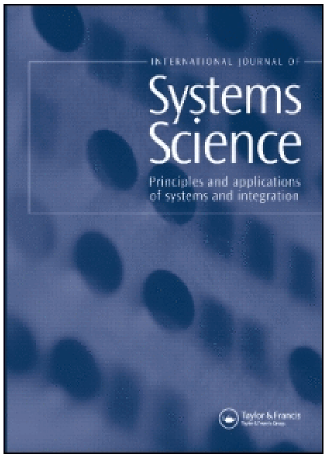

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Observation of linear systems with unknown inputs via high-order sliding-modes

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A high-order sliding-mode observer is designed for linear time invariant systems with single output and unknown bounded single input. It provides for the global observation of the state and the output under sufficient and necessary conditions of strong observability or strong detectability. The observation is finite-time-convergent and exact in the strong observability case. The accuracy of the proposed observation and identification schemes is estimated via the sampling step or magnitude of deterministic noises. The results are extended to the multi-input multi-output case.

Keywords: High order sliding modes; Observation; Identification

1. Introduction

1.1 *Antecedents and motivation*

Observation of system states in the presence of unknown inputs is one of the most important problems in the modern control theory. The standard conditions for such observation (Hautus 1983) are obtained under assumption that only the outputs are available without their derivatives. In particular, the unknown inputs need to match the known outputs. That requirement does not hold even for mechanical systems, when unknown forces are present, but only the position is available (Davila *et al*. 2006). An adaptive observer with convergence of the observation error to a bounded zone was proposed by Rapaport and Gouze (1999).

Sliding-mode-based robust state observation is developed successfully in the Variable Structure Theory in recent years (Edwards and Spurgeon 1998, Utkin *et al*. 1999, Barbot *et al*. 2002, Davila *et al*. 2005). The corresponding implementation issues were extensively studied in Poznyak (2003) and Edwards *et al*. (2002). The sliding-mode-based observation has such attractive features as

- insensitivity (more than robustness) with respect to unknown inputs;
- possibility to use the equivalent output injection in order to reveal additional information.

Step-by-step vector-state reconstruction by means of sliding modes is studied by Utkin *et al*. (1999), Xu and Hashimoto (1993), Ahmed-Ali and Lamnabhi-Lagarrigue (1999), Floquet and Barbot (2006). These observers are based on a system transformation to a triangular form and successive estimation of the state vector using the equivalent output injection. The corresponding *sufficient* conditions for observation of linear time invariant (LTI) systems with unknown inputs were obtained in Utkin *et al*. (1999) and Floquet and Barbot (2006).

Unfortunately, the realization of step-by-step observers is based on conventional sliding modes and requires filtration at each step due to imperfections of analog devices or discretization effects. The hierarchical observers based on super-twisting algorithm were recently developed (Bejarano *et al*. 2006) in order to

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avoid the filtration. These observers make successive use of the continuous super-twisting controller, based on second-order sliding mode (Levant 1993). A modified version of the super-twisting controller is also used in the step-by-step observer by Floquet and Barbot (2006). Unfortunately, also those observers are not free of drawbacks:

(i) The super-twisting algorithm provides for the best-possible asymptotic accuracy of the derivative estimation at each single realization step (Levant 1998). In particular, with discrete measurements the accuracy is proportional to the sampling step τ in the absence of noises, and to the square root of the input noise magnitude, if the above discretization error is negligible. The step-by-step and hierarchical observers use the output of the super-twisting algorithm as noisy input at the next step. As a result, the overall observation accuracy is of the order $\tau^{(1/2^{r-2})}$, where r is the observability index of the system. This means, for example, that in order to implement the fourth-order derivative observer with the 0.1 precision, and the unknown fifth derivative being less than 1 in its absolute value, the practically-impossible discretization step $\tau = 10^{-8}$ is needed.

(ii) Similarly, in the presence of the measurement noise with magnitude ε the estimation accuracy is proportional to $\varepsilon^{1/(2^{r-1})}$, which requires measurement noises not-exceeding 10^{-16} for the fourth-order derivative implementation under the above conditions.

(iii) The step-by-step observers Floquet and Barbot (2006) provide for semiglobal finite-time stability only, restricting the application of these observers to the class of the systems for which the upper bound of initial conditions might be estimated in advance.

At the same time the rth-order robust exact sliding-mode-based differentiator (Levant 2003) removes the first issue providing for the rth derivative accuracy proportional to the discretization step τ, and resolves the second one providing for the accuracy $\varepsilon^{1/(r+1)}$. Unfortunately, its straight-forward application requires the boundness of the unknown $(r+1)$th derivative. In practice it means that still only semiglobal observation of stable linear systems is allowed.

1.1.1 Main contribution. The observation and identification algorithms are developed for LTI systems with bounded unknown inputs, providing:

- global finite-time exact observation of the state vector of strongly-observable systems;
- exact finite-time identification of smooth unknown inputs of strongly observable systems;
- asymptotic estimation of unobservable states and unknown inputs is achieved for strongly detectable systems.

To realize this goal:

- Sufficient and necessary conditions of strong observability and strong detectability are formulated in the terms of the system relative degrees with respect to unknown inputs, which are also necessary in the case of single-input–single-output (SISO) systems;
- An additional Luenberger-like linear term is introduced ensuring the global convergence of the observer error to some bounded region;
- A modification of the robust exact sliding-mode-based differentiator (Levant 2003) is suggested providing for the finite time convergence of the observation error in the presence of unknown inputs;
- The asymptotic accuracy of the state observation and the unknown-input identification is estimated with respect to arbitrary bounded deterministic Lebesgue-measurable noises and discrete sampling.

1.1.2 Structure of this article. The considered SISO system is described in section 2. The problem statement is presented in subsection 2.1. Main notions on strong observability and strong detectability are discussed in terms of the relative degree in subsection 2.2. A global observer is designed for a strong observable system in section 3 based on high-order sliding-modes. Subsection 3.1 is devoted to the state observation algorithm, and subsection 3.2 deals with the unknown-input identification. A globally convergent observer based on high-order sliding modes is proposed for strongly detectable systems in section 4. The algorithm for state detection is presented in subsection 4.1. The algorithm for the asymptotic unknown-input identification is presented in subsection 4.2. A generalization to the multi-input-multi-output (MIMO) case is presented in section 5. Two illustrative examples are given in the section 6, for the cases of strongly observable and strongly detectable systems. Finite-time exact state observation and unknown input identification are demonstrated. Finally, section 7 contains the concluding remarks.

2. System description

2.1 *Problem statement*

Consider an LTI system

$$\begin{aligned} \dot{x} &= Ax + Bu + D\zeta(t), \quad D \neq 0 \\ y &= Cx, \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}$ are the system state and the output, $\zeta \in \mathbb{R}$ is the unknown input (disturbance), $u \in \mathbb{R}$ is the known control and the known matrixes A, B, C, D have suitable dimensions. The equations are understood in the Filippov sense (Filippov 1988) in order to provide for the possibility to use discontinuous signals in observers. Note that Filippov solutions coincide with the usual solutions, when the right-hand sides are continuous. It is assumed also that all considered inputs allow the existence and extension of solutions to the whole semi-axis $t \geq 0$.

The task is to build an observer providing for asymptotic (preferably finite-time convergent and exact) estimation of the states and the unknown input. Obviously, it can be assumed without loss of generality that the known input u is equal to zero (i.e. $u(t) = 0$).

2.2. *Strong observability, strong detectability and some of their properties*

The conditions for observability and detectability of LTI systems with unknown inputs are studied, for example, in Hautus (1983). Some necessary and sufficient conditions for strong observability and strong detectability are obtained in this section.

Definition 1: $s_0 \in \mathbb{C}$ is called an invariant zero of the triplet $\{A, D, C\}$ if $\operatorname{rank} R(s_0) < n + \operatorname{rank}(D)$, where $R(s)$ is the Rosenbrock matrix of system (1)

$$R(s) = \begin{bmatrix} sI - A, & -D \\ C, & 0 \end{bmatrix}. \tag{2}$$

It is assumed in the following definitions that $u = 0$.

Definition 2: System (1) is called (strongly) observable if for any initial state $x(0)$ and $\zeta(t) \equiv 0$ (any input $\zeta(t)$), $y(t) \equiv 0$ with $\forall t \geq 0$ implies that also $x \equiv 0$ (Hautus 1983).

The following statements are equivalent (Hautus 1983).

(i) The system (1) is strongly observable.
(ii) The triple $\{A, C, D\}$ has no invariant zeros.

Definition 3: The system is strongly detectable, if for any $\zeta(t)$ and $x(0)$ it follows from $y(t) \equiv 0$ with $\forall t \geq 0$ that $x \to 0$ with $t \to \infty$ (Hautus 1983).

The following statements are equivalent (Hautus 1983).

(i) The system (1) is strongly detectable.
(ii) The system (1) is minimum phase (i.e. the invariant zeroes of the triple $\{A, C, D\}$ satisfy $\operatorname{Re} s < 0$).

Obviously, in the case $D = 0$ the notions of strong observability and strong detectability coincide respectively with observability and detectability. Introduce the observability matrix

$$P = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}.$$

Recall that system (1) is observable (in the absence of the unknown input) if and only if the observability matrix P has the full rank. In that case the matrix $A - LC$ can be assigned any spectrum, choosing an appropriate column matrix parameter L. The rank n_O of the matrix P is called the observability index of the system.

Definition 4: Following Isidori (1996) the relative degree of system (1) with respect to the unknown input is the number r such that

$$CA^j D = 0, \quad j = 1, \ldots, r-2, \quad CA^{r-1} D \neq 0. \tag{3}$$

2.3. *Strong observability and strong detectability in terms of the relative degree with respect to the unknown input*

Let r be the relative degree of the system with respect to the unknown input. It is known that $r \leq n$, and in appropriate coordinates the system takes on the form

$$\begin{aligned} \dot{x}_1 &= A_{11}x_1 + A_{12}x_2 + B_1 u + D_1 \zeta, \\ \dot{x}_2 &= A_{21}x_1 + A_{22}x_2 + B_2 u, \\ y &= C_1 x_1 \end{aligned} \tag{4}$$

where the matrices A_{11}, A_{12}, C and D are of the form

$$A_{11} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ a_1 & a_2 & a_3 & \cdots & a_r \end{bmatrix},$$

$$A_{12} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ a_{r+1} & a_{r+2} & \cdots & a_n \end{bmatrix},$$

$$D_1 = \begin{bmatrix} 0 & \cdots & 0 & d \end{bmatrix}^T, \quad d \neq 0, \; C_1 = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}. \tag{5}$$

Let $u=0$. With $y\equiv 0$ the dynamics (4) is reduced to $\dot{x}_2 = A_{22}x_2$ which is called the zero-dynamics (Isidori 1996). The eigenvalues of A_{22} coincide with the invariant zeros of the system. The system is called minimum-phase if A_{22} is Hurwitz. In such a case $y\equiv 0$ implies $x\to 0$.

Theorem 2.1: *The system* (1) *is strongly observable if and only if the output of the system* (1) *has relative degree n with respect to the unknown input* $\zeta(t)$.

