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Decomposition of the min–max multi-model problem via integral sliding mode

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SUMMARY

The concept of the integral sliding mode (ISM) is revised and applied for robustification of a linear time invariant min–max multi-model problem with uncertainties. Modified version of ISM ensures the insensitivity of the designed min–max control law with respect to matched uncertainty, starting from the beginning of the process, and guarantees that *the unmatched part of uncertainties is minimized and not amplified*. Proposed ISM dynamics allows to *reduce the dimension* $[Nn]$ *of the min–max control design problem to the space of unmatched uncertainties only of* $[Nn-(N-1)m]$ size. A numerical example illustrates that the suggested modification of the ISM dynamics does not change the min–max control as well as the value of the corresponding performance index. Copyright © 2005 John Wiley & Sons, Ltd.

KEY WORDS: sliding mode control; optimal control; variable structure control

1. INTRODUCTION

1.1. Motivation

Sliding mode control is a powerful nonlinear control technique intensively developed during the last 35 years [1, 2]. A system motion in a sliding surface, named *sliding mode*, turns out to be robust with respect to disturbances and matched uncertainties by a control but seems to be sensitive to unmatched ones. The sliding mode approach consists of two steps [1]: first, the switching surface is constructed in such a manner that the system motion being in sliding mode

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satisfies the design specifications, and, second, a control function is designed to make the switching function attractive to the system dynamics.

The concept of the optimal sliding mode control, formulated in Reference [3], provides an optimal stabilization of sliding dynamics and ensures the insensitivity of designed the control law with respect to the matched uncertainties. In the case of unmatched uncertainties the problem of optimal sliding manifold design cannot be formulated, since an optimal control requires a complete knowledge of system dynamics. Therefore, in this situation another design concept must be developed. The corresponding optimization problem is usually treated as a min–max control dealing with different classes of partially known models [4, 5]. The min–max control problem can be formulated in such a way that the operation of the maximization is taken over a set of uncertainty and the operation of the minimization is taken over the control strategies within a given resource set. In view of this concept, the original system model is replaced by a finite set of dynamic models such that each model describes a particular uncertain case including exact realizations of possible dynamic equations as well as external bounded unmatched disturbances. In Reference [6] the authors developed the concept of *min–max sliding mode* control design for linear time variant multi-model system. This control design technique has the following disadvantages:

- the designed controller ensures optimality after the entrance point into the sliding mode only;
- the trajectory of the designed solution is not robust even with respect to the matched disturbances on a time interval preceding the sliding motion (within a reaching phase).

In References [1, 7–9] a new sliding mode design concept, namely integral sliding mode (ISM), *without any reaching phase* has been proposed. As a result, the robustness of the trajectory for a nominal system can be guaranteed throughout an entire response of the system starting from the initial time instant. The main disadvantage of ISM is the following: ISM does not have the decomposition property typical for sliding mode controllers since the trajectory robustification requires designing the control law in the complete state space.

In Reference [10] both ISM and min–max approaches are brought together for *linear* time variant *multi-model systems with uncertainties*. It allows

- to design the nominal system only taking into account the unmatched uncertainties;
- to ensure the insensitivity of the designed min–max control law with respect to matched uncertainties starting from the initial time instant.

On the other hand, the direct usage of ISM in Reference [10] requires designing the min–max control law in the space of extended variable with the dimension equal to the product of the state vector's dimension (**n**) multiplied by the number of scenarios (**N**), that is, the multi-model optimal problem was solved in the space of **nN**-order.

1.2. Main contribution

In this paper the concept of the ISM for *linear* ***time invariant*** *multi-model uncertain systems* is modified allowing

- *to reduce the dimension of min–max multi-model control design problem* (*originally equal to* $n \cdot N$) *up to the space of unmatched uncertainties by* $[Nn-(N-1)m]$*-dimension* (*m is the dimension of the control vector*);
- *to ensure insensitivity of the proposed controller with respect to unmatched uncertainties.*

Moreover, it is shown that proposed ISM design

- does not amplify the unmatched uncertainties in the sense that its Euclidian norm is not bigger than the Euclidian norm of the original unmatched perturbation;
- ensures that the Euclidian norm of the performed unmatched perturbation is minimal with respect to the class of the preliminary state-space transformations suggested in this paper.

2. PROBLEM STATEMENT

Let us consider a controlled linear uncertain system

$$\dot{x}(t) = Ax(t) + Bu(t) + \zeta(t), \quad x(0) = x^0 \tag{1}$$

where $x(t) \in R^n$ is the state vector at time $t \in [0, t_1]$, $u(t) \in R^m$ is a control action, ζ is external disturbance (or uncertainty). We will assume that

1. the matrix A may take a finite number of fixed and *a priori* known matrix functions, that is, $A \in \{A^1, A^2, \ldots, A^N\}$ which is supposed to be bounded, that is,
$$\sup_{\alpha=\overline{1,N}} \|A^\alpha\| \leqslant a \tag{2}$$
the constant matrix B is known, it has a full rank, that is, rank $B = m$ and $B^+ := [B^\mathsf{T} B]^{-1} B^\mathsf{T}$ exist;
2. N is a finite number of *possible dynamic scenarios*;
3. the external disturbances ζ may be represented in the following manner:
$$\zeta(t) = g(x,t) + \xi(t), \quad t \in [0, t_1] \tag{3}$$
where $g(x,t)$ is an unmeasured smooth uncertainty describing perturbations satisfying the so-called '*standard matching condition*', that is $g(x,t) \in \mathrm{span}B$, or, in other words, $g(x,t) \in \Omega$ where
$$\Omega := \{g(x,t) : g(x,t) = B\gamma(x,t), \|\gamma(x,t)\| \leqslant q\|x\| + p, \quad q,p > 0\} \tag{4}$$
and $\xi(t)$ is an unmatched disturbance taking a finite number of alternative functions, that is, $\xi(t) \in \{\xi^1(t), \ldots, \xi^N(t)\} := \Xi$, where $\xi^\alpha(t)$ $(\alpha = 1, \ldots, N)$ are known (smooth enough) bounded functions such that for all $t \in [0, t_1]$
$$\|\xi(t)\| \leqslant \xi^+ \tag{5}$$
So, for each concrete realization α of possible scenarios we obtain the following dynamics:
$$\dot{x}^\alpha(t) = A^\alpha x^\alpha(t) + Bu(t) + g(x^\alpha, t) + \xi^\alpha(t), \quad x^\alpha(0) = x^0 \tag{6}$$

2.1. The control design challenge

Now the control design problem can be formulated as follows: *design the control $u = u(t)$ in the following form*:

$$u(t) = u_0(t) + u_1(x,t) \tag{7}$$

where $u_1(x,t)$ is the '*integral sliding mode*' control part, providing:

- the complete compensation of the unmeasured matched uncertainly $g(x,t)$ *starting from initial time* $(t_{\mathrm{comp}} = 0)$,

- that the dynamics of the matched part of the system will depend on control component $u_0(t)$ only (the control function $u_0(t)$ is the control which minimized a performance index defined below).

