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Robust observation and identification of *n*DOF Lagrangian systems

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SUMMARY

A procedure to design a global exponentially stable, second-order, sliding-mode observer for *n*DOF Lagrangian systems is presented. The observer converges to the system state in spite of the existence of bounded disturbances or parameter uncertainties affecting the system dynamics. The generation of sliding modes permits the identification of disturbances using the equivalent output injection which, under some circumstances, can also be used to identify the system parameters *via* a continuous version of the last-square method. The proposed methodology is illustrated with some numerical examples and experiments.

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KEY WORDS: robust observers; variable structure systems; sliding modes; identification; Lagrangian systems

1. INTRODUCTION

Antecedents and motivation. The problem of observation of systems with unknown inputs has been one of the most important problems in control theory during the last two decades [1]. In [2, 3], sufficient and necessary conditions for observer robustness with respect to unknown inputs are established. These conditions require that the unknown inputs must be matched by known outputs, which turn out to be restrictive because they do not include the simplest class of

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mechanical systems with unknown inputs, where only the position is available. In [4], to cover this situation, an adaptive observer, ensuring an exponential convergence of the estimation error to a small neighbourhood of zero, was suggested.

The observation of systems with unknown inputs has been actively developed within variable structure theory using the sliding-mode approach. Sliding-mode observers are widely used due to their attractive features: (a) insensitivity with respect to unknown inputs; (b) possibility to use the equivalent output injection to identify unknown inputs; (c) finite time convergence to exact values of the state vectors (see, for example, the corresponding chapters in the textbooks [5, 6], and the recent tutorials [7–9]). In [10] a step-by-step design of sliding-mode observers was proposed. Such design is based on the transformation of a given system to a block observable form and the sequential estimation of each state by using the equivalent output injection. From the one hand, this scheme allows to formulate extended observability conditions for systems with unknown inputs, covering the observation of mechanical systems with measured positions. Such conditions were formulated in [10, 11] for the scalar case. On the other hand, realization of this scheme caused obligatory filtration due to the non-idealities during sliding-mode generation.

In [12, 13] a robust, exact, arbitrary-order differentiator ensuring finite time convergence to the values of the corresponding derivatives is proposed, and applications of higher-order sliding-mode algorithms were considered. A new generation of observers based on second-order sliding-mode algorithms has been recently designed and applied to some practical problems [14–19]. The main disadvantage of those observers is that they are semiglobal. Some new ideas of usage of equivalent output injections for parameter and disturbance identifications are suggested in [20, 21].

Main contribution. In this paper we present a robust, globally exponentially stable second-order sliding-mode observer for Lagrangian systems. Some specific contributions are enumerated below:

1. A robust globally asymptotically stable second-order sliding-mode observer for Lagrangian systems is proposed, ensuring convergence to the exact system state even under the presence of unknown inputs.
2. An algorithm for unknown input identification is proposed.
3. A modification of the least-square method allowing to identify the system parameters is presented.
4. Some simulations and experimental examples illustrating the main results are given.

Paper structure. The problem statement is given in the second section. In the third section a theorem establishing the stability properties of a class of discontinuous, second-order systems is discussed. This result is essential to prove the exponential convergence of the observer for *n*DOF systems. In the fourth section the observer design is presented, and in the next section the algorithms for perturbation and parameter identifications are described. Finally, in Section 6 the design technique is numerically and experimentally illustrated *via* the design and implementation of an observer for a simple pendulum.

2. PROBLEM STATEMENT

Consider a *n*DOF Lagrangian system described by

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) + \varphi(\ddot{q},\dot{q},q)\theta + \gamma(t) = \tau \tag{1}$$

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where $q \in \Re^n$ is the generalized position vector, $M(q)$ is the inertia matrix, $C(q, \dot{q})$ is the centrifuge and Coriolis matrix, $G(q)$ is the gravitational force vector, the term $\varphi(\ddot{q}, \dot{q}, q)\theta$ encloses the terms produced by the vector of parameter variations θ; $\varphi(\ddot{q}, \dot{q}, q)$ is a $n \times m$ matrix and θ is a $m \times 1$ vector. $\gamma(t) \in \Re^n$ is a bounded vector of external disturbances; finally, $\tau \in \Re^n$ is the generalized force input. All matrices and vectors are defined with the suitable dimensions. We consider that the measured variables are the generalized position q.

Defining the state variables $x_1 = q$, $x_2 = \dot{q}$, the state-space representation of system (1) is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ f(x) + g(x_1) + \xi(\cdot) + M^{-1}(\cdot)\tau \end{bmatrix} \tag{2}$$

$$y = x_1 \tag{3}$$

where

$$f(x) = -M^{-1}(\cdot)C(x_1, x_2)x_2$$

$$g(x_1) = -M^{-1}(\cdot)G(x_1)$$

$$\xi(\cdot) = -M^{-1}(\cdot)(\varphi(\ddot{q}, \dot{q}, q)\theta + \gamma(t))$$

We assume that the behaviour of system (2)–(3) is bounded for any bounded input τ and any bounded perturbation $\gamma(t)$.

In this paper we will design the globally asymptotically stable observer providing the exact value of variables x_2 based on the exact measurements of the variables x_1 and suggest the method for unknown perturbations and parameters identification.

3. GLOBAL EXPONENTIAL STABILITY OF A CLASS OF PERTURBED SECOND-ORDER SYSTEMS

In this section we present a preliminary result that will be useful to design the observer. Consider the following second-order system:

$$\begin{aligned} \dot{v}_1 &= v_2 \\ \dot{v}_2 &= -av_1 - bv_2 + \varepsilon(t) - c\,\mathrm{sign}(v_1) \end{aligned} \tag{4}$$

where a and b are positive constants, $\varepsilon(t)$ is an external perturbation with the bound

$$|\varepsilon(t)| \leqslant \rho_0 \tag{5}$$

where ρ_0 is a constant, c is a control parameter, and sign$(\cdot)$ is the signum function. Define the matrix A as

$$A = \begin{bmatrix} 0 & 1 \\ -a & -b \end{bmatrix} \tag{6}$$

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and the matrix P, which is the solution of the Lyapunov equation $A^{\mathrm{T}}P + PA = -I$ for the (Hurwitz) matrix A, as

$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \tag{7}$$

The stability properties of system (4) are given by the following theorem.

Theorem 1

For system (4), if

$$c > 2\lambda_{\max}(P)\sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}\left(\frac{a\rho_0}{\theta}\right) \tag{8}$$

for some $0<\theta<1$, then the origin of the state space is a globally asymptotically stable equilibrium point in Lyapunov sense.

Proof

The proof is divided in two parts. First we define the nominal system as (4) with $\varepsilon(t) \equiv 0$, and prove the stability of the origin using tools from variable structure systems. After that, we find the condition on c such that the stability properties are maintained for the perturbed system.

