m+2} \rightarrow R^m$ be a Caratheodory uncertain vector field and $f, g : R^{m+2} \rightarrow R$ be Caratheodory uncertain functions.

System (1)–(2) is driven by a linear rth order linear actuator, described by the following singularly perturbed system:

$$\mu \dot{\mathbf{z}} = \mathbf{C}\mathbf{z} + \mathbf{D}u, \qquad \mu > 0, \qquad z_1 = \mathbf{E}\mathbf{z} \tag{3}$$

where $u \in R$ is the actuator's input, $\mathbf{z} = [z_1, z_2, \ldots, z_r] \in R^r$, μ is a positive parameter, $\mathbf{D} = [0, 0, \ldots, 0, 1]^T$, $\mathbf{E} = [1, 0, \ldots, 0]$ and $\mathbf{C}$ is a constant Hurwitz matrix of appropriate dimension. Let the actuator (3) has unit zero-frequency gain.

The control aim is to define an actuator input profile u guaranteeing that $\omega = \infty$, i.e., the trajectories can be infinitely extended in time, and that the sliding variable and its derivative are steered in finite time into some invariant small vicinity of the second-order sliding set $s = \dot{s} = 0$.

Let the sliding variable s be the unique signal available for feedback and assume what follows:

A1) There exist positive constants $K_1, K_2 \in R^+$ such that for every $\mathbf{x}_0 \in R^m$ and every bounded and continuous $\boldsymbol{\xi}(t)$ the unique maximal solution of the initial value problem (1), with $\mathbf{x}(t_0) = \mathbf{x}_0$, has interval of existence R^+ and for any $t \in [t_1, t_2] \subseteq [t_0, \infty)$ satisfies the following condition:

$$\|\mathbf{x}(t)\| \leq \|\mathbf{x}(t_1)\| + K_1 \sup_{t_1 \leq t \leq t_2} |s| + K_2 \sup_{t_1 \leq t \leq t_2} |\dot{s}|$$
$$t_0 \leq t_1 \leq t \leq t_2 \leq \infty. \tag{4}$$

A2) There are known positive constants F_0, F_1, F_2, G_m, G_M such that

$$|f(\mathbf{x}, \boldsymbol{\xi})| \leq F_0 + F_1 \|\mathbf{x}\| + F_2 \|\boldsymbol{\xi}\|$$
$$\|\mathbf{x}\| = \sum_{i=1}^{m} |x_i| \quad \|\boldsymbol{\xi}\| = |\xi_1| + |\xi_2| \tag{5}$$
$$0 < G_m \leq g(\mathbf{x}, \boldsymbol{\xi}) \leq G_M. \tag{6}$$

A3) The initial conditions $\mathbf{x}(t_0)$ and $\boldsymbol{\xi}(t_0)$ belong to known compact domains $\overline{X}_0, \overline{\Omega}_0$

$$\mathbf{x}(t_0) \in \overline{X}_0 \quad \boldsymbol{\xi}(t_0) \in \overline{\Omega}_0. \tag{7}$$

A4) There is a positive constant K_3 such that

$$\|\mathbf{z}(t)\| \leq K_3 \sup_{t \geq t_0} |u|, \qquad t \geq t_0, \qquad K_3 \geq 1. \tag{8}$$

Assumption A1 involves the concept of input-to-state stability (ISS) (see [13, Sec. 4.9]). It can be said that the internal dynamics (1) is required to be input-to-state stable with linear gain. On the basis of Assumption A3, it can be evaluated the following constants:

$$X_0 = \sup_{\mathbf{x} \in \overline{X}_0} \|\mathbf{x}\| \quad S_0 = \sup_{\boldsymbol{\xi} \in \overline{\Omega}_0} |s| \quad \dot{S}_0 = \sup_{\xi \in \overline{\Omega}_0} |\dot{s}|. \tag{9}$$

Assumption A4 establishes the bounded-input bounded-state (BIBS) property of the actuator dynamics. The control algorithm dealt with in this note is such that the actuator input u has a constant magnitude $U_M (|u| \equiv U_M)$. Condition (8) can be therefore rewritten as

$$\|\mathbf{z}(t)\| \leq K_3 U_M, \qquad t \geq t_0. \tag{10}$$

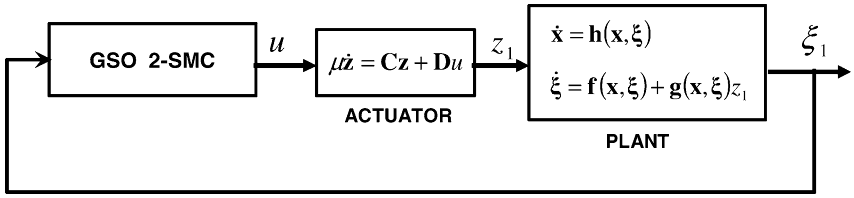

Fig. 1. Considered closed-loop system.

