INT. J. CONTROL, 2002, VOL. 75, NO. 7, 524–537

Slow periodic motions with internal sliding modes in variable structure systems

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Singularly perturbed relay systems (SPRS) in which the reduced systems have the stable periodic motions with internal sliding modes are studied. The slow motion integral manifold of such systems consists of the parts which correspond to the different values of relay control and the solutions may contain the jumps from one part of the slow manifold to another. For such systems a theorem about existence and stability of the periodic solutions is proved. An algorithm of asymptotic representation for this periodic solutions using boundary layer method is presented. It is proved that in the neighbourhood of the break away point the asymptotic representation starts with the first order boundary layer function.

1. Introduction

There are a wide class of relay control systems which work in periodic regimes (Sira Ramirez 1988, Johansson *et al*. 1999). Moreover, such regimes arise every time in relay control systems with time delays because a time delay does not allow an ideal sliding mode to be realized, but results in periodic oscillations (Drakunov and Utkin 1992, Fridman *et al*. 1993). In controllers of exhaust gases for fuel injector automotive control systems (see for example Choi and Hedrick 1996, Li and Yurkovitch 1999) the sensors can measure only the sign of the controlled variable with a delay. In such systems only oscillations around zero value can occur. In the controllers for stabilization of underwater manipulators it is possible to realize only oscillations because of the manipulators properties (see Bartolini *et al*. 1997).

Some relay systems work in periodic regimes with internal sliding modes. As the simplest modelling example of the periodic oscillations with the internal sliding modes we will consider the pendulum which has dry friction contact with an inclined uniformly rotating disc (see, for example, Rumpel 1996). First this pendulum is moving together with disc until returned point and returning back. In real relay control systems, every time we have some unmodelled dynamics which can correspond, for example, to the presence in system of fast actuators or inertial sensors. Usually such dynamics destroy the qualitative behaviour of control systems. The complicated model of sliding mode control systems taking into account the presence of fast and inertial sensors is described by singularly perturbed relay systems (SPRS).

SPRS describe the complete model of fuel injector systems taking into account the influence of the additional dynamics (the car motor). The knowledge of properties of SPRS it is necessary in the controllers for stabilization of the underwater manipulator fingers to take into account the influence of the elasticity of these fingers. In the simplest pendulum systems SPRS describe the influence of the second small pendulum on the oscillation of the first one.

SPRS was investigated by Utkin (1992), Fridman (1990) (stability), Bogatyrev and Fridman (1992) (existence of stable slow motion integral manifold), and Fridman (2001) (averaging and existence of stable periodic solutions). Some control algorithms for SPRS was suggested by Heck (1991), Su (1999) and Innocenti *et al*. (2000).

For *smooth* singularly perturbed systems there are two main classes of *slow* periodic solutions. Slow periodic solutions of the smooth singularly perturbed systems 'without jumps' are situated on slow motion integral manifold (see for example Wasov 1965). The other important class of such solutions are the relaxation solutions (see Mishchenko and Rosov 1980), which contain the 'jumps' from the neighbourhood of the one stable branch of slow motion manifold to the neighbourhood of another one.

Fridman (2000) was shown that the slow motion integral manifold of SPRS is discontinuous and consists at least of parts which correspond to the different values of control. This means that the desired periodic solution of SPRS should have the jumps from the small neighbourhood of the one sheet of integral manifold to the neighbourhood of another one. From this viewpoint the qualitative behaviour of this periodic solution will be nearer to the relaxation solution. The main specific feature of systems with relaxation oscillations is the following: at the moment of time corresponding to the jump from the neighbourhood of one branch of the stable integral manifold to the neighbourhood of another one, the value of the right-hand side is small. That is why in order to find the asymptotic representation of the relaxation solution it was necessary to make special asymptotic representations. The situation with SPRS is

Received 20 February 2001. Revised 20 November 2001.
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International Journal of Control ISSN 0020–7179 print/ISSN 1366–5820 online © 2002 Taylor & Francis Ltd
http://www.tandf.co.uk/journals
DOI: 10.1080/00207170110120026

different. The right-hand side of a SPRS is switches immediately after the switching moment and the right-hand side of fast equations in SPRS after this moment is very big. It allows us to use the standard boundary layer functions method (see Vasil'eva *et al*. 1995) for asymptotic representation of slow periodic solution of SPRS.

This paper is devoted to the investigation of the influence of additional dynamics on the periodic motion of the relay systems with internal sliding modes. We will consider the SPRS for which the reduced systems have the periodic solution with internal sliding mode. A theorem about existence and stability of slow periodic solutions for singularly perturbed relay systems with internal sliding mode is proved. The algorithm for asymptotic representation of the periodic solution is suggested and it is proved that there is no zero order boundary layer function in the asymptotic approximation of periodic solutions at break away point.

2. Problem statement

Consider the SPRS in the form

$$\left.\begin{aligned} \mu\,\mathrm{d}z/\mathrm{d}t &= g(z,\sigma,s,x,u), & \mathrm{d}s/\mathrm{d}t &= h_1(z,s,\sigma,x,u) \\ \mathrm{d}\sigma/\mathrm{d}t &= h_2(z,s,\sigma,x,u), & \mathrm{d}x/\mathrm{d}t &= h_3(z,s,\sigma,x,u) \end{aligned}\right\} \tag{1}$$

where $z \in R^m$, $s, \sigma \in R$, $x \in R^n$, $u(s) = \mathrm{sign}\,(s)$, $g, h_i\,(i = 1,2,3)$ are the smooth functions of their arguments, μ is the small parameter. Denote by Z, Σ, S, X the domains in which the variables (z,s,σ,x), (s,σ,x), (s,x) and x are defined. Suppose that $h_1, h_2, h_3, g \in C^2[\bar{Z} \times [-1,1]]$. Then putting $\mu = 0$ and expressing z from the equation

$$g(z_0, s, \sigma, x, u(s)) = 0$$

we have the reduced system

$$\left.\begin{aligned} z_0 &= \varphi(s,\sigma,x,u) \\ \mathrm{d}\bar{s}_0/\mathrm{d}t &= h_1(\varphi(\bar{s}_0,\bar{\sigma}_0,\bar{x}_0,u),\bar{s}_0,\bar{\sigma}_0,\bar{x}_0,u) \\ &= H_1(\bar{s}_0,\bar{\sigma}_0,\bar{x}_0,u) \\ \mathrm{d}\bar{\sigma}_0/\mathrm{d}t &= h_2(\varphi(\bar{s}_0,\bar{\sigma}_0,\bar{x}_0,u),\bar{s}_0,\bar{\sigma}_0,\bar{x}_0,u) \\ &= H_2(\bar{s}_0,\bar{\sigma}_0,\bar{x}_0,u) \\ \mathrm{d}\bar{x}_0/\mathrm{d}t &= h_3(\varphi(\bar{s}_0,\bar{\sigma}_0,\bar{x}_0,u),\bar{s}_0,\bar{\sigma}_0,\bar{x}_0,u) \\ &= H_3(\bar{s}_0,\bar{\sigma}_0,\bar{x}_0,u) \end{aligned}\right\} \tag{2}$$

Suppose that the measure of the sliding domain

$$\mathcal{S} = \{(z,\sigma,x) : \; h_1(z,0,\sigma,x,1) < 0, \; h_1(z,0,\sigma,x,-1) > 0\}$$

on the surface $s = 0$ in system (2) is non-zero in $\Sigma \times \{0\}$ and Γ is the border of $\mathcal{S}$ is described by equations

$$\begin{aligned} s = 0 \cap (\sigma = 0 \;\Leftrightarrow\; & u_{\mathrm{eq}}(z,0,x) \\ \equiv 1 \;\Leftrightarrow\; & h_1(z,0,0,x,1) \equiv 0) \end{aligned}$$

and moreover for all $(z,x) \in \Gamma \in \mathbf{R}^m \times \mathbf{R}^n$

$$h_1(z,0,0,x,-1) > 0, \qquad h_2(z,0,0,x,1) > 0$$

Suppose that the solution of system (1) in the sliding domain $\mathcal{S}$ is uniquely described by the equivalent control method (see for example Utkin 1992)

$$\left.\begin{aligned} \mu\,\mathrm{d}z/\mathrm{d}t &= g(z,\sigma,0,x,u_{\mathrm{eq}}(z,\sigma,x)) \\ \mathrm{d}\sigma/\mathrm{d}t &= h_2(z,0,\sigma,x,u_{\mathrm{eq}}(z,\sigma,x)) \\ \mathrm{d}x/\mathrm{d}t &= h_3(z,0,\sigma,x,u_{\mathrm{eq}}(z,\sigma,x)) \end{aligned}\right\} \tag{3}$$

where the equivalent control $u = u_{\mathrm{eq}}(z,\sigma,x)$ at all $(z,\sigma,x) \in \mathcal{S}$ is determined by equation

$$h_1(z,0,\sigma,x,u_{\mathrm{eq}}) = 0$$

and everywhere in $\mathcal{S}$ the inequality $|u_{\mathrm{eq}}(z,\sigma,x)| < 1$ is true.

The main specific feature of system (1) is the following: the zero approximation of the slow motion integral manifold for system (1) consists of three sheets $\bar{z}_0^{\pm} = \varphi^{\pm}(s,\sigma,x) = \varphi(s,\sigma,x,\pm 1)$, and

$$\bar{z}_0^{*} = \varphi^{*}(\sigma,x) = \varphi(0,\sigma,x,\bar{u}_{\mathrm{eq}}(\sigma,x))$$

corresponding to the value of relay control $u = \pm 1$ and $u = \bar{u}_{\mathrm{eq}}(\sigma,x)$, where $\bar{u}_{\mathrm{eq}}(\sigma,x)$ is the value of equivalent control determined by equation

$$H_1(s,\sigma,x,\bar{u}_{\mathrm{eq}}(\sigma,x)) = 0$$

It is obvious (see for example Heck 1991), that

$$u_{\mathrm{eq}}(\varphi(0,\sigma,x,\bar{u}_{\mathrm{eq}}(\sigma,x)),\sigma,x) = \bar{u}_{\mathrm{eq}}(\sigma,x)$$

For the description of periodic solution in the reduced system consider two auxiliary systems. The system

$$\left.\begin{aligned} \mathrm{d}\bar{s}_0^{+}/\mathrm{d}t &= H_1(\bar{s}_0^{+},\bar{\sigma}_0^{+},\bar{x}_0^{+},1) \\ \mathrm{d}\bar{\sigma}_0^{+}/\mathrm{d}t &= H_2(\bar{s}_0^{+},\bar{\sigma}_0^{+},\bar{x}_0^{+},1) \\ \mathrm{d}\bar{x}_0^{+}/\mathrm{d}t &= H_3(\bar{s}_0^{+},\bar{\sigma}_0^{+},\bar{x}_0^{+},1) \end{aligned}\right\} \tag{4}$$

describes the motions in (2) for $u = 1$. Consider the system

$$\left.\begin{aligned} \mathrm{d}\bar{\sigma}_0^{*}/\mathrm{d}t &= H_2(0,\bar{\sigma}_0^{*},\bar{x}_0^{*},\bar{u}_{\mathrm{eq}}) \\ \mathrm{d}\bar{x}_0^{*}/\mathrm{d}t &= H_3(0,\bar{\sigma}_0^{*},\bar{x}_0^{*},\bar{u}_{\mathrm{eq}}) \end{aligned}\right\} \tag{5}$$

corresponding to the motions in (2) in sliding mode on $s = 0$.