The *main goal* is to design the control law ensuring

- robustness of the system with respect to the matched perturbations. This is done by the control u_1;
- the reduction and minimization of the norm of the unmatched perturbations;
- u_0 dealing with the reduced dimension of the extended system without changing the value of the performance index.

Substitution of the control law (7) and (3) into system (1) yields

$$\dot{x}(t) = Ax(t) + Bu_0(t) + Bu_1(x,t) + g(x,t) + \xi(t), \quad x(0) = x^0 \tag{8}$$

2.2. *ISM control design*

2.2.1. Projection matrix design. Let us define the auxiliary sliding function

$$s = \sigma(t) + Gx(t) \tag{9}$$

where $\sigma(t)$ is some auxiliary variable and G is a projection matrix defined bellow. Then,

$$\dot{s}(t) = \dot{\sigma}(t) + G[Ax(t) + Bu_0(t) + Bu_1(x,t) + g(x,t) + \xi(t)]$$

Suppose that det $GB \neq 0$ and we wish to enforce the sliding mode on the surface $s = 0$ via ISM controller u_1. To find *ISM* dynamics one has

$$u_{1\text{eq}} = -[GB]^{-1}G[Ax(t) + \xi(t)] - u_0(t) - \gamma - [GB]^{-1}\dot{\sigma}(t)$$

The corresponding *ISM*-dynamics equation has the form

$$\dot{x}(t) = [I - B(GB)^{-1}G]\ [Ax(t) + \xi(t)] - B(GB)^{-1}\dot{\sigma}(t) \tag{10}$$

Let us design such a projection matrix G which

- does not amplify the unmatched uncertainties $\xi_{\text{eq}}(t) = [I - B(GB)^{-1}G]\xi(t)$ in the sense that its Euclidian norm is not bigger than the Euclidian norm of the original unmatched perturbation;
- ensures that the Euclidian norm $\xi_{\text{eq}}(t)$ of the performed unmatched perturbation is minimal over all admissible transformations G.

Lemma 1
B^+ is the matrix minimizing the norm of $\xi_{\text{eq}}(t)$, i.e.

$$B^+ = \arg \min_{G \in \Re^{m \times n}} \|[I - B(GB)^{-1}G]\xi(t)\|_2 \tag{11}$$

Proof
Remark that

$$\|[I - B(GB)^{-1}G]\xi(t)\|_2 = \|\xi(t) - B(GB)^{-1}G\xi(t)\|_2 = \|\xi(t) - B\varphi\|_2$$

where $\varphi = (GB)^{-1}G\xi(t)$. Thus, problem (11) can be rewritten in the form:

$$\varphi_0 = \arg \min_{\varphi \in \Re^m} \|\xi(t) - B\varphi\|_2$$

which has $\varphi_0 = B^+\xi(t)$ as the solution (see Reference [11]). Taking $G = B^+$ and in view of the relations $B^+BB^+ = B^+$ and $(B^+B)^{-1}B^+ = B^+$ we obtain

$$(GB)^{-1}G\xi(t) = B^+\xi(t) = \varphi_0$$

that implies (11). □

Lemma 2

The unmatched perturbation $\xi_{\text{eq}}(t) = [I - BB^+]\xi(t)$ is not amplified, i.e.

$$\|[I - BB^+]\|_2 = 1$$

Proof

Let $\mu(D)$ be the largest eigenvalue of D and $v(D)$ the smallest eigenvalue of D. Denote the Euclidian norm of a real matrix as

$$\|D\|_2 = (\text{largest eigenvalue of} D^{\mathrm{T}}D)^{1/2} = (\mu(D^{\mathrm{T}}D))^{1/2}$$

Then, we have

$$\|[I - BB^+]\|_2 = (\mu([I - BB^+]^{\mathrm{T}}[I - BB^+]))^{1/2}$$

Since $[I - BB^+]^{\mathrm{T}}[I - BB^+] = [I - BB^+]$, and in view of the properties of eigenvalues $(I + D)x = (1 + \lambda)x$ (λ is an eigenvalue of D) we get

$$\|[I - BB^+]\|_2 = (\mu(I - BB^+))^{1/2} = (1 - v(BB^+))^{1/2} \tag{12}$$

Now, let λ be any eigenvalue of BB^+ and, in view of the fact that the matrix $(B^{\mathrm{T}}B)^{-1}$ can be represented as $(B^{\mathrm{T}}B)^{-1/2}(B^{\mathrm{T}}B)^{-1/2}$, one may obtain the following:

$$\lambda x = BB^+x = B(B^{\mathrm{T}}B)^{-1}B^{\mathrm{T}}x$$

$$\lambda x^{\mathrm{T}}x = x^{\mathrm{T}}B(B^{\mathrm{T}}B)^{-1/2}(B^{\mathrm{T}}B)^{-1/2}B^{\mathrm{T}}x$$

$$= \|(B^{\mathrm{T}}B)^{-1/2}B^{\mathrm{T}}x\|^2 \geqslant 0$$

that means that $\lambda \geqslant 0$. The matrix BB^+ is singular, that is why at least one eigenvalue is equal to zero, hence, $v(BB^+) = 0$. Then, from (12) it follows that

$$\|[I - BB^+]\|_2 = 1$$ □

Remark 1

Lemmas 1 and 2 was firstly proved in Reference [9].