The nominal system has two structures: S_1 for $v_1 > 0$,

$$S_1 : \begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \end{bmatrix} = \begin{bmatrix} v_2 \\ -av_1 - bv_2 - c \end{bmatrix}$$

and S_2 for $v_1 < 0$,

$$S_2 : \begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \end{bmatrix} = \begin{bmatrix} v_2 \\ -av_1 - bv_2 + c \end{bmatrix}$$

Each structure has a different equilibrium point; $\bar{v}_{S_1} = (-c/a, 0)$ for S_1, and $\bar{v}_{S_2} = (c/a, 0)$ for S_2. Note that these equilibria are symmetrical and placed in the region where the system dynamics are given by the other structure (S_2 for $\bar{v}_{S_1}$, S_1 for $\bar{v}_{S_2}$). Each equilibrium point is globally asymptotically stable with the following Lyapunov functions; for S_1:

$$V_{S_1}(v) = v^{\mathrm{T}}Pv + 2v^{\mathrm{T}}P\varsigma + \left(\frac{c}{a}\right)^2 p_{11} \tag{9}$$

$$\dot{V}_{S_1}(v) = -v^{\mathrm{T}}v - 2v^{\mathrm{T}}\varsigma - \left(\frac{c}{a}\right)^2 \tag{10}$$

and for S_2

$$V_{S_2}(v) = v^{\mathrm{T}}Pv - 2v^{\mathrm{T}}P\varsigma + \left(\frac{c}{a}\right)^2 p_{11} \tag{11}$$

$$\dot{V}_{S_2}(v) = -v^{\mathrm{T}}v + 2v^{\mathrm{T}}\varsigma - \left(\frac{c}{a}\right)^2 \tag{12}$$

where $\varsigma = [c/a \;\; 0]^{\mathrm{T}}$. Figure 1 shows the graph of the Lyapunov functions defined by (9)–(11).

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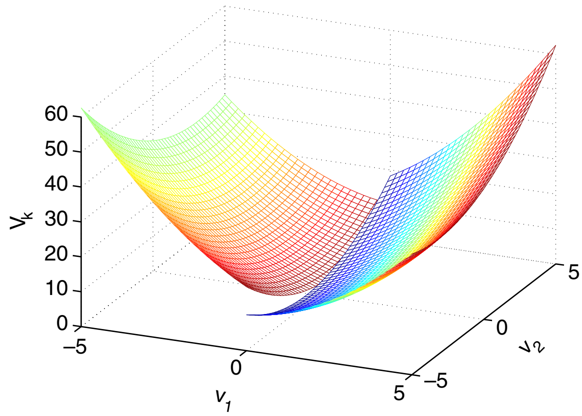

Figure 1. Lyapunov functions of each equilibrium point.

A direct application of the criterion given in [22] allows us to conclude that the discontinuity surface given by $\sigma = v_1 = 0$ is not a sliding surface. Note also that the solutions cross the line $v_1 = 0$ from quadrant II to quadrant I, and from quadrant IV to quadrant III. Functions $V_{S_i}(v)$ intersect at the origin with a value $V_{S_i}(0) = (c/a)^2 p_{11}$, for $i = 1, 2$. Define two neighbourhoods of the origin, Ω_ε with radio $\varepsilon > 0$, and Ω_β defined in the following form:

$$\Omega_\beta = \Omega_1 \cup \Omega_2$$

$$\Omega_1 = \{v \in \Re^2 | v_1 \geqslant 0, V_{S_1}(v) \leqslant \beta\}$$

$$\Omega_2 = \{v \in \Re^2 | v_1 < 0, V_{S_2}(v) \leqslant \beta\}$$

where $\beta > (c/a)^2 p_{11}$. Finally, define a neighbourhood Ω_δ with ratio $\delta < \varepsilon$ (δ can depend on ε and β; $\delta(\varepsilon, \beta)$) such that $\Omega_\delta \subset \Omega_\beta$. Define a set of times $T = \{t_1, t_2, \ldots, t_i, \ldots\}$, where t_i are the times where the system commutes its structures. We assume that $t_1 < t_2 < \cdots$. If $\|v(t_0)\| < \delta$ and $v(t_0) \in \Omega_k \subset \Omega_\beta$ for some $k = 1, 2$ (the kth structure is active), then the first change of structure appears at time t_1, and because $\dot{V}_{S_k} < 0$, we have $\|v(t_0)\| > \|v(t_1)\|$, then $V_{S_k}(v(t_0)) > V_{S_k}(v(t_1))$.

Now $v(t_1)$ is the initial condition for the next structure and, by construction of V_{S_i}, $V_{S_k}(v(t_1)) < V_{S_{k+1}}(v(t_1))$ by a factor $4|v_2(t_1)|p_{12}(c/a)$. The second commutation appears at time t_2; the system goes from Ω_{k+1} to Ω_k, $\|v(t_1)\| > \|v(t_2)\|$, $V_{S_{k+1}}(v(t_1)) > V_{S_{k+1}}(v(t_2))$ and $V_{S_{k+1}}(v(t_2)) > V_{S_k}(v(t_2))$ and so on for all $t_i \in T$; Figure 2 shows this phenomenon.

Then we see that the sequences $W_1 = \{V_{S_k}(t_1), V_{S_k}(t_3), \ldots\}$ and $W_2 = \{V_{S_{k+1}}(t_2), V_{S_{k+1}}(t_4), \ldots\}$ are strictly decreasing and lower bounded and converge to $(c/a)p_{11}$. Also, it is satisfied that $\|v(t_{i+1})\| < \|v(t_i)\| < \cdots < \|v(t_0)\| < \delta < \varepsilon \forall t > t_0$, $\forall i$.

For all $\varepsilon > 0$ and $\beta > (c/a)^2 p_{11}$ we can find a number δ so that the trajectories initiating in Ω_δ will remain within the neighbourhood Ω_ε for all $t \geqslant t_0$. Therefore, the origin is stable in the Lyapunov sense.

To demonstrate asymptotic stability it is enough to note that

$$\lim_{i\to\infty} V_{S_k}(v(t_i)) = \lim_{i\to\infty} V_{S_{k+1}}(v(t_i)) = \frac{c}{a} p_{11}$$

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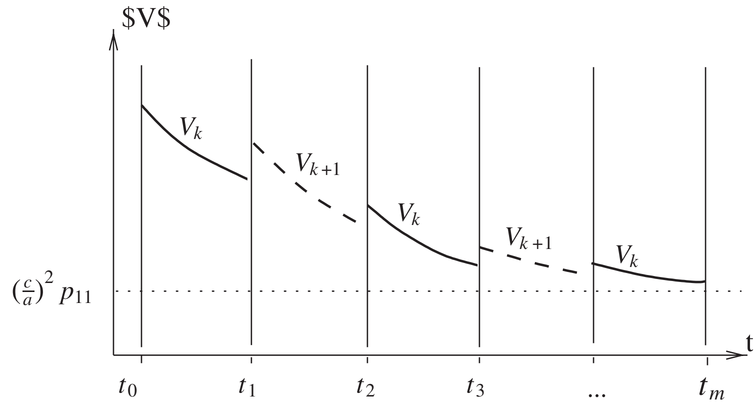

Figure 2. Behaviour of the value of the Lyapunov functions along the time.

this is the value that takes both Lyapunov functions at the origin; then

$$\lim_{t\to\infty} v(t) = 0$$

To demonstrate global asymptotic stability it is enough to note that the equilibrium point of each structure of the nominal system, Equation (4) with $\varepsilon(t) \equiv 0$, is global exponentially stable. Therefore, the behaviour described later will be valid for all initial conditions.

