

Output integral sliding mode control based on algebraic hierarchical observer

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The problem of the realization of integral sliding mode controllers based only on output information is discussed. The implementation of an output integral sliding mode controller ensures insensitivity of the state trajectory with respect to the matched uncertainties from the initial time moment. In the case when the number of inputs is more than or equal to the number of outputs, the closed loop system, describing the output integral sliding mode dynamics, is shown to lose observability. For the case when the number of inputs is less than the number of outputs, a hierarchical sliding mode observer is proposed. The realization of the proposed observer requires a filtration to obtain the equivalent output injections. Assigning the first order low-pass filter parameter small enough (during this filter realization), the convergence time and the observation error can be made arbitrarily small. The results obtained are illustrated by simulations.

1. Introduction

1.1 Antecedents and motivation

Various areas of control, such as robotics or optimal control, demand the compensation of arising uncertainty effects. In this situation the special sliding mode technique, namely, integral sliding mode (ISM), seems to be useful, see, e.g., Utkin and Shi (1996). It has two main properties: first, the ISM does not have reaching phase; and second, resulting from the first one, it ensures insensitivity of the desired trajectory with respect to matched uncertainties starting from the initial moment. These properties make attractive the study of ISM, see, for example, Utkin *et al.* (1999), Basin *et al.* (2002a, b, 2003, 2005), Poznyak *et al.* (2004), Shtessel *et al.* (2004), Fridman *et al.* (2005), Xu *et al.* (2005), and Castaños and Fridman (2006).

When only the output of a system is available, there are two possibilities for sliding mode control design. One is to use an output feedback control, i.e., design a sliding surface using the output of the system in such a way that the dynamics of the system, during the corresponding sliding motion, has a property required by the designer. This kind of controls can be seen in Edwards and Spurgeon (1998), Sira-Ramírez and Spurgeon (1996) and Bag *et al.* (1997). Another possibility is to design an observer and with the following use of the obtained estimates in a control law instead of the real states of the system. To construct an estimator, providing convergence of generated estimates to real states, the corresponding sliding surface

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However, the main problem related to the implementation of this ISM concept is the requirement of the complete knowledge of the state vector, including the initial one. Obviously, when dealing with ISM and only output (no states) information is available, it turns out to be useless when being applied directly. Here, we present a possible approach to the solution of this problem.

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should be specially designed. There are two main methods for designing sliding mode observers: one is aimed to get a zero tracking error between the outputs of the plant and the observer to be constructed (see, e.g., Edwards and Spurgeon (1998), Utkin et al. (1999), Edwards et al. (2002) and Sira-Ramírez and Fliess (2004)); the other one is to design several sliding surfaces to estimate the state step by step (Hashimoto et al. 1990, Barbot et al. 2003). Here we design a hierarchical observer which completely differs from the observers studied in Hashimoto et al. (1990), Barbot et al. (2003) and Barbot and Floquet (2004). We obtain a vector which is the result of multiplying an observability matrix by the state. Thus, at each k-level of the hierarchy we estimate a sub-block of such vector and so on until we get all the vector previously mentioned.

1.2 Main contribution

In this paper the OISM algorithm for uncertain linear time invariant systems based only on the output information is discussed. Some specific contributions are enumerated below.

- 1. We design an ISM controller, using only output information that compensates the matched uncertainties from the initial time of the control process.
- 2. It is shown that when the number of inputs is more than (or equal to) the number of outputs, the corresponding ISM dynamics always lose observability and therefore the application of ISM, based only on output information, is useless when state estimation is required.
- 3. We propose a hierarchical sliding mode observer for the case when the number of inputs is less than the number of outputs. The observation error can be made arbitrarily small for an arbitrary short time by the adjustment of the parameters of the filter required during the realization.

1.3 Structure of the paper

In section 2, the model description is presented and the purpose of the control law is formulated. Section 3 is devoted to the design of the OISM control and the design of the observer. In § 3.1 an output integral sliding mode (OISM) controller rejecting the matched uncertainty is proposed. Subsections 3.2 and 3.3 deal with the observer design. Subsection 3.4 is related to the realization of the observer. In § 3.5, the algorithm of design is formulated. A case of study related to an optimal control is given in section 4. Section 5 deals with some numerical illustrations. In Appendix A, it is shown that if an OISM is used, and the number of outputs is less than or equal to the number of inputs,

the nominal closed system becomes unobservable. In Appendix B the conditions, in terms of A, B and C, under which the proposed observer can be carried out, are formulated.

2. Model description and problem statement

2.1 Plant's model

Let us consider a linear time invariant system with uncertainties

$$\dot{x}(t) = Ax(t) + Bu(t) + g(x, t); \ x(0) = x^{0}$$

$$y(t) = Cx(t),$$
(1)

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control law and $y(t) \in \mathbb{R}^p$ $(1 \le p < n)$ is the output of the system. The pair $\{u(t), y(t)\}$ is assumed to be measurable (available) for all time $t \ge 0$. The current state x(t) and the initial state x(0) are supposed to be non-available. *A*, *B*, *C* are known matrices of appropriate dimension with rank B = m and rank C = p. All the solutions of the dynamic systems are defined in Filippov's sense (Filippov 1988).

