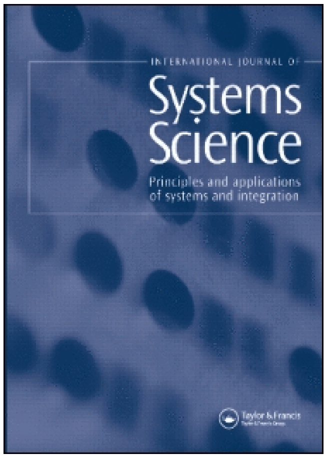

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Hierarchical second-order sliding-mode observer for linear time invariant systems with unknown inputs

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The problem of observability for systems with unknown inputs is revised. The sufficient and necessary conditions are used for the design of an observer for linear systems with bounded unknown inputs. To realize the observation of the state, a second-order sliding-mode observer is suggested to be applied. Such an observer provides a robust estimate of the state vector in a finite time, *without filtration*. The design is based on the concept of the hierarchical output injection maintaining zero value for output tracking error at each level of the hierarchy. The equivalent control is used to identify the unknown inputs. A numerical example illustrates the effectiveness of the suggested technique.

Keywords: Variable structure systems; Observation; Identification; State estimation

1. Introduction

1.1 *Antecedents and motivations*

The problem of state observation for systems with unknown inputs is one of the most important in modern control theory during the last two decades. The specific features of the majority of the suggested observers are:

(1) The number of unknown inputs of the system must be less than the number of outputs, moreover, other special conditions are required to achieve the robustness of such observer (see e.g., Hautus 1983, Zasadzinski *et al*. 1994). Those conditions turn out to be very restrictive because they cannot cover the simplest class of mechanical systems with unknown inputs, in which only the position is measurable. To cover this situation, in Rapaport and Gouze (1999), an adaptive observer that ensures an exponential convergence of the estimation error to a small neighborhood of zero was suggested.
(2) Usually the observers have an estimation error that converges asymptotically to zero (Nijmeijer 1999). Such observers force the designer either to check the stability of the closed-loop system (plant plus observer), or to prove that the, so-called, separation principle holds. In Engel and Kreisselmeier (2002), an idea concerning the finite-time state estimation *in the absence of uncertainties* was suggested. It consists in the simultaneous consideration of two asymptotic observers, where the second one contains a *delay* in its dynamics. In fact, this is some sort of a derivative approximation that makes such an approach unrealizable in the presence of unknown, even bounded, inputs (external perturbations).
(3) The asymptotic convergence of the observers means that the unknown inputs can be reconstructed in the best case approximately (see e.g., Rapaport and Gouze 1999).

The problem of observation has been actively developed within *Variable Structure Theory* using the

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Sliding Mode approach. Sliding-mode observers [(see, e.g., the corresponding chapters in the textbooks Edwards and Spurgeon (1998), Utkin *et al.* (1999), and the recent tutorials Edwards *et al.* (2002), Barbot *et al.* (2002), Xiong and Saif (2001) and Poznyak (2004)] are widely used due to their attractive features: (a) insensitivity (more than robustness!) with respect to unknown inputs; (b) possibilities to use the values of the equivalent output injection for the unknown inputs identification; (c) finite time convergence to the values of the state vectors. In Hashimoto *et al.* (1990), Utkin *et al.* (1999) and Floquet and Barbot (2006) a step-by-step form of sliding-mode observers was proposed. Such observers are based on the transformation of a given system to a block observable form and the sequential estimation of each state by using the value of the equivalent output injection. On the one hand, this scheme allows to formulate some sufficient conditions for the designing of an observer for linear time invariant systems with unknown inputs (LTISUI). Such conditions were formulated in Barbot *et al.* (1996), Ahmed-Ali and Lamnabhi-Lagarrigue (1999) and Utkin *et al.* (1999) for the scalar case. On the other hand, the realization of that scheme causes obligatory filtration due to the nonidealities and further it requires the system transformation to the triangular form.

In Levant (1998, 2003), a robust exact arbitrary order differentiator was designed ensuring finite time convergence to the values of the corresponding derivatives, and some applications of higher order sliding algorithms were considered. A new generation of observers based on second-order sliding-mode algorithms has been recently designed and applied to some practical applications (Alvarez *et al.* 2000, Orlov *et al.* 2003, Pisano and Usai 2004, Cannas *et al.* 2005, Davila *et al.* 2005).

1.2 *Main contribution*

In this article we proposed a scheme for the design of a robust observer, ensuring convergence in finite time to the state of the system. The advantages of the proposed observer are enumerated below:

(i) the conditions required for the design of the observer suggested are sufficient and necessary for the case when the unknown inputs are bounded;
(ii) a convergence in finite time to the exact system state, even in the presence of unknown inputs;
(iii) the transformation of the initial system to any canonical form is not needed; only in the case when the zero-input response is unstable, a stabilization term has to be included;
(iv) for the observer realization, the filtration is not obligatory;
(v) an algorithm for unknown input identification is proposed.

1.3 *Structure of this article*

In section 2, the model description and the problem formulation are presented. Section 3 deals with the design of the hierarchical sliding-mode concept. The subsection 3.3 is devoted to the specific algebraic form that takes the observer proposed here. Identification of the unknown inputs is studied in section 4. Finally, section 5 is related to a numerical illustration of the method proposed here.