Let us denote

$$\Delta = \{x : \; H_1(0,0,x,1) = 0; \qquad H_1(0,0,x,-1) > 0\}$$

as the border of sliding domain of system (2). Then the points $(0,0,x) \in \Delta$ are the points in which solutions of (2) are leaving the sliding domain. Suppose that for solution of system (4) with the initial conditions

$$s_0(0)=0, \quad \sigma_0(0)=0, \quad x_0(0)=x^0,\, x^0\in\Delta$$

the following conditions are true

(i) there exists $t=\theta(x^0)$ the smallest root of equation $\bar{s}_0^+(\theta)=0$, such that

- $h_1(\varphi^+(0,\bar{\sigma}_0^+(\theta), \quad \bar{x}_0^+(\theta)),0,\bar{\sigma}_0^+(\theta), \quad \bar{x}_0^+(\theta),1)<0$, $h_1(\varphi^+(0,\bar{\sigma}_0^+(\theta),\bar{x}_0^+(\theta)),0,\bar{\sigma}_0^+(\theta),\bar{x}_0^+(\theta),-1)>0$,
- $\bar{\sigma}_0^+(\theta)<0$;

(ii) for solution of system (5) with initial conditions

$$\bar{\sigma}_0^*(\theta)=\bar{\sigma}_0^+(\theta), \quad \bar{x}_0^*(\theta)=\bar{x}_0^+(\theta)$$

there exists $t=T(x^0)$ the smallest root of equation $\bar{\sigma}_0^*(T)=0$ such that

- for all $t\in[\theta,T)$

$$h_1(\varphi^*(0,\bar{\sigma}_0^*(t),\bar{x}_0^*(t)),0,\bar{\sigma}_0^*(t),\bar{x}_0^*(t),1)<0,$$
$$h_1(\varphi^*(0,\bar{\sigma}_0^*(t),\bar{x}_0^*(t)),0,\bar{\sigma}_0^*(t),\bar{x}_0^*(t),-1)>0;$$

- $H_2(0,0,\bar{x}_0^*(T),-1)>0$.

Now we can define the Poincare map Ψ: $x^0 \rightleftarrows \bar{x}_0^*(T(x^0))$ of the border of the sliding domain Δ generated by system (2) into itself (see figure 1).

The systems (1) and (2) are discontinuous, and consequently for investigation of stability for their periodic solution it is impossible to use equation in variations. That is why we will write down the conditions of existence and stability of periodic solutions for systems (1) and (2) in the form of the Poincare map properties.

Suppose that for the system (2) the following hypotheses are true:

(iii) there exists an isolated fixed point of the Paincare map $\Psi(x): \Psi(x_0^*)=x_0^*,\, x_0^*\in\Delta$, corresponding to the periodic solution of (2), such that $\det(\partial\Psi/\partial x)(x_0^*)\neq 0$;

(iv) $\|(\partial\Psi/\partial x)(x_0^*)\|<q<1$.

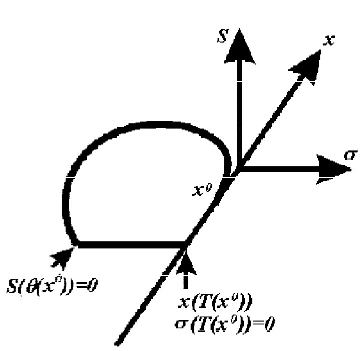

Figure 1. The Poincare map $\Psi(x^0)$.

Denote by $\theta_0=\theta(x_0^*)$, $T_0=T(x_0^*)$. Consider the broken line

$$\mathcal{L}_0(t)=\begin{cases}\varphi^+(s_0^+(t),\sigma_0^+(t),x_0^+(t)) & \text{for } t\in(0,\theta_0)\\ \varphi^*(\sigma_0^*(t),x_0^*(t)) & \text{for } t\in(\theta_0,T_0)\\ (1-\lambda)\varphi^+(0,\bar{\sigma}_0^+(\theta_0),\bar{x}_0^+(\theta_0)) & \\ \qquad +\lambda\varphi^*(\bar{\sigma}_0^+(\theta_0),\bar{x}_0^+(\theta_0)) & \\ \lambda\in[0,1] & \text{for } t=\theta_0\end{cases}$$

In this paper the sufficient conditions are found for existence of the isolated orbitally asymptotically stable periodic solution of system (1) with internal sliding modes near to the broken line

$$(\mathcal{L}_0(t),s_0(t),\sigma_0(t),x_0(t))$$

An algorithm for the asymptotic representation of this periodic solution is suggested. This solution consists of boundary layers at the break away point $t=0$ and at the point $t=\theta_0$ and it is proved there is no zero order boundary layer function in the asymptotic representation of periodic solution at break away point.

3. Existence of the slow periodic solution

We will consider only situations in which the fast motions in (1) are uniformly asymptotically stable. This means that for systems

$$\frac{dz}{d\tau}=g(z,s,\sigma,x,1) \tag{6}$$

$$\frac{dz}{d\tau}=g(z,0,\sigma,x,u_{\text{eq}}(z,\sigma,x)) \tag{7}$$

which describe the fast motions in (1) for $u=1$ and (3) respectively, for some $\alpha>0$, $\delta>0$ the following conditions are true:

(v) the matrix $(\partial g/\partial z)(z,s,\sigma,x,1)$ is stable on the set

$$\begin{aligned}Z^+=\{(z,s,\sigma,x):\ &(z,s,\sigma,x)\in Z, \qquad s>0\\ &|(z,s,\sigma,x)-(\varphi^+(\bar{s}_0^+(t),\bar{\sigma}_0^+(t),\bar{x}_0^+(t)),\bar{s}_0^+(t),\\ &\bar{\sigma}_0^+(t),\bar{x}_0^+(t))\|<\delta, \qquad t\in[0,\theta_0]\}\end{aligned}$$

and

$$\text{Re Spec}\,\frac{\partial g}{\partial z}(z,s,\sigma,x)<-\alpha<0$$

(vi) the matrix $(\partial g/\partial z)(z,0,\sigma,x,u_{\text{eq}}(z,0,\sigma,x))$ is stable on the set

$$Z^* = \{(z,0,\sigma,x) : (z,0,\sigma,x) \in Z, \|(z,0,\sigma,x) - (\varphi^*(\bar{\sigma}_0^*(t),\bar{x}_0^*(t),\bar{u}_{\mathrm{eq}}(\bar{\sigma}_0^*(t),\bar{x}_0^*(t))),\bar{\sigma}_0^*(t),\bar{x}_0^*(t))\| < \delta, \quad t \in [\theta_0, T_0]\}$$

and

$$\mathrm{Re\ Spec}\,\frac{\partial g}{\partial z}(z,0,\sigma,x) < -\alpha < 0$$

It is natural to suppose that at the time moment of input into the sliding mode the corresponding point of system (1) solution is situated in the interior of attractivity domain for slow motion integral manifold of system (3). Suppose that:

(vii) point $\varphi^+(0,\bar{\sigma}_0^+(\theta_0),\bar{x}_0^+(\theta_0))$ is an internal point of attractivity domain of $\varphi^*(\bar{\sigma}_0^+(\theta_0),\bar{x}_0^+(\theta_0))$ which is equilibrium point of system

$$\mathrm{d}z/\mathrm{d}\tau = g(z,0,\bar{\sigma}_0^+(\theta_0),\bar{x}_0^+(\theta_0),u_{\mathrm{eq}}(z,\bar{\sigma}_0^+(\theta_0),\bar{x}_0^+(\theta_0))$$

and at all points of segment connected the points $\varphi^+(0,\bar{\sigma}_0^+(\theta_0),\bar{x}_0^+(\theta_0))$ and $\varphi^*(\bar{\sigma}_0^+(\theta_0),\bar{x}_0^+(\theta_0))$ the sufficient conditions for sliding mode existence are true, which means that for all $\lambda \in [0,1]$

$$h_1((1-\lambda)\varphi^+(0,\bar{\sigma}_0^+(\theta_0),\bar{x}_0^+(\theta_0)) + \lambda\varphi^*(\bar{\sigma}_0^+(\theta_0),\bar{x}_0^+(\theta_0)),0,\bar{\sigma}_0^+(\theta_0),\bar{x}_0^+(\theta_0),1) < 0$$

$$h_1((1-\lambda)\varphi^+(0,\bar{\sigma}_0^+(\theta_0),\bar{x}_0^+(\theta_0)) + \lambda\varphi^*(\bar{\sigma}_0^+(\theta_0),\bar{x}_0^+(\theta_0)),0,\bar{\sigma}_0^+(\theta_0),\bar{x}_0^+(\theta_0),-1) > 0$$

For the proof of existence and stability of system (1) periodic solution consider the properties of the Poincare map $\Phi(z,x,\mu)$ of the border of sliding domain Γ into itself, generated by (1).

Denote by

$$(z(t,\mu),s(t,\mu),\sigma(t,\mu),x(t,\mu))$$

the solution of system (1) with initial conditions

$$z(0,\mu) = \eta; \quad s(0,\mu) = 0,$$
$$\sigma(0,\mu) = 0, \quad x(0,\mu) = \xi$$

Lemma 1: *Under conditions* (i)–(vii) *there exists the neighbourhood $\mathcal{U} \in \Gamma$ of the point $(\varphi^*(0,x_0^*),x_0^*)$, such that for all $(\eta,\xi) \in \mathcal{U}$ for sufficiently small μ there exist $0 < \theta(\eta,\xi,\mu) < T(\eta,\xi,\mu)$ such that*

$$s(\theta(\eta,\xi,\mu),\mu) = 0$$

$$\sigma(T(\eta,\xi,\mu),\mu) = 0 \Leftrightarrow h_1(z(T(\eta,\xi,\mu),\mu),0,\sigma(T(\eta,\xi,\mu),\mu), x(T(\eta,\xi,\mu),\mu),1) = 0,$$

and for all $t \in [\theta(\eta,\xi,\mu),T(\eta,\xi,\mu))$:

- $\sigma(t,\mu) < 0$
- $h_1(z(t,\mu),0,\sigma(t,\mu),x(t,\mu),1) < 0$
- $h_1(z(t,\mu),0,\sigma(t,\mu),x(t,\mu),-1) > 0$

Moreover $y(t,\mu) = (z(t,\mu),s(t,\mu),\sigma(t,\mu),x(t,\mu))$ exists and unique on $[0,T(\eta,\xi,\mu)]$ and

$$(z(T(\eta,\xi,\mu),\mu),x(T(\eta,\xi,\mu),\mu)) \in \mathcal{U}$$

Proof: The functions $(z(t,\mu),s(t,\mu),\sigma(t,\mu),x(t,\mu))$ is diifferentiable on η,ξ,μ at the points $t=\theta_0$, $T=T_0$ (see for example Strygin and Sobolev 1988). Then from implicit function theorem it follows that there exists the closed ball $\bar{B}(\alpha) \subset U \in R^n$ with radius α with the centre of x_0^* such that for every $\xi \in \bar{B}(\alpha)$:

- $\|(\partial\Psi/\partial x)(\xi)\| < q' < 1;$
- the point $\varphi^+(0,\bar{\sigma}_0^+(\theta(\xi)),\bar{x}_0^+(\theta(\xi)))$ is an internal point of attractivity domain for asymptotically stable equilibrium point $\varphi^*(\bar{\sigma}_0^+(\theta(\xi)),\bar{x}_0^+(\theta(\xi)))$ of system

$$\frac{\mathrm{d}z}{\mathrm{d}\tau} = g(z,0,\bar{\sigma}_0^+(\theta(\xi)),\bar{x}_0^+(\theta(\xi)),u_{\mathrm{eq}}(0,\bar{\sigma}_0^+(\theta(\xi)), \bar{x}_0^+(\theta(\xi)))$$

- $\bar{x}_0^*(T(\xi)) \in \bar{B}(q'\alpha);$
- the set $\bar{W} = \mathrm{co}\,\varphi^*(0,\bar{U}(\alpha))$ is situated in the interior of the attractivity domain of the equilibrium point $\varphi^+(0,0,x_0^*)$ for system (6).