Proof: Strong observability of the system requires its observability, and, therefore, rank $P=n$. The observability implies the existence of the relative degree r of the output y with respect to the unknown input ζ. Indeed, otherwise $PD=0$ and therefore $D=0$. Then the coordinate transformation $x_O = Px$ turns system (1) into

$$\begin{aligned}\dot{x}_O &= A_O x_O + B_O u(t) + D_O \zeta(t)\\ y(t) &= C_O x_O\end{aligned} \tag{6}$$

where

$$A_O = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & 1\\ -a_1 & -a_2 & -a_3 & \cdots & -a_n \end{bmatrix}, \tag{7}$$

$$D_O = [CD, \dots, CA^{n-2}D, CA^{n-1}D]^T, \tag{8}$$

$$C_O = [1, 0, \dots, 0], \tag{9}$$

a_j, $j=1,\dots,n$ are some constants; and the vector B_O does not have any specific form. Recall that u is assumed to be zero. When $r=n$ only the last component of D_O is not zero. It is obvious that in that case the identity $y\equiv 0$ implies $x_O\equiv 0$.

Now assume that $r<n$. This means that some nontrivial zero-dynamics exists, which corresponds to nontrivial solutions satisfying $y\equiv 0$ and contradicts the strong observability. This ends the proof of the theorem.

The theorem can be proved also algebraically. The determinant $p(s)$ of the Rosenbrock matrix is invariant with respect to coordinate transformations. It is computed as $p(s)=\det(\tilde{R})$, where

$$\tilde{R} = \begin{bmatrix} s & -1 & 0 & \cdots & 0 & -CD\\ 0 & s & -1 & \cdots & 0 & -CAD\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & -1 & -CA^{n-2}D\\ a_1 & a_2 & a_3 & \cdots & s+a_n & -CA^{n-1}D\\ 1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

Thus,

$$\begin{aligned} p(s) = CD\det &\begin{bmatrix} s & -1 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & -1\\ a_2 & a_3 & \cdots & s+a_n \end{bmatrix}\\ &+ CAD\det\begin{bmatrix} s & -1 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & -1\\ a_3 & a_4 & \cdots & s+a_n \end{bmatrix}\\ &+\cdots + CA^{n-2}D\det\big[\,s+a_n\,\big] + CA^{n-1}D. \end{aligned}$$

Assume that the characteristic polynomial of the matrix A be

$$s^n + a_n s^{n-1} + \cdots + a_2 s + a_1.$$

Then $p(s)=\det(\tilde{R})$ is rewritten as

$$\begin{aligned} p(s) &= (s^{n-1} + a_n s^{n-2} + \cdots + a_2)CD\\ &\quad + (s^{n-2} + a_n s^{n-3} + \cdots + a_3)CAD\\ &\quad + \cdots + (s+a_n)CA^{n-2}D + CA^{n-1}D. \end{aligned} \tag{10}$$

The zeros of the system coincide with the roots of the polynomial $p(s)$. According to the Definition 3, the system has no zeros if and only if it has the relative degree n. □

Theorem 2.2: *The system* (1) *is strongly detectable if and only if the relative degree with respect to the unknown input exists, and the system is minimum-phase. In that case also* $r\le n_O$ *is ensured.*

Proof: Assume first that the system is strongly detectable. Suppose first that the relative degree does not exist. It is possible only in the case, when $PD=0$, i.e. D belongs to the invariant subspace $Px=0$ of the unobservable states. In that case in appropriate coordinates the system takes on the form

$$\begin{aligned} \dot{x}_O &= A_O x_O,\\ \dot{x}_N &= A_{NO}x_O + A_N x_N + D_N\zeta,\\ y &= C_O x_O \end{aligned}$$

where the matrices A_O, and C_O are of the form

$$A_O = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_1 & -a_2 & -a_3 & \cdots & -a_{n_O} \end{bmatrix},$$

$$C_O = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix},$$

with $n_O = \operatorname{rank} P$. Thus $y \equiv 0$ does not impose any condition on the input ζ, and the induced dynamics in the unobservable subspace x_N has always non-zero constant solution with $\zeta = \text{const} \neq 0$, which contradicts the strong detectability. Hence, $r \leq n_O = \operatorname{rank} P$. Now the minimum-phase property of the system exactly corresponds to the requirement $x \to 0$ with $y \equiv 0$.

Assume now that $r \leq n_O$, and the system is minimum-phase. In that case, in appropriate coordinates, the system gets the form (4), (5) with a Hurwitz matrix A_{22}. The identity $y \equiv 0$ obviously implies $x_1 \equiv 0$ and $x_2 \to 0$. □

3. Observer design for the case of strong observability

3.1 *Observation of coordinates*

System (1) is supposed to satisfy the following assumptions.

Assumption 3.1: *The system* (1) *has the relative degree n with respect to the unknown input* $\zeta(t)$.

This assumption means that the system is strongly observable. The next assumption has two variants.

Assumption 3.2: *The unknown input* $\zeta(t)$ *is a bounded Lebesgue-measurable function,* $|\zeta(t)| \leq \zeta^+$.

Assumption 3.3: *The unknown input* $\zeta(t)$ *is a bounded function,* $|\zeta(t)| \leq \zeta^+$, *with successive derivatives up to the order k bounded by the same constant* ζ_1^+. *The kth derivative is a Lipschitzian function with the Lipshitz constant not exceeding* ζ_1^+. *Thus,* $\zeta^{(k+1)}(t)$ *exists almost everywhere and is a bounded Lebesgue-measurable function,* $|\zeta^{(k+1)}(t)| \leq \zeta_1^+$.

The latter assumption is needed for the estimation of the unknown input. The observer is built in the form

$$\dot{z} = Az + Bu + L(y - Cz), \tag{11}$$

$$\hat{x} = z + Kv, \tag{12}$$

$$\dot{v} = W(y - Cz, v), \tag{13}$$

where $z, \hat{x} \in \mathbb{R}^n$, $\hat{x}$ is the estimation of x, and the column matrix $L = [l_1, l_2, \ldots, l_n]^T \in \mathbb{R}^n$ is a correction factor chosen so that the eigenvalues of the matrix $A - LC$ have negative real parts. Such L exists due to Assumption 3.1 and Theorem 2.2. The matrix K, vector v and the nonlinear discontinuous function W are chosen differently depending on the Assumptions 3.2 or 3.3.

The proposed observer is actually composed of two parts. Equation (11) is a traditional Luenberger observer providing for the boundedness of the difference $z - x$ in the presence of the unknown bounded input ζ. System (13) is the high-order sliding modes differentiator and ensures the finite time convergence of the resulting estimation error to zero.

Suppose that only coordinates are to be estimated, and that Assumptions 3.1 and 3.2 hold, respectively. Note that in the simplest case when $n = 1$ the only observable coordinate coincides with the measured output and, therefore, only the input estimation problem makes sense, requiring Assumptions 3.1 and 3.3. The latter problem is considered in the next subsection. Thus assume that $n > 1$.

Since the pair (C, A) is observable, arbitrary stable values are assigned to the eigenvalues of the matrix $(A - LC)$, choosing an appropriate column gain matrix L (Chen 1984). Obviously the pair $(C, A - LC)$ is also observable, and its observability matrix

$$\tilde{P} = \begin{bmatrix} C \\ C(A - LC) \\ C(A - LC)^2 \\ \vdots \\ C(A - LC)^{n-1} \end{bmatrix} \tag{14}$$

is not singular. Set the gain matrix $K = \tilde{P}^{-1}$ and assign

$$\hat{x} = z + \tilde{P}^{-1} v. \tag{15}$$

The nonlinear part of the observer (13) is chosen in the form of the $(n-1)$th-order differentiator (Levant 2003)

$$\begin{aligned} \dot{v}_1 &= w_1 = -\alpha_n M^{1/n} |v_1 - y \\ &\quad + Cz|^{(n-1)/n} \operatorname{sign}(v_1 - y + Cz) + v_2, \\ \dot{v}_2 &= w_2 = -\alpha_{n-1} M^{1/(n-1)} |v_2 \\ &\quad - w_1|^{(n-2)/(n-1)} \operatorname{sign}(v_2 - w_1) + v_3, \\ &\ \ \vdots \\ \dot{v}_{n-1} &= w_{n-1} = -\alpha_2 M^{1/2} |v_{n-1} \\ &\quad - w_{n-2}|^{1/2} \operatorname{sign}(v_{n-1} - w_{n-2}) + v_n, \\ \dot{v}_n &= -\alpha_1 M \operatorname{sign}(v_n - w_{n-1}), \end{aligned} \tag{16}$$

where v_i, z_i and w_i are the components of the vectors v, $z \in \mathbb{R}^n$ and $w \in \mathbb{R}^{n-1}$ respectively. The parameter M is chosen sufficiently large, in particular $M > |d|\zeta^+$, where $d = CA^{n-1}D$. The constants α_i are chosen recursively sufficiently large as in Levant (2003). In particular, one of the possible choices is $\alpha_1 = 1.1$, $\alpha_2 = 1.5$, $\alpha_3 = 2$, $\alpha_4 = 3$, $\alpha_5 = 5$, $\alpha_6 = 8$, which is sufficient for $n \le 6$. Note that (16) has a recursive form, useful for the parameter adjustment. In any computer realization one has to calculate the internal auxiliary variables v_j and w_j, $j = 1, \dots, n$, using only the simultaneously-sampled current values of y, z and v_j.

Recall that $x_O = Px$ is the vector of canonical observation coordinates, and $e_O = P(\hat{x} - x)$ is the canonical observation error.

Theorem 3.4: *Let Assumptions* 3.1 *and* 3.2 *be satisfied and the output be measured with a noise, being a Lebesgue-measurable function of time with the maximal magnitude ε. Then with properly chosen α_j and M sufficiently large, the state x of the system is estimated in finite time by the observer* (11), (14)–(15). *With sufficiently small ε the observation errors $e_{Oi} = \hat{x}_{Oi} - x_{Oi} = CA^{i-1}(\hat{x} - x)$ are of the order of $\varepsilon^{(n-i+1)/n}$. That means that the inequalities $|e_{Oi}| \le \gamma_i \varepsilon^{(n-i+1)/n}$ hold for some constants $\gamma_i > 0$ depending only on the observer and system parameters, and on the input upper bound. The accuracy of the order of $\varepsilon^{1/n}$ is obtained in non-canonical coordinates due to the mix of coordinates. In particular, the state x is estimated **exactly** and in **finite time** in the absence of noises.*

Proof: Consider the linear Luenberger part of the observer (11). Denote $\tilde{e} = x - z$, $\tilde{e}_y = C\tilde{e}(t)$, then, replacing the noise by the whole segment $[-\varepsilon, \varepsilon]$ of its possible values, obtain

$$\dot{\tilde{e}} \in (A - LC)\tilde{e}(t) + L[-\varepsilon, \varepsilon] + D\zeta(t), \tag{17}$$

$$\tilde{e}_y \in C\tilde{e}(t) + [-\varepsilon, \varepsilon]. \tag{18}$$

Recall that the matrix $A - LC$ is Hurwitz. Let the Lyapunov function of the homogeneous system be

$$V = \frac{1}{2}\tilde{e}^T H \tilde{e},$$

where H is a symmetric positive-definite matrix. Its derivative

$$\begin{aligned}\dot{V} \in \tilde{e}^T(H(A-LC) + (A-LC)^T H)\tilde{e} + (\tilde{e}^T HD + D^T H\tilde{e})\zeta(t) \\ + (\tilde{e}^T HL + L^T H\tilde{e})[-\varepsilon, \varepsilon]\end{aligned}$$

is negative definite with $\zeta = \varepsilon = 0$. Thus, due to the boundedness of ζ, obtain that the estimation error $\tilde{e}$ converges to a bounded vicinity of the origin $\tilde{e} = 0$. Since that moment also $\dot{\tilde{e}}$ remains uniformly bounded.