2. Model description and problem formulation

2.1 *Model of the plant*

Let us consider the following linear time invariant system with unknown inputs

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + Dw(t), \quad x(0) = x_0 \\ y(t) &= Cx(t), \quad t \geq 0 \end{aligned} \tag{1}$$

where $x(t) \in R^n$ is the state vector, $u(t) \in R^m$ is the control, $y(t) \in R^p$ $(1 \leq p < n)$ is the output of the system. The pair $\{u(t), y(t)\}$ is assumed to be measurable (available) at any time $t \geq 0$. $w(t) \in R^q$ is the vector of the unknown inputs. The matrices A, B, C, and D are known matrices of suitable dimension with $\operatorname{rank}(C) = p$, and $\operatorname{rank}(D) = q$. The current state $x(t)$, as well as the initial state x_0 are not measurable (available).

Throughout this article it is assumed that:

A1. the triple (A, C, D) has no invariant **zeros**, that is

$$\text{for all } s \in C, \operatorname{rank} P(s) = n + q, P(s) := \begin{bmatrix} sI - A & -D \\ C & 0 \end{bmatrix} \tag{2}$$

A2. there exists a constant w^+ such that

$$\|w(t)\| \leq w^+ \tag{3}$$

2.2 *Necessary and sufficient conditions for the exact observation*

The following extended observer matching condition was formulated for Hashimoto *et al.* (1990), Barbot *et al.* (2002) and Rapaport and Gouze (1999)

$$\left[C^T (CA)^T \cdots \left(CA^{n-2}\right)^T\right] D = 0 \tag{3}$$

where n is the observability index.

The condition (3) may be less restrictive in some case than the standard conditions given in Hautus (1983) and Edwards *et al.* (2002) (the triple (A, C, D) has no invariant zeros and $\operatorname{rank}(CD) = q$). In fact the conditions established by Hautus

$$\operatorname{rank} CD = q \text{ and the triple } (A, D, C) \text{ has no invariant zeros}$$

are necessary and sufficient for designing an observer whose only input is the output of the original system. But we can design an observer satisfying the condition (3) since the sliding-modes works as a substitute of a differentiator, so such a sort of observer will note only use the output of the system but also use the derivatives of the output of the system. That is why our aim in this article is to design one observer having as the only restriction the absence of invariant zeros for the triple (A, C, D); even if the system does not satisfy any matching condition.

In the remainder of this subsection we will consider that $u(t) = 0$ since the effect of $u(t)$ can be compensated by any observer. That is, consider the following linear system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Dw(t), \quad x(0) = x_0 \\ y(t) &= Cx(t) \end{aligned} \tag{4}$$

We recall some definitions corresponding to properties of (1) (Molinari 1976, Hautus 1983, Hautus and Silverman 1983, Trentelman *et al.* 2001).

Definition 2.1: The system (4) is called strongly observable if, and only if for all initial condition x_0 and for every unknown input $w(t)$, the condition $y(t) = 0$ for all $t \geq 0$ implies that $x(t) = 0$ for all $t \geq 0$.

Definition 2.2: $\mathcal{V}$ is a null-output (A, D) invariant subspace if for every $x \in \mathcal{V}$ there exists some w such that $(Ax + Dw) \in \mathcal{V}$ and $Cx = 0$. $\mathcal{V}^*$ is the maximal null-output (A, D) invariant subspace, i.e. $\mathcal{V} \subseteq \mathcal{V}^*$ for each subspace $\mathcal{V}$.

The following statements are equivalent (see, e.g., Molinari 1976, Hautus 1983, Hautus and Siverman 1983, Trenteleman *et al.* 2001).

(i) The system (4) is strongly observable.
(ii) The triple (A, D, C) has no invariant zeros,
(iii) $\mathcal{V}^* = 0$.

It means that if the triple (A, C, D) has invariant zeros, then there exists an initial condition $x_0 = \xi$ and an unknown input $w(t)$ such that $y(t) = 0$ for all $t \geq 0$ and $x(t)$ being not equal to zero for all $t \geq 0$. So, in that case it would be impossible to make an estimation of the state $x(t)$ independently of $w(t)$. Therefore, the condition A1 is a necessary condition for designing any observer for which we can modify the degree of convergence of the estimation error. Now, we will see that condition A1 is also a sufficient condition to design an observer.

Consider the following algorithm that takes the decoupled part to the unknown inputs from the successive derivatives of the output. First, we make a definition. For any matrix $F \in R^{r \times q}$ having $\operatorname{rank} F = h$, we define $F^{\perp} \in R^{r-h \times r}$ as a matrix such that $F^{\perp} F = 0$ and $\operatorname{rank} F^{\perp} = r - h$.

Algorithm 1:

0. $y = Cx$, define $M_{A,1} := C$
1. $\mathrm{d}/\mathrm{d}t (CD)^{\perp} y(t) = (CD)^{\perp} CAx(t)$ and construct the extended vector

$$\begin{bmatrix} \dfrac{\mathrm{d}}{\mathrm{d}t} (CD)^{\perp} y(t) \\ y(t) \end{bmatrix} = \underbrace{\begin{bmatrix} (CD)^{\perp} CA \\ C \end{bmatrix}}_{\overset{\|}{M_{A,2}}} x(t)$$

k. $\mathrm{d}/\mathrm{d}t (M_{A,k} D)^{\perp} M_{A,k} x(t) = (M_k D)^{\perp} M_{A,k} A x(t)$ and construct the extended vector

$$\begin{bmatrix} \dfrac{\mathrm{d}}{\mathrm{d}t} \left(M_{A,k} D\right)^{\perp} M_{A,k} x(t) \\ y(t) \end{bmatrix} = \underbrace{\begin{bmatrix} \left(M_{A,k} D\right)^{\perp} M_k A \\ C \end{bmatrix}}_{\overset{\|}{M_{A,k+1}}} x(t)$$