III. Main Result

We consider the following control law, called the "*Generalized Sub-optimal*" (GSO) 2-SMC algorithm [4] shown in (11) at the bottom of the page where U_M, β are positive constant parameters and $t_{Mi} (i = 1, 2, \ldots)$ is the sequence of time instants at which $\dot{s} = 0$. The values $s(t_{Mi})$ are referred to as the "singular values" of s.

The analysis of the closed-loop system (1)–(3) satisfying assumptions A1)–A4) and driven by the GSO controller (11) (Fig. 1) is the matter of Theorem 1.

We show that if proper tuning rules of the GSO 2-SMC controller are adopted, and the actuator dynamics is "sufficiently fast," then the state trajectories of the controlled system are bounded. More precisely, we show that it can be computed a positive constant F such that function $f(\mathbf{x}, \boldsymbol{\xi})$ meets the following condition:

$$|f(\mathbf{x}, \boldsymbol{\xi})| \le F, \qquad t \ge t_0. \tag{12}$$

Furthermore, we prove that the following conditions are achieved after a finite transient T_f for some positive constants ρ_0 and ρ_1

$$|s| \le \rho_0 \mu^2, \quad |\dot{s}| \le \rho_1 \mu, \qquad t \ge T_f > t_0. \tag{13}$$

The *a priori* information concerning the plant and actuator dynamics is a set of positive constants which have been defined throughout this note: $K_1, K_2, F_0, F_1, F_2, G_m, G_M, X_0, S_0, \dot{S}_0, K_3$.

A major result of the present note relies on the computation of two constants, F^* and μ^*, according to the algorithm described as follows. The "inputs" of the algorithm are the above-cited constants.

Algorithm 1. Computation of F^* and μ^*:

1) Define

$$\eta > 1, \quad \varepsilon \in (0, 1), \quad \gamma \in (0, 1). \tag{14}$$

2) Compute N_ε according to (39) and to the procedure reported in the Appendix A
3) Set β according to the following condition

$$\beta \in \left(\frac{\eta + 1 + \frac{G_M(1+\varepsilon)}{G_m(1-\varepsilon)} - \gamma(\eta - 1)}{\eta + 1 + \frac{G_M(1+\varepsilon)}{G_m(1-\varepsilon)}}, 1 \right). \tag{15}$$

4) Compute the following constants.

$$\begin{aligned} H_0 &= [F_0 + F_1 X_0] \quad H_1 = [F_1 K_1 + F_2] \\ H_2 &= [F_1 K_2 + F_2] \end{aligned} \tag{16}$$

$$\begin{aligned} \theta_1 &= \left[1 + K_3 \eta \frac{G_M}{G_m(1-\varepsilon)} \right] \\ \theta_2 &= \left[1 + \eta \frac{G_M(1+\varepsilon)}{G_m(1-\varepsilon)} \right] \end{aligned} \tag{17}$$

$$\begin{aligned} q_0 &= \frac{1}{2} \theta_1 N_\varepsilon^2 \left(1 + \frac{\theta_1}{\eta - 1} \right) \quad q_1 = \frac{1}{\eta - 1} \dot{S}_0 \theta_1 N_\varepsilon \\ q_2 &= \frac{1}{2(\eta - 1)} \dot{S}_0^2 \end{aligned} \tag{18}$$

$$\begin{aligned} \lambda_0 &= H_0 + H_1 S_0 + H_2 \dot{S}_0 \quad \lambda_1 = H_2 N_\varepsilon (\theta_1 + \theta_2) \\ \lambda_2 &= H_2 \sqrt{2(1-\beta)\theta_2}. \end{aligned} \tag{19}$$

5) Compute μ^* and F^* as the unique positive solutions of the following equations:

$$H_1 q_0 \mu^{*2} + [\lambda_1 + \lambda_2 \sqrt{\varphi_0}] \mu^* = 1 \tag{20}$$

$$\begin{aligned} F^* = {} & \lambda_0 + H_1 q_0 F \mu^{*2} + H_1 q_1 \mu^* + \frac{H_1 q_2}{F^*} + \lambda_1 F^* \mu^* \\ & + \lambda_2 \sqrt{q_2 + S_0 F^* + q_0 F^{*2} \mu^{*2} + q_1 F^* \mu^*} \end{aligned} \tag{21}$$

Theorem 1.: Consider system (1)–(3) satisfying assumptions A1)–A4) and driven by the Generalized Sub-Optimal controller (11). Let parameters $\eta, \varepsilon, \gamma, \beta, \mu^*$ and F^* be evaluated according to the above Algorithm 1. Set the discontinuous control magnitude U_M as follows:

$$U_M = \frac{\eta F}{(1 - \varepsilon) G_m}, \qquad F \ge F^* \tag{22}$$

Then, provided that the following condition holds

$$\mu < \mu^* \tag{23}$$

the closed loop system trajectories will enter in finite time into the invariant domain

$$O_\mu \equiv \{(\mathbf{x}, s, \dot{s}, z_1) : |s| \le \rho_0 \mu^2, |\dot{s}| \le \rho_1 \mu\} \tag{24}$$

where ρ_0 and ρ_1 are positive constants independent of μ.