Then from the implicit functions theorem and lemmas 2 and 3 from the appendix it follows that at every point $(\eta,\xi) \in \bar{W} \times \bar{B}(\alpha)$ there exists $\mu_0(\eta,\xi)$, such that for every $\mu \in [0,\mu_0(\eta,\xi)]$

- for the solution $(z^+(t,\mu),s^+(t,\mu),\ \sigma^+(t,\mu), x^+(t,\mu))$ of system (1) for $u=1$

$$\mu\,\mathrm{d}z^+/\mathrm{d}t = g(z^+,s^+,\sigma^+,x^+,1)$$
$$\mathrm{d}s^+/\mathrm{d}t = h_1(z^+,s^+,\sigma^+,x^+,1)$$
$$\mathrm{d}\sigma^+/\mathrm{d}t = h_2(z^+,s^+,\sigma^+,x^+,1)$$
$$\mathrm{d}x^+/\mathrm{d}t = h_3(z^+,s^+,\sigma^+,x^+,1)$$

with initial condition (8), there exists $\theta(\eta,\xi,\mu)$ the smallest root of equation

$$s^+(\theta(\eta,\xi,\mu),\mu) = 0$$

such that

$$h_1(z^+(\theta(\eta,\xi,\mu),\mu),0,\sigma^+(\theta(\eta,\xi,\mu),\mu),$$
$$x^+(\theta(\eta,\xi,\mu),\mu),1) < 0$$
$$h_1(z^+(\theta(\eta,\xi,\mu),\mu),0,\sigma^+(\theta(\eta,\xi,\mu),\mu),$$
$$x^+(\theta(\eta,\xi,\mu),\mu),\mu),-1) > 0$$
$$\sigma^+(\theta(\eta,\xi,\mu),\mu) < 0$$

- the point $z^+(\theta(\eta,\xi,\mu),\mu),\mu)$ is situated in the attractivity domain of

$$\varphi^*(\bar{\sigma}_0^+(\theta(\eta,\xi,\mu),\mu),\bar{x}_0^+(\theta(\eta,\xi,\mu),\mu))$$

- for $(z^*(t,\mu),\sigma^*(t,\mu),x^*(t,\mu))$ the solution system (3) with initial conditions ensuring the continuity of solution in the input in the sliding mode

$$\left.\begin{array}{l} z^*(\theta(\eta,\xi,\mu),\mu) = z^+(\theta(\eta,\xi,\mu),\mu) \\ \sigma^*(\theta(\eta,\xi,\mu),\mu) = \sigma^+(\theta(\eta,\xi,\mu),\mu) \\ x^*(\theta(\eta,\xi,\mu),\mu) = x^+(\theta(\eta,\xi,\mu),\mu) \end{array}\right\} \tag{9}$$

there exists the break away point $T(\eta,\xi,\mu) > \theta(\eta,\xi,\mu)$ as the smallest positive root of equation $\sigma^*(T(\eta,\xi,\mu),\mu) = 0$ for which

$$\mathrm{d}\sigma^*/\mathrm{d}t(T,\mu) = h_2(z^*(T,\mu),0,0,x^*(T,\mu),1) > 0$$

solution $(z^*(t,\mu),s^*(t,\mu),x^*(t,\mu))$ is uniquely defined on

$$[\theta(\eta,\xi,\mu),T(\eta,\xi,\mu)]$$

and the point $(z^*(T(\eta,\xi,\mu),\mu),x^*(T(\eta,\xi,\mu),\mu))$ belongs to the set

$$(\varphi^*(0,U((1+q')\alpha/2)),U((1+q')\alpha/2))$$

Moreover

$$\Phi(\eta,\xi,0) = \lim_{\mu\to 0}\Phi(\eta,\xi,\mu) = (\varphi^*(0,x^+(T(\xi))),x^*(T(\xi)))$$

and for $\xi = x_0^*$ we have $\Phi(\varphi(0,x_0^*,1),\ x_0^*,0) = (\varphi^*(0,x_0{}^*),x_0^*)$. Now from the compactness of the set $\Upsilon = \bar{W}\times\bar{U}(\alpha)$ one can conclude that there exists such μ_0 such that for all $\mu\in[0,\mu_0]$

$$\begin{aligned}\Phi(\eta,\xi,\mu) &= (\Phi_1(\eta,\xi,\mu),\Phi_2(\eta,\xi,\mu)) \\ &= (z^*(T(\eta,\xi,\mu),\mu),x^*(T(\eta,\xi,\mu),\mu))\end{aligned}$$

the Poincare map of Υ, generated by system (1) transforms Υ into itself. This means that $\Phi(\eta,\xi,\mu)$ for all $\mu\in[0,\mu_0]$ has on the set Υ a fixed point corresponding to the periodic solution of system (1) in the neighbourhood of the broken line $(\mathcal{L}_0(t),\bar{s}_0(t),\bar{\sigma}_0(t),\bar{x}_0(t))$. □

4. Uniqueness of periodic solution and its stability

Theorem 1: *Under conditions* (i)–(vii) *for sufficiently small* μ *in the neighbourhood of the broken line* $(\mathcal{L}_0(t),\bar{s}_0(t),\bar{\sigma}_0(t),\bar{x}_0(t))$ *there exists orbitally asymptotically stable periodical solution with period* $T(\mu) = T_0 + O(\mu)$ *and boundary layers at* $t = 0$ *near to the point* $t = \theta_0$. *The zero order boundary layer function at* $t = 0$ *is equal zero.*

Proof: The derivative of the Poincare map Φ by η,ξ,μ is smoothly depending from the corresponding derivatives of functions

$$z^+(\theta(\eta,\xi,\mu),\mu),x^+(\theta(\eta,\xi,\mu),\mu),$$
$$z^*(T(\eta,\xi,\mu),\mu),x^*(T(\eta,\xi,\mu),\mu)$$

and $\theta(\eta,\xi,\mu)$, $T(\eta,\xi,\mu)$. The existence and continuity of this derivative follows from the theorems about existence and continuity on initial conditions and parameters for solutions of system differential equations at the end of the finite time interval (Strygin and Sobolev 1988).

Introduce the new variable $\chi = \eta - \varphi^*(0,x^*(T(\xi)))$. Consider now the conditions under which the fixed point of the map

$$\begin{aligned}\Lambda(\chi,\xi,\mu) &= (\Lambda_1(\chi,\xi,\mu),\Lambda_2(\chi,\xi,\mu)) \\ &= [\Phi_1(\chi+\varphi^*(0,x^*(T(\xi))),\xi,\mu) \\ &\quad -\varphi^*(0,x^*(T(\xi))), \\ &\quad \Phi_2(\chi+\varphi^*(0,x^*(T(\xi))),\xi,\mu)]\end{aligned}$$

It is necessary to take into account that at $\mu = 0$ at the point $(0,x_0^*)$ is the fixed point of the map Λ, and Λ for sufficiently small $\beta,\bar{\mu}$ transforms the set

$$M(\beta,\alpha,\bar{\mu}) = \{(\chi,\xi,\mu)\colon\ \|\chi\| < \beta, x\in\bar{B}(\alpha),\mu\in[0,\bar{\mu}]\}$$

into itself.

Let us find the derivative of Λ with respect to χ and ξ. For $\mu = 0$ the value $\Lambda(\chi,\xi,0)$ does not depend on χ and $\Lambda_1(\chi,\xi,0)$ does not depend on ξ. This means that

$$\frac{\partial\Lambda}{\partial(\chi,\xi)} = \begin{pmatrix} O(\mu) & O(\mu) \\ O(\mu) & \partial\Psi/\partial\xi(x_0^*) + O(\mu)\end{pmatrix}$$

Let us choose such $\beta,\bar{\mu} > 0$ that for some $q_1 (q_1 < 1)$

$$\sup_{M(\beta,\alpha,\bar{\mu})}\left\|\frac{\partial\Lambda}{\partial(\chi,\xi)}\right\| < q_1 < 1$$

This means that the Poincare map $\Lambda(\chi,\xi,\mu)$ is a contractive on $M(\beta,\alpha,\bar{\mu})$ and has a unique fixed point, corresponding to a desired periodic solution of system (1). The Poincare map Λ is contractive on $M(\beta,\alpha,\bar{\mu})$ and consequently a corresponding solution of system (1) is orbitally asymptotically stable. □

5. Algorithm of asymptotic representation for solution

Suppose that $h_1,h_2,h_3,g\in C^{k+3}[\bar{Z}\times[-1,1]]$ and conditions (i)–(vi) are true.

Denote by $y^{\mathrm{T}} = (z^{\mathrm{T}},s,\sigma,x^{\mathrm{T}})$ and $v^{\mathrm{T}} = (s,\sigma,x^{\mathrm{T}})$. Then the asymptotic representation of the point $\theta(\mu)$

and period the $T(\mu)$ of desired periodic solution of system (1) on interval $[0, \tilde{T}_{k+1}(\mu)]$ has the form

$$\left.\begin{aligned} Y_k(t,\mu) &= \sum_{i=0}^{k}[\bar{y}_i(t)+\Pi_i^* y(\tau_{k+1})]\mu^i + \sum_{j=1}^{k}\Pi_j^+ y(\tau)\mu^i \\ V_k(t,\mu) &= \sum_{i=0}^{k}\bar{v}_i(t)\mu^i + \sum_{i=2}^{k}\Pi_i^+ v(\tau)\mu^i + \sum_{i=1}^{k}\Pi_i^* v(\tau_k)\mu^i \\ &\tau = t/\mu,\ \tau_{k+1} = (t-\tilde{\theta}_{k+1}(\mu)))/\mu \\ &\tilde{\theta}_{k+1}(\mu) = \theta_0+\mu\theta_1+\cdots+\mu^{k+1}\theta_{k+1} \\ &\tilde{\Theta}_{k+1}(\mu) = \Theta_0+\mu\Theta_1+\cdots+\mu^{k+1}\Theta_{k+1} \\ &\tilde{T}_k(\mu) = T_0+\mu T_1+\cdots+\mu^k T_k \\ &\|\Pi_i{}^* y(\tau)\| < C\,\mathrm{e}^{-\gamma\tau}, C,\gamma>0,\ \Pi_i{}^* y(\tau)\equiv 0 \text{ for } \tau<0 \\ &\|\Pi_i^+ y(\tau_{k+1})\| < C\,\mathrm{e}^{-\gamma\tau_{k+1}},\ \Pi_i^+ y(\tau_{k+1})\equiv 0 \text{ for } \tau_{k+1}<0 \end{aligned}\right\} \tag{10}$$

Denote by

$$\bar{y}_0(t) = \begin{cases} \bar{y}_0^+(t) = (\varphi^+(\bar{s}_0^+(t),\bar{\sigma}_0^+(t),\bar{x}_0^+(t)),\bar{s}_0^+(t),\bar{\sigma}_0^+(t),\bar{x}_0^+(t)) \\ \qquad \text{for } t\in[0,\theta_0] \\ \bar{y}_0{}^*(t) = (\varphi^*(\bar{\sigma}_0{}^*(t),\bar{x}_0{}^*(t)),0,\bar{\sigma}_0{}^*(t),\bar{x}_0{}^*(t)) \\ \qquad \text{for } t\in[\theta_0,T_0] \end{cases}$$

$$\bar{v}_0(t) = \begin{cases} \bar{v}_0^+(t) = (\bar{s}_0^+(t),\bar{\sigma}_0^+(t),\bar{x}_0^+(t)) & \text{for} \quad t\in[0,\theta_0] \\ \bar{v}_0^*(t) = (0,\bar{\sigma}_0^*(t),\bar{x}_0^*(t)) & \text{for} \quad t\in[\theta_0,T_0] \end{cases}$$