Let meantime $\varepsilon = 0$. The output $\tilde{e}_y$ of the estimation error system (18) has the same relative degree n with respect to the unknown input as system (1). Indeed, for any i the equalities $CD = CAD = \cdots = CA^iD = 0$ imply that

$$\begin{aligned} C(A - LC)^{i+1}D &= C(A - LC)^i AD + C(A - LC)^i L(CD) \\ &= C(A - LC)^i AD = C(A - LC)^{i-1}A^2 D \\ &\quad + C(A - LC)^{i-1}L(CAD) \\ &= \cdots = CA^{i+1}D. \end{aligned}$$

Applying the transformation $\bar{e} = \tilde{P}\tilde{e}$ to system (17), (18), with $\tilde{P}$ selected according to (14), obtain that

$$\begin{aligned}\dot{\bar{e}} &= \bar{A}\bar{e} + \bar{D}\zeta(t), \\ \tilde{e}_y &= \bar{C}\bar{e}\end{aligned} \tag{19}$$

where the matrixes $\bar{A}$, $\bar{D}$ and $\bar{C}$ are of the form

$$\bar{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ a_1 & a_2 & a_3 & \cdots & a_n \end{bmatrix},$$

$$\bar{D} = [0, \dots, 0, d]^T, \quad \bar{C} = [1, 0, \dots, 0].$$

Notice that $d = C(A - LC)^{n-1}D = CA^{n-1}D$. By definition $\bar{e}_1 = \tilde{e}_y = y - Cz$, and due to (19) obtain that

$$\tilde{e}_y = \bar{e}_1, \ \dot{\tilde{e}}_y = \bar{e}_2, \dots, \tilde{e}_y^{(n)} = \bar{e}_n.$$

Denote $\sigma_i = v_i - (y - Cz)^{(i-1)} = v_i - \bar{e}_i$ and obtain

$$\begin{aligned} \dot{\sigma}_1 &= -\alpha_n M^{1/n}|\sigma_1|^{(n-1)/n}\mathrm{sign}(\sigma_1) + \sigma_2, \\ \dot{\sigma}_2 &= -\alpha_{n-1}M^{1/(n-1)}|\sigma_2 - \dot{\sigma}_1|^{(n-2)/(n-1)}\mathrm{sign}(\sigma_2 - \dot{\sigma}_1) + \sigma_3, \\ &\vdots \\ \dot{\sigma}_{n-1} &= -\alpha_2 M^{1/2}|\sigma_{n-1} - \dot{\sigma}_{n-2}|^{1/2}\mathrm{sign}(\sigma_{n-1} - \dot{\sigma}_{n-2}) + \sigma_n, \\ \dot{\sigma}_n &= -\alpha_1 M \mathrm{sign}(\sigma_n - \dot{\sigma}_{n-1}) - \dot{\bar{e}}_n. \end{aligned} \tag{20}$$

Now show that with sufficiently large M the dynamics of σ is finite-time stable in the absence of noise. Since, due to (17), starting from some moment, $\tilde{e}$ and $\dot{\tilde{e}}$ remain uniformly bounded, the same is true with respect to $\bar{e}$ and its derivative. Obviously, in the presence of noises not exceeding $\varepsilon \ge 0$ with

$M \geq \sup|\dot{\bar{e}}_n|$ obtain that dynamics (20) satisfies the differential inclusion

$$\begin{aligned}
\dot{\sigma}_1 &\in -\alpha_n M^{1/n}|\sigma_1 + [-\varepsilon, \varepsilon]|^{(n-1)/n}\mathrm{sign}(\sigma_1 + [-\varepsilon, \varepsilon]) + \sigma_2 \\
\dot{\sigma}_2 &= -\alpha_{n-1} M^{1/(n-1)}|\sigma_2 - \dot{\sigma}_1|^{(n-2)/(n-1)}\mathrm{sign}(\sigma_2 - \dot{\sigma}_1) + \sigma_3 \\
&\vdots \\
\dot{\sigma}_{n-1} &= -\alpha_2 M^{1/2}|\sigma_{n-1} - \dot{\sigma}_{n-2}|^{1/2}\mathrm{sign}(\sigma_{n-1} - \dot{\sigma}_{n-2}) + \sigma_n \\
\dot{\sigma}_n &\in -\alpha_1 M \mathrm{sign}(\sigma_n - \dot{\sigma}_{n-1}) + [-M, M]
\end{aligned} \tag{21}$$

The rest of the proof is based on the following lemma.

Lemma 3.5: *Suppose that $\alpha_n > 1$ and $\alpha_{n-1}, \ldots, \alpha_1$ are chosen sufficiently large in the list order. Then after finite time of the transient process any solution of* (21) *satisfies the inequalities $|\sigma_k| \leq \mu_k M^{(k-1)/n}\varepsilon^{(n-k+1)/n}$, $k = 1, 2, \ldots, n$, where $\mu_k > 1$ are some positive constants depending only on the choice of α_k.*

Proof: Denoting $\tilde{\sigma}_k = \sigma_k/M$ obtain that

$$\begin{aligned}
\dot{\tilde{\sigma}}_1 &\in -\alpha_n|\tilde{\sigma}_1 + M[-\varepsilon, \varepsilon]|^{(n-1)/n}\mathrm{sign}(\sigma_1 + M[-\varepsilon, \varepsilon]) + \tilde{\sigma}_2, \\
\dot{\tilde{\sigma}}_2 &= -\alpha_{n-1}|\tilde{\sigma}_2 - \dot{e}_1|^{(n-2)/(n-1)}\mathrm{sign}(\sigma_2 - \dot{\tilde{\sigma}}_1) + \tilde{\sigma}_3, \\
&\vdots \\
\dot{\tilde{\sigma}}_{n-1} &= -\alpha_2|\tilde{\sigma}_{n-1} - \dot{\tilde{\sigma}}_{n-2}|^{1/2}\mathrm{sign}(\tilde{\sigma}_{n-1} - \dot{\tilde{\sigma}}_{n-2}) + \tilde{\sigma}_n, \\
\dot{\tilde{\sigma}}_n &\in -\alpha_1 \mathrm{sign}(\tilde{\sigma}_n - \dot{\tilde{\sigma}}_{n-1}) + [-1, 1].
\end{aligned}$$

The lemma is now a direct consequence of Lemma 8 from Levant (2003). □

According to Lemma 3.5 the equalities

$$v_1 = \bar{e}_1 + O(\varepsilon),\ v_2 = \bar{e}_2 + O(\varepsilon^{(n-1)/n}), \ldots, \quad v_n = \bar{e}_n + O(\varepsilon^{1/n}) \tag{22}$$

are established in finite time. Since CL, CAL, $CA^2L, \ldots$ are scalars, it is easily shown by induction that there are numbers δ_{ij} such that the equalities

$$C(A - LC)^j = CA^j + \delta_{1j}CA^{j-1} + \cdots + \delta_{jj}C, \quad j = 1, 2, \ldots$$

are true. These numbers depend on C, A, L. Thus, multiplying by $\hat{x}_1 - x_1$ obtain that

$$\bar{e}_j = e_{Oj} + \delta_{1j}\, e_{Oj-1} + \cdots + \delta_{jj} e_{O1}, \quad j = 1, 2, \ldots. \tag{23}$$

which is inevitably equivalent to $\bar{e} = \tilde{P}\tilde{e}$. $v = \tilde{P}\tilde{e} = \tilde{P}x - \tilde{P}z$ with $\varepsilon = 0$. Inverting the equations by the Gauss procedure and taking into account (22) obtain by induction

$$e_{Oj} = v_j + \tilde{\delta}_{1j}v_{j-1} + \cdots + \tilde{\delta}_{jj}v_1 + O(\varepsilon^{(n-j+1)/n}), \quad j = 1, 2, \ldots \tag{24}$$

with some constants $\tilde{\delta}_{ij}$. Obtain from $v = \tilde{P}\tilde{e} = \tilde{P}x - \tilde{P}z$ with $\varepsilon = 0$ that

$$\hat{x} = z + \tilde{P}^{-1}v$$

with $\varepsilon = 0$. The accuracy stated in the theorem is obtained from (24) with $\varepsilon \neq 0$, $\varepsilon << 1$. □

Let now the output y be sampled at discrete times with the constant time step τ, $t \in (t_j, t_{j+1})$, $t_{j+1} - t_j = \tau$. Substituting $\bar{e}_1(t_j) = y(t_j) - Cz(t_j)$ for the term $\bar{e}_1 = y - Cz$ in (11), (13), obtain a discrete-sampling observer. Note that the simultaneous sampling of y and z when calculating the sampled value of $\bar{e}_1$ is important.

Theorem 3.6: *Let Assumptions* 3.1, 3.2 *be satisfied, the parameters be chosen as in Theorem* 3.4 *and the output be measured without measurement errors at discrete sampling times with a sufficiently small sampling interval τ. Then after a finite-time transient the canonical observation errors $e_{Oi} = CA^{i-1}(\hat{x} - x)$ are of the order of τ^{n-i+1}.*

Theorem 3.7: *Under the conditions of Theorem* 3.6 *let the nonlinear observer part* (16) *be realized* (*in a computer*) *by means of the Euler integration method with the integration time step being a constant part of τ. Then the statement of Theorem* 3.6 *is preserved.*

Proof: Let $t \in [t_j, t_{j+1})$, $t_{j+1} - t_j = \tau$. Then the input injection $LC\tilde{e}$ is based on the values measured at the moment t_j and (19) can be rewritten as

$$\dot{\tilde{e}}(t) = (A - LC)\tilde{e}(t) + LC(\tilde{e}(t) - \tilde{e}(t_j)) + \bar{D}\zeta(t). \tag{25}$$

Assuming that $\|\tilde{e}\| < 2Q$ obtain that the right-hand side of (25) is bounded in norm by some constant of the form $\lambda_1 Q + \lambda_2\zeta^+$, and

$$\|LC(\tilde{e}(t) - \tilde{e}(t_j))\| \leq (\lambda_3 Q + \lambda_4\zeta^+)\tau,$$

where $\lambda_1, \ldots, \lambda_4 > 0$ are some constants. Hence, (25) can be rewritten in the form

$$\dot{\tilde{e}} = (A - LC)\tilde{e} + \tilde{\zeta}, \quad \|\tilde{\zeta}\| \leq (\lambda_3 Q + \lambda_4\zeta^+)\tau + \lambda_5\zeta^+.$$

Thus, with any sufficiently large Q the inequality $\|\tilde{e}\| < Q$ is established and kept afterwards, if τ is sufficiently small. Since that time $\dot{\tilde{e}}$ is bounded. Applying the Lagrange theorem to the difference $\tilde{e}(t) - \tilde{e}(t_j)$

obtain from (25) that

$$\dot{\tilde{e}} = (A - LC)\tilde{e} + LC\dot{\tilde{e}}(t_j + \rho(t - t_j))\tau + \bar{D}\zeta(t), \ \ \rho \in (0, 1).$$

Therefore, due to the boundedness of $\dot{\tilde{e}}$, and independently of Q, the same bounds of $\tilde{e}$ are established as in the proof of Theorem 3.4 with sufficiently small τ. As follows from Lemma (3.5), differential inclusion (20) is finite-time stable. It is also homogeneous with the weights $n, \ldots, 1$ of $\sigma_1, \ldots, \sigma_n$ respectively and the homogeneity degree -1. Thus, the discrete sampling provides for the accuracy $|v_i - \bar{e}_i| = |\sigma_i| \leq \gamma_i \tau^{n-i+1}$ (Levant 2005), and Theorem 3.6 follows now from (23) with sufficiently small τ. The Euler integration scheme implementation can be considered as introduction of a variable delay bounded by some constant times τ. Thus, Theorem 3.7 also follows now from Levant (2005). □

Note that the linear part (11) of the observer can be realized using any advanced integration methods.