Proof: The proof is organized in the following five steps. Steps 1)–3) assume, temporarily, that constant F in (12) exists, while its existence is proven formally in Step 4). Detailed proof of Steps 1)–5) is given in the Appendix B.

Step 1) There exists a time instant $t_{M1} \ge t_0$ at which $\dot{s}(t_{M1}) = 0$.

Step 2) A sequence of " singular points" $P_i \equiv (s(t_{Mi}), 0) \equiv (s_{Mi}, 0) (i = 1, 2, \ldots)$ is enforced.

Step 3) There exist $\rho_0^* > 0$ such that whenever $|s_{Mi}| > \rho_0^* \mu^2$ the following contraction condition holds:

$$|s_{M,i+h}| \le \max\{\beta, \gamma\} |s_{Mi}|, \qquad h \in \{1, 2\}. \tag{25}$$

Step 4) A constant F can be computed such that condition (12) holds.

Step 5) There exist $\rho_0 > \rho_0^* > 0$ and $\rho_1 > 0$ such that the domain (24) is attracting and invariant.

$$u(t) = \begin{cases} -U_M \operatorname{sign}(s(t) - s(t_0)), & t_0 \le t \le t_{M1} \\ -U_M \operatorname{sign}(s(t) - \beta s(t_{Mi})), & t_{Mi} < t \le t_{Mi+1}, \qquad i = 1, 2, \ldots \end{cases} \tag{11}$$

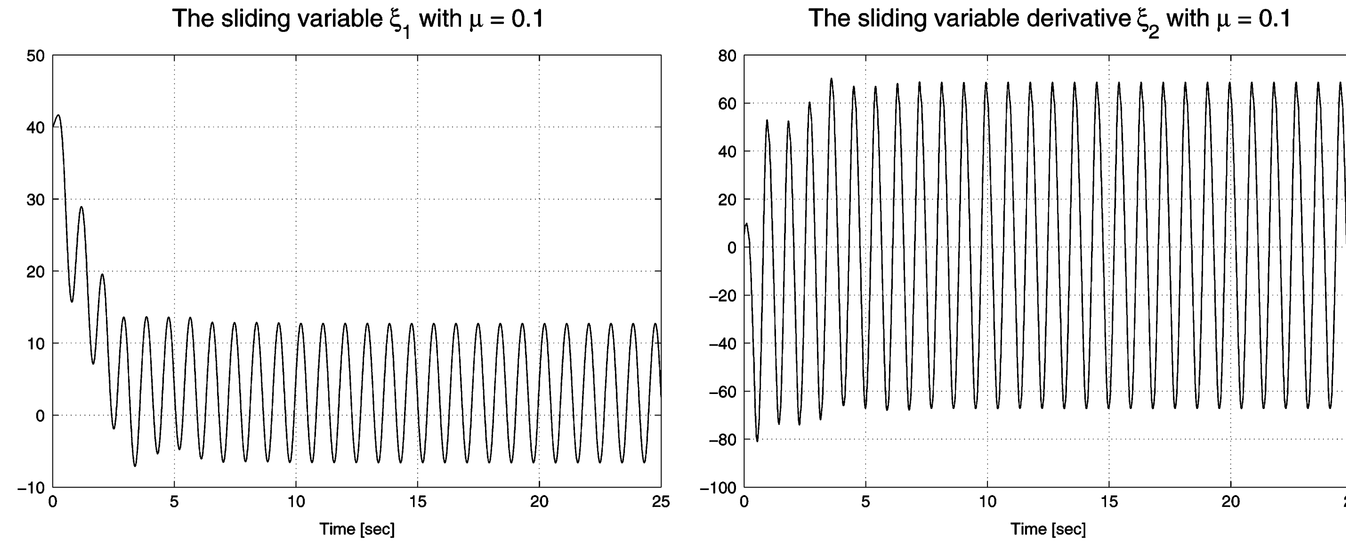

Fig. 2. First test: $\mu = 0.1$. The sliding variable ξ_1 (left) and its derivative ξ_2 (right).