2.2.2. Transformation of the state. Now, taking in account Lemmas 1, 2, let us transform system (8) into two subsystems using the co-ordinates corresponding to the matched and unmatched parts of uncertainties. Define the following non-singular transformation:

$$T := \begin{bmatrix} B^{\perp} \\ B^+ \end{bmatrix}$$

where $B^{\perp} \in \Re^{(n-m)\times n}$ is a matrix which is composed by the transposition of a basis of the orthogonal space of B. Since rank $(B) = m$, then rank $(B^{\perp}) = n - m$.

Applying the transformation T to system (8) one get

$$z(t) = \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} := Tx(t) = \begin{bmatrix} B^{\perp}x(t) \\ B^{+}x(t) \end{bmatrix}$$

and

$$\dot{z}(t) = \begin{bmatrix} \dot{z}_1(t) \\ \dot{z}_2(t) \end{bmatrix} = \begin{bmatrix} B^{\perp}\dot{x}(t) \\ B^{+}\dot{x}(t) \end{bmatrix} \tag{13}$$

2.2.3. ISM surface design. Redefine the auxiliary '*sliding*' function $s(x,t) \in R^m$ as

$$s(x,t) = \sigma(t) + z_2 = \sigma(t) + B^{+}x(t) \tag{14}$$

where $G = B^{+}$ and $\sigma(x,t)$ is an auxiliary variable which will be defined bellow. Then, since $B^{+}B = I$, it follows that

$$\dot{s}(x,t) = \dot{\sigma}(x,t) + B^{+}Ax(t) + u_0(t) + u_1(x,t) + B^{+}g(x,t) + B^{+}\xi(t) \tag{15}$$

The next step is to select the auxiliary variable σ as the solution to the following Cauchy problem:

$$\dot{\sigma}(t) = -u_0(t), \quad \sigma(0) = -z_2(0) \tag{16}$$

Then, the equation for the slack function $s(x,t)$ becomes as

$$\begin{aligned} \dot{s}(x,t) &= B^{+}[Ax(t) + Bu_1(x,t) + B\gamma(x,t) + \xi(t)] \\ s(x(0),0) &= 0 \end{aligned} \tag{17}$$

In order to realize a *sliding mode dynamics*, let us design the relay control in the form

$$\begin{aligned} u_1(x,t) &= -M(x)\frac{s(t)}{\|s(t)\|} \\ M(x) &= \bar{q}\|x(t)\| + \bar{p}, \quad \bar{q} \geqslant q + b^{+}a, \quad \bar{p} \geqslant p + b^{+}\xi^{+} \end{aligned} \tag{18}$$

(q, a, p, , ξ^{+} were defined in (2), (4) and (5), $b^{+} := \|B^{+}\|$), that implies

$$\dot{s}(x,t) = \gamma(x,t) - M(x)\frac{s(t)}{\|s(t)\|} + B^{+}(Ax(t) + \xi(t)) \tag{19}$$

2.2.4. ISM stability. For the Lyapunov function $V = \frac{1}{2}\|s\|^2$, in view of (4), (2) and (19), it follows that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}V &= \left(s, \gamma(x,t) - M(x)\frac{s(t)}{\|s(t)\|}\right) + (s, B^{+}(Ax(t) + \xi(t))) \\ &\leqslant -\|s\|\,[M(x) - \|\gamma(x,t)\|] - \|s\|\,[-b^{+}(\|A\|\,\|x(t)\| + \xi^{+})] \\ &\leqslant -\|s\|\,[(\bar{q} - q - b^{+}a)\,\|x(t)\|] - \|s\|\,[\bar{p} - p - b^{+}\xi^{+}] \leqslant 0 \end{aligned}$$

So, by (16), it follows that

$$V(s(x(t),t)) \leqslant V(s(x(0),0)) = \tfrac{1}{2}\|s(x(0),0)\|^2 = 0$$

that implies for all $t \in [0, t_1]$ the following identities:

$$s(t) = 0, \quad \dot{s}(t) = 0 \tag{20}$$

This means that *the ISM control* (18) *completely compensates the effect of the matched uncertainty g from the beginning of the process.*

2.3. Nominal system design

Taking into account (17), we will find the equivalent control (maintaining the dynamics over a sliding surface) for ISM dynamics as follows:

$$u_{1\text{eq}} = -B^{+}[Ax(t) + B\gamma(x(t),t) + \xi(t)]$$

Applying $u_{1\text{eq}}$ to (8) we obtain the nominal system as the ISM dynamics in the following form:

$$\dot{x}_0(t) = A_{\text{eq}}x_0(t) + Bu_0(t) + \xi_{\text{eq}}(t) \tag{21}$$

where

$$A_{\text{eq}} = [I - BB^{+}]A \quad \text{and} \quad \xi_{\text{eq}}(t) = [I - BB^{+}]\xi(t)$$

Applying $u_{1\text{eq}}$ to (13), one get

$$\begin{aligned}
\dot{z}_0(t) &= [TA_{\text{eq}}T^{-1}z_0(t) + TBu_0(t) + T\xi_{\text{eq}}(t)] \\
&= \begin{bmatrix} \dot{z}_{10}(t) \\ \dot{z}_{20}(t) \end{bmatrix} = \begin{bmatrix} [B^{\perp}AT^{-1}z_0(t) + B^{\perp}\xi(t)] \\ u_0(t) \end{bmatrix} \\
&:= \begin{bmatrix} A_{e1} & A_{e2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} z_{10}(t) \\ z_{20}(t) \end{bmatrix} + \begin{bmatrix} \xi_{e1}(t) \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ u_0(t) \end{bmatrix}
\end{aligned} \tag{22}$$

which may be called *the transformed nominal system*. One can see that the state vector $\dot{z}_{20}(t)$ in (22) does not depend on the state vector z_0, but depends on control u_0 only.