Throughout the paper we will assume the following.

- 1. The pair (A, B) is controllable and (A, C) is observable.
- 2. The plant (1) operates only under matched uncertainties, that is,

$$g(x,t) = B\gamma(x,t)$$

with the function $\gamma(x, t)$ being bounded, that is,

$$\left\|\gamma(x,t)\right\| \le q_a(y(t),t). \tag{2}$$

3. The vector x^0 is supposed to be unknown but belonging to a given ball, that is

$$\|x^0\| \le \mu. \tag{3}$$

4. $\operatorname{rank}(CB) = m$

2.2 Control challenge

Let the nominal state be

$$\dot{x}_0(t) = Ax_0(t) + Bu_0(t); \quad x_0(0) = x^0$$
 (4)

Now, for the system (1), we design the control law u to be

$$u = u_0 + u_1 \tag{5}$$

where the control $u_0 \in \mathbb{R}^m$ is the ideal control designed for the system (4) and $u_1 \in \mathbb{R}^m$ is designed to compensate the matched uncertainty g(x, t) from the initial time.

3. OISM control and design of the observer

This section deals first with the design of control u_1 . Then, a hierarchical integral sliding modes (HISM) observer is suggested and it is shown that the estimation error can be made arbitrarily small for an arbitrary short time by the adjustment of the filter parameter used during the realization of the observer.

3.1 Output integral sliding modes

Define the auxiliary affine sliding function $s: \mathbb{R}^p \to \mathbb{R}^m$ as follows:

$$s(y) := Gy + \sigma. \tag{6}$$

Here, the matrix $G \in \mathbb{R}^{m \times p}$ must satisfy the condition

$$det(GCB) \neq 0.$$

The term σ is a function of t and includes an integral term which will be defined below. Thus, for the time derivative \dot{s} we have

$$\dot{s} = GC[Ax + Bu_0 + Bu_1 + B\gamma] + \dot{\sigma}.$$

Define $\dot{\sigma}$ as

$$\dot{\sigma} = -GCA\hat{x} - GCBu_0, \quad \sigma(0) = -Gy(0). \tag{7}$$

The vector \hat{x} represents an observer and its specific form will be selected further. The substitution of $\dot{\sigma}$ in (\dot{S}) yields

$$\dot{s} = GCA(x - \hat{x}) + GCBu_1 + GCB\gamma; \quad s(0) = 0.$$
 (8)

We propose the control u_1 in the following form

$$u_1 = -\beta(t)D^{-1}\frac{s(t)}{\|s(t)\|}, \quad D := GCB$$
 (9)

with the $\beta(t)$ being a scalar gain that satisfies the condition

$$\beta(t) - \left(\|D\| q_a(y, t) + \|GCA\| \|x(t) - \hat{x}(t)\| \right) \ge \lambda > 0,$$

where λ is a constant. Selecting the Lyapunov function as $V = 1/2 ||s||^2$ and in view of (9) and (2), differentiating V yields

$$\dot{V} = (s, \dot{s}) = \left(s, GCA(x - \hat{x}) - \beta \frac{s}{\|s\|} + D\gamma\right)$$

$$\leq -\|s\|(\beta - \|GCA\|\|x - \hat{x}\| - \|D\|(q_a)) \leq -\|s\|\lambda \leq 0$$

 $((s, \dot{s}) := s^T \dot{s})$. It means that V does not increase in time and since s(0) = 0, this implies

$$\frac{1}{2} \| s(t) \| = V(s(t)) \le V(s(0)) = \frac{1}{2} \| s(0) \| = 0.$$

So, the identities

$$s(y(t)) = \dot{s}(y(t)) = 0$$
 (10)

hold for all $t \ge 0$, i.e., there is no reaching phase.

From (8) and in view of the equality in (10), the equivalent control maintaining the trajectories on the surface is

$$u_{1eq} = -(GCB)^{-1}GCA(x(t) - \hat{x}(t)) - \gamma.$$
(11)

The substitution of u_{1eq} in (1) yields the sliding mode equations

$$\dot{x}(t) = Ax(t) - B(GCB)^{-1}GCA[x(t) - \hat{x}(t)] + Bu_0$$

$$y(t) = Cx(t).$$
(12)

Define the matrix \tilde{A} as

$$\tilde{A} := \left[I - B(GCB)^{-1}GC\right]A.$$
(13)

Lemma 1: When the number of outputs is less than or equal to the number of inputs, the matrix \tilde{A} in (13) always belongs to the null space of the matrix C and, consequently, the pair (\tilde{A}, C) is not observable.

The proof of Lemma 1 is given in Appendix A.

Remark 1: Lemma 1 means that in the case when $p \le m$, the ISM control using only output information should not be realized.

The following lemma establishes the condition, in terms of A, B and C, providing the observability of the pair (\tilde{A}, C) .

Lemma 2: The pair (\tilde{A}, C) is observable if and only if the triple (A, B, C) does not have any invariant zeros, i.e.,

$$\{s \in \mathbb{C} : \operatorname{rank} (P(s)) < n + m\} = \emptyset, \tag{14}$$

where P(s) is the Rosenbrock's matrix system defined as

$$P(s) = \begin{bmatrix} sI - A & B \\ -C & 0 \end{bmatrix}.$$
 (15)

A proof of Lemma 2 is given in Appendix B.