Remark 1: The linear growth assumption A2) might be considered as rather restrictive. Indeed, general nonlinear system with a polynomial (e.g., quadratic) state-dependent drift term clearly violates such an assumption. The growth restriction (5) can be generalized provided that, as often happens in physical systems, the state is *a priori* confined into a bounded, possibly large, physical "operating region". From this viewpoint, let us consider the following more general growth condition:

$$|f(\mathbf{x}, \boldsymbol{\xi})| \leq \mathcal{F}(\|\mathbf{x}\|, \|\boldsymbol{\xi}\|) \tag{26}$$

where $\mathcal{F}$ is some bounded not-negative function with arbitrary growth rate. Let the state vector $(\mathbf{x}, \boldsymbol{\xi})$ be confined within a given, possibly large, compact domain $\mathcal{D}$, i.e., $(\mathbf{x}(t), \boldsymbol{\xi}(t)) \in \mathcal{D}$ for all $t > 0$. Then, by considering (12) and (26), constant F^* could be defined as follows, thereby skipping the rather complex procedure outlined in the Algorithm 1.

$$F^* = \sup_{(\mathbf{x}, \boldsymbol{\xi}) \in \mathcal{D}} \mathcal{F}(\|\mathbf{x}\|, \|\boldsymbol{\xi}\|) \tag{27}$$

Furthermore, condition (23) is no longer necessary. This conclusion comes from the developments presented in the Step 4 of the proof of Theorem 1. It is apparent that such restriction on μ was introduced just to guarantee the solvability of (21). Since (21) should not be solved anymore, also the restriction $\mu < \mu^*$ can be dispensed with.

Thus, for large classes of practically relevant systems (e.g., uncertain mechanical systems governed by interconnected second-order Lagrangian equations) restricted to evolve into their physical "operating region" the allowed class of uncertainties can be significantly enlarged. At the same time, the approach turns out to be simplified.

IV. Simulation Example

To validate the present analysis consider the following fourth-order nonlinear system:

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}(\xi_1 + \xi_2) \quad \mathbf{x} = [x_1, x_2]^T \\ \dot{\xi}_1 &= \xi_2, \dot{\xi}_2 = \frac{\xi_2}{1 + \xi_2^2} + \xi_1 + \xi_2 + \|\mathbf{x}\| \\ &\quad + (1 + 0.2\cos(x_1 + \xi_2) z_1 \end{aligned} \tag{28}$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{29}$$

with the second-order actuator

$$\mu \dot{\mathbf{z}} = \mathbf{C}\mathbf{z} + \mathbf{D}u \quad \mathbf{z} = [z_1, z_2]^T \tag{30}$$

$$\mathbf{C} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \quad \mathbf{D} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{31}$$

The sliding quantity ξ_1 is the unique signal available for measurement. The considered dynamics meets all the requirements stated in the Assumptions A1–A4. The initial conditions are: $[\xi_1(t_0), \xi_2(t_0)] = [20, 5], [x_1(t_0), x_2(t_0)] = [1, 1], [z_1(t_0), z_2(t_0)] = [0, 0]$. The controller parameters are set as $U_M = 200, \beta = 0.8$. The Euler integration algorithm with step $T_s = 0.001$ has been used in the Matlab-Simulink environment. In a first test the actuator's time constant was set to $\mu = 0.1$. Fig. 2 shows that the sliding variable and its derivative converge to a bounded neighborhood of the origin. To check whether the accuracy order (24) is actually achieved we performed a second test with the μ parameter reduced by a factor 10 ($\mu = 0.01$). By comparing Fig. 3 with Fig. 4 it is apparent that the sliding accuracy is improved by a factor 100, as for the sliding variable ξ_1, and by a factor 10 as for its derivative ξ_2. This is according to the expected order of accuracy stated in Theorem 1.

V. Conclusion

The effect of introducing a stable linear actuator into systems driven by the generalized suboptimal 2-SMC algorithm is investigated. It is shown that if the actuator dynamics are sufficiently fast, the system trajectories enter a small invariant boundary layer of the 2-sliding domain $s = \dot{s} = 0$. The analysis carried out proves that the actual system behavior is close to the one that would be obtained without the actuator. Quantitative assessment of performance degradation due to actuator introduction is provided.

Appendix

A. Analysis of the Actuator Response Under a Constant Input

Consider the response of the actuator (3) to the constant input $u = U_M \delta_{-1}(t - t_i), t_i$ being a generic time instant

$$\begin{aligned} z_1 = z_{1L} + z_{1F} &= \mathbf{E} e^{\frac{1}{\mu}\mathbf{C}(t - t_i)} \mathbf{z}(t_i) \\ &\quad + \frac{1}{\mu} U_M \mathbf{E} \int_{t_i}^{t} e^{\frac{1}{\mu}\mathbf{C}(t-\tau)} \mathbf{D} d\tau \quad t \geq t_i \end{aligned} \tag{32}$$