Now we analyse the perturbed system (4). Consider the structure S_1 of the system (the analysis of the structure S_2 is similar),

$$\begin{aligned}\dot{v}_1 &= v_2\\ \dot{v}_2 &= -av_1 - bv_2 + \varepsilon(t) - c\end{aligned}$$

and make the following change of variables $z_1 = v_1 + c/a$ and $z_2 = v_2$. The dynamics of system (4), in the new state space, is given by

$$\begin{aligned}\dot{z}_1 &= z_2\\ \dot{z}_2 &= -az_1 - bz_2 + \varepsilon(t)\end{aligned} \tag{13}$$

or in simplified form,

$$\dot{z} = Az + g \tag{14}$$

where $g = [0 \ \ \varepsilon(t)]^{\mathrm{T}}$. Note that the nominal system has the origin $z = 0$ as an exponentially stable equilibrium point with a Lyapunov function given by $V(z) = z^{\mathrm{T}}Pz$, where $P > 0$ is a solution of the Lyapunov equation $A^{\mathrm{T}}P + PA = -I$. Furthermore, because $\|\varepsilon(t)\| \leqslant \rho_0$, then we can apply the results given in [23] about the stability of perturbed systems.

Propose a Lyapunov function $V(z) = z^{\mathrm{T}}Pz$, where the matrices A and P are defined by Equations (6) and (7), respectively. The time derivative of V is given by

$$\dot{V}(z) = -z^{\mathrm{T}}z + 2z^{\mathrm{T}}Pg \leqslant -\|z\|^2 + 2|z^{\mathrm{T}}Pg| \leqslant -\|z\|^2 + 2\lambda_{\max}(P)\|z\|\rho_0$$

Because $a > 0$ and $b > 0$, we can apply Lemma 9.2 given in [23], and conclude that, for all $\|z(t_0)\| > \mu$, the solution $z(t)$ satisfies

$$\|z(t)\| \leqslant k \exp(-\zeta(t - t_0))\|z(t_0)\| \quad \forall t_0 \leqslant t < t_0 + t_{\mathrm{f}}$$

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and

$$\|z(t)\| \leqslant \mu \quad \forall t \geqslant t_0 + t_f$$

where t_f is a finite time, and

$$k = \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}, \quad \zeta = \frac{(1-\theta)}{2\lambda_{\max}(P)}, \quad \mu = 2\lambda_{\max}(P)\sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}\frac{\rho_0}{\theta}$$

for some θ, $0 < \theta < 1$. This part shows that the ball of radius μ, with centre located at $(-c/a, 0)$, is an attractor for structure S_1, denoted as B_{S_1}. Similarly, the solutions of the structure S_2

$$\begin{aligned} \dot{v}_1 &= v_2 \\ \dot{v}_2 &= -av_1 - bv_2 + \varepsilon(t, v) + c \end{aligned}$$

converge to the ball B_{S_2} of radius μ and centred at $(c/a, 0)$. Therefore, each structure of the perturbed system has an attractor (a ball) of radius μ, symmetrically located on the v_1-axis and at a distance $d = c/a$ from the origin. If this distance d is greater than μ, i.e. if

$$d = \frac{c}{a} > \mu = 2\lambda_{\max}(P)\sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}\left(\frac{\rho_0}{\theta}\right) \tag{15}$$

then the two attractor B_{S_1} and B_{S_2} do not intersect each other, and the behaviour of the solutions of the perturbed system (4) will be qualitatively similar to the behaviour of the nominal system, for which these attractors correspond to the equilibrium points $(\pm c/a, 0)$, that is, when $\mu = 0$. Therefore, the perturbed system converges to the origin in the same way than the nominal system. □

Also, we can prove that the convergence is exponential in a small vicinity of the origin; in fact, we can find exponential functions that are upper and lower bounds of the solutions near the origin; the following theorem presents this result.

Theorem 2

If $c > |\varepsilon(t)|$, $c > |(2/b)\, \mathrm{d}\varepsilon(t)/\mathrm{d}t - \varepsilon(t)|$ for all $t \in (0, \infty)$, system (4) is exponentially stable and the time of convergence is infinite, i.e. there exists a small neighbourhood Λ of the origin and constants $\sigma^-, \sigma^+, L^-, L^+$ such that

$$L^- \mathrm{e}^{-\sigma^- t}(|v_1(0)| + v_2^2(0)) < |v_1(t)| + v_2^2(t) < L^+ \mathrm{e}^{-\sigma^+ t}(|v_1(0)| + v_2^2(0))$$
$$v_1(0), v_2(0) \in \Lambda \quad \forall t \in (0, \infty)$$

Proof

Consider a locally positive definite Lyapunov function:‡

$$E = (c - \varepsilon(t)\,\mathrm{sign}(v_1))|v_1| + v_2^2/2 + bv_1v_2/2$$

In the small neighbourhood of origin the following inequalities are satisfied:

$$\sigma_1(|v_1| + v_2^2/2) \leqslant E \leqslant \sigma_2(|v_1| + v_2^2/2) \quad \text{for some } 0 < \sigma_1 < \sigma_2$$

‡This function is a generalization of the Lyapunov functions considered in [24, 25].

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Computing the derivative of the function E, we have:

$$\begin{aligned}\frac{\mathrm{d}E}{\mathrm{d}t} &= cv_2\,\mathrm{sign}(v_1) - \varepsilon(t)v_2 - v_1\frac{\mathrm{d}\varepsilon(t)}{\mathrm{d}t} - av_1v_2 - bv_2^2 + \varepsilon(t)v_2 \\ &\quad - cv_2\,\mathrm{sign}(v_1) + bv_2^2/2 - bav_1^2/2 - b^2v_1v_2/2 + bv_1\varepsilon(t)/2 - bc|v_1|/2 \\ &= -|v_1|\left(cb/2 + \frac{\mathrm{d}\varepsilon(t)}{\mathrm{d}t}\mathrm{sign}(v_1) - b\varepsilon(t)\,\mathrm{sign}(v_1)/2 + a\,\mathrm{sign}(v_1)v_2 + b^2v_2\,\mathrm{sign}(v_1)/2\right) \\ &\quad - bav_1^2/2 - bv_2^2/2\end{aligned}$$