$\Pi_0^+ z(\tau)\equiv 0$. Function $\Pi_0^* z(\tau^*)$ is defined by equation

$$\mathrm{d}\Pi_0^* z/\mathrm{d}\tau = g(\Pi_0^* z+\varphi^*(\bar{\sigma}_0^+(0),\bar{x}_0^+(0)),0,0,\bar{x}_0^+(0))$$

$$\Pi_0^+ z(0) = \varphi^+(0,\bar{\sigma}_0^+(\theta_0),\bar{x}_0^+(\theta_0)) - \varphi^*(\bar{\sigma}_0^+(\theta_0),\bar{x}_0^+(\theta_0))$$

To find $\bar{s}_1^+(t),\bar{\sigma}_1^+(t),\bar{x}_1^+(t),\bar{z}_1^+(t)$ and $\bar{\sigma}_1^*(t)$, $\bar{x}_1^*(t)$, $\bar{z}_1^*(t)$ we have the system of linear equations in form

$$\left.\begin{aligned} \bar{z}_1^+(t) &= -[g_z'^+]^{-1}(g_s'^+\bar{s}_1^+ + g_\sigma'+(t)\bar{\sigma}_1^+ + g_x'^+\bar{x}_1^+ \\ &\quad + g_1^+(t)) \\ \mathrm{d}\bar{s}_1^+/\mathrm{d}t &= h_{1z}'^+(t)\bar{z}_1^+(t) + h_{1s}'^+\bar{s}_1^+(t) + h_{1\sigma}'^+(t)\bar{\sigma}_1^+ \\ &\quad + h_{1x}'^+\bar{x}_1^+(t) \\ \mathrm{d}\bar{\sigma}_1^+/\mathrm{d}t &= h_{2z}'^+(t)\bar{z}_1^+(t) + h_{2s}'^+\bar{s}_1^+(t) + h_{2\sigma}'^+(t)\bar{\sigma}_1^+ \\ &\quad + h_{2x}'^+\bar{x}_1^+(t) \\ \mathrm{d}\bar{x}_1^+/\mathrm{d}t &= h_{3z}'^+(t)\bar{z}_1^+(t) + h_{3s}'^+\bar{s}_1^+(t) \\ &\quad + h_{3\sigma}'^+(t)\bar{\sigma}_1^+ h_{3x}' + \bar{x}_1^+(t) \\ \bar{z}_1^*(t) &= -[g_z'^*]^{-1}(g_\sigma'^*\bar{\sigma}_1^* + g_x'^*\bar{x}_1^* + g^{**}(t)) \\ \mathrm{d}\bar{\sigma}_1^*/\mathrm{d}t &= h_{2z}'^*(t)\,\bar{z}_1^*(t) + h_{2\sigma}'^*\bar{\sigma}_1^*(t) + h_{2x}'^*\bar{x}_1^*(t) \\ \mathrm{d}\bar{x}_1^*/\mathrm{d}t &= h_{3z}'^*(t)\bar{z}_1^*(t) + h_{3\sigma}'^*\bar{\sigma}_1^*(t) + h_{3x}'^*\bar{x}_1^*(t) \end{aligned}\right\} \tag{11}$$

Here the upper index + means that the values of corresponding functions are computed at the point

$$(\varphi^+(\bar{s}_0^+(t),\bar{\sigma}_0^+(t),\bar{x}_0^+(t)),\bar{s}_0^+(t),\bar{\sigma}_0^+(t),\bar{x}_0^+(t),1)$$

but index * means that the values of corresponding functions are computed at the point

$$(\varphi^*(\bar{\sigma}_0^*(t),\bar{x}_0^*(t)),0,\bar{\sigma}_0^*(t),\bar{x}_0^*(t),u_{\mathrm{eq}}(\varphi^+(0,\bar{\sigma}_0^*(\theta_0),\bar{x}_0^*(\theta_0)))$$

$\Pi_1^+ s\equiv 0, \Pi_1^+ x\equiv 0$. Then to find $\Pi_1^+ z$, $\Pi_1^* z, \Pi_1^* s, \Pi_1^* x$ one have the system

$$\mathrm{d}\Pi_1^+ z/\mathrm{d}\tau = g_z'\Pi_1^+ z$$

$$\mathrm{d}\Pi_1^* z/\mathrm{d}\tau = g_z'^*\Pi_1^* z + \Pi_1^* g(\tau)$$

$$\begin{aligned} \mathrm{d}\Pi_1^*\sigma/\mathrm{d}\tau = \Pi_0^* h_2 &= h_2(\bar{z}_0^*(\theta_0)+\Pi_0^* z,0,\bar{\sigma}_0^*(\theta_0),\bar{x}_0^*(\theta_0),u_{\mathrm{eq}}) \\ &\quad - h_2(\bar{z}_0^*(\theta_0),0,\bar{\sigma}_0^*(\theta_0),\bar{x}_0^*(\theta_0),u_{\mathrm{eq}}) \end{aligned}$$

$$\begin{aligned} \mathrm{d}\Pi_1^* x/\mathrm{d}\tau &= \Pi_0^* h_3 \\ &= h_3(\bar{z}_0^*(\theta_0)+\Pi_0^* z,0,\bar{\sigma}_0^*(\theta_0),\bar{x}_0^*(\theta_0),u_{\mathrm{eq}}) \\ &\quad - h_3(\bar{z}_0^*(\theta_0),0,\bar{\sigma}_0^*(\theta_0),\bar{x}_0^*(\theta_0),u_{\mathrm{eq}}) \end{aligned}$$

where

$$\bar{z}_0^+(0) = \varphi^+(0,0,x_0^*),\ \bar{z}_0^+(\theta_0) = \varphi(\bar{\sigma}_0^+(\theta_0),\bar{x}_0^+(\theta_0))$$

$u_{\mathrm{eq}} = u_{\mathrm{eq}}(\bar{z}_0^*(\theta_0)+\Pi_0^* z,\bar{\sigma}_0^*(\theta_0),\bar{x}_0^*(\theta_0))$, here the upper index + means that the values of derivatives of function g are computed at the point

$$(\bar{z}_0^+(0)+\Pi_0^+ z,0,0,\bar{x}_0(0),1)$$

but the corresponding derivative with upper index * is computed at the point

$$(\bar{z}_0^*(\theta_0)+\Pi_0^* z,0,\bar{\sigma}_0^*(\theta_0),\bar{x}_0^*(\theta_0),u_{\mathrm{eq}}(\varphi^+(0,\bar{\sigma}_0^*(\theta_0),\bar{x}_0^*(\theta_0)))$$

Now the initial conditions for boundary layer functions of the slow variables have the form

$$\Pi_1^*\sigma(0) = \int_\infty^0 \Pi_0^* h_2(\Theta)\,\mathrm{d}\Theta$$

$$\Pi_1^* x(0) = \int_\infty^0 \Pi_0^* h_3(\Theta)\,\mathrm{d}\Theta$$

Then $\bar{s}_1^+(0) = \bar{\sigma}_1^+(0) = 0$, $\bar{\sigma}_1^*(\theta_0) = -\Pi_1^*\sigma(0)$. Functions $\bar{v}_1^+(t), \bar{v}_1^*(t)$ could be uniquely defined from system (11), via the initial conditions $\bar{x}_1^+(0),\bar{\sigma}_1^*(\theta_0),\bar{x}_1^*(\theta_0)$.

For the first-order terms in asymptotic representations for $\sigma(T(\mu),\mu)=0$ and $s(\theta(\mu),\mu)=0$ we have

$$\left.\begin{aligned} \Theta_1 H_2(0,0,x_0^*,1) + \bar{\sigma}_1^*(T_0) = 0 \\ \theta_1 H_1(0,\bar{\sigma}_0^+(\theta_0),\bar{x}_0^+(\theta_0),1) + \bar{s}_1^+(\theta_0) = 0 \end{aligned}\right\} \tag{12}$$

It follows from condition (ii) that it is possible to express uniquely the constants θ_1 and Θ_1 via $\bar{s}_1^+(0)$ and $\bar{\sigma}_1^*(\theta_0)$ in form

$$\Theta_1 = -[H_2(0,0,x_0^*,1)]^{-1}\bar\sigma_1^*(T_0)$$
$$\theta_1 = -[H_1(0,\bar\sigma_0^+(\theta_0),\bar x_0^+(\theta_0),1)]^{-1}\bar s_1^+(\theta_0)$$

Substituting the expressions for Θ_1 and θ_1 in the conditions for input in the sliding mode (9)

$$\left.\begin{aligned}\bar\sigma_1^*(\theta_0) + \Pi_1^*\sigma(0) &= \bar\sigma_1^+(\theta_0) + \theta_1 H_2(0,\bar\sigma_0^+(\theta_0),\bar x_0^+(\theta_0),1)\\ \bar x_1^*(\theta_0) + \Pi_1^* x(0) &= \bar x_1^+(\theta_0) + \theta_1 H_3(0,\bar\sigma_0^+(\theta_0),\bar x_0^+(\theta_0),1)\end{aligned}\right\} \tag{13}$$

and periodicity conditions

$$\bar x_1^+(0) + \Pi_1^+ x(0) = \bar x_1^*(T_0) + \Theta_1 H_3(0,0,x_0^*,-1) \tag{14}$$

Let us express the values $\bar\sigma_1^+(\theta_0), \bar x_1^+(\theta_0), \bar x_1^*(T_0), \theta_1, \Theta_1$ via $\bar x_1^+(0), \bar x_1^*(\theta_0)$, form (11). After this it is possible to express $\bar x_1^*(\theta_0)$ via $\bar x_1^+(0)$ from equation (14) and substituting this expression in (13), we have the system of linear equations for $\bar x_1^+(0)$. Moreover the determinant of this system coincide with $\det \partial\Psi/\partial x(x_0^*) \neq 0$.

Then initial condition $\bar x_1^+(0), \bar\sigma_1^*(\theta_0), \bar x_1^*(\theta_0)$ are defined uniquely. Now to find $Y_1(t,\mu)$ it is necessary to define functions $\bar y_i^+(t), i = 0,1$ on segment $[0,\tilde T_1(\mu)]$ as

$$\bar y_i(t) = \begin{cases} \bar y_i^+(t) = (\bar z_i^+(t), \bar s_i^+(t), \sigma_i^+(t), \bar x_i^+(t)) \\ \qquad\qquad \text{for} \quad t \in [0,\tilde\theta_1(\mu)] \\ \bar y_i^*(t) = (\bar z_i^*(t), 0, \sigma_i^*(t), \bar x_i^*(t)) \\ \qquad\qquad \text{for} \quad t \in [\tilde\theta_1(\mu), \tilde T_1(\mu)] \end{cases}$$

The initial conditions $\Pi_1^+ z$ and $\Pi_1^* z$ are uniquely defined by equations

$$\bar z_1^+(0) + \Pi_1^+ z(0) = \bar z_1^+(T_0) + \Theta_1 \mathrm{d}\bar z_0^*/\mathrm{d}t(T_0)$$
$$\bar z_1^*(\theta_0) + \Pi_1^* z(0) = \bar z_1^+(\theta_0) + \theta_1 \mathrm{d}\bar z_0^+/\mathrm{d}t(\theta_0)$$

Now the design of the first approximation of asymptotic representation of system (1) periodic solution is finished. To design the first approximation of the fast variables it is necessary to find the value of θ_2 and substitute the result in function $\Pi_1^* z(\tau_2)$.

Suppose now that the functions

$$y_j^+(t), y_j^*(t), \Pi_j^+ y(\tau), \Pi_j^* y(\tau)$$

and constants $\theta_j, \Theta_j, j = 1,\dots,k-1$ was uniquely defined.