2.4. The corrected LQ-index

Let us apply the min–max approach [4, 5] to the nominal system (21), allowing to obtain the control $u_0(x)$ as *a control function minimizing the worst LQ-index over a finite horizon* t_1, *that is*

$$\min_{u_0 \in R^m} \max_{\alpha=\overline{1,N}} h^{\alpha} \tag{23}$$

where

$$\begin{aligned}
h^{\alpha} :=& \frac{1}{2}(x_0^{\alpha}(t_1), Lx_0^{\alpha}(t_1)) + \frac{1}{2}\int_{t=0}^{t_1} [(x_0^{\alpha}(t), Qx_0^{\alpha}(t)) \\
&+ [u_0(t) - (B^{+}A^{\alpha}x_0^{\alpha}(t)), R(u_0(t) - B^{+}A^{\alpha}x_0^{\alpha}(t))]]\,\mathrm{d}t
\end{aligned}$$

$$L = L^{\top} \geqslant 0, \quad Q = Q^{\top} \geqslant 0, \quad R = R^{\top} > 0$$

Since $z(t) = Tx(t)$ and $x(t) = T^{-1}z(t)$, the *LQ-index* h^{α} *can be represented as*

$$h^{\alpha} := \frac{1}{2}(z_0^{\alpha}(t_1), (T^{\top})^{-1}LT^{-1}z_0^{\alpha}(t_1)) + \frac{1}{2}\int_{t=0}^{t_1} [(z_0^{\alpha}(t), (T^{\top})^{-1}QT^{-1}z_0^{\alpha}(t)) + [u_0(t) - (B^{+}A^{\alpha}T^{-1}z_0^{\alpha}(t)), R(u_0(t) - B^{+}A^{\alpha}T^{-1}z_0^{\alpha}(t))]]\,\mathrm{d}t$$

2.5. *Minimax multi-model control design*

Following to [4, 5] consider the extended system

$$\dot{\mathbf{x}} = \mathbf{A}_{\mathrm{eq}}\mathbf{x} + \mathbf{B}u_0 + \mathbf{d} \tag{24}$$

where

$$\mathbf{x}^{\mathrm{T}} = [\,\mathbf{x}_0^{1\mathrm{T}} \;\cdots\; \mathbf{x}_0^{N\mathrm{T}}\,], \quad \mathbf{A}_{\mathrm{eq}} := \mathrm{diag}\;(A_{\mathrm{eq}}^1, \ldots, A_{\mathrm{eq}}^N), \quad \mathbf{x} \in R^{N\cdot n}$$
$$\mathbf{B}^{\top} := [\,B^{\top} \;\cdots\; B^{\top}\,], \quad \mathbf{d}^{\top} := [\,\xi_{\mathrm{eq}}^{1\mathrm{T}} \;\cdots\; \xi_{\mathrm{eq}}^{N\mathrm{T}}\,] \tag{25}$$

Using the previous extended system and according to Poznyak *et al.* [5, 10] the control u_0, denoted below by u_{0_x} to emphasize that it is designed before any state-space transformation application), is

$$u_{0_x} = -R^{-1}\mathbf{B}^{\mathsf{T}}[\mathbf{P}_{\lambda}\mathbf{x} + \mathbf{p}_{\lambda}] + \mathbf{B}^{+}\mathbf{A}\boldsymbol{\Lambda}\mathbf{x} \tag{26}$$

where the matrix $\mathbf{P}_{\lambda} = \mathbf{P}_{\lambda}^{\top} \in R^{nN\times nN}$ is the solution to the parametrized differential matrix Riccati equation:

$$\dot{\mathbf{P}}_{\lambda} + \mathbf{P}_{\lambda}(\mathbf{A}_{\mathrm{eq}} + \mathbf{B}\mathbf{B}^{+}\mathbf{A}\boldsymbol{\Lambda}) + (\mathbf{A}_{\mathrm{eq}} + \mathbf{B}\mathbf{B}^{+}\mathbf{A}\boldsymbol{\Lambda})^{\mathsf{T}}\mathbf{P}_{\lambda} - \mathbf{P}_{\lambda}\mathbf{B}R^{-1}\mathbf{B}^{\mathsf{T}}\mathbf{P}_{\lambda} + \boldsymbol{\Lambda}(\mathbf{Q}_{\mathrm{eq}} - (\mathbf{B}^{+}\mathbf{A})^{\top}R\mathbf{B}^{+}\mathbf{A}\boldsymbol{\Lambda}) = 0; \quad \mathbf{P}_{\lambda}(t_1) = \boldsymbol{\Lambda}\mathbf{L} \tag{27}$$

and the shifting vector $\mathbf{p}_{\lambda}$ satisfies

$$\dot{\mathbf{p}}_{\lambda} + (\mathbf{A}_{\mathrm{eq}} + \mathbf{B}\mathbf{B}^{+}\mathbf{A}\boldsymbol{\Lambda})^{\mathsf{T}}\mathbf{p}_{\lambda} - \mathbf{P}_{\lambda}\mathbf{B}R^{-1}\mathbf{B}^{\mathsf{T}}\mathbf{p}_{\lambda} + \mathbf{P}_{\lambda}\mathbf{d} = \mathbf{0}$$
$$\mathbf{p}_{\lambda}(t_1) = 0 \tag{28}$$

with the matrices defined as

$$\mathbf{A} := \mathrm{diag}\;(A^1, \ldots, A^N), \quad \mathbf{Q}_{\mathrm{eq}} := \mathrm{diag}\;(Q^1, \ldots, Q^N)$$
$$\mathbf{L} := \mathrm{diag}\;(L, \ldots, L), \quad \boldsymbol{\Lambda} := \mathrm{diag}\;(\lambda_1 I_{n\times n}, \ldots, \lambda_N I_{n\times n})$$
$$Q^{\alpha} = Q + [B^{+}(t)A^{\alpha}(t)]^{\top}RB^{+}(t)A^{\alpha}(t)$$

Now consider the extend system using $z_0(t)$

$$\dot{\mathbf{z}} = \mathbf{T}\mathbf{A}_{\mathrm{eq}}\mathbf{T}^{-1}\mathbf{z} + \mathbf{T}\mathbf{B}u_0 + \mathbf{T}\mathbf{d} \tag{29}$$

where

$$z^{\mathrm{T}} = \left[\,z_0^{1\mathrm{T}} \;\cdots\; z_0^{N\mathrm{T}}\,\right], \quad \mathbf{T} = \begin{bmatrix} T & 0\cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & T \end{bmatrix}$$