3.2. *Identification of the unknown input*

Suppose that assumptions 3.1 and 3.3 hold. Let now $v \in \mathbb{R}^{n+k+1}$ satisfy the nonlinear differential equation (13) in the form

$$\begin{aligned}
\dot{v}_1 &= w_1 = -\alpha_{n+k+1} M^{1/(n+k+1)} |v_1 - y(t) \\
&\quad + Cz|^{(n+k)/(n+k+1)} \mathrm{sign}(v_1 - y(t) + Cz) + v_2, \\
\dot{v}_2 &= w_2 = -\alpha_{n+k} M^{1/(n+k)} |v_2 \\
&\quad - w_1|^{(n+k-1)/(n+k)} \mathrm{sign}(v_2 - w_1) + v_3, \\
&\ \vdots \\
\dot{v}_n &= w_n = -\alpha_{k+2} M^{1/(k+2)} |v_n \\
&\quad - w_{n-1}|^{(k+1)/(k+2)} \mathrm{sign}(v_n - w_{n-1}) + v_{n+1}, \\
&\ \vdots \\
\dot{v}_{n+k} &= w_{n+k} = -\alpha_2 M^{1/2} |v_{n+k-1} \\
&\quad - w_{n+k-2}|^{1/2} \mathrm{sign}(v_{n+k-1} - w_{n+k-2}) + v_{n+k}, \\
\dot{v}_{n+k+1} &= -\alpha_1 M \mathrm{sign}(v_{n+k+1} - w_{n+k}),
\end{aligned} \tag{26}$$

where M is a sufficiently large constant. As previously, (39) has a recursive form, and the parameters α_i are chosen in the same way (Levant 2003). In particular, one of the possible choices is $\alpha_1 = 1.1$, $\alpha_2 = 1.5$, $\alpha_3 = 2$, $\alpha_4 = 3$, $\alpha_5 = 5$, $\alpha_6 = 8$, which is sufficient for $n + k \leq 5$. In any computer realization one has to calculate the internal auxiliary variables w_j, $j = 1, \ldots, n + k$, using only the simultaneously-sampled current values of y, z_1 and v_j.

The equality $\bar{e} = \omega$ is established in finite time, where ω is the truncated vector

$$\omega = (v_1, \ldots, v_n)^T. \tag{27}$$

Thus, the corresponding observer equation

$$\hat{x} = z + \tilde{P}^{-1}\omega \tag{28}$$

comes instead of (12) and $\tilde{P}$ is defined by (12). The estimation of the input ζ is defined as

$$\begin{aligned}
&\hat{\zeta} = \frac{1}{d}(v_{n+1} - (a_1 v_1 + a_2 v_2 + \cdots + a_n v_n)), \\
&s^n - a_n s^{n-1} - \cdots - a_1 = (-1)^n \det(A - LC - sI)
\end{aligned} \tag{29}$$

where the second line defines the characteristical polynomial of the matrix $A - LC$.

Theorem 3.8: *Let Assumptions* 3.1 *and* 3.3 *be satisfied and the output be measured with a noise, being a Lebesgue-measurable function of time with the maximal magnitude ε. Then with properly chosen α_j and M sufficiently large, in the absence of noises* (*i.e. with $\varepsilon = 0$*) *the observable state x_O of the system is estimated exactly in finite time by the observer* (11), (26)–(29). *With any sufficiently small ε the canonical observation errors e_{Oi} are obtained of the order of $\varepsilon^{(n-i+k+2)/(n+k+1)}$. The estimation error of the input ζ is of the order of $\varepsilon^{(k+1)/(n+k+1)}$. In particular, in the absence of noises the estimations of the unknown input and coordinates are exact after a finite-time transient. The accuracy of the order of $\varepsilon^{(k+1)/(n+k+1)}$ is obtained in non-canonical coordinates due to the mix of coordinates.*

Proof: The proof is very similar to that of Theorems 3.4, 3.6, but the differentiation order can now be increased. Exactly, as in the proof of Theorem 3.4 obtain that starting from some moment $\bar{e}$ and $\dot{\bar{e}}$ remain uniformly bounded due to (17) and the boundedness of ζ. Let now $\bar{e}_i = (Cz - y(t))^{(i-1)}$, $i = 1, \ldots, n + k$. [According to Lemma 3.5, $\bar{e} = \omega$.] Obviously, $(Cz - y(t))^{(n+k+1)} = \bar{e}_n^{(k+2)}$ is a linear combination of $\bar{e}_1, \ldots, \bar{e}_n, \zeta, \ldots, \zeta^{(k+1)}$ and is, therefore, bounded. Taking $M > \sup|(Cz - y(t))^{(n_O+k+1)}|$ obtain finite-time convergence of (26) (Levant 2003). This means that equalities $v_i = \bar{e}_i$, $i = 1, \ldots, n$, $v_{n+1} = \dot{\bar{e}}_n$ are established with $\varepsilon = 0$. The theorem statement with $\varepsilon = 0$ is obtained due to the equality

$$\hat{\zeta} = \frac{1}{d}(v_{n+1} - (a_1\bar{e}_1 + a_2\bar{e}_2 + \cdots + a_n\bar{e}_n)) \tag{30}$$

which is established now due to (19). The estimation error is obtained from the homogeneity reasoning as in the proof of Theorem 3.4. □

Similarly to Theorems 3.6, 3.7 the following theorems are obtained.

Theorem 3.9: *Let Assumptions* 3.1, 3.3 *be satisfied, the parameters be chosen as in Theorem* 3.8 *and the output be measured without measurement errors at discrete sampling times with a sufficiently small sampling interval* τ. *Then after a finite-time transient the canonical observation errors* e_{Oi} *are of the order of* $\tau^{n+k-i+2}$, *and the estimation error of the input* ζ *is of the order of* τ^{k+1}.

Theorem 3.10: *Under the conditions of Theorem* 3.8 *let the nonlinear observer part* (39) *be realized* (*in a computer*) *by means of the Euler integration method with the integration time step being a constant part of* τ. *Then the statements of Theorem* 3.9 *are preserved.*

The linear part (11) of the observer can be realized using any advanced integration methods.

Example: Consider the simplest case when $n=1$, and

$$\dot{x} = ax + bu(t) + d\zeta(t),$$

where $y = x$ is measured. Suppose that the unknown input $\zeta(t)$ (perturbation) is a Lipschitzian function (Assumption 3.3, $k=0$). Then the observer obtains the form

$$\begin{aligned}
&\dot{z} = az + bu(t) + l(x(t) - z),\\
&\dot{v}_1 = w_1 = -1.5M^{1/2}|v_1 - x(t) + z|^{1/2}\operatorname{sign}(v_1 - x(t) + z) + v_2,\\
&\dot{v}_2 = -1.1M\operatorname{sign}(v_2 - w_1),\\
&\hat{x} = z + v_1,\ \hat{\zeta} = \frac{1}{d}(v_2 - (a-l)v_1),
\end{aligned}$$

where $l > a$, and M is a sufficiently large constant, $M > |d|\zeta_1^+$. Note that excluding w_1 the third equation can be rewritten as $\dot{v}_2 = -1.1M\operatorname{sign}(v_1 - x(t) + z)$. The proposed estimation $\hat{x}$ of x can be considered as a smoothed value of the measured output $x(t)$. In the absence of noises such estimation is obviously redundant.

4. Observer design for the strong-detectability case

4.1 *Observation of coordinates*

Introduce a new assumption generalizing Assumption 3.1.

Assumption 4.1 System (1) has relative degree r with respect to the unknown input, $r < n$. It is also minimum phase.

Remark 1: This Assumption is equivalent to the stability of the invariant zeros of the system, and also means that the system is strongly detectable in the sense of Definition 3. Note also that Assumption 3.1 is obtained with $r = n$.

Let $r \le n$ be the relative degree of system (1) with respect to the unknown input $\zeta(t)$, which means that

$$\begin{bmatrix} CD \\ CAD \\ \vdots \\ CA^{r-2}D \end{bmatrix} = 0, \quad CA^{r-1}D \neq 0. \tag{31}$$

As previously, define

$$\dot{z} = Az + Bu + L(y - Cz). \tag{32}$$

It is easy to prove that neither the relative degree, nor the observability index change for the error system matrix triplet $(A - LC, D, C)$. Indeed, the invariancy of the relative degree is shown exactly as in the proof of Theorem 3.4, and the observability index is preserved, since the unobservable subspace does not change. Let the transformation $(x_1^T, x_2^T)^T = Tx = [T_1\ T_2]^T x$ transfer the system $(A - LC, D, C)$ into the form (4), (5). This means that

$$\begin{aligned}
A - LC &= T^{-1}\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}T, \quad B = T^{-1}\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}\\
D &= T^{-1}\begin{bmatrix} D_1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} C_1 & 0 \end{bmatrix}T
\end{aligned} \tag{33}$$

where $A_{21} \in \mathbb{R}^{(n-r)\times r}$, $A_{22} \in \mathbb{R}^{(n-r)\times(n-r)}$, matrices $A_{11} \in \mathbb{R}^{r\times r}$, $A_{12} \in \mathbb{R}^{r\times(n-r)}$, and C_1^T, $D_1 \in \mathbb{R}^r$ are of the form

$$A_{11} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ a_1 & a_2 & a_3 & \cdots & a_r \end{bmatrix},$$

$$A_{12} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ a_{r+1} & a_{r+2} & \cdots & a_n \end{bmatrix},$$

$$D_1 = \begin{bmatrix} 0 & \cdots & 0 & d \end{bmatrix}^T, \quad d \neq 0,\ C_1 = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}. \tag{34}$$