Now, suppose that for some l there is a matrix $M_{A,l}$ generated by the previous algorithm such that $\operatorname{rank} M_{A,l} = n$. It would mean that the algebraic equation

$$\begin{bmatrix} \dfrac{\mathrm{d}}{\mathrm{d}t} \left(M_{A,l-1} D\right)^{\perp} M_{A,l-1} x(t) \\ y(t) \end{bmatrix} = M_{A,l} x(t)$$

would have a unique solution for $x(t)$ that could be found by means of the pre-multiplication of both sides of the equation by $M_{A,l}^{+} := (M_{A,l}^T M_{A,l})^{-1} M_{A,l}^T$. That is,

$$M_{A,l}^{+} \begin{bmatrix} \dfrac{\mathrm{d}}{\mathrm{d}t} \left(M_{A,l-1} D\right)^{\perp} M_{A,l-1} x(t) \\ y(t) \end{bmatrix} = x(t) \tag{5}$$

It means that under the assumption that $\operatorname{rank} M_{A,l} = n$, the state $x(t)$ could be estimated using a linear combination of the output and its derivatives. Next we will see that the existence of a constant l satisfying $\operatorname{rank} M_{A,l} = n$ is equivalent to the nonexistence of invariant zeros for the triple (A, C, D).

Algorithm 2 (Molinari 1976)**:**

0. Define $M_{A,1} = C$
k. Define

$$M_{A,k+1}(A) = \begin{bmatrix} (M_{A,k}D)^{\perp} M_{A,k}A \\ C \end{bmatrix}, \quad k = 1, \ldots, n-1. \tag{6}$$

The following statement was proved in Molinari (1976).

$$\mathcal{V}^* = \ker M_{A,n} \tag{7}$$

Due to the equivalences (ii) and (iii), it follows that

$$(A, D, C) \text{ has no invariant zeros iff } \operatorname{rank} M_{A,n} = n. \tag{8}$$

Thus, from A1 and (8), we conclude that in our case $\operatorname{rank} M_{A,n} = n$.

3. Hierarchical second-order sliding-mode concept

3.1 *Auxiliary dynamic systems*

Firstly, design the following dynamic system

$$\begin{aligned} \dot{\tilde{x}}(t) &= A\tilde{x}(t) + Bu(t) + K(y(t) - C\tilde{x}(t)), \\ \tilde{x}(0) &= C^{+}y(0) \end{aligned} \tag{9}$$

where $C^{+} = (C)K$ must be designed such that

A3. the eigenvalues of $\tilde{A} := (A - KC)$ have negative real part.

Let $r(t) = x(t) - \tilde{x}(t)$, then, from (1) and (9), the dynamic equations governing $r(t)$ are

$$\dot{r}(t) = [A - KC]r(t) + Dw(t) = \tilde{A}r(t) + Dw(t)$$

Since A2 is fulfilled, it is well known that $r(t)$ is of bounded norm, i.e., there exist some constants $\gamma, \eta, \mu > 0$ such that

$$\|r(t)\| \leq \gamma \exp(-\mu t)\|r(0)\| + \eta w^{+} \tag{10}$$

Thus, the inequality in (10) yields the following statement. For any constant r^{+} satisfying $r^{+} > \eta w^{+}$ we have

$$\|r(t)\| < r^{+} \quad \text{for all } t > -\frac{1}{\mu}\ln\frac{r^{+} - \eta w^{+}}{\gamma\|r(0)\|} \tag{11}$$

This means that if $r^{+} > \eta w^{+}$, from a finite time, $r(t)$ will be bounded by r^{+} and it will remain in that situation for all the next time.

Since we are going to use the super twisting algorithm (Levant 1993), we will need a bound of $\|r(t)\|$, thus, (11) ensures that we can always satisfy such a requirement.

3.2 *Procedure to recover $M_k x(t)$*

As was shown in (5), the state can be recovered by a linear combination of the output and its derivatives independently of the unknown inputs. In this article we proposed to use the super twisting algorithm as a substitute of a differentiator.

Due to the insertion of the term $\tilde{x}$, in the procedure described below appears $\tilde{A}$ instead of A. That is, we will recover the vectors $M_{\tilde{A},k}x(t)$ instead of the vectors $M_{A,k}x(t)$, where $M_{\tilde{A},k}$ is the matrix obtained using the algorithm 2 with $\tilde{A}$ instead of A. Thus, we will show that the invariant zeros of (A, D, C) are the same as the invariant zeros of $(\tilde{A}, D, C)$, indeed,

$$\begin{aligned} \operatorname{rank}\begin{bmatrix} sI - \tilde{A} & -D \\ C & 0 \end{bmatrix} &= \operatorname{rank}\left\{\begin{bmatrix} I & K \\ 0 & I \end{bmatrix}\begin{bmatrix} sI - A & -D \\ C & 0 \end{bmatrix}\right\} \\ &= \operatorname{rank}\begin{bmatrix} sI - A & -D \\ C & 0 \end{bmatrix} \quad \text{for all } s \in C \end{aligned}$$