By (29) the control u_0 (denoted by u_{0_z} to emphasize that it is designed after the T-transformation application), is as follows:

$$u_{0_z} = -R^{-1}(\mathbf{TB})^{\mathsf{T}}[\mathbf{S}_\lambda \mathbf{z} + \mathbf{s}_\lambda] + \mathbf{B}^{+}\mathbf{A}\mathbf{T}^{-1}\mathbf{\Lambda z} \tag{30}$$

where the matrix $\mathbf{S}_\lambda = \mathbf{S}_\lambda^{\mathsf{T}} \in R^{nN\times nN}$ is the solution to the parametrized differential matrix Riccati equation:

$$\begin{aligned}\dot{\mathbf{S}}_\lambda + \mathbf{S}_\lambda(\mathbf{TA}_{\mathrm{eq}}\mathbf{T}^{-1} + \mathbf{TBB}^{+}\mathbf{AT}^{-1}\mathbf{\Lambda}) + (\mathbf{TA}_{\mathrm{eq}}\mathbf{T}^{-1} + \mathbf{TBB}^{+}\mathbf{AT}^{-1}\mathbf{\Lambda})^{\mathsf{T}}\mathbf{S}_\lambda \\ - \mathbf{S}_\lambda\mathbf{TB}R^{-1}(\mathbf{TB})^{\mathsf{T}}\mathbf{S}_\lambda + \mathbf{\Lambda}(\mathbf{T}^{\top})^{-1}(\mathbf{Q}_{\mathrm{eq}} - (\mathbf{B}^{+}\mathbf{A})^{\top}R\mathbf{B}^{+}\mathbf{A\Lambda})\mathbf{T}^{-1} = 0 \\ \mathbf{S}_\lambda(t_1) = \mathbf{\Lambda}(\mathbf{T}^{\top})^{-1}\mathbf{LT}^{-1}\end{aligned} \tag{31}$$

$$\begin{aligned}\dot{\mathbf{s}}_\lambda + (\mathbf{TA}_{\mathrm{eq}}\mathbf{T}^{-1} + \mathbf{TBB}^{+}\mathbf{AT}^{-1}\mathbf{\Lambda})^{\mathsf{T}}\mathbf{s}_\lambda - \mathbf{S}_\lambda\mathbf{TB}R^{-1}(\mathbf{TB})^{\mathsf{T}}\mathbf{s}_\lambda + \mathbf{S}_\lambda\mathbf{Td} = \mathbf{0} \\ \mathbf{s}_\lambda(t_1) = 0\end{aligned} \tag{32}$$

Lemma 3

The controls u_{0_x} (26), designed for system (24) and u_{0_z} for systems (29) and (30), are identical, that is

$$u_{0_z} = u_{0_x} \triangleq u_0 \tag{33}$$

Proof

Equation (33) is true if

$$-R^{-1}\mathbf{B}^{\mathsf{T}}[\mathbf{P}_\lambda\mathbf{x} + \mathbf{p}_\lambda] + \mathbf{B}^{+}\mathbf{A\Lambda x} = -R^{-1}(\mathbf{TB})^{\mathsf{T}}[\mathbf{S}_\lambda\mathbf{z} + \mathbf{s}_\lambda] + \mathbf{B}^{+}\mathbf{AT}^{-1}\mathbf{\Lambda z}$$

Since $\mathbf{T\Lambda} = \mathbf{\Lambda T}$ by the triangularity of both multipliers, it implies

$$\mathbf{P}_\lambda = \mathbf{T}^{\mathsf{T}}\mathbf{S}_\lambda\mathbf{T} \quad \text{and} \quad \mathbf{p}_\lambda = \mathbf{T}^{\mathsf{T}}\mathbf{s}_\lambda \tag{34}$$

and, of course, if (34) is true, then equality (33) is satisfied. That is why, in order to prove (33) it is necessary and sufficient to prove (34). Premultiplying (31) by $\mathbf{T}^{\mathsf{T}}$ and postmultiplying by $\mathbf{T}$ we obtain

$$\begin{aligned}\mathbf{T}^{\mathsf{T}}\dot{\mathbf{S}}_\lambda\mathbf{T} + \mathbf{T}^{\mathsf{T}}\mathbf{S}_\lambda\mathbf{T}(\mathbf{A}_{\mathrm{eq}} + \mathbf{BB}^{+}\mathbf{A\Lambda}) + (\mathbf{A}_{\mathrm{eq}} + \mathbf{BB}^{+}\mathbf{A\Lambda})^{\mathsf{T}}\mathbf{T}^{\mathsf{T}}\mathbf{S}_\lambda\mathbf{T} \\ - \mathbf{T}^{\mathsf{T}}\mathbf{S}_\lambda\mathbf{TB}R^{-1}\mathbf{B}^{\top}\mathbf{T}^{\mathsf{T}}\mathbf{S}_\lambda + \mathbf{T} + \mathbf{\Lambda}(\mathbf{Q}_{\mathrm{eq}} - (\mathbf{B}^{+}\mathbf{A})^{\top}R\mathbf{B}^{+}\mathbf{A\Lambda}) = 0 \\ \mathbf{T}^{\mathsf{T}}\mathbf{S}_\lambda(t_1)\mathbf{T} = \mathbf{\Lambda L}\end{aligned}$$

This differential Riccati equation is equal to (27) with $\mathbf{P}_\lambda = \mathbf{T}^{\mathsf{T}}\mathbf{S}_\lambda\mathbf{T}$. Now postmultiplying (32) by $\mathbf{T}^{\mathsf{T}}$ gives

$$\begin{aligned}\mathbf{T}^{\mathsf{T}}\dot{\mathbf{s}}_\lambda + (\mathbf{A}_{\mathrm{eq}} + \mathbf{BB}^{+}\mathbf{A\Lambda})^{\mathsf{T}}\mathbf{T}^{\mathsf{T}}\mathbf{s}_\lambda - \mathbf{T}^{\mathsf{T}}\mathbf{S}_\lambda\mathbf{TB}R^{-1}\mathbf{B}^{\mathsf{T}}\mathbf{T}^{\mathsf{T}}\mathbf{s}_\lambda + \mathbf{T}^{\mathsf{T}}\mathbf{S}_\lambda\mathbf{Td} = \mathbf{0} \\ \mathbf{T}^{\mathsf{T}}s_\lambda(t_1) = 0\end{aligned}$$