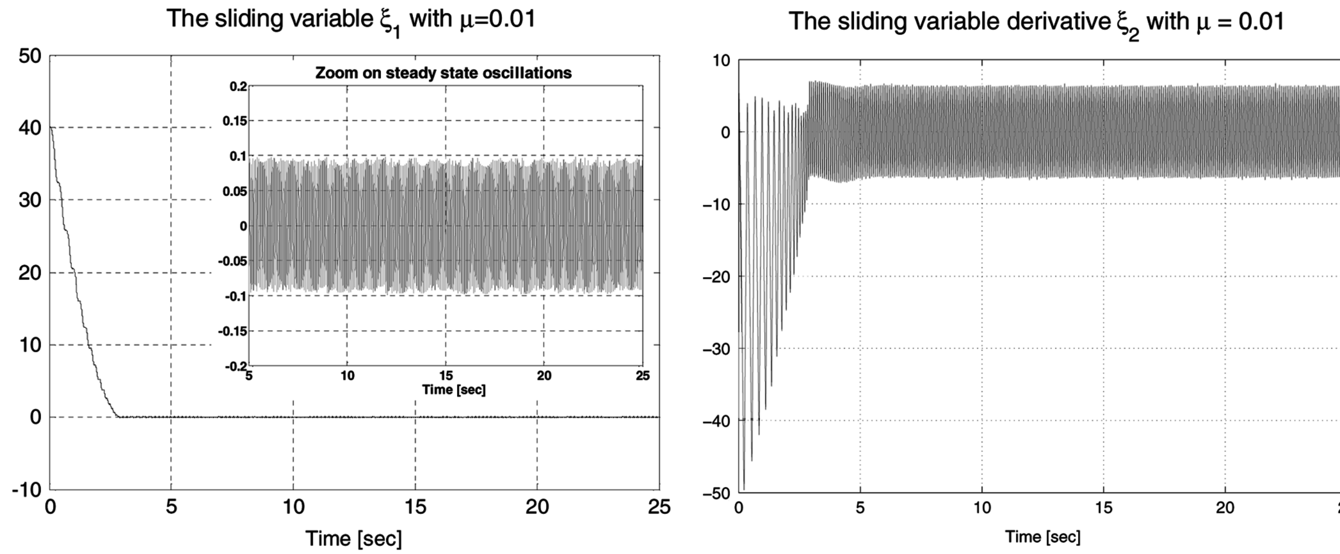

Fig. 3. Second test: $\mu = 0.01$. The sliding variable ξ_1 (left) and its derivative ξ_2 (right).

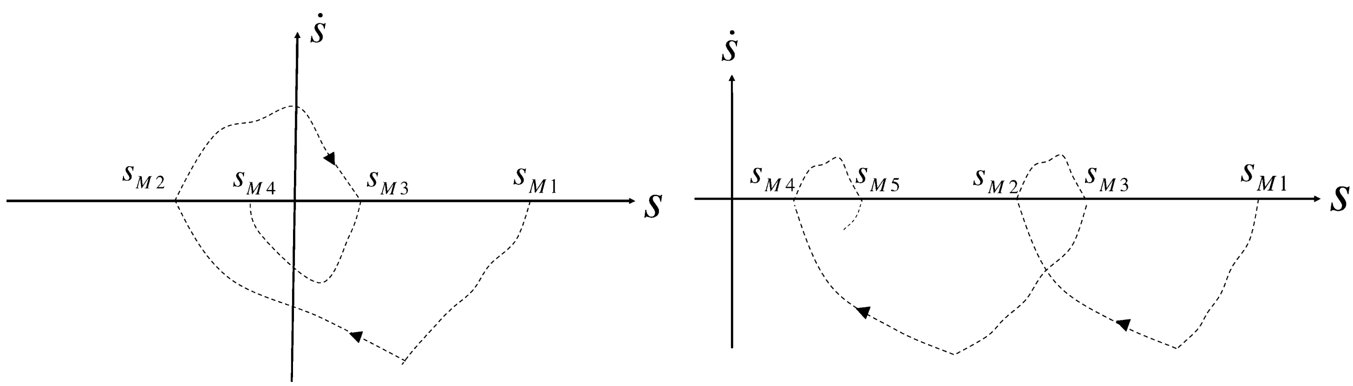

Fig. 4. The possible transient trajectories.