The nonlinear part of the observer (13) is chosen as

$$
\begin{aligned}
\dot{v}_1 &= w_1 = -\alpha_r M^{1/r}|v_1 - y(t) + Cz|^{(r-1)/r}\operatorname{sign}(v_1 - y(t) \\
&\quad + Cz) + v_2, \\
\dot{v}_2 &= w_2 = -\alpha_{r-1} M^{1/(r-1)}|v_2 \\
&\quad - w_1|^{(r-2)/(r-1)}\operatorname{sign}(v_2 - w_1) + v_3, \\
&\vdots \\
\dot{v}_{r-1} &= w_{r-1} = -\alpha_2 M^{1/2}|v_{r-1} \\
&\quad - w_{r-2}|^{1/2}\operatorname{sign}(v_{r-1} - w_{r-2}) + v_r, \\
\dot{v}_r &= -\alpha_1 M \operatorname{sign}(v_r - w_{r-1}),
\end{aligned} \tag{35}
$$

v_i, z_i and w_i being the components of the vectors v, $z \in \mathbb{R}^r$ and $w \in \mathbb{R}^{r-1}$ respectively. The parameter M is chosen sufficiently large, in particular $M > |CA^{r-1}D|\zeta^+$. In order to estimate the rest of coordinates introduce

$$\omega_1 = (v_1, \dots, v_r)^T, \quad \omega_2 \in \mathbb{R}^{n-r}, \tag{36}$$

set the gain matrix $K = T^{-1}$, and let

$$
\begin{aligned}
\dot{\omega}_2 &= A_{21}\omega_1 + A_{22}\omega_2, \\
\hat{x} &= z + T^{-1}\begin{bmatrix}\omega_1 \\ \omega_2\end{bmatrix}.
\end{aligned} \tag{37}
$$

Let $n_O = \operatorname{rank} P$, $r \le n_O \le n$, and let P_{n_O} be the submatrix of P consisting of its first n_O rows. Recall that $x_O = P_{n_O}x$ is the vector of the canonical observation coordinates. The vector $e_O = P_{n_O}(\hat{x} - x)$ is naturally called the canonical observation error. Unfortunately, this time only the first r canonical coordinates are estimated exactly, which coincide with the first r components of Tx.

Theorem 4.2: *Let Assumptions* 4.1 *and* 3.2 *be satisfied and the output be measured with a noise, being a Lebesgue-measurable function of time with the maximal magnitude ε. Then with properly chosen α_j and M sufficiently large, the state x of the system is asymptotically estimated by the observer* (32)–(37). *In particular, the first r canonical observation coordinates are estimated exactly and in finite time in the absence of noises. In the presence of a small noise with the magnitude ε the estimation error e_{Oi} of the order $\varepsilon^{(r-i+1)/r}$ is obtained with $1 \le i \le r$. All other coordinate observation errors tend asymptotically to zero with $\varepsilon = 0$ and, after some transient, are of the order of $\varepsilon^{1/r}$ in the presence of small noises. The same limit coordinate estimation accuracy of the order of $\varepsilon^{1/r}$ is ensured for any nonspecific coordinate system.*

Note that Theorem 3.4 is actually a particular case of Theorem 4.2.

Proof: The proof is developed in three steps.

Step 1: The stability of the estimation error of (32) is analyzed. Define $\tilde{e} = x - z$. The error dynamics is described by (17), (18). The boundedness of the estimation error $\tilde{e}$ of the Luenberger observer (11) follows from the stability of the matrix $A - LC$ exactly as in the proof of Theorem 3.4.

Step 2: The error system $(A - LC, D, C)$ is minimum-phase. Indeed its zero-dynamics is obviously the same as of the system (A, D, C).

Step 3: Show the convergence of (36) and (37) to the real values of the state. Denote $\theta = T\tilde{e}$, $\theta = (\theta_1^T, \theta_2^T)^T$, $\theta_1 = x_1 - T_1 z \in \mathbb{R}^r$, $\theta_2 = x_2 - T_2 z \in \mathbb{R}^{n-r}$. Then the error θ_1 satisfies the equation

$$
\begin{aligned}
\dot{\theta}_1 &= A_{11}\theta_1 + A_{12}\theta_2 + D_1\zeta, \\
y &= C_1\theta_1
\end{aligned} \tag{38}
$$

with the matrices of the form (5). Similarly to the proof of Theorem 3.4 after a finite-time transient the equalities

$$v_1 = \theta_{11}, \; v_2 = \theta_{12}, \cdots v_r = \theta_{1r}$$

are established in the absence of noises, in other words $\omega_1 = v = \theta_1$ is obtained. As a result, the convergence is ensured of the resulting error e_{Oi}, $i = 1, \dots, r$ to zero in finite time in the absence of noises. Similarly to Theorem 3.4 the estimation errors e_{Oi}, $i = 1, \dots, r$, of the order of $\varepsilon^{(r-i+1)/r}$ are obtained in the presence of noises of the magnitude ε.

Denote $\rho_1 = \theta_1 - v = (e_{O1}, \dots, e_{Or})^T \to 0$, $\rho_2 = x_2 - \omega_2$. Subtracting the first equation of (37) from (4), obtain that ρ_2 satisfies the equation

$$\dot{\rho}_2 = A_{21}\rho_1 + A_{22}\rho_2,$$

where $\rho_1 \to 0$ in finite time, and A_{22} is Hurwitz due to Step 1. Thus also $\rho_2 \to 0$, which proves the theorem in the absence of noises. In the presence of noises, after the finite-time transient the accuracy $O(\varepsilon^{1/r})$ is obtained in estimation of ρ_1, which ends the proof. □

Let the output y be sampled with the constant time step τ. Substituting $y(t_j)$ for y in (11), (12), $t \in [t_j, t_{j+1})$, $t_{j+1} - t_j = \tau$, obtain a discrete-sampling observer. The following theorem is proved similarly to Theorem 3.6.

Theorem 4.3: *Let Assumptions* 4.1, 4.2 *be satisfied, the parameters be chosen as in Theorem* 3.4 *and the output be measured without measurement errors at discrete sampling times with a sufficiently small sampling* *interval τ. Then after a finite-time transient the canonical observation errors $e_{Oi} = CA^{i-1}(\hat{x} - x)$, $i = 1, \dots, r$, are of the order of τ^{r-i+1}. The estimation error of the other coordinates is of the order of τ after some transient. The same limit coordinate estimation accuracy is ensured for any nonspecific coordinate system.*

A theorem is also true similar to Theorem 3.7 (see also the subsection 4.2).

Remark 2: In the presence of noises or with discrete sampling the finite-time convergence means that for each initial conditions the transient time converges to some constant value with $\varepsilon \to 0$ or $\tau \to 0$, contrary to the asymptotic convergence, when the transient time tends to infinity.

4.2. *Observation of the unknown input*

Let Assumptions 4.1 and 3.3 hold. Let $v \in \mathbb{R}^{r+k+1}$ satisfy the nonlinear differential equation

$$\begin{aligned}
\dot{v}_1 &= w_1 = -\alpha_{r+k+1} M^{1/(r+k+1)} |v_1 - y(t) \\
&\quad + Cz|^{(r+k)/(r+k+1)} \operatorname{sign}(v_1 - y(t) + Cz) + v_2, \\
\dot{v}_2 &= w_2 = -\alpha_{r+k} M^{1/(n+k)} |v_2 \\
&\quad - w_1|^{(r+k-1)/(n+k)} \operatorname{sign}(v_2 - w_1) + v_3, \\
&\vdots \\
\dot{v}_r &= w_r = -\alpha_{k+2} M^{1/(k+2)} |v_r \\
&\quad - w_{r-1}|^{(k+1)/(k+2)} \operatorname{sign}(v_r - w_{r-1}) + v_{r+1}, \\
&\vdots \\
\dot{v}_{r+k} &= w_{r+k} = -\alpha_2 M^{1/2} |v_{r+k-1} \\
&\quad - w_{r+k-2}|^{1/2} \operatorname{sign}(v_{r+k-1} - w_{r+k-2}) + v_{r+k}, \\
\dot{v}_{r+k+1} &= -\alpha_1 M \operatorname{sign}(v_{r+k+1} - w_{r+k}),
\end{aligned} \tag{39}$$

where M is a sufficiently large constant. As previously (39) has recursive form, and the parameters α_i are chosen in the same way (Levant 2003). In particular, one of the possible choices is $\alpha_1 = 1.1$, $\alpha_2 = 1.5$, $\alpha_3 = 2$, $\alpha_4 = 3$, $\alpha_5 = 5$, $\alpha_6 = 8$, which is sufficient for $r + k \leq 5$. The estimation of the input ζ is defined as

$$\hat{\zeta} = \frac{1}{d}(v_{r+1} - (a_1 v_1 + a_2 v_2 + \cdots + a_r v_r)) \tag{40}$$

The following theorems are obtained similarly to previously formulated analogous theorems.

Theorem 4.4: *Let Assumptions* 4.1, 3.3 *be satisfied. Then the observer* (32) *to* (34), (39), (36), (37), (40) *provides with any sufficiently small ε for the accuracy e_{Oi} of the order $\varepsilon^{(r+k-i+2)/(r+k+1)}$, $1 \leq i \leq r$, which is established in finite time. All other coordinate observation errors tend asymptotically to zero with $\varepsilon = 0$ and, after some transient, are of the order of $\varepsilon^{(k+2)/(r+k+1)}$ in the presence of small noises. The same limit accuracy of the order of $\varepsilon^{(k+2)/(r+k+1)}$ is obtained in estimation of any nonspecific coordinates. The unknown input is asymptotically exactly estimated with $\varepsilon = 0$, and, after some transient, the error is of the order of $\varepsilon^{(k+1)/(r+k+1)}$ in the presence of small noises.*

The proof is very similar to the proof of Theorem 4.2. The only difference is that (38) implies the identity

$$\begin{aligned}
\dot{\theta}_{1r} &= a_1\theta_{11} + a_2\theta_{12} + \cdots + a_r\theta_{1r} + a_{r+1}\theta_{21} \\
&\quad + \cdots + a_n\theta_{2,n-r} + d\zeta.
\end{aligned}$$

Taking into account that $v_i - \theta_{1i} \to 0$, $i = 1, \dots, r$, $v_{r+1} - \dot{\theta}_{1r} \to 0$, obtain (40).

Theorem 4.5: *Let Assumptions* 3.3, 4.1 *be satisfied, the parameters be chosen as in Theorem* 4.4 *and the output be measured without measurement errors at discrete sampling times with a sufficiently small sampling interval τ. Then after a finite-time transient the observation errors e_{Oi} are of the order of $\tau^{r+k-i+2}$, $1 \leq i \leq r$, and the estimation error of the input ζ is of the order of τ^{k+1} after some transient process. The estimation error of other coordinates is of the order of τ^{k+2} after some transient. The same limit coordinate estimation accuracy is ensured for any nonspecific coordinate system.*

Theorem 4.6: *Under the conditions of Theorem* 4.5 *let the nonlinear observer part* (39) *be realized* (*in a computer*) *by means of the Euler integration method with the integration time step being a constant part of τ. Then the statements of Theorem* 4.5 *are preserved.*

The linear part (11) of the observer can be realized using any advanced integration methods.