The previous equation is equal to (28) with $\mathbf{p}_\lambda = \mathbf{T}^{\mathsf{T}}\mathbf{s}_\lambda$ and $\mathbf{P}_\lambda = \mathbf{T}^{\mathsf{T}}\mathbf{S}_\lambda\mathbf{T}$. Hence, (34) and, therefore, (33) are proven. □

Since $z^{\alpha}(0) = z^0$ and $z^{\alpha}_{20} = z_{20}$, system (29), by rearranging the components order, can be represented as follows:

$$\dot{\mathbf{z}}_r = \mathbf{A}_r\mathbf{z}_r + \mathbf{B}_r u_0 + \mathbf{d}_r \tag{35}$$

$$\mathbf{z}_r = \begin{bmatrix} z^1_{10} \\ \vdots \\ z^N_{10} \\ z_{20} \end{bmatrix}, \quad \mathbf{A}_r := \begin{bmatrix} A^1_{e1} & 0\ldots & 0 & A^1_{e2} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & 0\ldots & A^N_{e1} & A^N_{e2} \\ 0 & 0\ldots & 0 & 0 \end{bmatrix}$$

$$\mathbf{B}_r^{\top} = [0 \quad \cdots \quad 0 \quad I_{m\times m}], \quad \mathbf{z}_r \in R^{N(n-m)+m}$$

$$\mathbf{d}_r^{\top} = [\,\xi^{1\top}_{e1} \quad \cdots \quad \xi^{N\top}_{e1} \quad 0\,] \tag{36}$$

We note that in (36) we reduce the original (nN)-dimension of the state vector up to $Nn - (N-1)m$. Hence, we can design the control u_0 using system (25), or, using system (36) that seems to be much simpler from the computational point of view.

According to Poznyak *et al.* [4, 5, 10] this control is as follows:

$$u_0 = -R^{-1}\mathbf{B}_r^{\mathsf{T}}[\bar{\mathbf{P}}_{\lambda}\mathbf{z}_r + \bar{\mathbf{p}}_{\lambda}] + \mathbf{F\Lambda z}_r \tag{37}$$

where the matrix $\bar{\mathbf{P}}_{\lambda} = \bar{\mathbf{P}}_{\lambda}^{\top} \in R^{[N(n-m)+m]\times[N(n-m)+m]}$ is the solution to the following parametrized differential matrix Riccati equation:

$$\overset{\Delta}{\bar{\mathbf{P}}}\,\lambda + \bar{\mathbf{P}}_{\lambda}(\mathbf{A_r} + \mathbf{B_r F\Lambda}) + (\mathbf{A_r} + \mathbf{B_r}F\mathbf{\Lambda})^{\mathsf{T}}\bar{\mathbf{P}}_{\lambda} - \bar{\mathbf{P}}_{\lambda}\mathbf{B}_r R^{-1}\mathbf{B_r^{\mathsf{T}}}\bar{\mathbf{P}}_{\lambda}$$
$$+ (\mathbf{\Lambda Q_r} - \mathbf{\Lambda F}^{\top}\mathbf{RF\Lambda}) = 0; \quad \bar{\mathbf{P}}_{\lambda}(t_1) = \mathbf{\Lambda L} \tag{38}$$

and the shifting vector $\bar{\mathbf{p}}_{\lambda} \in R^{N(n-m)+m}$ satisfies

$$\overset{\Delta}{\bar{\mathbf{p}}}\,\lambda + (\mathbf{A_r} + \mathbf{B_r F\Lambda})^{\mathsf{T}}\bar{\mathbf{p}}_{\lambda} - \bar{\mathbf{P}}_{\lambda}\mathbf{B}_r R^{-1}\mathbf{B}_r^{\mathsf{T}}\bar{\mathbf{p}}_{\lambda} + \bar{\mathbf{P}}_{\lambda}\mathbf{d}_r = \mathbf{0}$$
$$\bar{\mathbf{p}}_{\lambda}(t_1) = 0 \tag{39}$$

with

$$\mathbf{F} = [\,F^1_1 \;\ldots\; F^N_1 \quad \lambda_1 F^1_2 + \cdots + \lambda_N F^N_2\,]$$
$$F^{\alpha} := [\,F^{\alpha}_1 \quad F^{\alpha}_2\,] = B^{+}A^{\alpha}T^{-1}, \quad F^{\alpha}_2 \in \Re^{m\times m}$$

and

$$Q := \begin{bmatrix} Q_1 & Q_2 \\ Q_2^{\mathsf{T}} & Q_3 \end{bmatrix}, \quad Q^{\alpha} := \begin{bmatrix} Q^{\alpha}_1 & Q^{\alpha}_2 \\ (Q^{\alpha}_2)^{\top} & Q^{\alpha}_3 \end{bmatrix}$$

$$L := \begin{bmatrix} L_1 & L_2 \\ L_2^{\mathsf{T}} & L_3 \end{bmatrix}$$

$$Q_1,\ L_1 \in R^{(n-m)\cdot(n-m)},\ Q_3,\ L_3 \in R^{m\cdot m}$$

$$Q_1^\alpha = Q_1 + (B_2^{-1}A_{21}^\alpha)^\top R(B_2^{-1}A_{21}^\alpha)$$

$$Q_2^\alpha = Q_2 + (B_2^{-1}A_{21}^\alpha)^\top R(B_2^{-1}A_{22}^\alpha)$$

$$Q_3^\alpha = Q_3 + (B_2^{-1}A_{22}^\alpha)^\top R(B_2^{-1}A_{22}^\alpha)$$

$$\Lambda := \begin{bmatrix} \lambda_1 I_{(n-m)} & 0\ldots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & 0\ldots & \lambda_N I_{(n-m)} & 0 \\ 0 & 0\ldots & 0 & I_{m\times m} \end{bmatrix}$$

$$\Lambda Q_r := \begin{bmatrix} \lambda_1 Q_1^1 & 0\ldots & 0 & \lambda_1 Q_2^2 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & 0\ldots & \lambda_N Q_1^N & \lambda_N Q_2^N \\ \lambda_1 (Q_2^1)^\top & \ldots & \lambda_N (Q_2^N)^\mathrm{T} & \lambda_1 Q_3^1 + \cdots + \lambda_N Q_3^N \end{bmatrix}$$

$$\mathbf{\Lambda L} := \begin{bmatrix} \lambda_1 L_1 & 0\ldots & 0 & \lambda_1 L_2 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & 0\ldots & \lambda_N L_1 & \lambda_N L_2 \\ \lambda_1 L_2^\mathrm{T} & \ldots & \lambda_N L_2^\mathrm{T} & L_3 \end{bmatrix} \tag{40}$$