Then one can find the functions $\bar s_k^+(t), \bar x_k^+(t), \bar z_k^+(t)$ from the linear systems

$$\left.\begin{aligned}
\bar z_k^+(t) &= -[g_z'^+]^{-1}(g_s'^+\bar s_k^+ + g_\sigma'^+\bar\sigma_k^+ + g_x'^+\bar x_k^+ \\ &\quad + g_k^+(t))\\
\mathrm{d}\bar s_k^+/\mathrm{d}t &= h_{1z}'^+(t)\bar z_k^+(t) + h_{1s}'^+\bar s_k^+(t) + h_{1\sigma}'^+\bar\sigma_k^+(t)\\ &\quad + h_{1x}'^+\bar x_k^+(t) + h_{1k}^+(t)\\
\mathrm{d}\bar\sigma_k^+/\mathrm{d}t &= h_{2z}'^+(t)\bar z_k^+(t) + h_{2s}'^+\bar s_k^+(t) + h_{2\sigma}'^+\bar\sigma_k^+(t)\\ &\quad + h_{2x}'^+\bar x_k^+(t) + h_{2k}^+(t)\\
\mathrm{d}\bar x_k^+/\mathrm{d}t &= h_{3z}'^+(t)\bar z_k^+(t) + h_{3s}'^+\bar s_k^+(t) + h_{3\sigma}'^+\bar\sigma_k^+(t)\\ &\quad + h_{3x}'^+\bar x_k^+(t) + h_{3k}^+(t)\\
\bar z_k^*(t) &= -[g_z'^*]^{-1}(g_\sigma'^*\bar\sigma_k^* + g_x'^*\bar x_k^* + g_k^*(t))\\
\mathrm{d}\bar\sigma_k^*/\mathrm{d}t &= h_{2z}'^*(t)\bar z_k^*(t) + h_{1s\sigma}'^*\bar\sigma_k^* + h_{1x}'^*\bar x_k^*(t)\\ &\quad + h_{1k}^*(t)\\
\mathrm{d}\bar x_k^*/\mathrm{d}t &= h_{3z}'^*(t)\bar z_k^*(t) + h_{3s}'^*\bar s_k^*(t) + h_{3x}'^*\bar x_k^*(t)\\ &\quad + h_{2k}^*(t)
\end{aligned}\right\} \tag{15}$$

Here the upper index + means that the values of corresponding functions are computed at the point

$$(\varphi^+(\bar s_0^+(t), \bar\sigma_0^+(t), \bar x_0^+(t)), \bar s_0^+(t), \bar\sigma_0^+(t), \bar x_0^+(t), 1)$$

but index * means that the values of corresponding functions are compute at the point

$$(\varphi^*(\bar\sigma_0^*(t), \bar x_0^*(t)), 0, \bar\sigma_0^*(t), \bar x_0^*(t), \bar u(\bar\sigma_0^*(t), \bar x_0^*(t)))$$

$g_k^+(t), h_{ik}^+(t),\ i = 1,2,3; g_k^*(t), h_{jk}^*(t),\ j = 2,3$ are uniquely defined functions depending only on

$$\bar z_j^+(t), \bar s_j^+(t), \bar\sigma_j^+(t), \bar x_j^+(t), \bar z_j^*(t), \bar\sigma_j^*(t), \bar x_j^*(t), \theta_j,\ \Theta_j,$$
$$j = 1,\dots,k-1$$

The functions $\Pi_k^+ y$ and $\Pi_k^* y$ are defined by system

$$\mathrm{d}\Pi_k^+ z/\mathrm{d}\tau = g_z'^+\Pi_k^+ z + g_s'^+\Pi_k^+ s + g_x'^+\Pi_k^+ x + \Pi_k^+ g(\tau)$$
$$\mathrm{d}\Pi_k^+ s/\mathrm{d}\tau = \Pi_{k-1}^+ h_1; \quad \mathrm{d}\Pi_k^+\sigma/\mathrm{d}\tau = \Pi_{k-1}^+ h_2$$
$$\mathrm{d}\Pi_k^+ x/\mathrm{d}\tau = \Pi_{k-1}^+ h_3$$
$$\mathrm{d}\Pi_k^* z/\mathrm{d}\tau = g_z'^*\Pi_k^* z + g_\sigma'^*\Pi_k^*\sigma + g_x'^*\Pi_k^* x + \Pi_k^* g(\tau)$$
$$\mathrm{d}\Pi_k^*\sigma/\mathrm{d}\tau = \Pi_{k-1}^* h_2; \quad \mathrm{d}\Pi_k^* x/\mathrm{d}\tau = \Pi_{k-1}^* h_3$$

where the upper index + means that the values of derivatives of function g are computed at the point

$$(\bar z_0^+(0) + \Pi_0^+ z, 0, 0, \bar x_0, 1)$$

but the corresponding derivative with upper index * is computed at the point

$$(\bar z_0^*(\theta_0) + \Pi_0^* z, 0, \bar\sigma_0^*(\theta_0), \bar x_0^*(\theta_0),$$
$$u_{\mathrm{eq}}(\bar z_0^*(\theta_0) + \Pi_0^* z, \bar\sigma_0^*(\theta_0), \bar x_0^*(\theta_0))$$

The functions $\Pi_{k-1}^* h_2, \Pi_{k-1}^* h_3$ depend on $\Pi_j^* z(\tau), \Pi_j^*\sigma(\tau), \Pi_j^* x(\tau), j = 1,\dots,k-1$ only.

The initial conditions for boundary layer functions for the slow variables are defined by equations

$$\Pi_k^+ s(0) = \int_\infty^0 \Pi_{k-1}^+ h_1(\Theta)\,d\Theta$$

$$\Pi_k^+ \sigma(0) = \int_\infty^0 \Pi_{k-1}^+ h_2(\Theta)\,d\Theta$$

$$\Pi_k^+ x(0) = \int_\infty^0 \Pi_{k-1}^+ h_3(\Theta)\,d\Theta$$

$$\Pi_k^* \sigma(0) = \int_\infty^0 \Pi_{k-1}^* h_2(\Theta)\,d\Theta$$

$$\Pi_k^* x(0) = \int_\infty^0 \Pi_{k-1}^* h_3(\Theta)\,d\Theta$$

Consequently $\bar{s}_k^+(0) = -\Pi_k^+ s(0)$; $\bar{\sigma}_k^*(\theta) = -\Pi_k^* \sigma(0)$. System (15) is uniquely defined in the functions $\bar{x}_k^+(t), \bar{x}_k^*(t)$.

Considering kth order members in asymptotic representation of $\sigma(T(\mu),\mu) = 0$ and $s(\theta(\mu),\mu) = 0$ we have

$$\left.\begin{aligned} &\Theta_k H_2(0, x_0^*, 1) + \bar{\sigma}_k^*(T_0) + \Sigma_k^* = 0 \\ &\theta_k H_1(0, \bar{x}_0^+(\theta_0), 1) + \bar{s}_k^+(\theta_0) + \mathcal{S}_k^+ = 0 \end{aligned}\right\} \qquad (16)$$

Let us substitute the expressions for Θ_k and θ_k obtaining from equations (16) in the conditions for input in the sliding mode (9) and conditions of periodicity we have

$$\left.\begin{aligned} \bar{\sigma}_k^*(\theta_0) + \Pi_k^*\sigma(0) &= \bar{\sigma}_k^+(\theta_0) + \theta_k H_2(0, \bar{x}_0^+(\theta_0), \\ &\quad \bar{\sigma}_0^+(\theta_0), 1) + \Sigma_k^+ \\ \bar{x}_k^*(\theta_0) + \Pi_k^* x(0) &= \bar{x}_k^+(\theta_0) + \theta_k H_3(0, 0, \bar{\sigma}_0^+(\theta_0), \\ &\quad \bar{x}_0^+(\theta_0), 1) + \mathcal{X}_k^+ \end{aligned}\right\} \qquad (17)$$

$$\bar{x}_k^+(0) + \Pi_k^+ x(0) = \bar{x}_k^*(T_0) + \Theta_k H_2(0, 0, x_0^*, 1) + \mathcal{X}_k^* \qquad (18)$$

Here $\Sigma_k^+, \mathcal{X}_k^+$ are the functions dependent only on

$$\bar{s}_j^+(\theta_0),\ \bar{\sigma}_j^+(\theta_0),\ \bar{x}_j^+(\theta_0), \bar{\sigma}_j^*(T_0), \bar{x}_j^*(T_0), j = 1, \ldots, k-1$$

Now from condition (i) one can conclude that the values θ_k, Θ_k it is possible to express uniquely via $\bar{x}_k^+(0)$ $\bar{\sigma}_k^*(\theta_0)$ in the form

$$\Theta_k = -[H_2(0, 0, x_0^*, 1)]^{-1}[\bar{\sigma}_k^*(T_0) + \Sigma_k^*]$$

$$\theta_k = -[H_1(0, \bar{\sigma}_0^+(\theta_0), \bar{x}_0^+(\theta_0), 1)]^{-1}[\bar{s}_k^+(\theta_0) + \mathcal{S}_k^+]$$

The values $\bar{x}_k^+(\theta_0), \bar{\sigma}_k^*(T_0), \bar{x}_k^*(T_0)$ can be uniquely expressed via $\bar{x}_k^+(0), \bar{x}_k^*(\theta_0)$ from equations (15). Substituting θ_k, Θ_k into (17) and (18), one can have the system of linear equations on $\bar{x}_k^+(0), \bar{x}_k^*(\theta_0)$. Then expressing $\bar{x}_k^*(\theta_0)$ via $\bar{x}_k^+(0)$ from equation (18) and substituting this expression in (17), one has the system of linear equations for $\bar{x}_k^+(0)$. Moreover the determinant of this system coincides with det $(\partial\Psi/\partial x)(x_0^*) \neq 0$.

This means that the initial conditions

$$\bar{s}_k^+(0), \bar{\sigma}_k^+(0), \bar{x}_k^+(0), \bar{\sigma}_k^*(\theta_0), \bar{x}_k^*(\theta_0)$$

are uniquely defined. Now to find $Y_k(t,\mu)$ it is necessary to define functions $\bar{y}_i^+(t), i = 0, \ldots, k$ on segment $[0, \tilde{T}_k(\mu)]$ as

$$\bar{y}_i(t) = \begin{cases} \bar{y}_i^+(t) = (\bar{z}_i^+(t), \bar{s}_i^+(t), \bar{\sigma}_i^+(t), \bar{x}_i^+(t)) \\ \qquad\qquad \text{for } t \in [0, \tilde{\theta}_k(\mu)] \\ \bar{y}_i^*(t) = (\bar{z}_i^*(t), 0, \bar{\sigma}_i^*(t), \bar{x}_i^*(t)) \\ \qquad\qquad \text{for } t \in [\tilde{\theta}_k(\mu), \tilde{T}_k(\mu)] \end{cases}$$

The initial conditions for $\Pi_k^+ z, \Pi_k^* z$ are uniquely defined from equations

$$\bar{z}_k^+(0) + \Pi_k^+ z(0) = \bar{z}_k^*(T_0) + \Theta_k\, d\bar{z}_0^*/dt(T_0) + \mathcal{Z}_k^*$$

$$\bar{z}_k^*(\theta_0) + \Pi_k^* z(0) = \bar{z}_k^+(\theta_0) + \theta_k\, d\bar{z}_0^+/dt(\theta_0) + \mathcal{Z}_k^+$$

where $\mathcal{Z}_k^+, \mathcal{Z}_k^*$ are functions depending only on $\bar{z}_j^+(\theta_0), \bar{z}_j^*(T_0),\ j = 1, \ldots, k-1$.

Now the design of the kth approximation for the slow part of asymptotic representation for system (1) periodic solution is finished. To design the kth approximation of the fast variables it is necessary to find the value of θ_{k+1} and substitute the result in function $\Pi_1^* z(\tau_{k+1})$.