Remark 2: Note that \tilde{A} defined in (13) depends on a matrix G, which can be designed in a non-unique form. However, due to the Lemma 2, the observability of the pair (\tilde{A} , C) depends only on the matrices A, B and C. In other words, the design of G does not affect the observability of (\tilde{A} , C).

3.2 Design of the observer

Define G as $G = (CB)^+ := [(CB)^T (CB)]^{-1} (CB)^T$ which is the pseudo-inverse of CB. Substitution of G in (12) leads to the following expression:

$$\dot{x}(t) = \tilde{A}x(t) + Bu_0 + B(CB)^+ CA\hat{x}(t)$$

$$y(t) = Cx(t),$$
(16)

where the matrix \tilde{A} in (13) becomes

$$\tilde{A} = \left[I - B(CB)^+ C\right] A.$$

Now, it is assumed that the pair (\tilde{A}, C) is observable.

The observer will be based on the recovery of the vectors Cx(t), $C\tilde{A}x(t)$ and so on until get $C\tilde{A}^{l-1}x(t)$. After arranging the vectors $C\tilde{A}^kx(t)$, we will have obtained the vector f(t) := Hx(t), where

$$H = \begin{bmatrix} C \\ C\tilde{A} \\ \vdots \\ C\tilde{A}^{l-1} \end{bmatrix}, \quad H \in \mathbb{R}^{nl \times n}.$$
(17)

By definition l (the observability index) is the least positive integer such that rank (H) = n. Since (\tilde{A}, C) is observable, such a index l always exists, see, e.g., Chen (1999). So to obtain the vector x(t), we only need to solve the set of algebraic equations f(t) = Hx(t).

Before designing the observer we find a bound that we will need later. Design the following dynamic system

$$\dot{\tilde{x}}(t) = \tilde{A}\tilde{x}(t) + Bu_0(t) + B(CB)^+ CA\hat{x}(t) + K(y(t) - C\tilde{x}(t)),$$
(18)

where K must be designed such that the eigenvalues of $\hat{A} := (\tilde{A} - KC)$ have negative real part.

Let $r(t) = x(t) - \tilde{x}(t)$, then, from (16) and (18), the dynamic equations governing r(t) are

$$\dot{r}(t) = \left[\tilde{A} - KC\right]r(t) = \hat{A}r(t).$$
(19)

Since the eigenvalues of \hat{A} have negative real part, the equation (19) is exponentially stable i.e., there exist some constants γ , $\eta > 0$ such that

$$\|r(t)\| \le \gamma \exp(-\eta t) \|r(0)\|$$

$$\le \gamma \exp(-\eta t)(\mu + \|\tilde{x}(0)\|).$$
(20)

Below, it is shown that in the design of the observer we need a bound of ||r(t)||. Thus (20) ensures that we can always satisfy such a requirement.

3.2.1 Auxiliary dynamic systems and output injections. The principal goal in the design of the observer is to recover the vectors

$$C\tilde{A}^i x, \ i = \overline{1, l-1}$$

where *l* is defined as the observability index see e.g. Chen (1999). First, to recover $C\tilde{A}x(t)$, let us introduce an auxiliary state vector $x_a^{(1)}(t)$ governed by

$$\dot{x}_{a}^{(1)}(t) = \tilde{A}\tilde{x}(t) + B[u_{0}(t) + (CB)^{+}CA\hat{x}(t)] + L(CL)^{-1}v^{(1)}(t), \qquad (21)$$

where $x_a^{(1)}(0)$ satisfies

 $Cx_{a}^{(1)}(0) = y(0).$

For the variable $s^{(1)} \in \mathbb{R}^p$ defined by

$$s^{(1)}(y(t), x_{a}^{(1)}(t)) = Cx(t) - Cx_{a}^{(1)}(t)$$
(22)

we have

$$\dot{s}^{(1)}(t) = C\tilde{A}(x(t) - \tilde{x}(t)) - v^{(1)}(t)$$
(23)

with $v^{(1)}(t)$ defined as

$$v^{(1)} = \begin{cases} M_1 \frac{s^{(1)}}{\|s^{(1)}\|} & \text{if } s^{(1)} \neq 0\\ 0 & \text{if } s^{(1)} = 0. \end{cases}$$

Here the gain scalar M_1 should satisfy the condition

$$\|C\tilde{A}\| \|x - \tilde{x}\| < M_1 \tag{24}$$

to obtain the sliding mode regime. From (20),

$$M_1 = \|C\tilde{A}\| \left[\gamma \exp(-\eta t) \left(\mu + \|\tilde{x}(0)\|\right)\right]$$

satisfies (24). Then, repeating the same procedure as in $\S 3.1$, we get

$$s^{(1)}(t) = 0, \ \dot{s}^{(1)}(t) = 0, \quad \forall t \ge 0.$$
 (25)

Thus, in view of (25) and (22) we have

$$Cx(t) = Cx_{a}^{(1)}(t)$$
 (26)

and from (25) and (23), the equivalent output injection is

$$v_{\rm eq}^{(1)}(t) = C\tilde{A}x(t) - C\tilde{A}\tilde{x}(t), \quad \forall t > 0.$$