Thus, this statement and (8) yield

$$(\tilde{A}, D, C) \text{ has no invariant zeros iff } \operatorname{rank} M_{\tilde{A},n} = n \tag{12}$$

Since $\operatorname{rank} M_{\tilde{A},n} = n$, we can define l as the least positive integer such that $\operatorname{rank} M_{\tilde{A},l} = n$. From (6) we can infer that for the recovery of $M_{\tilde{A},k+1}x(t)$, one needs first to recover $M_{\tilde{A},k}x(t)$ for $k = 1, \ldots, l-1$; therefore, this subsection is devoted to the recovery of the vectors $M_{\tilde{A},k}x(t)$ $(k = 1, \ldots, l)$. The first vector $M_{\tilde{A},1}x(t)$ is already known since $M_{\tilde{A},1} = C$.

The recovery of $M_{\tilde{A},2}x(t)$ will be based on the design of a *sliding surface* $s^{(1)}$ and its corresponding *output injection* $v^{(1)}$ using the *"super-twisting" algorithm* (Levant 1993). The components of $v^{(1)}$ are defined as

$$v_i^{(1)} = z_i^{(1)} + \lambda_1 \left|s_i^{(1)}\right|^{1/2} \operatorname{sign}\left(s_i^{(1)}\right), \quad \dot{z}_i^{(1)} = \alpha_1 \operatorname{sign}\left(s_i^{(1)}\right) \tag{13}$$

the variable $s^{(1)}$ is given by the formula

$$s^{(1)}(y(t),\tilde{x}(t)) = \begin{pmatrix} (CD)^{\perp}[y(t) - C\tilde{x}(t)] \\ \int_{\tau=0}^{t}[y(\tau) - C\tilde{x}(\tau)]\mathrm{d}\tau \end{pmatrix} - \int_{\tau=0}^{t} v^{(1)}(\tau)\,\mathrm{d}\tau \tag{14}$$

Here, we note that all the solutions of the dynamic systems are defined in Filippov's sense (Filippov 1988). The dimension of the vector $v^{(1)}$ is the same as the dimension of $s^{(1)}$ and this is equal to the number of rows of $(CD)^{\perp}$ that depends on the specific values that the matrices of the triple (A, C, D) take. Thus, in view of (1), (9), and (6) the time derivative of $s^{(1)}$ is

$$\dot{s}^{(1)}(t) = M_{\tilde{A},2}[x(t) - \tilde{x}(t)] - v^{(1)}(t) \tag{15}$$

Now, choose the scalar gains λ_1, α_1 so that the following conditions are fulfilled

$$\begin{aligned} &\alpha_1 > \beta_1 \geq M_{\tilde{A},2}\left(\|\tilde{A}\| r^{+} + \|D\| w^{+}\right) \\ &\lambda_1 > \frac{(\alpha_1 + \beta_1)(1+\theta)}{(1-\theta)}\sqrt{\frac{2}{\alpha_1 - \beta_1}}, \quad 0 < \theta < 1 \end{aligned} \tag{16}$$

where r^{+} satisfies (11). In view of (11), one can always satisfy (16) in finite time. It was shown in Levant (1993) that if the bounds in (16) are fulfilled, then there exists a finite time t_1 such that the following equalities

$$s^{(1)}(t) = \dot{s}^{(1)}(t) = 0, \quad t \geq t_1 \tag{17}$$

hold, where t_1 is the reaching time. From (13), it is clear that if $s^{(1)} = 0$, then $v^{(1)} \equiv z^{(1)}$, so the vector $M_{\tilde{A},2}x(t)$ can be recovered from (15) in the following manner:

$$M_{\tilde{A},2}x(t) - M_{\tilde{A},2}\tilde{x}(t) = z^{(1)}(t) \quad \text{for } t \geq t_1 \tag{18}$$

Now, for recovering $M_{\tilde{A},3}x(t)$ we design a sliding surface $s^{(2)}$ and its corresponding *output injection* $v^{(2)}$

$$v_i^{(2)} = z_i^{(2)} + \lambda_1\left|s_i^{(2)}\right|^{1/2}\operatorname{sign}\left(s_i^{(2)}\right), \quad \dot{z}_i^{(2)} = \alpha_1 \operatorname{sign}\left(s_i^{(2)}\right)$$

The variable $s^{(2)}$ is given by the formula

$$s^{(2)}\left(y(t), z^{(1)}(t)\right) = \begin{pmatrix} \left(M_{\tilde{A},2}D\right)^{\perp} z^{(1)}(t) \\ \int_{\tau=0}^{t}(y(\tau) - C\tilde{x}(\tau))\mathrm{d}\tau \end{pmatrix} - \int_{\tau=0}^{t} v^{(2)}(\tau)\mathrm{d}\tau \tag{19}$$

By the substitution of (18) into (19), $s^{(2)}$ takes the form

$$s^{(2)}\left(y(t), z^{(1)}(t)\right) = \begin{pmatrix} \left(M_{\tilde{A},2}D\right)^{\perp} M_{\tilde{A},2}[x(t) - \tilde{x}(t)] \\ \int_{\tau=0}^{t}(y(\tau) - C\tilde{x}(\tau))\mathrm{d}\tau \end{pmatrix} - \int_{\tau=0}^{t} v^{(2)}(\tau)\mathrm{d}\tau$$