Theorem 2: *Under conditions* (i)–(vii)

$$|\tilde{T}_k(\mu) - T(\mu)| < C\mu^{k+1}$$

and uniformly on $t \in [0, \hat{T}(\mu)]$, *where* $\hat{T}(\mu) = \max\{T(\mu); \tilde{T}_{k+1}(\mu)\}$, *the following inequalities hold*

$$\|y(t,\mu) - Y_k(t,\mu)\| < C\mu^{k+1};$$

$$\|v(t,\mu) - V_k(t,\mu))\| < C\mu^{k+1} \qquad (19)$$

The proof of this theorem is in the Appendix.

6. Example

Let us show the existence and stability and design the asymptotic representation for slow periodic solution with internal sliding mode for SPRS in form

$$\left.\begin{aligned} &\mu\, dz/dt = -z + u; \quad ds/dt = 2s + \sigma + 5 - 5u \\ &d\sigma/dt = -6s - \sigma + x + 4z \\ &dx/dt = -x + \mu z, \quad u = \text{sign}\,[s(t)] \end{aligned}\right\} \qquad (20)$$

where $z, s, \sigma, x \in R, \mu$ is the small parameter. Let us show that for system (20) the conditions of theorem 1 and 2 are true. For $\mu = 0$ system (20) takes the form

$$\left.\begin{aligned} &\bar{z}_0 = u, \quad d\bar{s}_0/dt = 2\bar{s}_0 + \bar{\sigma}_0 + 5 - 5u \\ &d\bar{\sigma}_0/dt = -6\bar{s}_0 - \bar{\sigma}_0 + \bar{x}_0 + 4u,\ d\bar{x}_0/dt = -\bar{x}_0 \end{aligned}\right\} \qquad (21)$$

Than for system $s > 0$ instead of (21) one has

$$\left.\begin{aligned} &\mathrm{d}\bar{s}_0^+/\mathrm{d}t = 2\bar{s}_0^+ + \bar{\sigma}_0^+, \\ &\mathrm{d}\bar{\sigma}_0^+/\mathrm{d}t = -6\bar{s}_0^+ - \bar{\sigma}_0^+ + \bar{x}_0^+ + 4, \\ &\mathrm{d}\bar{x}_0^+/dt = -\bar{x}_0^+ \end{aligned}\right\} \tag{22}$$

The set

$$\mathcal{S} = \{\sigma\colon -10 < \sigma < 0\}$$

is a stable sliding mode domain for system (21). The motions into $\mathcal{S}$, are described

$$\mathrm{d}\sigma_0^*/\mathrm{d}t = -\frac{\bar{\sigma}_0^*}{5} + \bar{x}_0^* + 4, \quad \mathrm{d}\bar{x}_0^*/\mathrm{d}t = -\bar{x}_0^* \tag{23}$$

Then for the solution of the system (22) with initial conditions

$$\bar{s}_0^+(0) = 0,\ \bar{\sigma}_0^+(0) = 0,\ \bar{x}_0^+(0) = \xi$$

we have

$$\begin{aligned} \bar{s}_0^+(t,\xi) &= 1 + \frac{\mathrm{e}^{-t}\xi}{6} + \left(\frac{\sqrt{15}}{15} + \xi\sqrt{15}\right)\mathrm{e}^{t/2}\sin\frac{\sqrt{15}t}{2} \\ &\quad - \left(\frac{\xi}{6} + 1\right)\mathrm{e}^{t/2}\cos\frac{\sqrt{15}t}{2} \\ \bar{\sigma}_0^+(t,\xi) &= -2 - \frac{\mathrm{e}^{-t}\xi}{2} + \left(\frac{2\sqrt{15}}{5} + \xi\frac{\sqrt{15}}{30}\right)\mathrm{e}^{t/2}\sin\frac{\sqrt{15}t}{2} \\ &\quad + \left(2 + \frac{\xi}{2}\right)\mathrm{e}^{t/2}\cos\frac{\sqrt{15}t}{2} \end{aligned}$$

$$\bar{x}_0^+(t,\xi) = \mathrm{e}^{-t}\xi$$

The last equation of (21) is independent and only the solution of the equation $\bar{x}_0(t) \equiv 0$ can correspond to the periodic solution of (21). Then to find θ_0 as the input moment into the sliding mode we have the equation

$$\bar{s}_0^+(t,0) = 1 + \frac{\sqrt{15}}{15}\,\mathrm{e}^{t/2}\sin\frac{\sqrt{15}t}{2} - \mathrm{e}^{t/2}\cos\frac{\sqrt{15}t}{2} = 0$$

Then $\theta_0 \approx 2.45$, $\bar{\sigma}_0^+(\theta_0,0) \approx -7.03$.

The solution of system (21) on the switching surface takes the form

$$\begin{aligned} \bar{\sigma}_0^*(t,\bar{\sigma}_0^+(\theta_0,0)) &= 20 - (20 - \bar{\sigma}_0^+(\theta_0,0) \\ &\quad - \tfrac{5}{4}\bar{x}_0^+(\theta_0,0))\ \mathrm{e}^{-(t-\theta_0)/5} \\ &\quad - \tfrac{5}{4}\bar{x}_0^+(\theta_0,0)\ \mathrm{e}^{-(t-\theta_0)} \\ \bar{x}_0^*(t,\bar{\sigma}_0^+(\theta_0,0)) &= \bar{x}_0^+(\theta_0,0)\ \mathrm{e}^{-(t-\theta_0)} \end{aligned}$$

Now the period of system (21) periodic solution is defined by equation

$$\bar{\sigma}_0^*(T_0,\bar{\sigma}_0^+(\theta_0,0)) = 20 - (20 - \bar{\sigma}_0^+(\theta_0,0))\ \mathrm{e}^{-(T_0-\theta_0)/5} = 0$$

And consequently

$$T_0 \approx 3.96, \quad \frac{\partial \bar{x}_0^*}{\partial \xi}(0) = \mathrm{e}^{-T_0} \approx \mathrm{e}^{-3.96} \approx 0.019 \neq 0$$

This means that for system (20) the conditions of Theorems 1 and 2 are true.

To finish with zero approximation of the desired periodic it is necessary to define

$$\bar{z}_0(t) = \begin{cases} \bar{z}_0^+(t) = 1 & \text{for } 0 \le t \le \theta_0 \\ \bar{z}_0^*(t) = \dfrac{\bar{\sigma}_0^*(t,\bar{\sigma}_0^+(\theta_0,0)) + 5}{5} & \text{for } \theta_0 \le t \le T_0 \end{cases}$$

Then

$$\mathrm{d}\Pi_0^+ z/\mathrm{d}\tau = -\Pi_0^+ z; \quad \Pi_0^+ z(0) = -\frac{\bar{\sigma}_0^+(\theta_0,0)}{5};$$

$$\Pi_0^+ z(\tau) = -\frac{\bar{\sigma}_0^+(\theta_0,0)}{5}\mathrm{e}^{-\tau}$$

Let us compute the first approximation of desired periodic solution. Equations for the slow part of first approximation for $u = 1$ have the form

$$\left.\begin{aligned} &\bar{z}_1^+ = 0; \quad \mathrm{d}\bar{s}_1^+/\mathrm{d}t = 2\bar{s}_1^+ + \bar{\sigma}_1^+ \\ &\mathrm{d}\bar{\sigma}_1^+/\mathrm{d}t = -6\bar{s}_1^+ - \bar{\sigma}_1^+ + \bar{x}_1^+, \quad \mathrm{d}\bar{x}_1^+/\mathrm{d}t = -\bar{x}_1^+ + 1 \end{aligned}\right\} \tag{24}$$

Than the solution of (24) with initial conditions

$$\bar{s}_1^+(0) = \bar{\sigma}_1^+(0) = 0, \quad \bar{x}_1^+(0) = x_1^*$$

takes the form

$$\begin{aligned} \bar{s}_1^+(t) &= \frac{1}{4} + \frac{1}{6}\mathrm{e}^{-t}(x_1^* - 1) \\ &\quad + \left(\frac{-\sqrt{15}}{60} + \frac{\sqrt{15}}{30}x_1^*\right)\mathrm{e}^{t/2}\sin\frac{\sqrt{15}t}{2} \\ &\quad - \left(\frac{x_1^*}{6} + \frac{1}{12}\right)\mathrm{e}^{t/2}\cos\frac{\sqrt{15}t}{2} \\ \bar{\sigma}_1^+(t,x_1^*) &= -\frac{1}{2} - \frac{1}{2}\mathrm{e}^{-t}(x_1^* - 1) \\ &\quad + \frac{\sqrt{15}}{30}\left(x_1^* + 2\right)\mathrm{e}^{t/2}\sin\frac{\sqrt{15}t}{2} \\ &\quad + \frac{x_1^*}{2}\mathrm{e}^{t/2}\cos\frac{\sqrt{15}t}{2} \\ \bar{x}_1^+(t,x_1^*) &= (x_1^* - 1)\mathrm{e}^{-t} + 1 \end{aligned}$$

Then taking into account that we have $\theta_0 \approx 2.45$, $\bar{s}_1^+(\theta_0) \approx 0.45 - 0.45x_1^*$

$$\bar{\sigma}_1^+(\theta_0) \approx -1.34 - 0.42x_1^*; \quad \bar{x}_1^+(\theta_0) \approx 0.91 + 0.09x_1^*$$

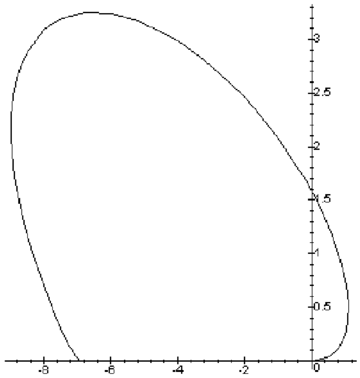

Figure 2. Periodic solution of the reduced system.

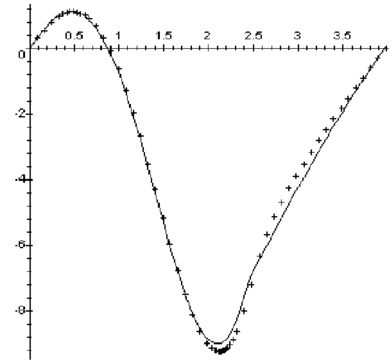

Figure 3. σ coordinate for the periodic solution of the original system (line) and its first order asymptotic representation (points) for $\mu = 0.2$.