By the exponential stability property of the actuator dynamics, two constants $\lambda > 1$ and $\alpha > 0$ exist such that the free and forced responses satisfy the following inequalities:

$$\begin{aligned} |z_{1L}| &\le \lambda e^{-\frac{\alpha}{\mu}(t-t_i)}\|\mathbf{z}(t_i)\| \\ |z_{1F} - U_M| &\le \lambda e^{-\frac{\alpha}{\mu}(t-t_i)} U_M \end{aligned} \tag{33}$$

Considering (10), we can rewrite (33) as follows:

$$|z_{1L}| \le \lambda K_3 U_M e^{-\frac{\alpha}{\mu}(t-t_i)} \tag{34}$$

There is a *finite* transient after which the overall actuator response enters an arbitrarily small vicinity of the steady-state value U_M. More specifically, for any $\varepsilon \in (0,1)$ there exists $T_\varepsilon > 0$ such that

$$|z_1 - U_M| \le \varepsilon U_M \quad t \ge t_i + T_\varepsilon \tag{35}$$

A sufficient condition for the occurrence of (35) is the simultaneous satisfaction of the following conditions:

$$|z_{1L}| \le \frac{1}{2}\varepsilon U_M \quad |z_{1F} - U_M| \le \frac{1}{2}\varepsilon U_M \tag{36}$$

which are equivalent, considering (33) and (34), to

$$\begin{aligned} \lambda K_3 U_M e^{-\frac{\alpha}{\mu}(t-t_i)} &\le \frac{1}{2}\varepsilon U_M \Rightarrow t - t_i \ge T_{\varepsilon 1} \\ &= \frac{1}{\alpha}\mu \log\frac{2\lambda K_3}{\varepsilon} \end{aligned} \tag{37}$$

$$\begin{aligned} \lambda U_M e^{-\frac{\alpha}{\mu}(t-t_i)} &\le \frac{1}{2}\varepsilon U_M \Rightarrow t - t_i \ge T_{\varepsilon 2} \\ &= \frac{1}{\alpha}\mu \log\frac{2\lambda}{\varepsilon} \end{aligned} \tag{38}$$

Since $K_3 > 1$, the overall transient time T_ε can be upper bounded as follows:

$$T_\varepsilon \le T_{\varepsilon 1} \equiv N_\varepsilon \mu, \qquad N_\varepsilon = \frac{1}{\alpha}\log\frac{2\lambda K_3}{\varepsilon} \tag{39}$$

Note that, by taking into account (22), condition (35) implies the following:

$$|z_1| > \frac{F}{G_m} \quad t \ge t_i + N_\varepsilon \mu \tag{40}$$

Since, as shown in the proof of Theorem 1, condition (12) holds during the entire control time interval $t \ge t_0$, inequality (40) represents the "dominance condition" guaranteeing that the sign of z_1 affects the sign of $\ddot{s}$ in accordance with the following:

$$z_1 \ge U_M(1-\varepsilon) \Rightarrow \ddot{s} \ge (\eta - 1)F \tag{41}$$

$$z_1 \le -U_M(1-\varepsilon) \Rightarrow \ddot{s} \le -(\eta - 1)F \tag{42}$$

B. Proof of Theorem 1

Step 1) Considering the first of (11) the actuator input is such that

$$u = -U_M \mathrm{sign}(\dot{s}(t)) = -U_M \mathrm{sign}(\dot{s}(t_0)), \qquad t_0 < t \le t_{M1}. \tag{43}$$

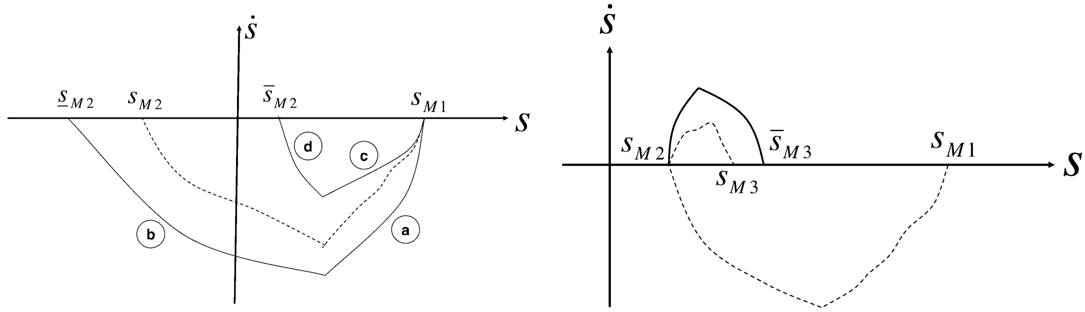

Fig. 5. Continuous lines: the limiting arcs. Dotted lines: possible actual trajectories.

Thus, by (40)–(42), the following "reaching condition" will be satisfied:

$$\ddot{s}\dot{s} \leq -(\eta - 1)F|\dot{s}|, \qquad t_0 + N_\varepsilon \mu \leq t \leq t_{M1} \tag{44}$$

which proves the claim.

Step 2) The initial condition is the first "singular point" $P_1 \equiv (s(t_{M1}), 0) \equiv (s_{M1}, 0)$. Let $\ddot{s}_{M1} = \ddot{s}(t_{M1})$ and assume, without loss of generality, that $s_{M1} > 0$.