Thus, the derivative of $s^{(2)}$ is

$$\begin{aligned} \dot{s}^{(2)}\left(y(t), z^{(1)}(t)\right) &= \begin{pmatrix} \left(M_{\tilde{A},k}D\right)^{\perp} M_{\tilde{A},2}\tilde{A}[x(t) - \tilde{x}(t)] \\ y(t) - C\tilde{x}(t) \end{pmatrix} - v^{(2)}(t) \\ &= M_{\tilde{A},3}x(t) - M_{\tilde{A},3}\tilde{x}(t) - v^{(2)}(t) \end{aligned}$$

Now, choosing the scalar gains λ_2, α_2 in the following form

$$\begin{aligned} &\alpha_2 > \beta_2 \geq M_{\tilde{A},3}\left(\|\tilde{A}\| r^{+} + \|D\| w^{+}\right) \\ &\lambda_2 > \frac{(\alpha_2 + \beta_2)(1+\theta)}{(1-\theta)}\sqrt{\frac{2}{\alpha_2 - \beta_2}}, \quad 0 < \theta < 1 \end{aligned}$$

one gets that there exists a finite time t_2 such that the following equalities are true

$$s^{(2)}(t) = \dot{s}^{(2)}(t) = 0, \quad t \geq t_2 \geq t_1 \tag{20}$$

Therefore, since in the sliding-motion $v^{(2)}(t) \equiv z^{(2)}(t)$, we have

$$M_{\tilde{A},3}x(t) - M_{\tilde{A},3}\tilde{x}(t) = z^{(2)}(t) \quad \text{for } t \geq t_1$$

We can follow the same procedure recursively to obtain $M_{\tilde{A},k}x(t)$, $k = 1, \ldots, l-1$. Subsequently, we give the general design of the auxiliary system and the sliding surfaces with their corresponding output injection.

(a) Design the *output injection* $v^{(k)}$ *at the k-th level as a "super-twisting" controller* (Levant 1993):

$$v_i^{(k)} = z_i^{(k)} + \lambda_k \left|s_i^{(k)}\right|^{1/2} \operatorname{sign}\left(s_i^{(k)}\right), \quad \dot{z}_i^{(k)} = \alpha_k \operatorname{sign}\left(s_i^{(k)}\right) \tag{21}$$

where λ_k and α_k are constants satisfying

$$\left.\begin{aligned} &\alpha_k > \beta_k \geq M_{k+1}\left(\left\|\tilde{A}\right\| r^+ + \|D\| w^+\right) \\ &\lambda_k > \frac{(\alpha_k+\beta_k)(1+\theta)}{(1-\theta)}\sqrt{\frac{2}{\alpha_k-\beta_k}}, \quad 0<\theta<1 \end{aligned}\right\} 1 \leq k < l-1$$
$$\left.\begin{aligned} &\alpha_{l-1} > \beta_{l-1} \geq \left\|\tilde{A}\right\| r^+ + \|D\| w^+ \\ &\lambda_{l-1} > \frac{(\alpha_{l-1}+\beta_{l-1})(1+\theta)}{(1-\theta)}\sqrt{\frac{2}{\alpha_{l-1}-\beta_{l-1}}}, \quad 0<\theta<1 \end{aligned}\right\}, \quad k = l-1 \tag{22}$$

where r^+ should satisfy (11).

(b) The variables $s^{(k)}$ and $z^{(k)}$ are related as

$$s^{(k)}\left(y, z^{(k-1)}\right) = \begin{cases} \begin{pmatrix} \left(M_{\tilde{A},1}D\right)^{\perp}[y(t) - C\tilde{x}(t)] \\ \int_{\tau=0}^{t} (y(\tau) - C\tilde{x}(\tau))\mathrm{d}\tau \end{pmatrix} - \int_{\tau=0}^{t} v^{(1)}(\tau)\mathrm{d}\tau, & k=1 \\ \begin{pmatrix} \left(M_{\tilde{A},k}D\right)^{\perp} z^{(k-1)} \\ \int_{\tau=0}^{t} (y(\tau) - C\tilde{x}(\tau))\mathrm{d}\tau \end{pmatrix} - \int_{\tau=0}^{t} v^{(k)}(\tau)\mathrm{d}\tau, & 1<k<l-1 \\ M_{\tilde{A},l}^{+}\begin{pmatrix} \left(M_{\tilde{A},l-1}D\right)^{\perp} z^{(l-2)} \\ \int_{\tau=0}^{t} (y(\tau) - C\tilde{x}(\tau))\mathrm{d}\tau \end{pmatrix} - \int_{\tau=0}^{t} v^{(l-1)}(\tau)\mathrm{d}\tau, & k=l-1 \end{cases} \tag{23}$$

Notice that, since $\operatorname{rank} M_{\tilde{A},l} = n$ the matrix $M_{\tilde{A},l}^{+} = [M_{\tilde{A},l}^{T} M_{\tilde{A},l}]^{-1} M_{\tilde{A},l}^{T}$ is well defined and $M_{\tilde{A},l}^{+} M_{\tilde{A},l} = I$. That is why we included $M_{\tilde{A},l}^{+}$ in the variable $s^{(l-1)}$, this allows obtained directly a representation of the state $x(t)$ that is shown in the following lemma.

The following lemma establishes how the vectors $M_{\tilde{A},k}x(t)$ can be recovered by the second-order sliding motions ($s^{(k)} = \dot{s}^{(k)} = 0$).