Thus, $C\tilde{A}x(t)$ is recovered by means of the following representation:

$$C\tilde{A}x(t) = C\tilde{A}\tilde{x}(t) + v_{\text{eq}}^{(1)}(t), \quad \forall t > 0.$$
(27)

Now, the next step is to recover the vector $C\tilde{A}^2x(t)$. To do that, let us design the second auxiliary state vector $x_a^{(2)}(t)$ generated by

$$\dot{x}_{a}^{(2)}(t) = \tilde{A}^{2}\tilde{x}(t) + \tilde{A}B[u_{0}(t) + (CB)^{+}CA\hat{x}(t)] + L(CL)^{-1}v^{(2)}(t),$$

where $x_a^{(2)}(0)$ satisfies

$$v_{\rm eq}^{(1)}(0) + C\tilde{A}\tilde{x}(0) - Cx_{\rm a}^{(2)}(0) = 0.$$

Again, for $s^{(2)} \in \mathbb{R}^p$ defined by

$$s^{(2)}\left(v_{eq}^{(1)}(t), x_{a}^{(2)}(t)\right) = C\tilde{A}\tilde{x}(t) + v_{eq}^{(1)}(t) - Cx_{a}^{(2)}(t)$$

in view of (27), it follows that

$$s^{(2)}\left(v_{\rm eq}^{(1)}(t), x_{\rm a}^{(2)}(t)\right) = C\tilde{A}x(t) - Cx_{\rm a}^{(2)}(t)$$
(28)

and hence, the time derivative of $s^{(2)}$ is

$$\dot{s}^{(2)}(t) = C\tilde{A}^2(x(t) - \tilde{x}(t)) - v^{(2)}(t).$$
⁽²⁹⁾

Take the output injection $v^{(2)}(t)$ as

$$v^{(2)} = \begin{cases} M_2 \frac{s^{(2)}}{\|s^{(2)}\|} & \text{if } s^{(2)} \neq 0\\ 0 & \text{if } s^{(2)} = 0\\ \|C\tilde{A}^2\| \|x - \tilde{x}\| < M_2. \end{cases}$$
(30)

Again, from (20),

$$M_2 = \|C\tilde{A}^2\|[\gamma \exp(-\eta t)(\mu + \|\tilde{x}(0)\|)]$$

satisfies (30). So, the procedure followed in §3.1 yields

$$s^{(2)}(t) = \dot{s}^{(2)}(t) = 0.$$
 (31)

From (31) and (29) the equivalent output injection $v_{eq}^{(2)}(t)$ could be represented in form

$$v_{\rm eq}^{(2)}(t) = C\tilde{A}^2(x(t) - \tilde{x}(t))$$

and the vector $C\tilde{A}^2 x(t)$ can be recovered by means of the equality:

$$C\tilde{A}^2 x(t) = C\tilde{A}\tilde{x}(t) + v_{eq}^{(2)}(t), \qquad t > 0.$$
 (32)

Thus, iterating the same procedure, all the vectors $C\tilde{A}^{i}x$ can be retrieved. The above mentioned procedure could be summarized as follows.

(a) The dynamics of the auxiliary state $x_a^{(k)}(t)$ at the *k*th level is governed by

$$\dot{x}_{a}^{(k)}(t) = \tilde{A}^{k}\tilde{x}(t) + \tilde{A}^{k-1}B[u_{0}(t) + (CB)^{+}CA\hat{x}(t)] + L(CL)^{-1}v^{(k)},$$
(33)

where $L \in \mathbb{R}^{n \times p}$ is a matrix so that $\det(CL) \neq 0$ for all k and the output injection $v^{(k)}$ at the kth level is

$$v^{(k)} = \begin{cases} M_k \frac{s^{(k)}}{\|s^{(k)}\|} & \text{if } s^{(k)} \neq 0\\ 0 & \text{if } s^{(k)} = 0. \end{cases}$$

$$\|C\tilde{A}\|\|r\| < M_k; r = x - \tilde{x} \end{cases}$$
(34)

 M_k is selected as $M_k \ge \|C\tilde{A}^k\| [\gamma \exp(-\eta t)(\mu + \|\tilde{x}^0\|)] + \lambda, \lambda > 0.$

(b) Define the sliding surface s^(k) at the k-level of the hierarchy as

$$s^{(k)} \left(v_{eq}^{(k-1)}(t), x_{a}^{(k)}(t) \right)$$

$$= \begin{cases} y(t) - Cx_{a}^{(1)}(t) & \text{for } k = 1 \\ v_{eq}^{(k-1)}(t) + C\tilde{A}^{k-1}\tilde{x}(t) - Cx_{a}^{(k)}(t) & \text{for } k > 1, \end{cases}$$
(35)

where $v_{eq}^{(k-1)}$ is the equivalent output injection whose general expression will be obtained in the lemma below, but $v_{eq}^{(k-1)}(0)$ and $s^{(k)}(0)$ should satisfy

$$s^{(k)}(0) = \begin{cases} Cy(0) - Cx_{a}^{(1)}(0) = 0 & \text{for } k = 1\\ v_{eq}^{(k-1)}(0) + C\tilde{A}^{k-1}\tilde{x}(0) - Cx_{a}^{(k)}(0) = 0 & \text{for } k > 1 \end{cases}$$
 (36)

Here, $v^{(k)}(t)$ is treated as a "sliding mode" output injection. The equivalent output injection of $v^{(k)}_{eq}(t)$ is given in the next lemma.