Remark 3: Note that with $r = n_O$ the matrix A_{12} in (33), (34) can be zeroed (the canonical observation form). As a result the unknown input is estimated in finite time. That means that while the limit accuracy of the input estimation stated in the Theorems remains the same, it is obtained after finite-time transient process.

Remark 4: The parameter identification algorithms presented by Davila *et al.* (2006) could be directly applied in the absence of perturbations.

5. Multiple-unknown input-multiple-output case

Consider the system with m outputs and the same number of unknown inputs

$$\dot{x} = Ax + Bu + D\zeta, \quad y = Cx, \tag{41}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^q$, $\zeta, y \in \mathbb{R}^m$, the matrices are of the suitable dimensions. The observability matrix for system (41) takes on the form

$$P = \begin{bmatrix} c_1 \\ c_1 A \\ \vdots \\ c_1 A^{n-1} \\ \vdots \\ c_m \\ c_m A \\ \vdots \\ c_m A^{n-1} \end{bmatrix}$$

where c_i, $i = 1, \ldots, m$ are the rows of the matrix C. Recall that the vector output $y = Cx$ is said to have the vector relative degree $(r_1, \ldots, r_m)$ with respect to the input ζ, if

$$c_i A^s D_j = 0, \quad i, j = 1, 2, \ldots, m, \; s = 0, 1, \ldots, r_i - 2 \tag{42}$$

and

$$\det Q \neq 0, \quad Q = \begin{bmatrix} c_1 A^{r_1-1} D_1 & \ldots & c_1 A^{r_1-1} D_m \\ & \ldots & \\ c_m A^{r_m-1} D_1 & \ldots & c_m A^{r_m-1} D_m \end{bmatrix}. \tag{43}$$

Lemma 5.1: *Let the output y of (41) have the vector relative degree $r = (r_1, \ldots, r_m)$ with respect to the unknown input ζ. Then the vectors $c_1, \ldots, c_1 A^{r_1-1}, \ldots, c_m, \ldots, c_m A^{r_m-1}$ are linearly independent.*

Proof: Suppose that the contrary is true, i.e.

$$\lambda_{11} c_1 + \lambda_{12} c_1 A + \cdots + \lambda_{1r_1} c_1 A^{r_1-1} + \cdots + \lambda_{m1} c_m + \lambda_{m2} c_m A + \cdots + \lambda_{mr_m} c_m A^{r_m-1} = 0. \tag{44}$$

Prove that $\lambda_{ij} = 0$. Multiply (44) by $D_1, \ldots, D_m$ and obtain m equalities. Due to (42) obtain that the rows of the matrix Q are linearly dependent with the dependence coefficients $\lambda_{1r_1}, \ldots, \lambda_{mr_m}$, which contradicts to (43). Thus $\lambda_{1r_1} = \cdots = \lambda_{mr_m} = 0$. Now multiply (44) by $AD_1, \ldots, AD_m$ and obtain new m equalities. Taking into account (42) obtain new linear dependence of the rows of Q with the coefficients $\lambda_{1r_1-1}, \ldots, \lambda_{mr_m-1}$. In that case the rows $(c_i A^{r_i-1} D_1, \ldots, c_i A^{r_i-1} D_m)$ corresponding to $r_j = 1$ do not appear. [Continuing this process, we obtain that all $\lambda_{ij} = 0$.] □

We restrict ourselves to the case when the relative degree exists. In that case the minimum-phase property is equivalent to strong detectability. The strong-observability case is simpler, and is similarly considered.

Assumption 5.2: *The vector relative degree with respect to the unknown inputs exists and equals $(r_1, \ldots, r_m)$. The system is minimum phase.*

This Assumption implies the strong detectability of the system. According to Lemma 5.1, the total relative degree $r = r_1 + \cdots + r_m$ with respect to the unknown input ζ does not exceed the observability index $n_O = \operatorname{rank} P$. Moreover, the same is component-wise true for some vector observation index.

Assumption 5.3: *Let $r_M = \max r_i$. Each unknown input (perturbation) $\zeta_i(t)$ is a bounded function, $|\zeta_i(t)| \leq \zeta_i^+$, with bounded $(r_M - r_i + k)$ successive derivatives, the last one being Lipschitzian. This means that its derivative $\zeta_i^{(r_M - r_i + k + 1)}(t)$ exists almost everywhere and is a bounded Lebesgue-measurable function, $|\zeta_i^{(r_M - r_i + k + 1)}(t)| \leq \zeta_{1i}^+$. Other derivatives are also supposed to be bounded by the same constant.*

Choose a matrix L such that $A - LC$ be a Hurwitz matrix. Such a matrix exists due to the detectability of the system (Assumption 5.2). As previously, the linear Luenberger part of the observer takes on the form

$$\dot{z} = Az + Bu + L(y - Cz). \tag{45}$$

It is shown exactly as in the scalar case that the system $(A - LC, D, C)$ keeps the relative degree, observability index, unobservable subspace $Px = 0$ and the minimum-phase property of the original system (A, D, C). The corresponding error system is

$$\dot{e} = (A - LC)e + D\zeta,$$

where $e = x - z$. Then in some new coordinates $\left[e_C^T \; e_N^T\right]^T = [T_C^T \; T_N^T]^T e = Te$, this system takes on the standard form

$$\begin{matrix} \dot{e}_C = A_C e_C + A_{CN} e_N + D_C \zeta \\ \dot{e}_N = A_{NC} e_C + A_N e_N \end{matrix}, \quad y = C_C e_C,$$

where the canonical observation errors $e_{Ci} = (e_{Ci1}^T, \ldots, e_{Cir_i}^T)^T \in \mathbb{R}^{r_i}$ are calculated as $e_{Cij} = c_i A^{j-1}(x - z)$, and $e_N \in \mathbb{R}^{n-r}$. The corresponding matrices have the form

$$\begin{bmatrix} A_C & A_{CN} \\ A_{NC} & A_N \end{bmatrix} = T(A - LC)T^{-1}, \tag{46}$$

$$A_C = \begin{bmatrix} A_{11} & \cdots & A_{1,m} \\ & \ddots & \\ A_{m,1} & \cdots & A_{mm} \end{bmatrix}, \quad A_{CN} = \begin{bmatrix} A_{C1} \\ \vdots \\ A_{Cm} \end{bmatrix}$$
$$D_C = \begin{bmatrix} D_{11} & \dots & D_{m1} \\ & \ddots & \\ D_{1m} & \dots & D_{mm} \end{bmatrix} \tag{47}$$

$$A_{ii} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \\ a_{ii,1} & a_{ii,2} & \cdots & a_{ii,r_i} \end{bmatrix}, \tag{48}$$

$$A_{i,j} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \\ a_{ij,1} & a_{ij,2} & \cdots & a_{ij,r_j} \end{bmatrix}, \ i \neq j,$$
$$A_{C,j} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \\ a_{Cj,1} & a_{Cj,2} & \cdots & a_{Cj,n-r} \end{bmatrix}, \tag{49}$$

$$y_j = C_{Cj}e_{Cj}, \quad C_{Ci} = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix},$$
$$D_{ij} = \begin{bmatrix} 0 & \cdots & 0 & d_{ij} \end{bmatrix}^T. \tag{50}$$

Here $d_{ij} = c_i A^{r_i - 1} D_j$, and other matrices do not have any specific form. The matrix A_N is Hurwitz similarly to the proof of Theorem 4.2 (Assumption 5.2). The nonlinear observer part takes on the form

$$\dot{v}_i = W_i(v_i, y_i(t) - C_i z,),$$
$$v = (v_1, \dots, v_m)^T \in \mathbb{R}^{m(r_M+k)}. \tag{51}$$

The auxiliary variable v_i is a solution of the discontinuous vector differential equation

$$\dot{v}_{i,1} = w_{i,1} = -\alpha_{r_M+k+1} M_i^{1/(r_M+k+1)} |v_{i,1} - y_i(t) + C_i z|^{(r_M+k)/(r_M+k+1)} \times \mathrm{sign}(v_{i,1} - y_i(t) + z_{i,1}) + v_{i,2},$$
$$\dot{v}_{i,2} = w_{i,2} = -\alpha_{r_M+k} M_i^{1/(r_M+k)} |v_{i,2} - w_{i,1}|^{(r_M+k-1)/(r_M+k)} \mathrm{sign}(v_{i,2} - w_{i,1}) + v_{i,3},$$
$$\vdots$$
$$\dot{v}_{i,r_M+k} = w_{i,r_M+k-1} = -\alpha_2 M_i^{1/2} |v_{i,r_M+k} - w_{i,r_M+k-1}|^{1/2} \times \mathrm{sign}(v_{i,r_M+k-1} - w_{i,r_M+k-2}) + v_{i,r_M+k},$$
$$\dot{v}_{i,r_M+k+1} = -\alpha_1 M_i \mathrm{sign}(v_{i,r_M+k} - w_{i,r_M+k}), \tag{52}$$

where M_i are sufficiently large constants, and the parameters α_i are chosen in the same way as in Levant (2003). Denote

$$\omega_{1i} = (v_{i,1}, \dots, v_{i,r_i})^T, \quad \omega_1 = (\omega_{11}^T, \dots, \omega_{1m}^T)^T \in \mathbb{R}^r,$$
$$\omega_2 \in \mathbb{R}^{n-r},$$
$$\bar{v} = (v_{1,r_1+1}, \dots, v_{m,r_m+1})^T. \tag{53}$$

then the system for the observation of e_N takes on the form

$$\dot{\omega}_2 = A_{CN}\omega_1 + A_N\omega_2. \tag{54}$$

The coordinates are estimated as

$$\hat{x} = z + T^{-1}\begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}. \tag{55}$$

The unknown inputs ζ_i, $i = 1, 2, \dots, m$, are estimated by $\hat{\zeta}_i$,

$$\hat{\zeta} = \bar{D}_C^{-1}(\bar{v} - \bar{A}_C\omega_1) \tag{56}$$

where the matrices $\bar{A}_C$, $\bar{A}_N$ are composed of the bottom rows of $A_{i,j}$ and $A_{C,j}$ composing the block matrices A_C and A_{CN} respectively, $\bar{D}_C$ is built of the nonzero elements of D_C, $\det D_C \neq 0$ (Assumption 5.2),

$$\bar{A}_C = \begin{bmatrix} a_{11} & \dots & a_{1m} \\ & \ddots & \\ a_{m1} & \dots & a_{mm} \end{bmatrix}, \quad a_{ij} = (a_{ij,1}, \dots, a_{ij,r_i})^T,$$

$$\bar{A}_N = \begin{bmatrix} a_{C1,1} & \dots & a_{C1,n-r} \\ & \ddots & \\ a_{C,m,1} & \dots & a_{C,m,n-r} \end{bmatrix}, \quad \bar{D}_C = \begin{bmatrix} d_{1,1} & \dots & d_{1,m} \\ & \ddots & \\ d_{m,1} & \dots & d_{m,m} \end{bmatrix}. \tag{57}$$

Recall once more that $x_O = Px$ is the vector of the canonical observation coordinates (not all of them are independent). The vector $e_O = P(\hat{x} - x)$ is naturally called the canonical observation error. Its linearly independent part naturally includes the coordinates $e_{Oij} = C_i A^{j-1}(\hat{x} - x)$, $j = 1, \dots, r_i$, $i = 1, \dots, m$.