The function $\mathrm{d}E/\mathrm{d}t$ is locally negative definite under condition $c + (2/b)\,\mathrm{d}\varepsilon(t)/\mathrm{d}t - \varepsilon(t) > 0$ and moreover in some small neighbourhood of the origin we will have

$$\sigma_3(|v_1| + v_2^2/2) \leqslant -\frac{\mathrm{d}E}{\mathrm{d}t} \leqslant \sigma_4(|v_1| + v_2^2/2)$$

and for some $0 < \sigma_5 < \sigma_6$ we will have $\sigma_5 E \leqslant -\mathrm{d}E/\mathrm{d}t \leqslant \sigma_6 E$, concluding the proof of the theorem. □

4. OBSERVER DESIGN

We propose an observer for system (2)–(3) as

$$\begin{bmatrix}\dot{\hat{x}}_1 \\ \dot{\hat{x}}_2\end{bmatrix} = \begin{bmatrix}\hat{x}_2 \\ f(\hat{x}) + g(x_1) + M^{-1}(\cdot)\tau\end{bmatrix} + H(y - \hat{y}) \tag{16}$$

$$\hat{y} = \hat{x}_1 \tag{17}$$

where the vector $H(y - \hat{y})$ has the form

$$H(y - \hat{y}) = \begin{bmatrix}C_1(y - \hat{y}) \\ C_2(y - \hat{y}) + C_3\,\mathrm{sign}(y - \hat{y})\end{bmatrix} \tag{18}$$

where C_1, C_2 and C_3 are definite positive diagonal matrices. Note that $C_3 = 0$ gives a design of a classical Luenberger observer for nonlinear systems.

Define the error variables $e_1 = x_1 - \hat{x}_1$, $e_2 = x_2 - \hat{x}_2$, hence the error dynamics is described by

$$\begin{bmatrix}\dot{e}_1 \\ \dot{e}_2\end{bmatrix} = \begin{bmatrix}e_2 - C_1e_1 \\ f(x) - f(x - e) + \xi(\cdot) - C_2e_1 - C_3\,\mathrm{sign}(e_1)\end{bmatrix} \tag{19}$$

Because function $f(\cdot)$ is Lipschitz, then

$$\|f(x) - f(x - e)\| \leqslant \rho_1\|e\|$$

where ρ_1 is a finite positive constant. We consider that

$$\|\Psi(\cdot)\| = \|f(x) - f(\hat{x}) + \xi(\cdot)\| \leqslant \rho_0 + \rho_1\|e\| \quad \forall e, t \tag{20}$$

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Proposition 3
For system (19) it is possible to find a set of matrices C_1, C_2 and C_3 such that the origin of the error space will be a global exponentially stable equilibrium point. Then the system defined by (16) and (17) is an observer for the system defined by (2) and (3).

Proof
We make a change of variables $v_1 = e_1$ and $v_2 = e_2 - C_1 e_1$. The dynamics of system (19) in the new state space is given by

$$\begin{aligned} \dot{v}_1 &= v_2 \\ \dot{v}_2 &= -C_2 v_1 - C_1 v_2 + \Psi(\cdot) - C_3 \,\mathrm{sign}(v_1) \end{aligned} \tag{21}$$

This system can be seen as a set of two-dimensional subsystems, with the states $v_{1,i}$, $v_{2,i}$, $i = 1, \ldots, n$, with the same form given by (4). Therefore, if $\rho_1 < 1/(2\lambda_{\max}(P))$, where P is a 2×2 matrix that is the solution of the Lyapunov equation for the nominal system of (21) then we can apply Theorem 1 to find the conditions on C_1, C_2 and C_3 such that the origin of system (21) is an exponential stable equilibrium point. □

We can see that the proof is straightforward by taking the result presented in the last section. The matrix inputs $c_{3,i}$ must satisfy Equation (8) $\forall\ i = 1, \ldots, n$.

5. PERTURBATIONS AND PARAMETERS IDENTIFICATION

System (21) has a discontinuity surface in $v_1 = 0$ and the term $C_3 \,\mathrm{sign}(v_1)$ produces a second-order sliding mode, i.e. the discontinuous output injection appears until the second time derivative of the function defining the discontinuity surface

$$\ddot{v}_1 = f(x) - f(\hat{x}) + \xi(\cdot) - C_2 v_1 - C_1 v_2 - u_{\mathrm{eq}} = 0$$

Then, the equivalent output injection is present at $v_1 = v_2 = 0$, which implies that $e_1 = e_2 = 0$ and $x = \hat{x}$; therefore, the equivalent output injection u_{eq} is given by

$$\begin{aligned} u_{\mathrm{eq}} &= \xi(\cdot) \\ &= -M^{-1}(\cdot)(\varphi(\ddot{q}, \dot{q}, q)\theta + \gamma(t)) \end{aligned}$$

We define

$$\Sigma(\cdot) \equiv -M(\cdot) u_{\mathrm{eq}} = \varphi(\ddot{q}, \dot{q}, q)\theta + \gamma(t)$$

We can see that, the equivalent output injection gives the disturbance terms and, as we know, it is the average of the term $C_3 \,\mathrm{sign}(v_1)$ when the trajectories stay at the origin. In this case, the convergence to the discontinuity surface is asymptotic; therefore, we can approximate the perturbation term in asymptotic form as

$$\lim_{t \to \infty} \overline{C_3 \,\mathrm{sign}(z_1(t))} = u_{\mathrm{eq}}$$

where the upper bar denotes the average. A way to estimate this average is by filtering the discontinuous term. A possible implementation is given in Figure 3. If the term $\xi(\cdot)$ does not

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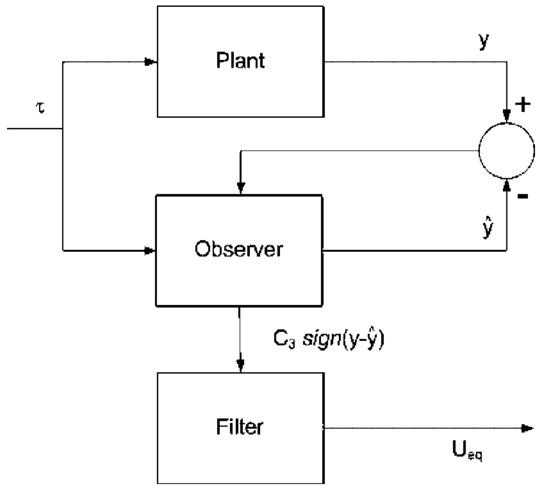

Figure 3. Block diagram to obtain the equivalent output injection.

depend on the parameters, i.e.

$$\xi(\cdot) = \gamma(t)$$

then the disturbance $\gamma(t)$ can be estimated directly from the equivalent output injection, that is,

$$\gamma(t) = -M(\cdot)u_{\mathrm{eq}} = -M(\cdot)\lim_{t\to\infty}\overline{C_3\,\mathrm{sign}(y(t)-\hat{y}(t))}$$

Another case considers that the term $\Sigma(\cdot)$ depends only on parameters, i.e.