Lemma 3.1: *Under the assumptions A1–A3, if the auxiliary state vector* $x_a^{(k)}$ *and the variable* $s^{(k)}$, *for all* $k = 1, \ldots, l-1$, *are designed as in* (21)–(23), *then, from some finite time* t_k, *one has*

$$\begin{aligned} M_{\tilde{A},k+1}x(t) &= M_{\tilde{A},k+1}\tilde{x}(t) + z^{(k)}(t), \quad \text{for } k = 1, \ldots, l-2 \\ x(t) &= \tilde{x}(t) + z^{(k)}(t), \quad \text{for } k = l-1 \end{aligned} \tag{24}$$

Proof: We prove the lemma by induction. For $k = 1$, as it was shown above, there exists a finite time t_1 such that $M_{\tilde{A},2}x(t)$ is recovered by the equation

$$M_{\tilde{A},2}x(t) = M_{\tilde{A},2}\tilde{x}(t) + z^{(1)}(t), \quad t \geq t_1$$

Now, suppose that there exist a finite time t_{j-1} such that (24) is true for some intermediate $k = j-1$. Thus, according to (23), $s^{(j)}$ is in the form

$$s^{(j)}\left(y(t), z^{(j-1)}(t)\right) = \begin{pmatrix} \left(M_{\tilde{A},j}D\right)^{\perp} z^{(j-1)} \\ \int_{\tau=0}^{t} (y(\tau) - C\tilde{x}(\tau))\mathrm{d}\tau \end{pmatrix} - \int_{\tau=0}^{t} v^{(j)}(\tau)\mathrm{d}\tau \tag{25}$$

Substitution of $z^{(j-1)}(t)$, from (24), into (25) yields

$$s^{(j)}(t) = \begin{pmatrix} \left(M_{\tilde{A},j}D\right)^{\perp} M_{\tilde{A},j}[x(t) - \tilde{x}(t)] \\ \int_{\tau=0}^{t} [y(\tau) - C\tilde{x}(\tau)]\mathrm{d}\tau \end{pmatrix} - \int_{\tau=0}^{t} v^{(j)}(\tau)\mathrm{d}\tau$$

for $t \geq t_{j-1}$. Thus, from (1), (9), and (6), the derivative of $s^{(j)}$ is obtained by the equation

$$\dot{s}^{(j)}(t) = M_{\tilde{A},j+1}[x(t) - \tilde{x}(t)] - v^{(j)}(t) \tag{26}$$

Again, as it was shown in Levant (1993), if the condition (22) is satisfied, then *a second-order sliding-mode* is obtained, i.e.

$$s^{(j)}(t) = \dot{s}^{(j)}(t) = 0, \quad t \geq t_j \tag{27}$$

where t_j is the reaching time. Thus, from the structure of $v^{(j)}$, $s^{(j)} = 0$ implies $v^{(j)} \equiv z^{(j)}$. Then in view of (27), the equality (24) for $k=j$ is deduced from (26). In the same form, since $M^{+}_{\tilde{A},l}M_{\tilde{A},l}=I$, designing $s^{(l)}$ and $v^{(l)}$ as in (21)–(23) we can prove that $x(t)=\tilde{x}(t)+z^{(l-1)}(t)$. □

3.3 *Design of the observer*

From (24) we have that

$$x(t)=\tilde{x}(t)+z^{(l-1)}(t) \quad \text{for all } t \geq t_{l-1} \tag{28}$$

where t_{l-1} is the reaching time. Since the right hand side of (28) is known, the observer is designed as

$$\hat{x}(t)=\tilde{x}(t)+z^{(l-1)}(t) \tag{29}$$

Now, we can resume the previous result in the following theorem.

Theorem 3.2: *Under assumptions A1–A3,*

$$\hat{x}(t)=x(t) \quad \textit{for all } t \geq t_{l-1} \tag{30}$$

Proof: It follows immediately from (28) and (29). □

We conclude this section with the algorithm for the design of the observer given in (29).

Algorithm 1 (Design of the observer)**:**

A. Find the matrices M_k according to the algorithm 2. Determine the value of the least positive integer l so that rank $M_l=n$.
B. Design $\tilde{x}$ according to (9) with the gain K satisfying A3.
C. Design $l-1$ sliding surface $s^{(k)}$ according to (23) and design the output injections $v^{(k)}$ following (21), each one fulfilling (22).
D. Design the observer according to (29).

Remark 1: It should be noticed that the control u is not supposed to be smooth and, in general, the design of the observer does not depend on u.

4. Uncertainty identification

Since now $x(t)$ is available, the uncertainty $w(t)$ can be identified. To this goal, we will use the equivalent control approach (Utkin 1992, Ahmed-Ali *et al.* 2004). Let us design the variable $\bar{x}$ satisfying the following equation

$$\dot{\bar{x}}(t)=A\hat{x}(t)+Bu(t)+D\bar{u}(t)$$

Define a sliding surface $\sigma(t)$ in the form

$$\sigma(t)=D^{+}(\hat{x}(t)-\bar{x}(t))$$

where $D^{+}:=(D^TD)^{-1}D^T$. Since $\hat{x}(t)=x(t)$ for $t \geq t_{l-1}$, and from (1), the time derivative of $\sigma(t)$ is as follows

$$\dot{\sigma}(t)=w(t)-\bar{u}(t) \quad \text{for } t \geq t_{l-1} \tag{31}$$

Thus, $\bar{u}(t)$ is designed in the following form

$$\bar{u}(t)=\frac{\sigma(t)}{\|\sigma(t)\|}$$

with the condition $\varkappa > w^{+}$. Thus, under A2, the following equations hold