Theorem 5.4: *Under Assumptions* 5.2, 5.3 *with properly chosen parameters observer* (45)–(57) *provides after finite-time transient for the accuracies*

$$|e_{Oij}| \sim \varepsilon^{(r_M+k+2-j)/(r_M+k+1)}, \quad j = 1, 2, \dots, r_i,$$
$$\|\hat{x}_N - x_N\| \sim \varepsilon^{2/(r_M+k+1)}$$

with ε being the magnitude of the measurement noise, and for the accuracies

$$|e_{Oij}| \sim \tau^{(r_M+k+2-j)}, \quad j = 1, 2, \dots, r_i, \ \|\hat{x}_N - x_N\| \sim \tau^{k+2}$$

with the sampling interval τ in the absence of noises. All other coordinates are estimated asymptotically with the accuracies $\varepsilon^{(k+2)/(r_M+k+1)}$ and τ^{k+2} respectively. The inputs ζ_i are asymptotically estimated with the accuracies of the order $\varepsilon^{(r_M+k+1-r_i)/(r_M+k+1)}$ and $\tau^{(r_M+k+1-r_i)}$ respectively. Exact estimations of the canonical observable coordinates, as well as asymptotically exact estimations of the unobservable coordinates and the inputs are obtained with continuous measurements and $\varepsilon = 0$.

The proof is very similar to the previous theorems and is omitted. Also here finite-time convergent exact observation of the inputs is possible with $r = \operatorname{rank} P$. The accuracy estimation can be refined taking into account that each input may have its own smoothness order.

6. Examples

6.1 *Example 1*

Consider the system

$$\begin{aligned} \dot{x} &= Ax + D\zeta(t) + Bu, \\ y &= Cx \end{aligned} \tag{58}$$

with matrices

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 6 & 5 & -5 & -5 \end{bmatrix}$$
$$B = D = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^T, \quad C_O = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$$

and initial conditions $x_O(0) = [1 \ \ 0 \ \ 1 \ \ 1]$. Its eigenvalues are $-3, -2, -1, 1$. The relative degree r with respect to the unknown input equals 4. In consequence, the system is strongly observable. Note that A is not stable. The "unknown" input

$$\zeta = \cos 0.5t + 0.5 \sin t + 0.5$$

is taken, being obviously a bounded smooth function with bounded derivatives. It is taken for simplicity that $u = 0$.

6.1.1 State estimation.

In this subsection only the state observation problem will be considered. Let the output of the system be affected by a deterministic noise

$$0.1 \ \sin(1037|\cos(687t)|)$$

of amplitude $\varepsilon = 0.1$. The correction factor $L = \begin{bmatrix} 5 & 5 & 5 & 5 \end{bmatrix}^T$ provides for the eigenvalues $-1, -2, -3, -4$ of the matrix $A - LC$. The gain matrix K in (12) is chosen as

$$K = \begin{bmatrix} C \\ C(A - LC) \\ C(A - LC)^2 \\ C(A - LC)^3 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 5 & 1 & 0 & 0 \\ 5 & 5 & 1 & 0 \\ 5 & 5 & 5 & 1 \end{bmatrix}$$

The parameters $\alpha_1 = 1.1$, $\alpha_2 = 1.5$, $\alpha_3 = 2$, $\alpha_4 = 3$, $M = 2$ are chosen, and the nonlinear observer part (13) takes on the form

$$\begin{aligned} \dot{v}_1 &= w_1 = -3 \cdot 2^{1/4} |v_1 - y(t) \\ &\quad + Cz|^{3/4} \operatorname{sign}(v_1 - y(t) + Cz) + v_2, \\ \dot{v}_2 &= w_2 = -2 \cdot 2^{1/3} |v_2 - w_1|^{2/3} \operatorname{sign}(v_2 - w_1) + v_3, \\ \dot{v}_3 &= w_3 = -1.5 \cdot 2^{1/2} |v_3 - w_2|^{1/2} \operatorname{sign}(v_3 - w_2) + v_4, \\ \dot{v}_4 &= -1.1 \cdot 2 \operatorname{sign}(v_4 - w_3). \end{aligned}$$

The observer performance and finite-time convergence for the sampling time interval $\tau = 0.001$ are depicted in figure 1. Figure 2 shows the detail of the state convergence graph. Note that in the

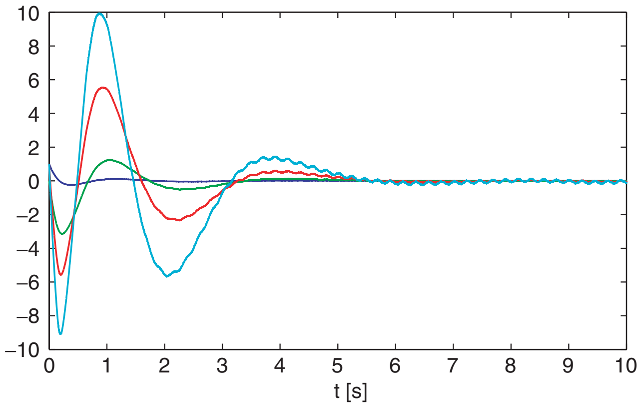

Figure 1. State estimation errors in the presence of a deterministic noise of amplitude 10^{-1}.

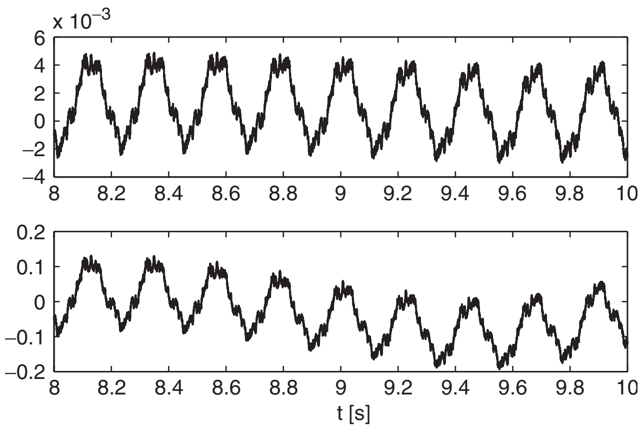

Figure 2. Detail of observer error graphs. Estimation error of x_2 (above). Estimation error of x_4 (below).

correspondence with the proof of Theorem 3.4 the estimation error of x_2 converges to a bounded region of order $5 \cdot 10^{-3}$, while the estimation error in x_4 converges to a bounded region of order $2 \cdot 10^{-1}$. The transient process is shown in figure 3 for the states x_1 and x_4. It is seen from figure 4 that the system trajectories and their derivatives of any order tend to infinity. Thus, the differentiator could not alone perform the observation. Figure 5 shows the effect of discretization in observation. The sampling time intervals $\tau = 0.0001$ and $\tau = 0.01$ were taken in the absence of noises.

6.1.2 Unknown input estimation. Consider now the input ζ as a bounded function with a Lipschitzian derivative, $k = 1$. Both the state x and the disturbance ζ are now estimated.

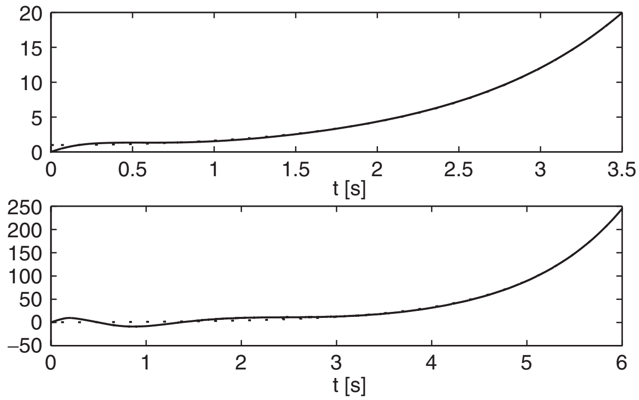

Figure 3. Convergence of $\hat{x}_1$, $\hat{x}_4$ to x_1 and x_4.

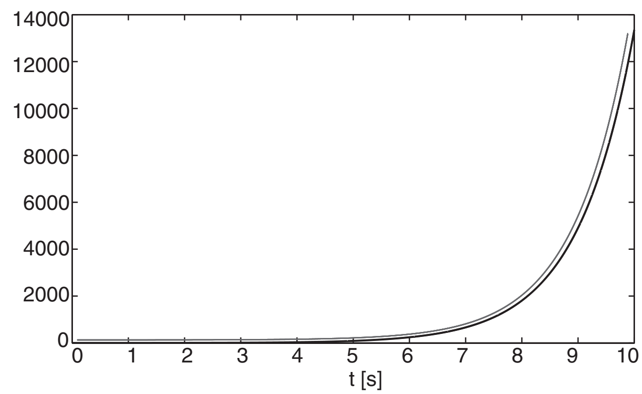

Figure 4. System coordinates.

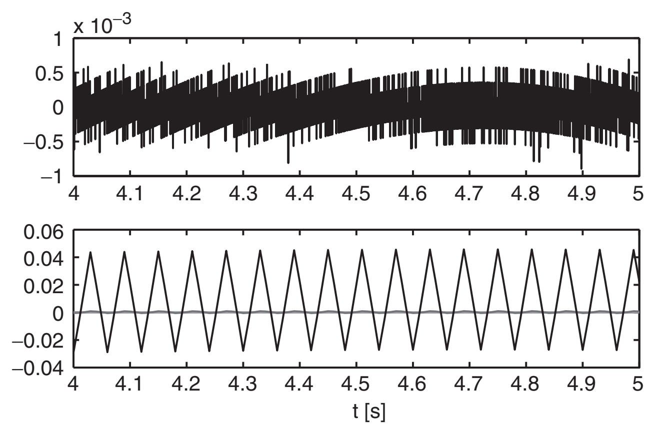

Figure 5. Observer errors (detail) with sampling intervals $\tau = 0.0001$ (above) and $\tau = 0.01$ (below).

The state observer (11), (12) is designed in the same way with $L = \begin{bmatrix} 5 & 5 & 5 & 5 \end{bmatrix}^T$, providing for the eigenvalues $-1, -2, -3, -4$ of the matrix $A - LC$. The gain matrix K in (12) is also the same.

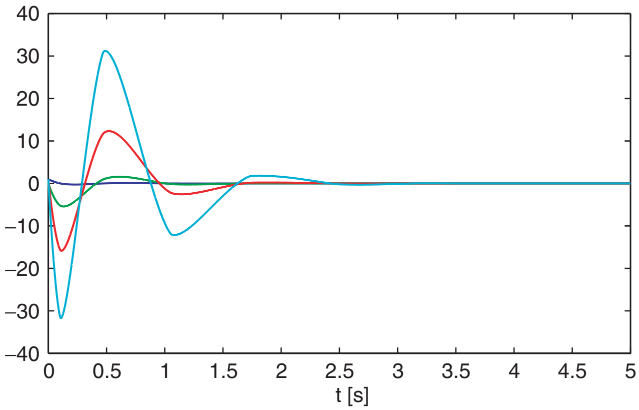

Figure 6. Observer errors for the unknown input estimation case.