The matrix $\mathbf{\Lambda} = \mathbf{\Lambda}(\lambda^*)$ is defined by (40) with the weight vector $\lambda = \lambda^*$ solving the following finite-dimensional optimization problem:

$$\lambda^* = \arg\min_{\lambda \in S^N} J(\lambda) \tag{41}$$

$$J(\lambda) := \max_{\alpha=\overline{1,N}} h^\alpha = \frac{1}{2}\mathbf{z}_r^\top(0)\bar{\mathbf{P}}_\lambda(0)\mathbf{z}_r(0) + \mathbf{z}_r^\top(0)\bar{\mathbf{p}}_\lambda(0)$$

$$+\frac{1}{2}\max_{i=\overline{1,N}}\left[\int_0^{t_1} [x_0^{i\mathrm{T}}(t)Q^i x_0^i(t) + 2x_0^{i\mathrm{T}}(t) \times (F^i)^\top(\mathbf{B}_r^\mathrm{T}[\bar{\mathbf{P}}_\lambda \mathbf{z}_r + \bar{\mathbf{p}}_\lambda]\right.$$

$$-R\mathbf{F}\mathbf{\Lambda}\mathbf{z}_r)]\,\mathrm{d}t + x_0^{i\mathsf{T}}(t_1)Lx_0^i(t_1)\Bigg] - \frac{1}{2}\sum_{i=1}^{N}\lambda_i\left[\int_0^{t_1}[x_0^{i\mathsf{T}}(t)Q^ix_0^i(t) + 2x_0^{i\mathsf{T}}(t)\right.$$

$$\left.\times(F^i)^\top(\mathbf{B}_r^\mathsf{T}[\bar{\mathbf{P}}_\lambda\mathbf{z}_r + \bar{\mathbf{p}}_\lambda] - R\mathbf{F}\mathbf{\Lambda}\mathbf{z}_r)]\,\mathrm{d}t + x_0^{i\mathsf{T}}(t_1)Lx_0^i(t_1)\right]$$

$$+\frac{1}{2}\int_{t=0}^{t_1}\bar{\mathbf{p}}_\lambda^\mathsf{T}[2\mathbf{d}_r - \mathbf{B}_rR^{-1}\mathbf{B}_r^\mathsf{T}\bar{\mathbf{p}}_\lambda]\,\mathrm{d}t$$

$$S^N = \left\{\boldsymbol{\lambda}\in\Re^N : \lambda_\alpha \geqslant 0,\quad \sum_{\alpha=1}^{N}\lambda_\alpha = 1\right\}$$

2.6. *Control algorithm description*

We can summarize the designed control algorithm as follows:

1. For a fixed control u_0, construct the, so-called, extended nominal system in the form (29).
2. Create the corrected LQ-index.
3. Design the control u_0 using the extended system (36) and (40).
4. Design the ISM law u_1 completely compensating the matched part of the uncertainties from the beginning of the process.
5. Apply the control $u = u_0 + u_1$ to the closed-loop system (1).

3. EXAMPLE

Let us consider the following system:

$$\dot{x}^\alpha(t) = A^\alpha x^\alpha(t) + Bu(x^\alpha, t) + g(x^\alpha, t) + \xi^\alpha(t)$$

with three possible scenarios ($N = 3$), where

$$A^1 = \begin{bmatrix} -1 & 2 \\ 1.2 & -1.5 \end{bmatrix},\quad A^2 = \begin{bmatrix} 1 & -2 \\ 1.5 & 1 \end{bmatrix},\quad A^3 = \begin{bmatrix} 0.5 & 2.5 \\ -1.5 & 1 \end{bmatrix}$$

$$B^\top = [1 \quad 1],\quad g^\top = [0.8x_1 \quad 0.8x_1],\quad (\xi^1)^\top = [0.25 \quad 0.15] \tag{42}$$

$$(\xi^2)^\top = [0.12 \quad 0.57],\quad (\xi^3)^\top = [0.45 \quad 0.25]$$

Step 1. The nominal system has the following parameters and unmatched uncertainties:

$$\dot{z}_0(t) = [TA_{\mathrm{eq}}T^{-1}z_0(t) + TBu_0(t) + T\xi_{\mathrm{eq}}(t)]$$

where

$$T := \begin{bmatrix} B^\perp \\ B^+ \end{bmatrix} = \begin{bmatrix} -0.7071 & 0.7071 \\ 0.5 & 0.5 \end{bmatrix}$$

$$TA_{\rm eq}^{1}T^{-1} = \begin{bmatrix} -2.85 & -0.9192 \\ 0 & 0 \end{bmatrix}, \quad [T(\xi_{\rm eq}^{1})]^{\rm T} = [\,-0.0707 \;\; 0\,]$$

$$TA_{\rm eq}^{2}T^{-1} = \begin{bmatrix} 1.25 & 2.4749 \\ 0.0 & 0.0 \end{bmatrix}, \quad [T(\xi_{\rm eq}^{2})]^{\rm T} = [\,0.3182 \;\; 0\,]$$

$$TA_{\rm eq}^{3}T^{-1} = \begin{bmatrix} 0.25 & -2.4749 \\ 0.0 & 0.0 \end{bmatrix}, \quad [T(\xi_{\rm eq}^{3})]^{\rm T} = [\,-0.1414 \;\; 0\,]$$