Lemma 3: If the auxiliary state vector $x_a^{(k)}$ and the variable $s^{(k)}$ are designed as in (33) and (35), respectively, then for all $t \ge 0$

$$v_{\rm eq}^{(k)}(t) = C\tilde{A}^k x(t) - C\tilde{A}^k \tilde{x}(t)$$
(37)

and each $k = \overline{1, l-1}$.

Proof: As it was shown before, the following identity holds

$$v_{\rm eq}^{(1)}(t) = C\tilde{A}x(t) - C\tilde{A}\tilde{x}(t), \quad \forall t > 0$$

Now, suppose that the equivalent output injection $v_{eq}^{(k-1)}$ is as in (37). Then substitution of $v_{eq}^{(k-1)}$ in (35) gives

$$s^{(k)}\left(v_{eq}^{(k-1)}(t), x_{a}^{(k)}(t)\right) = C\tilde{A}^{k-1}x(t) - Cx_{a}^{(k)}(t).$$
(38)

The derivation of (38) yields

$$\dot{s}^{(k)}\left(v_{eq}^{(k-1)}(t), x_{a}^{(k)}(t)\right) = C\tilde{A}^{k}(x(t) - \tilde{x}(t)) - v^{(k)}(t).$$
(39)

Thus, selecting the Lyapunov function $V = \frac{1}{2} ||s^{(k)}||^2$ and $v^{(k)}(t)$ as in (34), for any $t \ge 0$ one gets

$$s^{(k)}(t) \equiv 0, \quad \dot{s}^{(k)}(t) \equiv 0.$$
 (40)

Therefore, from (40) and (39) it follows that

$$v_{\rm eq}^{(k)}(t) \equiv C\tilde{A}^k x(t) - C\tilde{A}^k \tilde{x}(t)$$

3.3 Observer in the algebraic form

From (26) and (37), we have the following set of equations

$$Cx(t) = C\tilde{x}(t) + (Cx_{a}^{(1)}(t) - C\tilde{x}(t))$$

$$C\tilde{A}x(t) = C\tilde{A}\tilde{x}(t) + v_{eq}^{(1)}$$

$$\vdots$$

$$C\tilde{A}^{l-1}x(t) = C\tilde{A}^{l-1}\tilde{x}(t) + v_{eq}^{(l-1)}$$

$$(41)$$

or, in a matrix representation

$$Hx(t) = H\tilde{x}(t) + v_{eq}(t), \quad \forall t > 0,$$
(42)

where

$$H = \begin{bmatrix} C\\ C\tilde{A}\\ \vdots\\ C\tilde{A}^{l-1} \end{bmatrix}, \quad v_{eq} = \begin{bmatrix} Cx_a^{(1)} - C\tilde{x}(t)\\ v_{eq}^{(1)}\\ \vdots\\ v_{eq}^{(l-1)} \end{bmatrix}.$$
(43)

Since the pair (\tilde{A}, C) is observable, the matrix *H* has rank *n*. Thus, the left multiplication of (42) by $H^+ := [H^T H]^{-1} H^T$ implies

$$x(t) \equiv \tilde{x}(t) + H^+ v_{eq}(t), \quad \forall t > 0$$
(44)

That is why an observer, based on the *Hierarchical ISM* can be suggested as follows

$$\hat{x}(t) := \tilde{x}(t) + H^+ v_{eq}(t)$$
 (45)

Remark 3: Notice, that in general,

$$x^* := \arg \min_{x \in \mathbb{R}^n} ||f - Hx||^2 = H^+ f$$

where the limit $H^+ = \lim_{\delta \to 0} (\delta^2 I + H^T H)^{-1} H^T$ always exists (see Albert 1972) and, moreover,

$$||f - Hx^*||^2 = ||(I - HH^+)f||^2.$$

This norm is not obligatory equal to zero. In the particular case, when f = Hx, one has

$$\min_{z \in \mathbb{R}^n} \|f - Hz\|^2 = \|f - Hx^*\|^2 = \|(I - HH^+)f\|^2$$
$$= \|(I - HH^+)Hx\|^2 = \|(H - HH^+H)x\|^2 = 0$$

for any z.

Now we are ready to formulate the main result.

Theorem 1: Under the assumptions 1–4 and supposing the ideal output integral sliding mode exists, the following identity holds:

$$\hat{x}(t) \equiv x(t) \quad \forall t > 0 \tag{48}$$

Proof: It follows directly from (44) and (45). \Box

Remark 4: The realization of the observer (45) requires filters whose parameters affect convergence time of the observer.