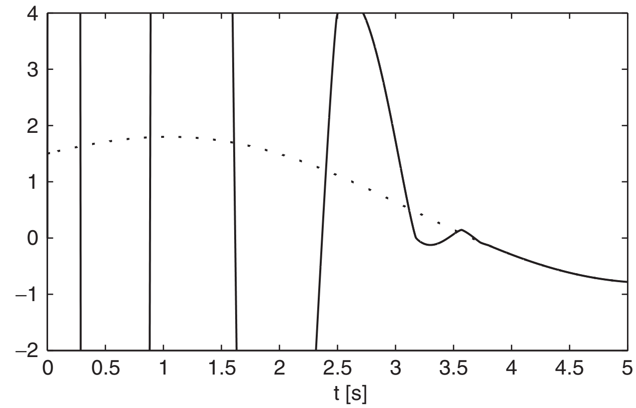

Figure 7. Unknown input estimation.

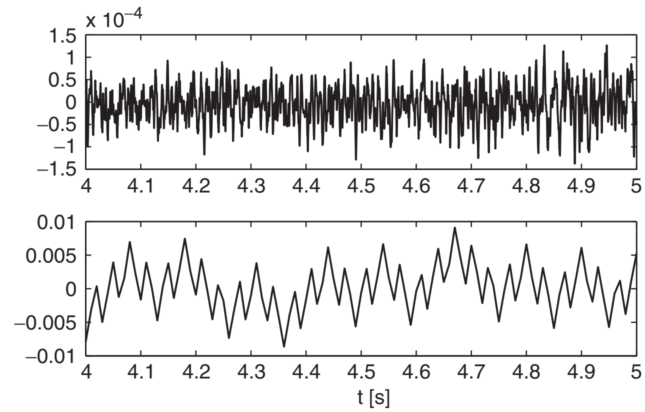

Figure 8. Unknown input estimation error with $\tau = 0.0001$ (above) and 0.01 (below).

The parameters of (26) $\alpha_1 = 1.1$, $\alpha_2 = 1.5$, $\alpha_3 = 2$, $\alpha_4 = 3$, $\alpha_5 = 5$, $\alpha_6 = 8$, $M = 1$ are chosen, so that

$$
\begin{aligned}
\dot{v}_1 &= w_1 = -8|v_1 - y(t) + Cz|^{5/6}\operatorname{sign}(v_1 - y(t) + Cz) + v_2,\\
\dot{v}_2 &= w_2 = -5|v_2 - w_1|^{4/5}\operatorname{sign}(v_2 - w_1) + v_3,\\
\dot{v}_3 &= w_3 = -3|v_3 - w_2|^{3/4}\operatorname{sign}(v_3 - w_2) + v_4,\\
\dot{v}_4 &= w_4 = -2|v_4 - w_3|^{2/3}\operatorname{sign}(v_4 - w_3) + v_5,\\
\dot{v}_5 &= w_5 = -1.5|v_5 - w_4|^{1/2}\operatorname{sign}(v_5 - w_4) + v_6,\\
\dot{v}_6 &= -1.1\operatorname{sign}(v_6 - w_5).
\end{aligned}
$$

The finite-time convergence of estimated states to the real states is shown in figure 6 with the sampling interval $\tau = 0.001$. The unknown-input estimation is obtained using the relation (29). It is demonstrated in figure 7. The effects of discretization are shown in figure 8 for the sampling intervals $\tau = 0.0001$ and $\tau = 0.01$.

6.2. *Example 2*

Consider the system

$$
\begin{aligned}
\dot{x}(t) &= Ax(t) + D\zeta(t) + Bu,\\
y &= Cx(t)
\end{aligned}
\tag{59}
$$

with matrices

$$
A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 7 & -17 & 17 & -2 & 0 & 0\\ 1 & 3 & 4 & 5 & -1 & 3\\ 1 & 3 & 4 & 5 & 0 & -3 \end{bmatrix}
$$

$$
B = D = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}^T,
$$

$$
C_O = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}
$$

and initial conditions $x(0) = [1, 1, 1, 1, 1, 1]^T$. The observability matrix has rank $n_O = 4$. The relative degree with respect to the unknown input equals 3, $r = 3$. Note that A is unstable. The zeroes of the system are located at $-1, -2, -3$; Assumption 4.1 is satisfied, i.e. the system is strongly detectable. The "unknown" input

$$
\zeta = \cos 0.5t + 0.5 \sin t + 0.5
$$

is taken, being obviously a bounded smooth function with bounded derivatives. It is taken for simplicity that $u = 0$.

6.2.1 State estimation. The correction factor $L = \begin{bmatrix} 8 & 36 & 97 & 313 & 0 & 0 \end{bmatrix}^T$ provides for the eigenvalues $-1, -3, -1, -3, -4, -2$ of the matrix $A - LC$. The gain

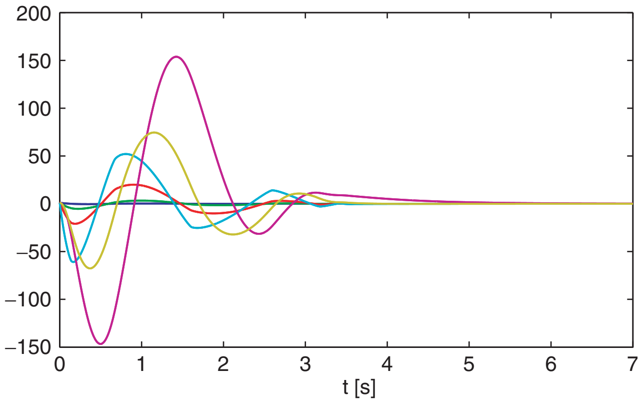

Figure 9. Observer errors with unstable A.

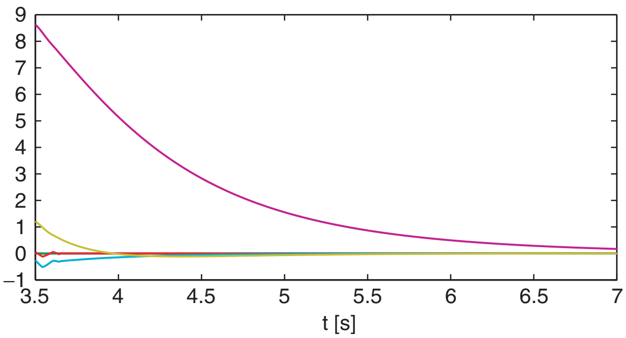

Figure 10. Observer errors with unstable A (Detail).

matrix K in (12) is chosen as

$$K = T^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 8 & 1 & 0 & 0 & 0 & 0 \\ 36 & 8 & 1 & 0 & 0 & 0 \\ 97 & 36 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The parameters of the nonlinear observer part (13) are as follows: $\alpha_1 = 1.1$, $\alpha_2 = 1.5$, $\alpha_3 = 2$, $M = 2$. The nonlinear part of the observer is designed as

$$\begin{aligned} \dot{v}_1 &= w_1 = -2 \cdot 2^{1/3} |v_1 - y(t) + Cz|^{2/3} \operatorname{sign}(v_1 - y(t) \\ &\quad + Cz) + v_2, \\ \dot{v}_2 &= w_2 = -1.5 \cdot 2^{1/2} |v_2 - w_1|^{1/2} \operatorname{sign}(v_2 - w_1) + v_3, \\ \dot{v}_3 &= -1.1 \cdot 2 \operatorname{sign}(v_3 - w_2). \end{aligned}$$

The observer performance and finite-time convergence for a sample time $\tau = 0.001$ can be seen from figure 9, see the detail in figure 10. The transient process

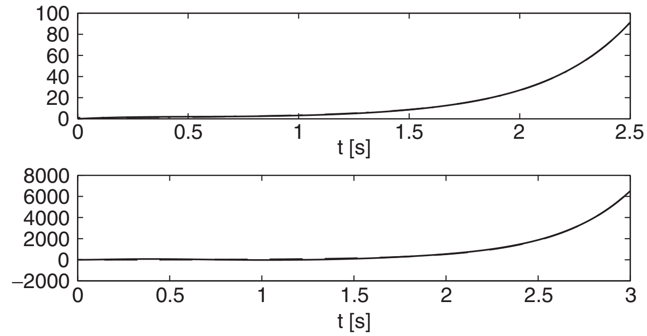

Figure 11. Convergence of $\hat{x}_1$, $\hat{x}_6$ to x_1 and x_6 with unstable A.

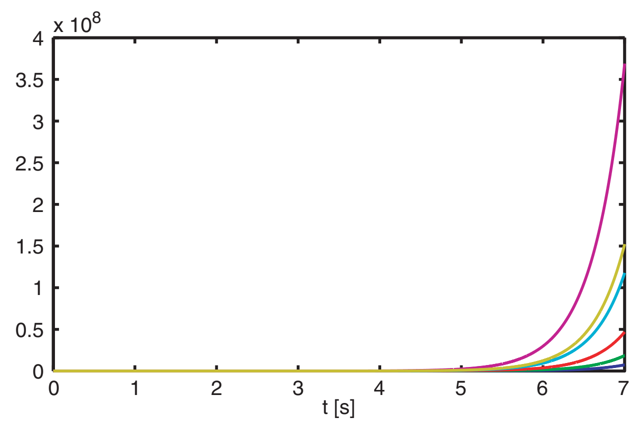

Figure 12. System coordinates with unstable A.

is shown in figure 11 for the states x_1 and x_6. It is seen from figure 12 that the system trajectories and their derivatives of any order tend to infinity. Thus, the differentiator could not alone perform the observation.

7. Conclusions

Sufficient and necessary conditions of the strong observability and strong detectability are reformulated in the terms of the plant relative degree with respect to the unknown input. High-order-sliding-mode observers are proposed for LTI SISO systems with unknown inputs under such conditions. The global finite-time-convergent exact observation of the state is provided under sufficient and necessary conditions of strong observability and the boundedness of unknown inputs. In the case of the strong detectability only a part of the states are observed exactly, while other estimations are asymptotically exact. An additional Luenberger-like linear term is introduced ensuring global convergence of the observer error to a bounded neighborhood of zero. The robust exact sliding-mode-based differentiator (Levant 2003) is applied providing for the finite-time convergence of the observation error in the presence of unknown inputs for the strong observable systems. Moreover, the identification algorithm is proposed, ensuring global finite-time exact identification of the smooth unknown inputs in the case of strongly observable systems. Global asymptotic convergence of estimations to the exact values of the system states and unknown inputs is obtained in the case of strong detectability. The effects of bounded deterministic Lebesgue-measurable noises and discrete sampling are estimated in the terms of the accuracy of the proposed observer. The results are extended to the MIMO case.

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