$$\Sigma(\cdot) = \varphi(q,\dot{q},\ddot{q})\theta$$

where $\varphi(q,\dot{q},\ddot{q})$ is a $n \times m$ matrix and θ is a $m \times 1$ vector. We can estimate the vector θ from the equivalent output injection using the least-square method (see, for example, [26]). We want to find the vector θ that minimizes

$$J = \frac{1}{t}\int_0^t (v_{\mathrm{eq}} - \Psi(q,\dot{q},\ddot{q})\theta)^{\mathrm{T}}(v_{\mathrm{eq}} - \Psi(q,\dot{q},\ddot{q})\theta)\,\mathrm{d}t$$

where v_{eq} is a $n \times 1$ vector. The optimal solution is given by

$$\theta = \left[\int_0^t \Psi^{\mathrm{T}}(\cdot)\Psi(\cdot)\,\mathrm{d}t\right]^{-1}\int_0^t \Psi^{\mathrm{T}}(\cdot)v_{\mathrm{eq}}\,\mathrm{d}t \tag{22}$$

where the matrix

$$\int_0^t \Psi^{\mathrm{T}}(q,\dot{q},\ddot{q})\Psi(q,\dot{q},\ddot{q})\,\mathrm{d}t$$

must be non-singular. Define a new variable

$$\Gamma_t = \left[\int_0^t \Psi^{\mathrm{T}}(q,\dot{q},\ddot{q})\Psi(q,\dot{q},\ddot{q})\,\mathrm{d}t\right]^{-1} \tag{23}$$

Using the following identities:

$$\Gamma_t^{-1}\Gamma_t = I$$

$$\Gamma_t^{-1}\dot{\Gamma}_t + \dot{\Gamma}_t^{-1}\Gamma_t = 0$$

we have

$$\dot{\Gamma}_t = -\Gamma_t \Psi^{\mathrm{T}}(\cdot)\Psi(\cdot)\Gamma_t \tag{24}$$

A parameter identification algorithm based on the equivalent output injection is given, taking into account (22)–(24), by

$$\dot{\theta} = \Gamma_t \Psi^{\mathrm{T}}(\cdot)(v_{\mathrm{eq}} - \Psi(\cdot)\theta) \tag{25}$$

Taking into account that the matrix $\Gamma_t \Psi^{\mathrm{T}}(\cdot)\Psi(\cdot)$ is Hurwitz, we can conclude that Equations (24) and (25) provide the actual values of parameters.

6. A SIMPLE PENDULUM EXAMPLE

Consider a simple pendulum[§] given by the model

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -ax_2 - b\sin(x_1) + c\tau + \gamma(t) \\ y &= x_1 \end{aligned} \tag{26}$$

where $a = 2.9996^{-2}$, $b = 67.912$, $c = 55.549$ and $\gamma(t)$ is a perturbation term that satisfies the following bound:

$$|\gamma(t)| \leqslant \rho$$

where ρ is a constant. In this case we suppose that $\rho = 1$. Now, the state observer for system (26) is proposed to be

$$\begin{aligned} \dot{\hat{x}}_1 &= \hat{x}_2 + h_1 \\ \dot{\hat{x}}_2 &= -a\hat{x}_2 - b\sin(x_1) + c\tau + h_2 \\ \hat{y} &= \hat{x}_1 \end{aligned}$$

where h_1 and h_2 are given by

$$\begin{aligned} h_1 &= c_1 e_1 \\ h_2 &= c_2 e_1 + c_3\,\mathrm{sign}(e_1) \end{aligned}$$

and the error dynamics between the plant and the observer is

$$\begin{aligned} \dot{e}_1 &= -c_1 e_1 + e_{2[t} \\ \dot{e}_2 &= -c_2 e_1 - ae_2 + \gamma(t) - c_3\,\mathrm{sign}(e_1) \end{aligned}$$

[§]These values were taken from an approximated model of a pendulum manufactured by Mechatronics Systems Inc.

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The change of variables $y_1 = e_1$, $y_2 = -c_1e_1 + e_2$, leads to

$$\dot{y}_1 = y_2$$
$$\dot{y}_2 = -(c_2 + ac_1)y_1 - (c_1 + a)y_2 + \gamma(t) - c_3\,\mathrm{sign}(y_1)$$

We chose $c_1 = 2$, $c_2 = 10$, and $c_3 = 10$. The following figures show some numerical results for system (26) for $\gamma(t) = 0.5\sin(10t)$ and a sine signal as input torque. Figure 4 shows the behaviour of the plant (continuous graphs) and the observer (dashed graphs) with $c_3 = 0$, that is, for the classical Luenberger observer. As we can see, the angle and velocity error are large. When $c_3 = 10$ (Figure 5), the errors go near to zero after a transient. There are small errors due to the discontinuous nature of the observer, Figure 6 shows these errors. We identified two kinds of perturbations: a sine and a square waves (see Figures 7 and 8). We note a good approximation of these perturbations. Finally, we introduced a parameter variation $\Delta_a = -10$ in the parameter a. Figure 9 shows a good identification of this uncertainty Δ_a.

Experimental results. The observer was applied to the mechanical pendulum as shown in Figure 10. The experimental results are shown in the following figures. For $c_3 = 0$ (classical Luenberger observer) the error between the real and the observed angles is, similar to the numerical case, very large, see Figure 11. In this and the following figures the vertical line indicates the time where the input torque τ was applied to the pendulum.

Figure 12 shows the experimental results for the proposed discontinuous observer, with a gain $c_3 = 10$. As we can see, after a transient due to the initial condition, the error is almost zero; it remains in the band of $\pm 2 \times 10^{-3}$ rad, see Figure 13. Note that the magnitude of this error is similar to the numerical simulations. We also applied the estimation procedure to this system. It

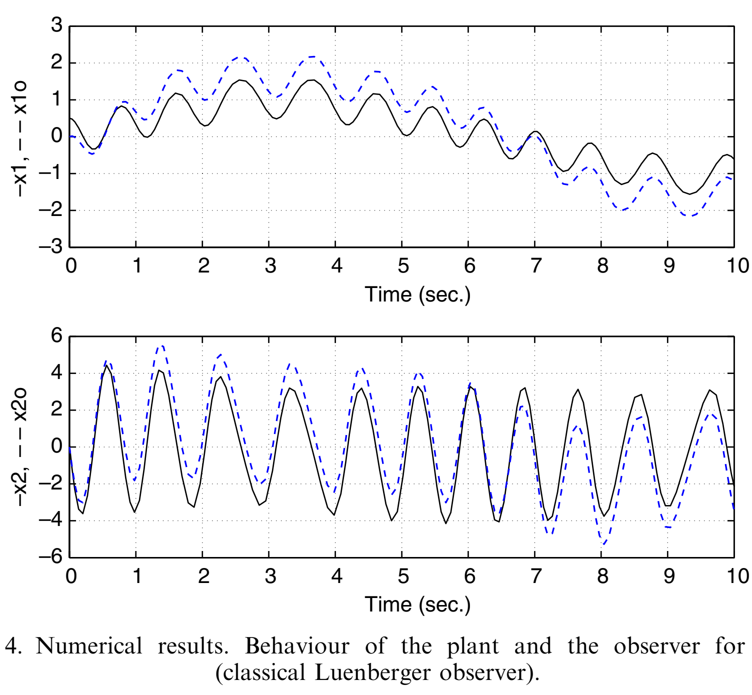

Figure 4. Numerical results. Behaviour of the plant and the observer for $c_3 = 0$ (classical Luenberger observer).