3.4 Observer realization

To carry out the observer in the form (45), the surface $s^{(k)}$ must be realizable. Thus, to guarantee the realization of $s^{(k)}$, the equivalent output injection $v_{eq}^{(k)}$ must be available. However, the non-idealities in the implementation of $v^{(k)}$ cause the, so-called, chattering phenomenon. So, we will have a high frequency signal. Therefore, $v_{eq}^{(k)}$ cannot be directly obtained from $v^{(k)}$. Nevertheless, $v_{eq}^{(k)}$ could be computed via filtration. Namely, the first order low-pass filter

$$\tau \dot{v}_{av}^{(k)}(t) + v_{av}^{(k)}(t) = v^{(k)}(t); \quad v_{av}^{(k)}(0) = 0$$
(47)

gives an approach of $v_{eq}^{(k)}$ (Utkin 1992). Or, in other words,

$$\lim_{t \to 0 \atop a_{t} \to 0} v_{av}^{(k)}(t) = v_{eq}^{(k)}(t), \quad t > 0$$

where Δ is proportional to the sampling time (the time that $v^{\alpha,k}$ lasts to pass from one state (*M*) to other (-M)). So, selecting $\tau = \Delta^{\eta}$ ($0 < \eta < 1$), we have the following conditions to realize the OISM observer.

- 1. Use a sampling interval Δ very small.
- 2. Substitute $v_{eq}^{(k)}(t)$ in (35) by $v_{av}^{(k)}(t)$.
- 3. Substitute $v_{av}^{(k)}(0)$ in (36) by $v_{av}^{(k)}(0) \equiv 0$, i.e., the initial conditions $x_a^{(k)}(0)$ should satisfy the equations

$$C\tilde{A}^{k-1}x_{a}^{(k-1)}(0) - Cx_{a}^{(k)}(0) = 0 \quad \text{for } k > 1,$$

$$Cy(0) - Cx_{a}^{(1)}(0) = 0 \quad \text{for } k = 1.$$

So, the realization of the observer in (45) takes the form

$$\hat{x}(t) := \tilde{x}(t) + H^{+} v_{av}(t)$$

$$v_{av} = \left[\left(C x_{a}^{(1)} - C \tilde{x}(t) \right)^{T} (v_{av}^{(1)})^{T} \cdots (v_{av}^{(l-1)})^{T} \right]^{T} \right]$$
(48)

3.5 OISM algorithm

The proposed OISM algorithm can be summarized as follows.

- 1. Design the matrix K such that the eigenvalues of $\hat{A} := (\tilde{A} KC)$ have negative real part.
- 2. Compute the scalar gain $\beta(t)$ as in (9).
- 3. Design the auxiliary systems $x_a^{(k)}$ as in (33) with the sliding surfaces $s^{(k)}$ as in (35) and compute the constants M_k , k = 1, ..., l-1. Recall that $v_{eq}^{(k)}(t)$ should be substituted in (35) by $v_{av}^{(k)}(t)$.

4. Run simultaneously the observer \hat{x} according to (48) and the controllers u_0 , u_1 according to (52) and (9) respectively.

4. Case of study: LQ control law

In order to show an application of the OISM suggested in this paper, we propose to design the nominal control u_0 as an optimal control based on the standard LQ-index for a finite horizon. In view of (46) and (12), the sliding dynamics equations for the state x have the form

$$\dot{x}(t) = Ax(t) + Bu_0, \quad x(0) = x^0$$
 (49)

Note that now the dynamic equations (49) of the state x are the same as those ones of the nominal state (4). Here, u_0 is an admissible control (belonging to a set U_{adm} of piecewise continuous functions) which minimizes the following standard LQ-index:

$$J_{t_f}(u_0(\cdot)) := x^\top(t_f) Fx(t_f) + \int_{t=0}^{t_f} (x^\top(t)Qx(t) + u_0^\top(t)Ru_0(t)) dt,$$

where $F = F^{\top} \ge 0$, $Q = Q^{\top} \ge 0$, $R = R^{\top} > 0$. Thus, the aim of the control u_0 is: to minimize the index $J(u(\cdot))$, i.e.,

$$u_0^*(\cdot) = \arg\min_{u_0 \in U_{adm}} J_{t_f}(u_0(\cdot)).$$
(50)

Thus, the control law solving (50) for (49) (e.g. see Anderson and Moore (1990)) is of the form

$$u_0^*(x(t)) = -R^{-1}B^{\top}P(t)x(t)$$

with $P(t) \in \mathbb{R}^{n \times n}$ satisfying the differential Riccati equation

$$\frac{\dot{P}(t) + P(t)A + A^{\top}P(t) - P(t)BR^{-1}B^{\top}P(t) + Q = 0}{P(t_f) = F}$$
(51)

From (46), the estimated state \hat{x} is used to realize the control u_0 , i.e., the control u_0 should be designed as

$$u_0(t) = -R^{-1}B^{+}P(t)\hat{x}(t)$$
(52)

with $\hat{x}(t)$ being designed as in (45). That is, since we have compensated the matched uncertainties and we can ensure the estimation error being arbitrarily small after an arbitrarily small time, we can design the control u_0 for the nominal system but being applied to the system (1).