$$\sigma(t)=\dot{\sigma}(t)=0 \quad \text{for all } t \geq t_{l-1} \tag{32}$$

Thus, from (31) and (32) one gets the following identity

$$w(t)=\bar{u}_{\text{eq}}(t) \quad \forall t \geq t_{l-1} \tag{33}$$

It is known that the control $\bar{u}(t)$ is a high-frequency signal. To overcome this obstacle, $\bar{u}_{\text{eq}}$ should be substituted by the output of the first-order filter

$$\tau\dot{\bar{u}}_{\text{av}}+\bar{u}_{\text{av}}=\bar{u}, \quad \tau>0$$

For $\tau \to 0$ very small, the output of the filter approaches to the equivalent control $\bar{u}_{\text{eq}}$, i.e., $\lim_{\tau\to 0}\bar{u}_{\text{av}}=\bar{u}_{\text{eq}}$ (see Utkin 1999 and Fridman 2001). That is why the identification of $w(t)$ can be done by means of the substitution of $\bar{u}_{\text{eq}}$ by u_{av} in (33), in other words, the identification of w is achieved by means of the signal $\hat{w}(t)=\bar{u}_{\text{av}}(t)$, where $\lim_{\tau\to 0}(w(t)-\hat{w}(t))=0 \ \forall t \geq t_{l-1}$.

5. Example

Now, we have an academic example that illustrates the method suggested in this manuscript. Thus, the matrices

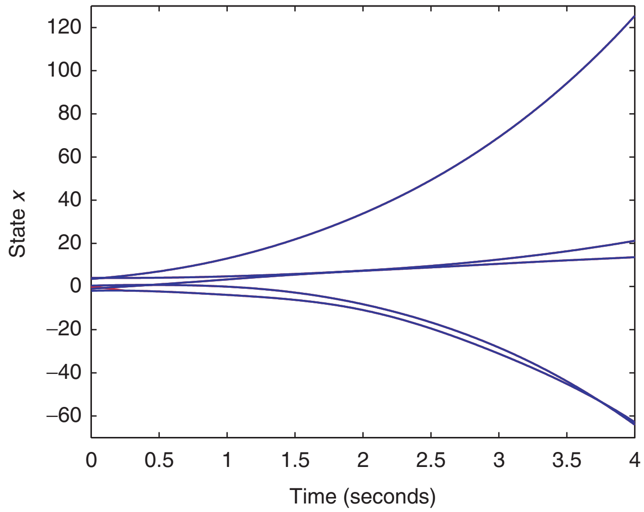

Figure 1. Trajectories of the state x.

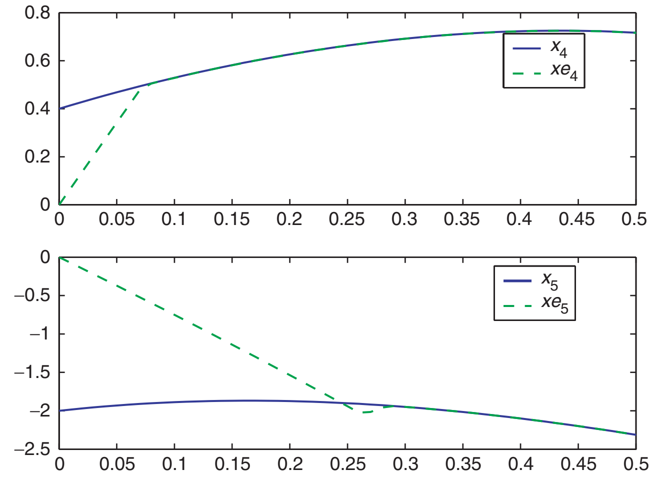

Figure 2. Zoom of the trajectories of the state x and the observer $\hat{x}$.

in (1) take the form

$$A = \begin{bmatrix} 0.52 & 0.78 & 0.22 & 0.62 & -0.43 \\ 0.13 & -0.26 & -0.28 & 0.23 & -0.32 \\ -0.25 & 0.83 & -0.36 & 0.44 & -0.83 \\ -0.77 & 0.58 & -0.25 & -0.24 & -0.47 \\ -0.55 & 0.25 & -0.58 & 0.36 & -0.87 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 \\ 0.6 \\ 1 \\ 1 \\ 0.5 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & -1 \\ 0 & -1 \\ 0 & -1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix},$$

$$u(t) = 3\sin(t), \quad w(t) = \begin{bmatrix} 2\sin(3t) + 0.63 \\ 0.5\cos(t) + 0.23 \end{bmatrix}$$

It can be verified that the triple (A, C, D) has no invariant zeros. Following the algorithm 2, we have that the matrix $M_{\tilde{A},2}$ has rank equal to 5.

$$M_{\tilde{A},1} = C,$$

$$M_{\tilde{A},2} = \begin{bmatrix} -0.842 & 0.4338 & 0.7008 & 0.2327 & 0.1184 \\ -0.1922 & 1.1757 & -0.7073 & 0.1485 & -0.3606 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Thus, we must design only one sliding surface for the construction of the observer. Since in this case $M^{+}_{\tilde{A},2} = M^{-1}_{\tilde{A},2}$, the output injection $v_i^{(1)}$ and the sliding surface take the form