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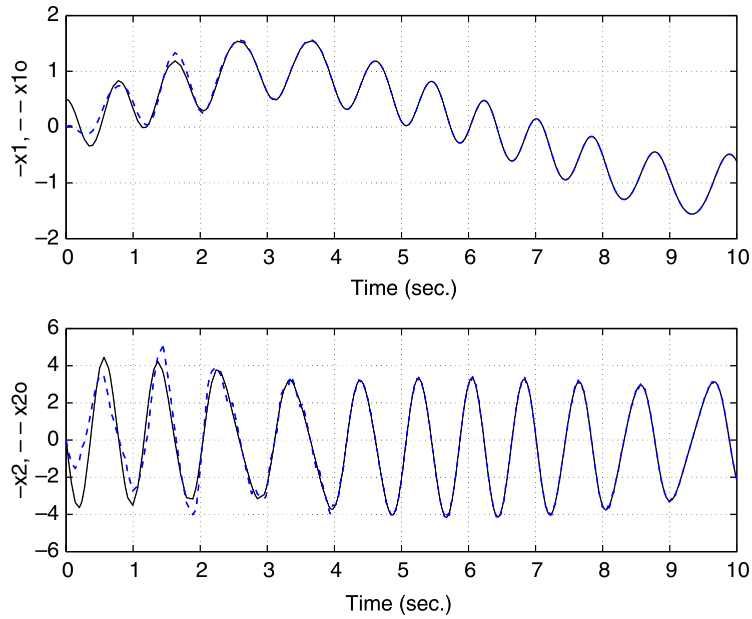

Figure 5. Numerical results. Behaviour of the plant and the observer for $c_3 = 10$ (proposed observer).

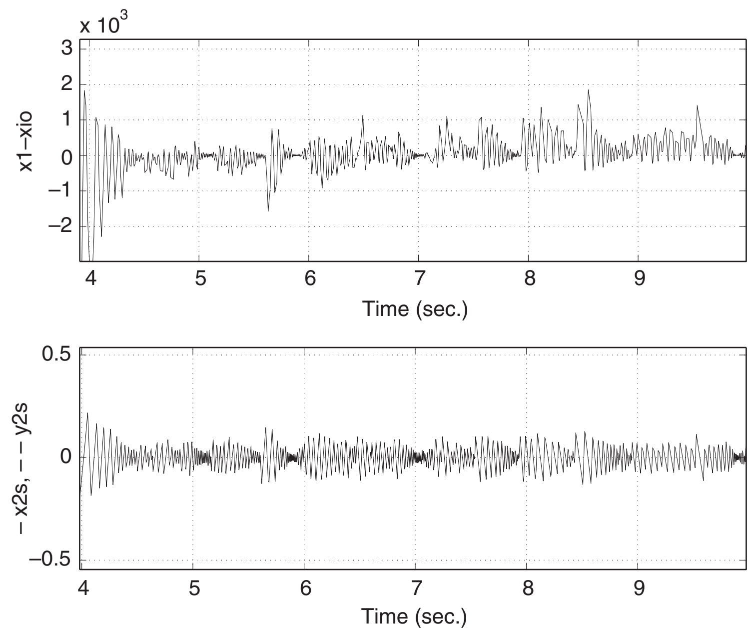

Figure 6. Numerical results. Behaviour of the error between the plant and the observer state for $c_3 = 10$.

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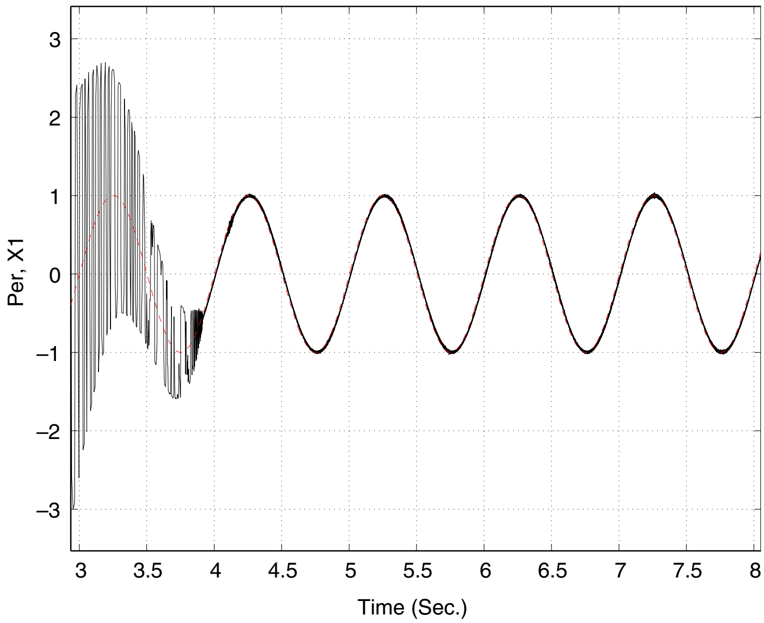

Figure 7. Sinusoidal perturbation signal and perturbation estimation.

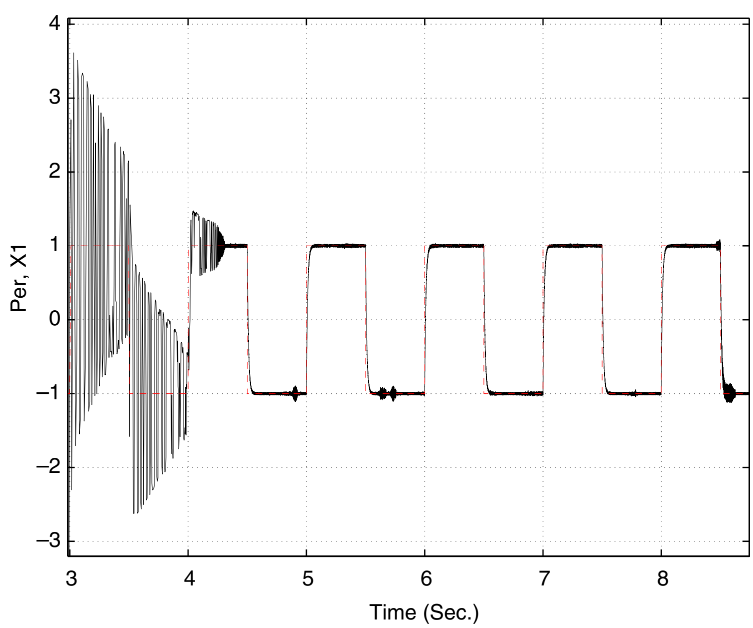

Figure 8. Square perturbation signal and perturbation estimation.