5. Example

To illustrate the procedure given above, let us take the linearized model of a inverted pendulum with a trolley considered in Utkin *et al.* (1999, pp. 90–91). A control force is applied to the cart so that the pendulum remains in a vertical line. The motions equations are

$$\dot{x}(t) = Ax(t) + B(u_0 + u_1) + B\gamma(x, t) y(t) = Cx(t)$$
(53)

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1.2586 & 0 & 0 \\ 0 & 7.5514 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 0.1905 \\ 0.1429 \end{bmatrix},$$
$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
$$\gamma(t) = \begin{cases} -0.4 & n-5 \le t < n-2.5 \\ 0.4 & n-2.5 < t \end{cases} \quad n = 5, 10, \dots$$

The vector state x consist of four state variables: x_1 is the distance between a reference point and the center of inertia of the trolley; x_2 represents the angle between the vertical and the pendulum; x_3 represents the linear velocity of the trolley; finally, we have that x_4 is equal to the angular velocity of the pendulum. As can be verified, the pair (A, C) has no invariant zeros. Lemma 2, implies that (\tilde{A}, C) is observable $(\tilde{A} = [I - B(CB)^+ C]A)$.

The initial conditions are considered as $x(0) = \begin{bmatrix} 0.3 & 0.2 & 0.1 & -0.1 \end{bmatrix}^{\top}$; and as a consequence we have $y(0) = \begin{bmatrix} 0.3 & -0.1 \end{bmatrix}^{\top}$. The matrix \tilde{A} takes the form

$$\tilde{A} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -8.81 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

As it can be verified, the pair (\tilde{A}, C) is observable. The matrix *K* was calculated as follows:

$$K = \begin{bmatrix} 4.6234 & -0.3148 \\ -1.3423 & 0.5548 \\ 10.2373 & -1.7542 \\ -0.3148 & 0.9492 \end{bmatrix}.$$

The weighing matrices Q, R and F were chosen as Q = 20I, R = 0.5 and F = 20I.

The simulations were carried out with two sampling steps: $\Delta = 2 \cdot 10^{-5}$ and $\Delta = 2 \cdot 10^{-4}$. In both cases, as the filter constant, the value τ was chosen as $\tau = 150\Delta^{4/5}$. The trajectories of the state vector, when \hat{x} (called *xe* in the graph) is used in the control *u*, and when *x* is used in the control *u*, are depicted in figures 1 and 2.



Figure 1. Trajectories of x using $\Delta = 2 \times 10^{-5}$. Trolley position (TP), pendulum position (PP), trolley velocity (TV) and pendulum angular velocity (PAV).



Figure 2. Trajectories of x using $\Delta = 2 \times 10^{-4}$. Trolley position (TP), pendulum position (PP), trolley velocity (TV) and pendulum angular velocity (PAV).

To realize the suggested observer, the filters suggested in (47) must be used. The simulations show that those filters are not affecting too much the observation process (see the observation error $e(t) = x(t) - \hat{x}(t)$ in figures 3 and 4. As we can see in those figures, the convergence to zero is better when Δ is smaller, i.e., the convergence depends only on Δ .

6. Conclusions

In this paper we discussed the possibilities to realize the output integral sliding mode control, ensuring





Figure 3. Observation error $e = x - \hat{x}$ using $\Delta = 2 \times 10^{-5}$.

Figure 4. Observation error $e = x - \hat{x}$ using $\Delta = 2 \times 10^{-4}$.

insensitivity of the state trajectory (optimal in this paper) with respect to matched uncertainties from the initial time.

It was shown that the use of output integral sliding mode control, for the case when the number of inputs is more than or equal to the number of outputs, immediately causes the loss of observability, and so the output integral sliding mode control could not be realized if any observation process is required. It was shown that, for the case when the number of inputs is less than the number of outputs, the use of output integral sliding mode allows: first, the matched uncertainties from the initial time (independently of the observation process) to be compensated, and second, design of the hierarchical sliding mode observer reconstructing the system states. Using a low-pass filter for the observer realization, we showed that the estimation error depends only on the sampling time and the filter time constant. It is proved that time of convergence of the observation error can be made arbitrarily small after a short time by decreasing the sampling step and filter time constant.

Appendix

A. Proof of Lemma 1

Proof: Consider system (1) with $p \le m$ and rank(*CB*) = *p*. Suppose that the control law *u* is designed in the following way

$$u=u_0+u_1,$$

where u_0 is the nominal control used after the compensation of the perturbation g and u_1 is designed to compensate the perturbation g. At first we will consider the case when p = m and next the case when p < m.

1. Consider the case when p = m.

Define auxiliary function *s* as follows:

$$s(y) := G(y+z), \ s \in \mathbb{R}^m$$
(54)

the matrix $G \in \mathbb{R}^{m \times m}$ must satisfy rank (GCB) = m, but this is only satisfied when det(G) $\neq 0$. Following the same process as in § 3.1, one has

$$u_{1eq} = -(GCB)^{-1}GCAx - \gamma.$$

Substitution of u_{1eq} in the system (1) yields

$$\dot{x}(t) = \left[I - B(GCB)^{-1}GC\right]Ax(t) + Bu_0$$

$$y(t) = Cx(t)$$

recall $\tilde{A} = [I - B(GCB)^{-1}GC]A$, then pre-multiply \tilde{A} by *GC* one gets

$$GC\tilde{A} = GC[I - B(GCB)^{-1}GC]A = 0.$$

This means \tilde{A} belongs to the null space of GC and since G is a non-singular matrix, then \tilde{A} belongs to the null space of C and it implies that (\tilde{A}, C) is not observable.