Then

$$\theta_1 = -[\bar{\sigma}_0^+(\theta_0)]^{-1}\bar{s}_1^+(\theta_0) \rightleftarrows \theta_1 \approx 0.063 - 0.063x_1^*$$

$$\Pi_1^*\sigma(\tau) = 4\int_\infty^\tau \Pi_0^+ z(\Theta)\, \mathrm{d}\Theta = 4\frac{\bar{\sigma}_0^+(\theta_0, 0)}{5}\mathrm{e}^{-\tau}$$

$$\Pi_1^*\sigma(0) = 4\frac{\bar{\sigma}_0^+(\theta_0, 0)}{5}$$

$$\Pi_1^* x(\tau) \equiv 0$$

The initial conditions for the first approximation of slow variables on the sliding surface are defined by equations

$$\bar{\sigma}_1^*(\theta_0, x_1^*) + \Pi_1^*\sigma(0) = \bar{\sigma}_1^+(\theta_0, x_1^*) + \theta_1(x_1^*)\frac{\bar{\sigma}_0^+}{\mathrm{d}t}(\theta_0, 0)$$

$$\rightleftarrows \bar{\sigma}_1^*(\theta_0, x_1^*) = \bar{\sigma}_1^+(\theta_0, x_1^*) + \theta_1(x_1^*)(4 - \bar{\sigma}_0^+(\theta_0, 0)) - 4\frac{\bar{\sigma}_0^+(\theta_0, 0)}{5}$$

$$\bar{x}_1^*(\theta_0, x_1^*) = \bar{x}_1^+(\theta_0, x_1^*) - \theta_1(x_1^*)\bar{x}_0^+(\theta_0)$$

$$\bar{\sigma}_1^*(\theta_0, x_1^*) \approx -1.12x_1^* + 4.98;$$

$$\bar{x}_1^*(\theta_0, x_1^*) \approx 0.086x_1^* + 0.91$$

At the same time the slow coordinates of system (21) periodic solution are describing by equations

$$\mathrm{d}\bar{\sigma}_1^*/\mathrm{d}t = -\frac{\bar{\sigma}_1^*}{5} + \bar{x}_1^* - 4\mathrm{d}\bar{z}_0^*(t)/\mathrm{d}t;$$

$$\mathrm{d}\bar{x}_1^*/\mathrm{d}t = -\bar{x}_1^* + \bar{z}_0^*$$

Now

$$\bar{\sigma}_1^*(t, x_1^*) = 25 - (3.34 + 0.11x_1^*)\mathrm{e}^{-(t-\theta_0)} - (1.01x_1^* + 16.67 + 11.08t)\,\mathrm{e}^{-(t-\theta_0)/5}$$

$$\bar{x}_1^*(t) = 5 - 6.76\,\mathrm{e}^{-(t-\theta_0)/5} + \mathrm{e}^{-(t-\theta_0)}(2.67 + 0,086x_1^*)$$

Taking into account that $t = T_0$ we have

$$\bar{\sigma}_1^*(T_0, x_1^*) \approx 4.38 - 0.77x_1^*,$$

$$\bar{x}_1^*(T_0, x_1^*) \approx 0.59 + 0.019x_1^*$$

The value x_1^* is determined by equation $\bar{x}_1^*(T_0) = x_1^*$, which means that

$$x_1^* \approx 0.60, \quad \theta_1 \approx 0.025, \quad \Theta_1 \approx 0.22, \quad T_1 \approx 0.25$$

7. Conclusions

Singularly perturbed relay systems (SPRS) for which the reduced systems have stable periodic motions with internal sliding modes are studied. For such systems a theorem about existence and stability of the periodic solutions is proved. The algorithm for the asymptotic representation of these periodic solutions using boundary functions method is presented. It is proved that in the asymptotic representation of periodic solutions with internal sliding modes there are two boundary layers:

- the boundary layer at the point of input into the sliding mode which corresponds to the jump of solution to the small neighbourhood of the slow motion integral manifold of singularly perturbed system describing the behaviour of original SPRS into the sliding domain;
- the boundary layer at the break away point in which the solution is leaving the sliding domain.

It is proved that the zero order boundary function in the asymptotic representation of the periodic solution at the break away point is equal to zero because the zero approximation of the slow motion integral manifold at this point is continuous.

Acknowledgements

The author would like to thank Klaus Schneider and Rainer Rumpel for the helpful discussions. This research was supported under the grant of Consejo Nacional de Ciencia y Tecnologia (CONACYT) N990704.

Appendix: Asymptotic representation for solutions of SPRS

A.1. *Asymptotic representations for solutions of singularly perturbed systems with perturbed initial conditions*

Consider the smooth singularly perturbed system in form

$$\mu\,\mathrm{d}a/\mathrm{d}t = F(a,b,t), \quad \mathrm{d}b/\mathrm{d}t = f(a,b,t), \tag{25}$$

where $(a,b) \in \Omega \subset \mathbf{R}^n \times \mathbf{R}^m$, $\mu \in [0,\mu_0]$ is the small parameter, $t \in [\bar{t}, T]$, $\bar{t} = \min\{t_0, t_0 + \min_{\mu\in[0,\mu_0]} \hat{t}(\mu)\}$, $t_0 < T$. For system (1) consider two different initial problems

$$a(t_0,\mu) = a^0, \quad b(t_0,\mu) = b^0 \tag{26}$$

$$a(t_0 + \hat{t}(\mu),\mu) = a^0 + \hat{a}(\mu), \quad b(t_0,\mu) = b^0 + \hat{b}(\mu) \tag{27}$$

where $\hat{a}(\mu), \hat{b}(\mu), \hat{t}(\mu)$ are smooth functions on $\mu \in [0,\mu_0]$.

Assume the following hypothesis:

(A1) $F, f \in \mathbf{C}^{k+2}(\Omega \times [\bar{t}, T])$, $k \geq 0$.

(A2) Equation $F(a,b,t) = 0$ has the unique isolated solution $a_0 = \phi(b,t)$ with the same smoothness as F.

(A3) The reduced system

$$\mathrm{d}\bar{b}_0/\mathrm{d}t = f(\phi(\bar{b}_0,t),\bar{b}_0,t), \quad \bar{b}_0(t_0) = b^0 \tag{28}$$

has unique solution

$$(\phi(\bar{b}_0(t),t),\bar{b}_0(t),t) \in \Omega \times [\bar{t}, T].$$

(A4) There is a $\gamma > 0$ such that for all $t \in [\bar{t}, T]$

$$\mathrm{Re\ Spec}\,\frac{\partial F}{\partial z}(\phi(\bar{b}_0(t),t),\bar{b}_0(t),t) < -\gamma < 0$$

(A5) Let a^0 be in interior of the attraction domain of the asymptotically stable equilibrium point $\phi(b^0,\bar{t})$ of the associated system $\mathrm{d}a/\mathrm{d}\tau = F(a,b^0,t_0)$.

(A6) $\hat{a}(\mu) = O(\mu^{k+1})$, $\hat{b}(\mu) = O(\mu^{k+1})$, $\hat{t}(\mu) = O(\mu^{k+2})$.

Lemma 2: *Assume hypotheses* (A1)–(A6). *Let us denote by*

$$L(t,\mu) = (A(t,\mu),B(t,\mu)), \quad N(t,\mu) = (\hat{A}(t,\mu),\hat{B}(t,\mu))$$

the solutions of the Cauchy problems (25), (26) *and* (25), (27). *Then for sufficiently small* μ_1 *the functions* $L(t,\mu)$ *and* $N(t,\mu)$ *are the unique solutions of the corresponding Cauchy problems and uniformly on* $t \in [\bar{t}, T]$ *it holds*

$$\|L_k(t,\mu) - N(t,\mu)\| = O(\mu^{k+1})$$

where

$$L_k(t,\mu) = \Sigma\mu^i(\bar{l}_i(t) + \Pi_i l(\tau))$$

is the asymptotic representation of Cauchy problem (25), (26) *according boundary function method Vasil'eva* et al. *(1995).*

Proof: Assume that $\bar{t} = t_0$. Existence and uniqueness of L and N follow from the Tikhonov theorem (see for example Vasil'eva *et al*. 1995). Moreover from hypotheses (A1) it follows that it is possible to continue N on $[t_0, T]$.

Then for all $t \in [t_0, t_0 + \hat{t}(\mu)]$ we get

$$\|A(t_0+t,\mu) - a^0\| \leq \left\|\frac{1}{\mu}\int_{t_0}^{t_0+t} F(A(\xi,\mu),B(\xi,\mu),\xi)\mathrm{d}\xi\right\| \leq K_1\,O(\mu^{k+1})$$

$$\|B(t_0+t,\mu) - b^0\| \leq \left\|\int_{t_0}^{t_0+t} f(A(\xi,\mu),B(\xi,\mu),\xi)\,\mathrm{d}\xi\right\| \leq K_2\,O(\mu^{k+2})$$

where

$$K_1 = \sup_{(a,b,t)\in\bar{\Omega}\times[\bar{t},T]} \|F(a,b,t)\|,$$

$$K_2 = \sup_{(a,b,t)\in\bar{\Omega}\times[\bar{t},T]} \|f(a,b,t)\|.$$

Then it is possible to conclude that for every $t \in [t_0, t_0 + \hat{t}(\mu)]$

$$L(t,\mu) = N(t,\mu) + O(\mu^{k+1})$$

Hence asymptotical representations for L and N coincide on $[t_0, t_0 + \hat{t}(\mu)]$ and consequently uniformly on $[\bar{t}_0, T]$

$$\|L_k(t,\mu) - N(t,\mu)\| \leq \|L_k(t,\mu) - N_k(t,\mu)\| + \|N_k(t,\mu) - N(t,\mu)\| = O(\mu^{k+1})$$

The proof for the case $\hat{t}(\mu) < t_0$ can be make analogously. □

A.2. *Asymptotic representations for solutions of singularly perturbed systems in small neighbourhood of slow motion manifold*

Lemma 3: *Assume hypothesis* (A1)–(A5) *but in hypotheses* (A6) $\hat{t}(\mu) = O(\mu^{k+1})$. *Let us denote*

$$P(t,\mu) = (\bar{A}(t,\mu),\bar{B}(t,\mu)), \quad R(t,\mu) = (\tilde{A}(t,\mu),\tilde{B}(t,\mu))$$

the solutions of Cauchy problems for system (25) *with initial condition*

$$a(t_0,\mu) = \phi(b^0,t_0),\ b(t_0,\mu) = b^0 \tag{29}$$

and

$$a(t_0+\hat{t}(\mu),\mu)=\phi(b^0,t_0)+\hat{a}(\mu),\ b(t_0,\mu)=b^0+\hat{b}(\mu) \tag{30}$$

Then for sufficiently small μ_2 the functions $P(t,\mu)$ and $R(t,\mu)$ are the unique solutions of corresponding Cauchy problems and uniformly on $t\in[\bar{t},T]$ it holds

$$\|P_k p(t,\mu)-R(t,\mu)\|=O(\mu^{k+1})$$

where

$$P_k(t,\mu)=\sum_{i=0}^{k}\bar{p}_i(t)\mu^i+\sum_{i=1}^{k}\Pi_i p(\tau)\mu^i$$

is the asymptotic representation of the Cauchy problem (25) *and* (29) *according boundary function method (Vasil'eva* et al. *1995).*

Proof: Assume that $\bar{t}=t_0$. Existence and uniqueness of P and R follow from the Tikhonov theorem (see for example Vasil'eva *et al.* 1995). Moreover, from the hypothesis (A.1) it is possible to continue R on $[t_0,T]$. It follows from the boundary function method that the zero representation in $\Pi_0 c(\tau)\equiv 0$. Then for all $t\in[t_0,t_0+\hat{t}(\mu)]$ we get

$$\begin{aligned}\|\bar{B}(t_0+t,\mu)-b^0\| &\le \left\|\int_{t_0}^{t_0+t} f(\bar{A}(\xi,\mu),\bar{B}(\xi,\mu),\xi)\mathrm{d}\xi\right\| \\ &\le K_2\,O(\mu^{k+1}) \\ \|\bar{A}(t,\mu)-\phi(b^0,t_0)\| &= \|\bar{A}(t,\mu)-P_k p(t,\mu)\| \\ &\quad+\|P_k p(t,\mu)-\phi(b^0,t_0)\| \\ &\le O(\mu^{k+1})+\|\phi(B(t_0+t,\mu),t_0+t) \\ &\quad-\phi(b^0,t_0)\|=O(\mu^{k+1}).\end{aligned}$$

This means that

$$\|\hat{A}(t_0+\hat{t}(\mu),\mu)-\bar{A}(t_0+\hat{t}(\mu),\mu)\|=O(\mu^{k+1})$$

and consequently the asymptotic representations of $\hat{A}(t_0+\hat{t}(\mu),\mu)$ and $\bar{A}(t_0+\hat{t}(\mu),\mu)$ coincide up to the kth order. □

A.3. *Transition into the sliding mode*

Consider the Cauchy problems

$$\begin{aligned}\mu\,\mathrm{d}a/\mathrm{d}t&=F(a,s,b,u(s),t)\\ \mathrm{d}s/\mathrm{d}t&=h(a,s,b,u(s),t)\\ \mathrm{d}b/\mathrm{d}t&=f(a,s,b,u(s),t)\end{aligned} \tag{31}$$

$$a(t_0)=a^0,\ s(t_0)=s^0>0,\quad b(t_0)=b^0 \tag{32}$$

where

$$(a,s,b)\in\Xi\subset\mathbf{R}^n\times\mathbf{R}\times\mathbf{R}^m,$$

$$t\in[t_0,T],\mu\in[0,\mu_0],u(s)=\text{sign}\,(s).$$

Suppose that for system (31) the following hypothesis are true:

(B_0) $(a_0,\ b_0,\ s_0,\ t_0)\in\Xi^+\times[t_0,\bar{T}]$, $\Xi^+=\{(a,s,b)\in\Xi,\ s>0\},\bar{T}<T$.