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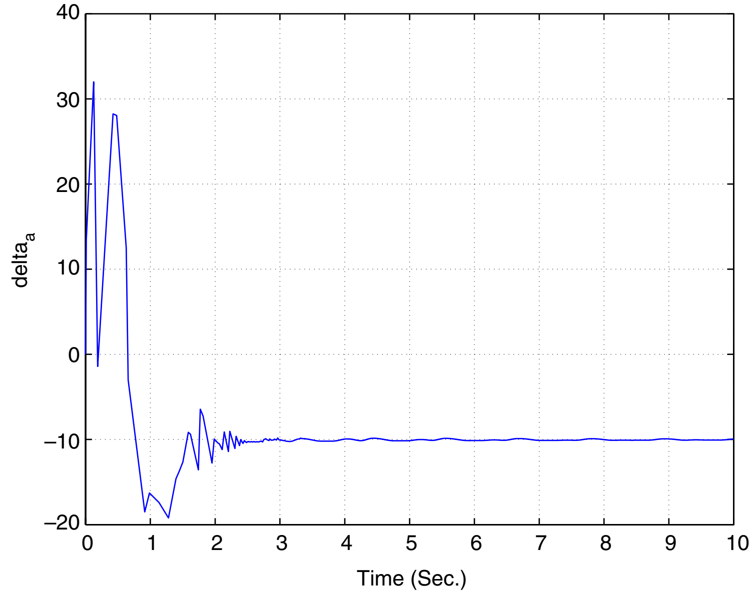

Figure 9. Δ_a estimation.

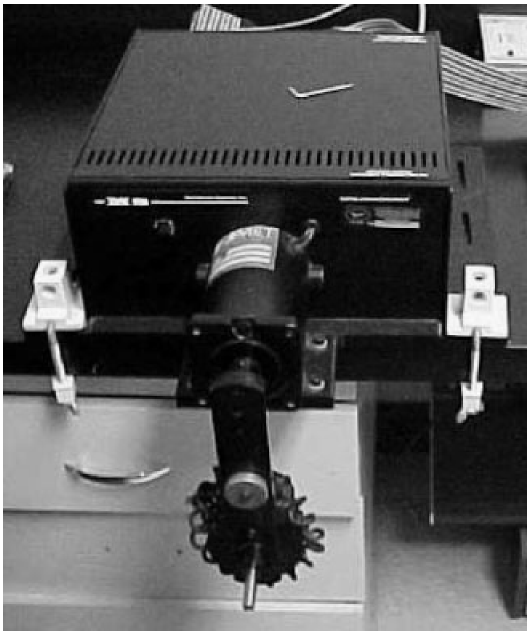

Figure 10. Simple pendulum.

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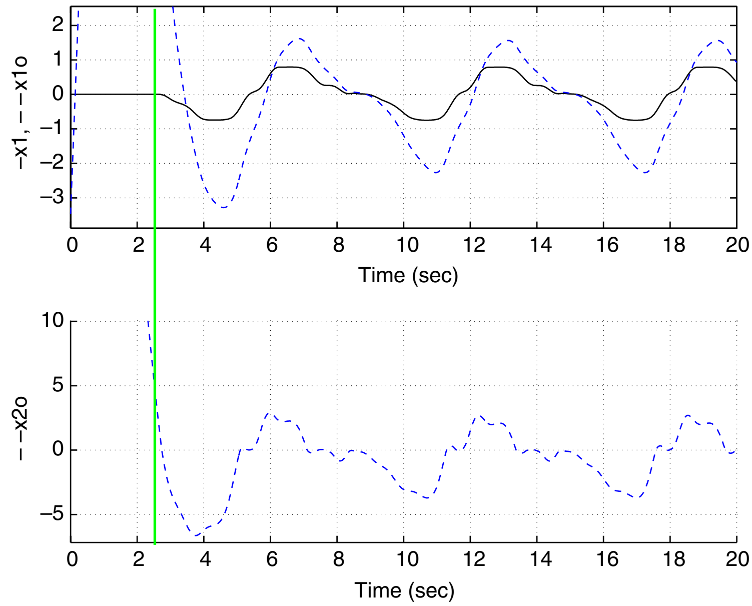

Figure 11. Experimental results. Behaviour of the plant and the observer for $c_3 = 0$ (classical observer).

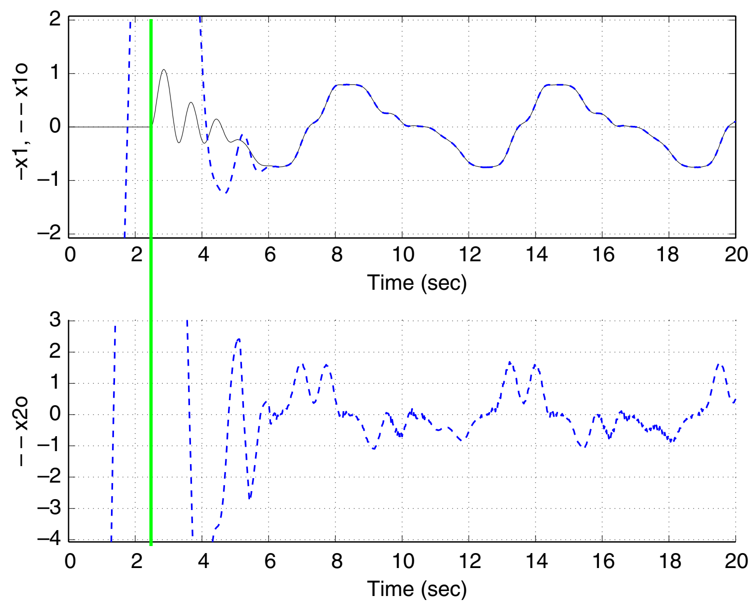

Figure 12. Experimental results. Behaviour of the plant and the observer for $c_3 = 10$ (proposed observer).

is important to note that the real system has parametric uncertainties and non-modelled dynamics as the Coulomb friction which produce an intrinsic perturbation term. We identified this term first, see Figure 14. This term will be added to the perturbations applied artificially in