If $\ddot{s}_{M1} > 0$ then s is increasing during the time interval $t \in [t_{M1}, t_{M2})$ and the sign of the switching function $s - \beta s_{M1}$ does not change. Hence, the actuator input u is given by $u(t) = -U_M, t_{M1} \leq t < t_{M2}$. Taking into account (40) it follows that $z_1 \leq -(1-\varepsilon)U_M, t_{M1} + N_\varepsilon \mu \leq t \leq t_{M2}$, which implies, according to (42), that $\ddot{s}(t) \leq -(\eta - 1)F, t_{M1} + N_\varepsilon \mu \leq t \leq t_{M2}$ thereby guaranteeing the existence of $t_{M2} > t_{M1}$ such that $\dot{s}_1(t_{M2}) = 0$ and $\ddot{s}(t_{M2}) < 0$.

If $\ddot{s}_{M1} < 0$ then s is decreasing during the time interval $t \in [t_{M1}, t_{M2})$. This implies that there exists $t_{c1} > t_{M1}$ such that $s(t_{c1}) = \beta s_{M1}$, and the actuator input u will be given correspondingly by

$$u = \begin{cases} -U_M, & t_{M1} \leq t < t_{c1} \\ U_M, & t_{c1} \leq t < t_{M2} \end{cases}. \tag{45}$$

By analogous considerations as before, according to (41) the following condition holds $\ddot{s}(t) \geq (\eta - 1)F, t_{c1} + N_\varepsilon \mu \leq t \leq t_{M2}$, which guarantees the existence of t_{M2}.

Step 3) Singular point P_i such that $s_{Mi}\ddot{s}_{Mi} < 0$ is reached in finite time and can be considered as the initial point of the present step of analysis. We shall prove that there exists $\rho_0^* > 0$ such that, as long as $|s_{Mi}| > \rho_0^* \mu^2$, then:

$$\begin{aligned} s_{Mi}s_{M,i+1} < 0 \Longrightarrow |s_{M,i+1}| \leq \max\{\gamma, \beta\}|s_{Mi}|, \\ s_{M,i+1}\ddot{s}_{M,i+1} < 0 \end{aligned} \tag{46}$$

$$\begin{aligned} s_{Mi}s_{M,i+1} > 0 \Longrightarrow |s_{M,i+1}| < |s_{M,i+2}| \leq \max\{\gamma, \beta\}|s_{Mi}| \\ s_{M,i+2}\ddot{s}_{M,i+2} < 0. \end{aligned} \tag{47}$$

Conditions (46)–(47) tell us that during convergence different qualitative behaviors may occur, depending on the sign of the encountered singular points. If they have alternating sign then the transient trajectory is of the type in Fig. 4-left, which features the contraction condition (46). If they have the same sign, the transient trajectory is of the type in Fig. 4, right, which features the contraction condition (47).

Consider without loss of generality, the case in which $s_{Mi} > 0$ and $\ddot{s}_{Mi} < 0$. Let $i \equiv 1$.

The system evolution between points P_1 and P_2 is confined between limit curves of parabolic shape similar to those depicted in the next Fig. 5, left.

The next singular point P_2, achieved at $t = t_{M2}$ when $\dot{s}(t_{M2}) = 0$, is such that

$$\underline{s}_{M2} \leq s_{M2} \leq \overline{s}_{M2} \tag{48}$$

$$\begin{aligned} \underline{s}_{M2} &= \beta s_{M1} - \frac{(1-\beta)(F + G_M(1+\varepsilon)U_M)}{G_m(1-\varepsilon)U_M - F} s_{M1} \\ &\quad - \varphi_4 \mu \sqrt{s_{M1}} - \varphi_3 \mu^2 \\ \overline{s}_{M2} &= \beta s_{M1} - \frac{(1-\beta)(G_m(1-\varepsilon)U_M - F)}{F + G_M K_3 U_M} s_{M1} \end{aligned} \tag{49}$$

with φ_3 and φ_4 being positive constants.

The contraction condition (25), with $i, h = 1$, is then equivalent to

$$\underline{s}_{M2} \geq -\gamma s_{M1}. \tag{50}$$

Considering (48) and (49), (50) can be rewritten as

$$\begin{aligned} \frac{(1-\beta)[F + G_M(1+\varepsilon)U_M]}{G_m(1-\varepsilon)U_M - F} s_{M1} + \varphi_4 \mu \sqrt{s_{M1}} + \varphi_3 \mu^2 \\ \leq (\beta + \gamma)s_{M1}. \end{aligned} \tag{51}$$

To solve (51) let us introduce the new variable $\rho_0 = (s_{M1})/(\mu^2)$ and rewrite (51) as

$$\begin{aligned} \Omega_1 \rho_0 + \varphi_4 \sqrt{\rho_0} + \varphi_3 \leq (\beta + \gamma)\rho_0, \\ \Omega_1 = \frac{(1-\beta)[F + G_M(1+\varepsilon)U_M]}{G_m(1-\varepsilon)U_M - F}. \end{aligned} \tag{52}$$

If, for sufficiently large ρ_0, the slope of the right-hand side of (52) is less than $\beta + \gamma$ then a solution interval of (52) exists. Considering the higher order terms in ρ_0 it yields the following :

$$\Omega_1 < \beta + \gamma \tag{53}$$

Manipulating (53) one obtains directly the second of (22). The resulting solution interval of (52) is $\rho_0 \geq \underline{\rho}_0$ where $\underline{\rho}_0$ is computed easily by finding the unique positive root of equation $\Omega_1 \rho_0 + \varphi_4 \sqrt{\rho_0} + \varphi_3 = (\beta + \gamma)\rho_0$.