(B_1) $F,h,\ f\in\mathbf{C}^{k+2}(\bar{\Xi}\times[-1,1]\times[t_0,T]),\ k\ge 0$.

(B_2) The equation $F(a,s,b,1,t)=0$ has an isolated solution $z=\phi^+(s,b,t)$, F and $\phi^+(s,b,t)$ have the same smoothness.

(B_3) The Cauchy problem

$$\left.\begin{aligned}\frac{\mathrm{d}\bar{s}_0^+}{\mathrm{d}t}&=h(\phi^+(\bar{s}_0^+,\bar{b}_0^+,t),\bar{s}_0^+,\bar{b}_0^+,1,t)\\ \frac{d\bar{b}_0^+}{dt}&=f(\phi^+(\bar{s}_0^+,\bar{b}_0^+,t),\bar{s}_0^+,\bar{b}_0^+,1,t)\\ \bar{s}_0^+(t_0)&=s^0,\bar{b}_0^+(t_0)=b^0\end{aligned}\right\} \tag{33}$$

has the unique solution on $t\in[t_0,\bar{T}_0]$.

(B_4) There is a $\gamma>0$ such that for all $t\in[t_0,\bar{T}]$

$$\text{Re Spec}\,\frac{\partial F}{\partial z}(\phi^+(\bar{b}_0^+(t),t),\bar{b}_0^+(t),t)<-\gamma<0.$$

(B_5) a^0 is located in the interior of the domain of attraction of the asymptotically stable equilibrium point $\phi^+(s^0,b^0,t)$ of the associated system

$$\frac{\mathrm{d}a}{\mathrm{d}\tau}=F(a,s^0,b^0,1,t_0)$$

(B_6) There exists a moment $t=\theta_0\in[t_0,\bar{T}]$, where the trajectory arrives on surface $s=0$ with

- $\bar{s}_0^+(\theta_0)=0$;
- $h(\phi^+(0,\bar{b}_0^+(\theta_0),\theta_0),0,\bar{b}_0^+(\theta_0),1,\theta_0)<0$;
- $h(\phi^+(0,\bar{b}_0^+(\theta_0),\theta_0),0,\bar{b}_0^+(\theta_0),-1,\theta_0)>0$.

Equation

$$h(a^*,0,b^*,u_{\mathrm{eq}},t)=0$$

has a unique, smooth solution $u=u_{\mathrm{eq}}(a^*,b^*,t)$ for $\Xi\times[\theta_0,T]$. The behaviour of system (31) into sliding domain is uniquely defined by equivalent control method and the sliding motions are described by the system

$$\left.\begin{aligned}&\mu\,\mathrm{d}a^*/\mathrm{d}t=F(a^*,0,b^*,u_{\mathrm{eq}}(a^*,b^*,t),t)\\ &\mathrm{d}b^*/\mathrm{d}t=f(a,0,b,u_{\mathrm{eq}}(a,b,t),t)\\ &h(a^*,0,b^*,u_{\mathrm{eq}}(a^*,b^*,t),t)=0\end{aligned}\right\} \tag{34}$$

Suppose that the equation $F(a^*,0,b^*,u_{\mathrm{eq}}(a^*,b^*,t),t)=0$ has a unique solution $a^*=\phi^*(b^*,t)$ and the reduced system for system (34) takes the form

$$\mathrm{d}\bar{b}_0^*/\mathrm{d}t=f(\phi^*(\bar{b}^*{}_0,t),0,\bar{b}^*{}_0,\bar{u}_{\mathrm{eq}},t), \tag{35}$$

where $\bar{u}_{\mathrm{eq}}=u_{\mathrm{eq}}(\phi^*(t,\bar{b}^*{}_0),\bar{b}^*,t)$.

(B_7) The Cauchy problem for system (35) with the initial conditions $\bar{b}_0^*(\theta_0) = \bar{b}_0^+(\theta_0)$ has a unique solution $\bar{b}_0^*(t), t \in [\theta_0 - \delta, T]$ and

$$h(\phi^*(\bar{b}_0^*(t), t), 0, \bar{b}_0^*(t), 1, t) < 0,$$
$$h(\phi^*(\bar{b}_0^*(t), t), 0, \bar{b}_0^*(t), -1, t) > 0$$

(B_8) For $t \in [\theta_0, T]$ there exists $\gamma > 0$ such that

$$\mathrm{Re\ Spec}\ \frac{\partial F}{\partial z}(\phi^*(\bar{b}_0^*(t), t), 0, \bar{b}_0^*(t), \bar{u}_{\mathrm{eq}}, t) < -\gamma < 0$$

(B_9) The point $\phi^+(0, \bar{b}_0^+(\theta_0), \theta_0)$ is situated in the interior of attractivity domain of equilibrium point $\phi^*(\bar{b}_0^+(\theta_0), \theta_0)$ for system

$$\mathrm{d}a^*/\mathrm{d}\tau = F(a^*, 0, b^*, u_{\mathrm{eq}}, t)$$

where $\tau = (t - \theta_0)/\mu$, and moreover for all $\lambda \in [0, 1]$ it holds

$$h(\lambda(\phi^+(0, \bar{b}_0^+(\theta_0), \theta_0) + (1 - \lambda) \times (\phi^*(\bar{b}_0^+(\theta_0), \theta_0), 0, \bar{b}_0^+(\theta_0), 1, \theta_0) < 0$$
$$h(\lambda(\phi^+(\theta_0, 0, \bar{b}_0^+(\theta_0), \theta_0) + (1 - \lambda) \times (\phi^*(\bar{b}_0^+(\theta_0), \theta_0), 0, \bar{b}_0^+(\theta_0), -1, \theta_0) > 0$$

Theorem 3: *Let system* (31) *satisfy the conditions* (B_1)–(B_9). *Then there exists a* μ_2 *such that for all* $\mu \in [0, \mu_2]$ *there is a unique solution* $c(t, \mu) = (a(t, \mu), s(t, \mu), b(t, \mu))$ *of Cauchy problem* (31), (32) *and uniformly on* $[t_0, T]$ *the following estimation holds*

$$\|c(t, \mu) - C_k(t, \mu)\| = O(\mu^{k+1})$$

$$C_k(t, \mu) = \sum_{i=0}^{k} [\bar{c}_i(t) + \Pi_i^* c(\tau_{k+1})]\mu^i + \sum_{j=1}^{k} \Pi_j^+ c(\tau)\mu^i, \qquad \tau_{k+1} = (t - \bar{\theta}_{k+1}(\mu)/\mu$$

where $\bar{\theta}_{k+1}(\mu) = \theta_0 + \theta_1\mu + \cdots + \theta_{k+1}\mu^{k+1}$ *is the* $(k+1)$th *order approximation of the switching moment* $\theta(\mu)$.

Proof: Assumptions (B_0)–(B_5) allows one to apply the Tikhonov theorem and boundary functions method for Cauchy problem (31), (32) and at least for $s(t, \mu) > 0$. Taking into account the condition (B_6) and using the implicit function theorem one can conclude that there exists arrival time moment $t = \theta(\mu)$ for which

$$s(\theta(\mu), \mu) = 0, \qquad \frac{\mathrm{d}s}{\mathrm{d}t}(\theta(\mu), \mu) \neq 0, \qquad \lim_{\mu \to 0} \theta(\mu) = \theta_0$$

Conditions (B_7)–(B_9) together with the Tikhonov theorem and boundary functions method ensure the existence and uniqueness of solutions of (31) into the sliding mode on $s = 0$. This means that we can reduce the solution of the Cauchy problem (31), (32) on the segment $t \in [0, T]$ to the solution of two consequent Cauchy problems. First it is necessary to find the solution $c^+(t, \mu)$ problem (31), (32) on $[t_0, \theta(\mu)]$. Then it is necessary to find $\theta(\mu)$ and the solution of the Cauchy problem for system (35) on $t \in [\theta(\mu), T]$ with initial conditions

$$a^*(\theta(\mu), \mu) = a^+(\theta(\mu), \mu), \quad b^*(\theta(\mu), \mu) = b^+(\theta(\mu), \mu) \tag{36}$$

This means that to find the asymptotic representations of the Cauchy problem (31), (32) on $[t_0, T]$ it is necessary to find first the asymptotic representation for $\theta(\mu)$. Suppose the we have found $\bar{s}_i^+$, $i = 0, \ldots, k+1$ the regular terms of asymptotic representation of $s(t, \mu)$ according boundary functions method for $t \in [t_0, \theta(\mu)]$. Then to find asymptoic representation $\theta(\mu)$ we have

$$\bar{s}_0^+(\theta_0 \cdots + \mu^i\theta_i + \cdots) + \mu\bar{s}_1^+(\theta_0 + \cdots + \mu^i\theta_i + \cdots) + \cdots + \mu^i\bar{s}_i^+(\theta_0 + \cdots + \mu^i\theta_i + \cdots) + \cdots = 0$$

(the terms $\Pi^+ s(\theta(\mu)/\mu)$ are exponentially small). Expanding functions $\bar{s}_i^+$ as series near $t = \theta_0$ and considering ith order members we have the equations for defining θ_i in the form

$$\theta_i h(\phi^+(0, \bar{b}_0^+(\theta_0), \theta_0), 0, \bar{b}_0^+(\theta_0), 1, \theta_0) + p_i(\theta_0, \theta_1, \ldots, \theta_{i-1}) = 0$$

where p_i is the function depending only on $\theta_0, \theta_1, \ldots, \theta_{i-1}$. From the condition (B_6) it follows that

$$h(\phi^+(0, \bar{b}_0^+(\theta_0), \theta_0), 0, \bar{b}_0^+(\theta_0), 1, \theta_0) < 0$$

which means it is possible to define $\theta_0, \theta_1, \ldots, \theta_i, \ldots$ uniquely.

Suppose that we have found $\theta_0, \theta_1, \ldots, \theta_{k+1}$ and the coefficients of regular parts of asymptotic representations of Cauchy problem (31), (32) $\bar{c}_i^+(t)$, $(i = 1, \ldots, k)$ for $[t_0, \theta(\mu)]$. From lemma 2 it follows that to find asymptotic representation of the initial time moment in (36) it is necessary to use the value $\bar{\theta}_{k+1}(\mu)$ instead of $\theta(\mu)$. But for asymptotic representation of state variable it is enough to have kth order approximation of the regular part for asymptotic representation of value $c^+(\bar{\theta}_k(\mu), \mu)$ in the form

$$\bar{C}_k^+(\tilde{\theta}_k(\mu), \mu) = \sum_{i=0}^{k} \bar{c}_i^+(\bar{\theta}_k(\mu))\mu^i$$

Moreover, for asymptotic representation of the state variable in (36) it is reasonable to use only the terms of orders $\mu^0, \mu, \mu^2, \ldots, \mu^k$ in the expansion of

$$\bar{C}_k^+(\bar{\theta}_k(\mu), \mu) = \bar{c}_0^+(\theta_0) + \mu(\bar{c}_1^+(\theta_0) + \theta_1 \mathrm{d}\bar{c}_0^+/\mathrm{d}t(\theta_0)) + \cdots$$

instead of $\bar{C}_k^+(\bar{\theta}_k(\mu), \mu)$.

Now Theorem 3 follows from the boundary functions method and Lemma 2. □

The proof of Theorem 2 follows from Lemma 3 and Theorem 3.

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