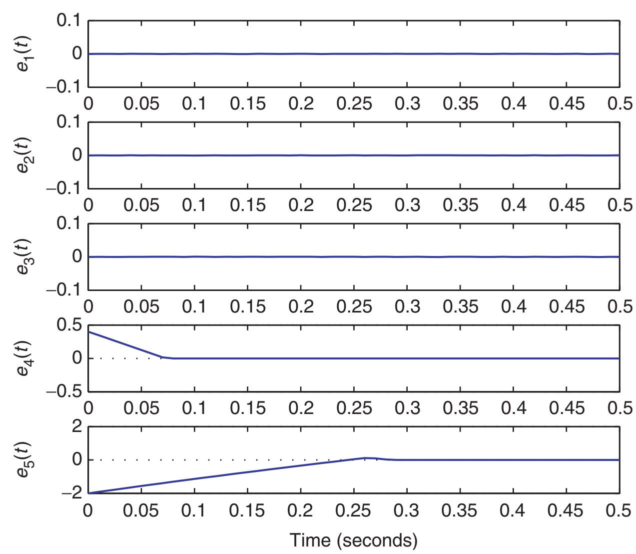

Figure 3. Error of estimation $e(t) = x(t) - \hat{x}(t)$.

$$v_i^{(1)} = z_i^{(1)} + 5\left|s_i^{(1)}\right|^{1/2} \operatorname{sign}\left(s_i^{(1)}\right), \quad \dot{z}_i^{(1)} = 7\operatorname{sign}\left(s_i^{(1)}\right)$$

$$s^{(1)}(t) = M^{-1}_{\tilde{A},2} \begin{pmatrix} (CD)^{\perp} z^{(l-2)} \\ \int_{\tau=0}^{t} (y(\tau) - C\tilde{x}(\tau))\mathrm{d}\tau \end{pmatrix} - \int_{\tau=0}^{t} v^{(1)}(\tau)\mathrm{d}\tau$$

The figure 1 shows the trajectories of the state x. A zoom of the trajectories of the state and the observer is shown in figure 2; here we only depicted the last two states since the three first states are already known. Figure 3 shows

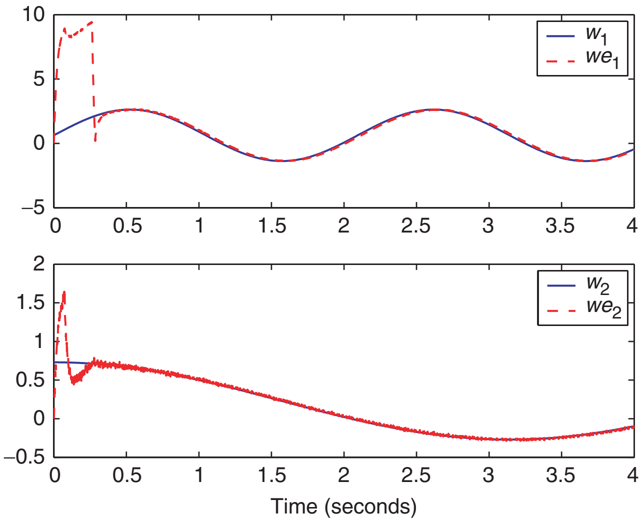

Figure 4. Unknown input w and its estimation $\hat{w}$ (called w_e in the graph).

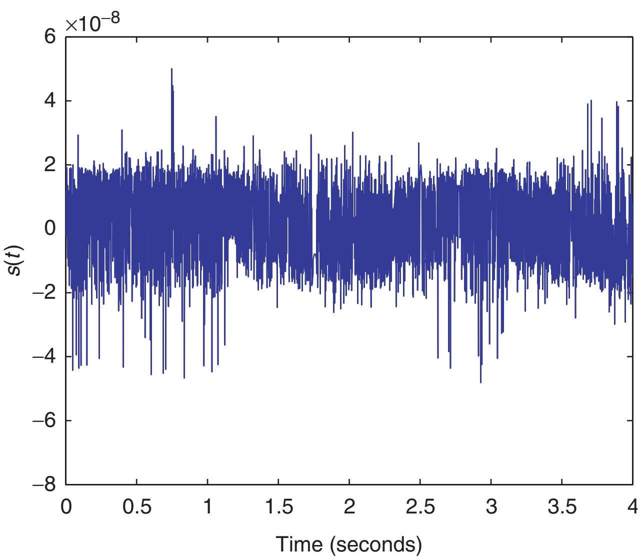

Figure 5. Sliding surface $s^{(1)}(t)$.

the error of state estimation, and we can see that such an error converges to zero in a very short time. The estimation of the unknown input w is shown in figure 4. Finally, figure 5 shows the sliding surface.

6. Conclusions

A new hierarchical approach for the problem of LTISUI is suggested. In the framework of the proposed approach, we show that necessary and sufficient conditions for the design of the observer suggested is the absence of invariant zeros. The observer suggested

- follows a scheme that uses the output of the system and a linear combination of its derivatives to recover the state of the system. In this article we proposed to use the super twisting algorithm as a substitute of a differentiator;
- ensures the insensitivity of the observer with respect to the unknown inputs,
- provides a global convergence of the estimation error to the zero value;
- identifies the unknown inputs since we recover the exact state in afinite time;
- provides the finite time estimation of the states of the system; therefore the "separation principle" (concerning the independence of state estimation and control processes) is fulfilled.

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