Step 2. Then, now the objective is to design the control u_0 such that

$$\min_{u_0 \in R^m} \max_{\alpha = \overline{1,3}} h^{\alpha}$$

selecting $R = 1,\ Q = I,\ L = I,\ t_1 = 10$. The LQ-index becomes

$$h^{\alpha} := \frac{1}{2}(x_0^{\alpha}(10), x_0^{\alpha}(10)) + \frac{1}{2}\int_{t=0}^{10} [(x_0^{\alpha}(t), x_0^{\alpha}(t)) + (K^{\alpha}x_0^{\alpha}, K^{\alpha}x_0^{\alpha}) + (u_0(t), u_0(t)) - 2(K^{\alpha}x_0^{\alpha}, u_0(t))]\,{\rm d}t$$

$$K^1 := [\,0.1061 \;\; 0.3500\,]x_0^1, \quad K^2 := [\,-1.2374 \;\; 0.7500\,]x_0^2$$

$$K^3 = [\,1.5910 \;\; 1.2500\,]x_0^3$$

Step 3. The control u_0 is designed using the following extended system:

$$\dot{\mathbf{z}}_r = \mathbf{A}_r\mathbf{z}_r + \mathbf{B}_r u_0(\mathbf{z}_r, t) + \mathbf{d}_r$$

$$\mathbf{z}_r^{\top} = [\,z_{10}^1 \;\; z_{10}^2 \;\; z_{10}^3 \;\; z_{20}\,], \quad \mathbf{B}_r^{\top} = [\,0 \;\; 0 \;\; 0 \;\; 1\,]$$

$$\mathbf{A}_r = \begin{bmatrix} -2.85 & 0 & 0 & -0.9192 \\ 0 & 1.25 & 0 & 2.4749 \\ 0 & 0 & 0.25 & -2.4749 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{d}_r^{\top} = [\,-0.0707 \;\; 0.3182 \;\; -0.1414 \;\; 0\,]$$

One can see in Figure 1 that the optimal weights are $\lambda_1^* = 0, \lambda_2^* = 0.1, \lambda_3^* = 0.9$ and the performance index $J(\lambda^*) = 594.6517$.

The control u_0 was calculated as in (24) and also as in (29). In both cases it turned out to be the same. This confirms that the proposed decomposition scheme does not affect the value of $J(\lambda^*)$. In this example the dimension of the extended state vector $\mathbf{z}_r$ of the previous extended system is 4, while the dimension of the state vector $\mathbf{z}$ of the extended system (25) is 6.

Step 4. Design the ISM law of control with $M = (2\|x\| + 0.5)$ (this is only an option, the choose of M depends on the knowledge of the bound of the matched uncertainty), that implies $u_1 = -(2\|x\| + 0.5)\text{sign}[s(t)]$. Note that in $\|x\|, x$ represent the state variable of the realization of system (1).

Step 5. Applying the control $u = u_0 + u_1$ to each one within the set of the different given scenarios we obtain the corresponding state variable dynamics and the control law which are depicted at Figures 1 and 2.

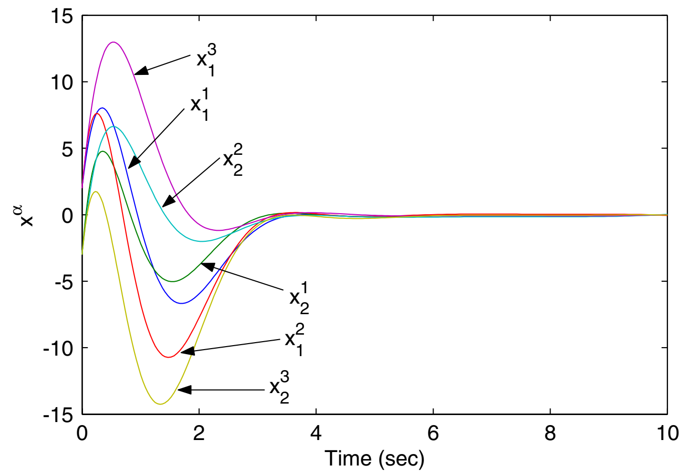

Figure 1. Performance index J and Trajectories of the states variables for system (42).

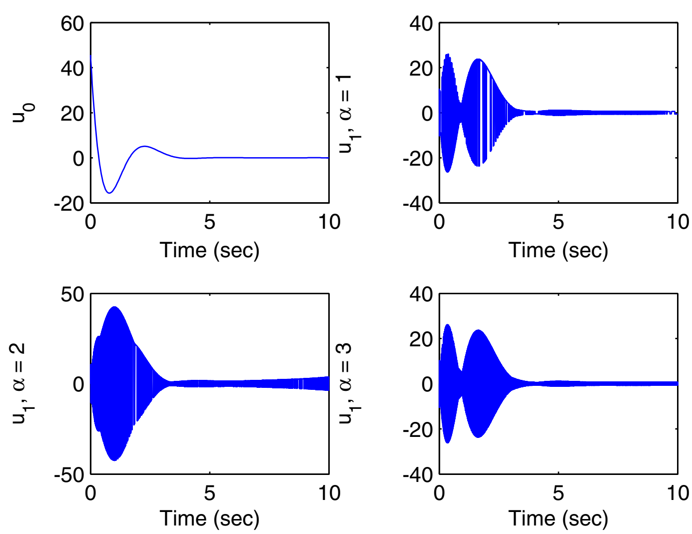

Figure 2. Controls u_0 and u_1 for $\alpha = 1, \alpha = 2$ and $\alpha = 3$.

4. CONCLUSIONS

The *decomposition problem* for the *robust* optimal (min–max) control design is considered for a stationary linear multi-model system with bounded disturbances and uncertainties which are assumed to be partially known. In view of this the methods of integral sliding mode control and min–max robust optimal control are modified. The suggested control law consists of two terms: the integral sliding-mode component and multi-model min–max optimal controller.

The integral sliding-mode component:

- compensates the matching part of the uncertainty *right from the start point of the process*, that is, from $t = 0$;
- does not amplify the modified unmatched perturbation in the sense that its Euclidian norm is not bigger than the Euclidian norm of the original unmatched perturbation;
- minimizes (over all admissible state transformations) the Euclidian norm of the performed unmatched perturbation.

The proposed sliding dynamics design allows:

- to apply the min–max control design taking into account only *the projection of possible perturbations on the space of unmatched uncertainties*,
- *to reduce the original order* $[Nn]$ *of the extended system into* $[Nn - (N-1)m]$ for the min–max problem design.

It is proven and illustrated by the presented example that the suggested version of the ISM dynamics does not modify the robust optimal control and, consequently, the value of the performance index is not modified by the use of the extended system in a lower dimension to designed the control u_0. Sure, that the proposed procedure demands an extra work on verification of the decomposition properties in more complex situations (for example, when only the output of the system is available or when there are external noises in output observations) that might present a formidable problem for the further investigations.

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