If $s_{M1}s_{M2} < 0$ then point P_{M2} is such that $s_{M2}\ddot{s}_{M2} < 0$. This means that the same considerations can be iterated and s_{M3} will be contractive with respect to s_{M2}.

If $s_{M1}s_{M2} > 0$ then $s_{M2}\ddot{s}_{M2} > 0$ and the obtained behavior is different (see Fig. 5, right). It can be computed $\underline{\rho}_1 > 0$ such that $s_{M3} \in [s_{M2}, s_{M2} + \underline{\rho}_1 \mu^2]$. By considering (48)–(49) it follows that as long as

$$|s_{M1}| \geq \rho_0^* \mu^2 \quad \rho_0^* = \max\left(\underline{\rho}_0, \frac{\underline{\rho}_1 \theta_1}{(1-\beta)(\eta - 1)}\right) \tag{54}$$

with θ_1 defined in (17), then $|s_{M3}|$ is contractive with respect to $|s_{M1}|$ according to (47).

The convergence takes place in finite time since there is a known $k > 0$ such that

$$t_{M,i+1} - t_{M,i} \leq k\sqrt{|s_{Mi}|} \tag{55}$$

and analogous arguments as those used in [3] allow for computing a finite upper-bound to the overall transient length.

Step 4) Define $\overline{s} = \sup_{t \geq t_0} |s|$ and $\overline{\dot{s}} = \sup_{t \geq t_0} |\dot{s}|$. By combining (5) and (4) it follows that $|f(\mathbf{x}, \boldsymbol{\xi})| \leq H_0 + H_1\overline{s} + H_2\overline{\dot{s}}$. A constant F, overestimating $|f(\mathbf{x}, \xi)|$ for any $t \geq t_0$, exists iff the following inequality admits the semi-infinite solution interval $F \in (F^*, \infty)$:

$$F \geq H_0 + H_1\overline{s}(F) + H_2\overline{\dot{s}}(F). \tag{56}$$

After algebraic manipulations one can write

$$\overline{s} \leq S_0 + q_0 F\mu^2 + q_1\mu + \frac{q_2}{F} \tag{57}$$

$$\overline{\dot{s}} \leq \dot{S}_0 + \theta_1 N_\varepsilon \mu F + \sqrt{2(1-\beta)\theta_2 F}\sqrt{\overline{s}} + N_\varepsilon \theta_2 \mu F. \tag{58}$$

Manipulating (56) in light of (57) and (58) yields

$$F \geq \lambda_0 + H_1 q_0 F\mu^2 + H_1 q_1 \mu + \frac{H_1 q_2}{F} + \lambda_1 F\mu + \lambda_2\sqrt{q_2 + S_0 F + q_0 F^2\mu^2 + q_1 F\mu} \tag{59}$$

Inequality (59) is solvable provided that the slope of its right-hand side, viewed as a function of the variable F, is less than one for sufficiently large F. This condition yields the following:

$$H_1 q_0 \mu^2 + [\lambda_1 + \lambda_2\sqrt{q_0}]\mu < 1 \tag{60}$$

Condition (60) yields $\mu \leq \mu^*$, where μ^* is the unique positive solution of (20).

Step 5) Consider the worst-case evolution starting from one of the neighbors (says the right one) of the attracting domain $\mathcal{B}_1(\rho_0^*) \equiv \{(\mathbf{x}, s, \dot{s}, z_1) : |s| \leq \rho_0^*\mu^2, \dot{s} = 0\}$. The analysis performed in step 4 can be applied by setting $S_0 = \rho_0^*\mu^2$ and $\dot{S}_0 = 0$ in (57) and (58). By evaluating the corresponding values of $\overline{s}$ and $\overline{\dot{s}}$ and considering the contraction condition (25), it can be concluded that the following relationships will never be violated after that the system has entered the set $\mathcal{B}_1(\rho_0^*)$:

$$|s| \leq \rho_0^* + q_0 F\mu^2 \equiv \rho_0\mu^2 \quad |\dot{s}| \leq \rho_1\mu. \tag{61}$$

Explicit computation of the constant ρ_1 is skipped for brevity. This concludes the proof. □

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