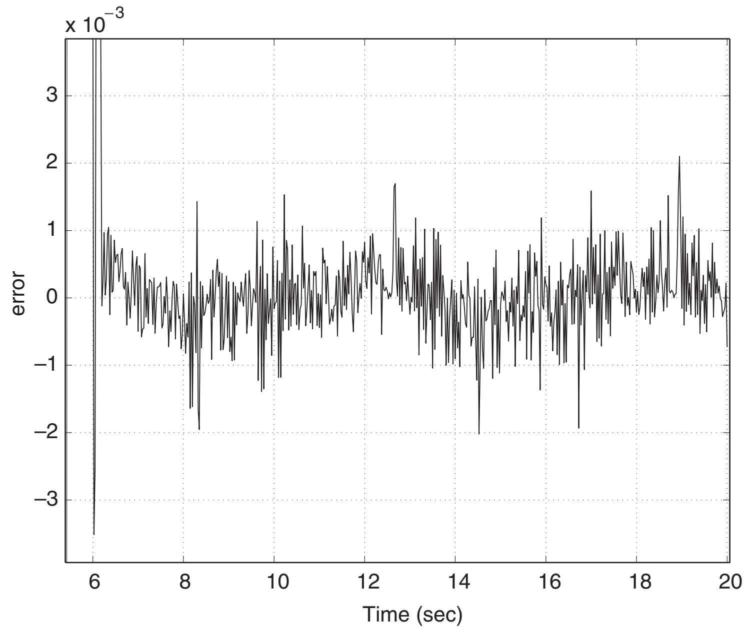

Figure 13. Experimental results. Behaviour of the error between the plant and the observer state for $c_3 = 10$ (proposed observer).

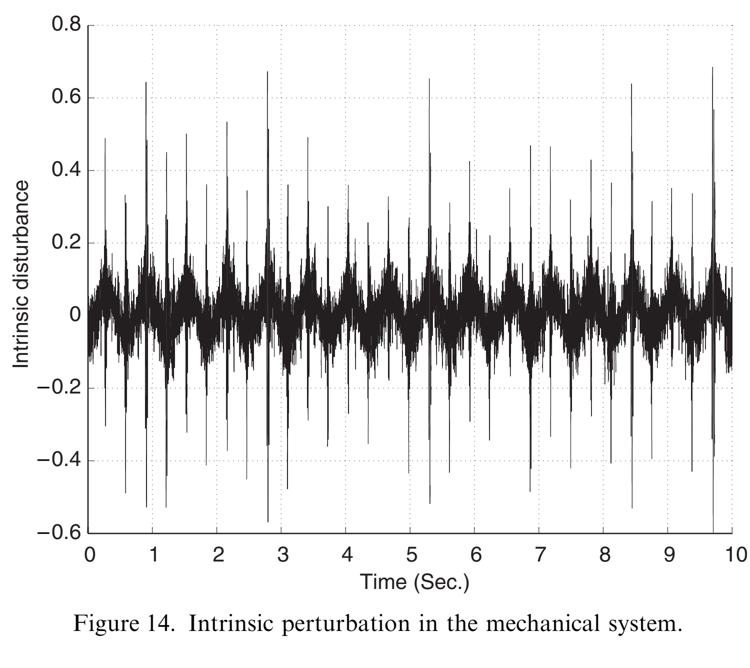

Figure 14. Intrinsic perturbation in the mechanical system.

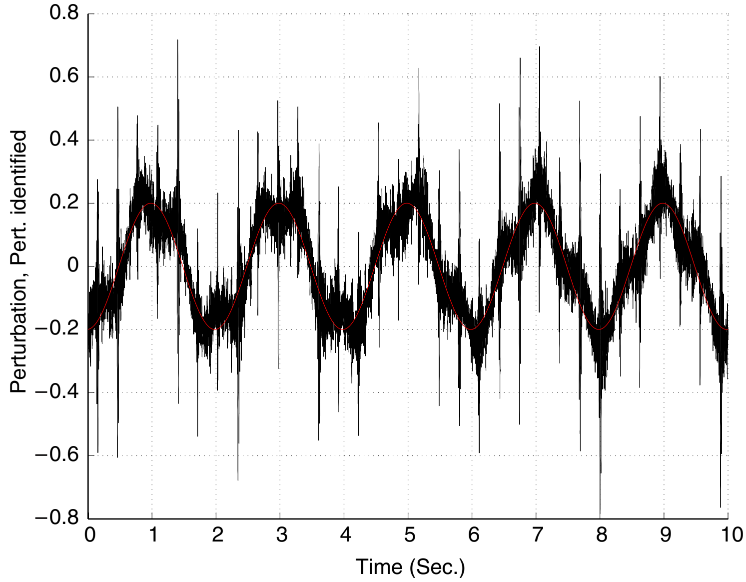

Figure 15. Identification of a sine perturbation plus the intrinsic disturbance.

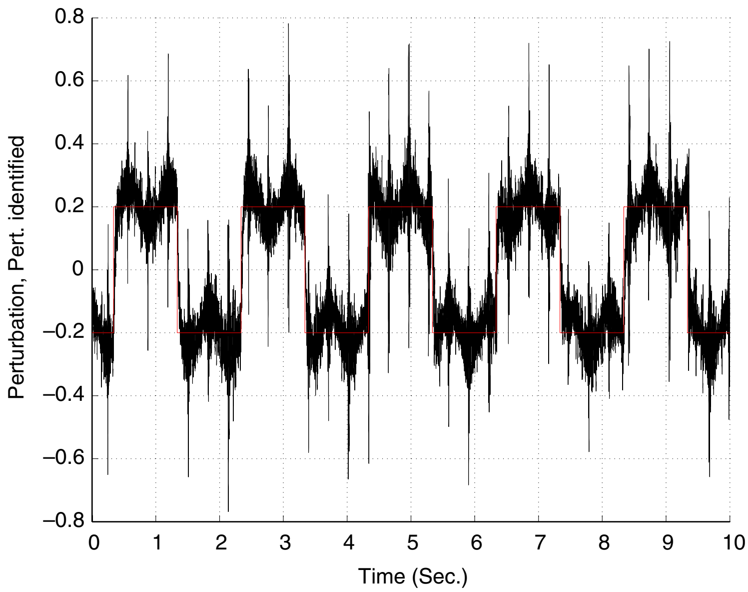

Figure 16. Identification of a square perturbation plus the intrinsic disturbance.

the following experiments. In a second experiment we applied an external perturbation $\gamma(t)$ with sine signal form, Figure 15 shows this perturbation (light line) and the identified perturbation (black line), which is the sine signal plus the intrinsic perturbation. In a final experiment we

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applied perturbation $\gamma(t)$ with a square signal form. Figure 16 shows the results, where we can see that the identified perturbation is the square signal plus the intrinsic perturbation.

7. CONCLUSION

The main contribution of this work is the design of a globally asymptotically stable second-order sliding-mode observer for a class of Lagrangian systems. The observer displays good characteristics of robustness to bounded parametric variation and external perturbations. For the case of plants with perturbations this observer has better performance than the observer proposed in [4] because we can guarantee convergence to zero error.

Due to its discontinuous nature, the observer state vector displays chattering; however, in the experimental results chattering was not an important problem, as we can see in Figure 13, where the chattering has very small amplitude.

This observer ensures exponential rate of convergence to the state of the plant in spite of the existence of non-vanishing bounded perturbations with bounded derivative. The performance of the observer has been tested experimentally and results match with the theory.

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