2. Now suppose that p < m.

Let the auxiliary function s as in (83) but, since rank (CB) = p and p < m, then there is not any matrix $G \in \mathbb{R}^{m \times p}$ satisfying rank (GCB) = m. That is why the sliding surface s(y) can not be designed in a space of dimension bigger than p. Let us define s(y) in the space \mathbb{R}^p , that is,

$$s(y) := G(y+z),$$

where the matrix $G \in \mathbb{R}^{p \times p}$. Thus, the time derivative \dot{s} is as follows

$$\dot{s} = GC[Ax + Bu_0 + Bu_1 + B\gamma] + G\dot{z}.$$

Define \dot{z} as follows

$$\dot{z} = -CBu_0, \ z(0) = y(0)$$

the substitution of \dot{z} in \dot{s} gives as result

$$\dot{s} = GCAx + GCBu_1 + GCB\gamma.$$

Now, in order to produce the sliding mode, the control u_1 should be designed as $u_1 := F\bar{u}$, where the matrix $F \in \mathbb{R}^{m \times p}$ should satisfy rank (GCBF) = p. Thus BF can be considered as the new matrix of input distribution B, and \bar{u} as the new control u_1 . In this form, we can consider that the number of inputs is p, that is we have the same number of inputs as number of outputs. Hence, we can follow the same proof used for case 1.

B. Proof of Lemma 2

Proof: Recall $\tilde{A} := [I - B(GCB)^{-1}GC]A$, hence, Lemma 2 asserts that for every complex scalar *s* the equivalence

$$\operatorname{rank} \begin{bmatrix} sI - A & B \\ -C & 0 \end{bmatrix}$$
$$= n + m \quad \text{if and only if } \operatorname{rank} \begin{bmatrix} sI - \tilde{A} \\ C \end{bmatrix} = n$$

is satisfied. So, define the matrices V and U in the following form

$$V := \begin{bmatrix} B^{\perp} \\ (GCB)^{-1}GC \end{bmatrix},$$

$$V^{-1} = \begin{bmatrix} [I - B(GCB)^{-1}GC]B^{\perp +} & B \end{bmatrix}$$

$$U := \begin{bmatrix} (CB)^{\perp} \\ G \end{bmatrix},$$

$$U^{-1} = \begin{bmatrix} [I - CB(GCB)^{-1}GC](CB)^{\perp +} & CB(GCB)^{-1} \end{bmatrix}.$$

Now, before to prove the required equivalence we need to express the following matrices into the expanded form

$$VAV^{-1} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad UCV^{-1} = \begin{bmatrix} C_1 & 0 \\ 0 & GCB \end{bmatrix},$$
(55)

where $A_{11} \in \mathbb{R}^{n-m \times n-m}$ and $C_1 \in \mathbb{R}^{p-m \times n-m}$. We obtain

$$V\tilde{A}V^{-1} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} - \begin{bmatrix} 0 \\ I \end{bmatrix} \begin{bmatrix} A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & 0 \end{bmatrix}.$$
 (56)

Then, from (55), (56), and since $det(GCB) \neq 0$ we have the following equivalences

$$\operatorname{rank} \begin{bmatrix} sI - A & B \\ -C & 0 \end{bmatrix} = n + m \Leftrightarrow \operatorname{rank} \begin{bmatrix} sI - VAV^{-1} & VB \\ -UCV^{-1} & 0 \end{bmatrix}$$
$$= n + m \Leftrightarrow \operatorname{rank} \begin{bmatrix} sI - A_{11} & -A_{12} & 0 \\ -A_{21} & sI - A_{22} & I \\ -C_1 & 0 & 0 \\ 0 & -GCB & 0 \end{bmatrix}$$
$$= n + m \Leftrightarrow \operatorname{rank} \begin{bmatrix} sI - A_{11} \\ -C_1 \end{bmatrix}$$
$$= n - m \Leftrightarrow \operatorname{rank} \begin{bmatrix} sI - A_{11} & -A_{12} \\ 0 & sI \\ -C_1 & 0 \\ 0 & -GCB \end{bmatrix}$$
$$= n \Leftrightarrow \operatorname{rank} \begin{bmatrix} sI - V\tilde{A}V^{-1} \\ -UCV^{-1} \end{bmatrix}$$
$$= n \Leftrightarrow \operatorname{rank} \begin{bmatrix} sI - V\tilde{A}V^{-1} \\ -UCV^{-1} \end{bmatrix}$$
$$= n \Leftrightarrow \operatorname{rank} \begin{bmatrix} I_n & 0 \\ 0 & U \end{bmatrix} \begin{bmatrix} V & 0 \\ 0 & I_p \end{bmatrix} \begin{bmatrix} sI - \tilde{A} \\ -C \end{bmatrix} V^{-1}$$
$$= n \Leftrightarrow \operatorname{rank} \begin{bmatrix} sI - \tilde{A} \\ -C \end{bmatrix} = n \qquad (57)$$

and so the Lemma is proven.

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