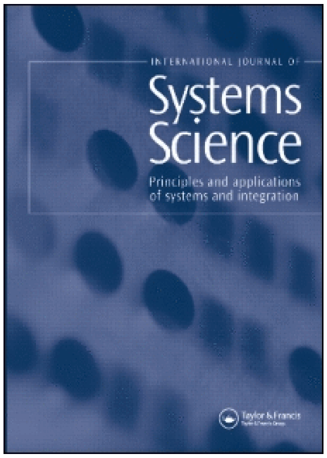

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Sliding mode identification and control for linear uncertain stochastic systems

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This article presents the integral sliding mode technique applied to identify disturbances and robustify the optimal linear quadratic Gaussian controller for linear uncertain stochastic systems, which is compared to the conventional sliding mode approach. The obtained identifier/controller provides a method for estimating uncertainty values and ensures robustness of the system against matched uncertainties, starting from the initial time instant. Numerical simulations illustrating the obtained results are given for the inverted pendulum.

Keywords: Sliding mode observers; Identification; Stochastic noises

1. Introduction

The identification and robust control design using the sliding mode approach has been receiving considerable attention in recent years. A particular interest is given to the integral sliding mode (ISM) technique, which maintains the state trajectory on the sliding manifold from the initial time moment, thus assuring insensitivity against matched uncertainties. The ISM robust control design was initiated in Utkin and Shi (1996) and has been continued for linear deterministic regulators in Basin *et al*. (2003) and linear stochastic filters in Basin *et al*. (2002, 2005). Other recent applications of the sliding mode approach to stochastic systems can be found in Azemi and Yaz (2000), Poznyak (2003), Niu *et al*. (2005), Shi *et al*. (2006). In this article, the sliding mode approach is directed to identify matched disturbances and design a robust controller for linear uncertain stochastic systems.

This article studies application of two sliding mode techniques, ISM and conventional sliding mode (SM), to the state estimation/control and perturbation identification problem for linear stochastic systems with random Gaussian noises and deterministic uncertainties. The algorithm includes the following steps. In both ISM and SM cases, the unmeasured state is estimated using the Kalman–Bucy filter and the linear quadratic Gaussian (LQG) control is constructed as for the nominal system without disturbances. Simultaneously, one of the sliding mode techniques is applied to suppress the matched uncertainties. The sliding mode disturbance estimation method based on a first-order filter (see Uktin 1992) is then used for identifying the matched uncertainties.

The ISM approach rejects the matched uncertainties in the nominal deterministic system from the initial time moment, if the system state is measured, while the conventional SM does the same after a reaching phase. However, since the original system state depends on stochastic noises and is not measured, its real value is unknown. Hence, the control design is based on the asymptotically converging Kalman–Bucy estimate, and both ISM and SM disturbance compensation approaches become asymptotical. As the inverted pendulum example shows, the estimates steadily converge to the real state values in both cases, providing optimization of the quadratic criterion up to the minimum values reachable for the nominal system. A certain advantage of the ISM performance, in comparison to the conventional SM, can be observed for linear stochastic systems as well. Moreover, although

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the disturbance estimation technique also becomes asymptotical, identification of a sinusoidal signal is very clear, especially in the ISM case.

This article is organized as follows. Section 2 presents the problem statements for the optimal and robust LQG control problems, outlines the design principles of the ISM and conventional SM approaches, and recalls the disturbance estimation method involving a first-order filter. In section 3, two problem solutions are obtained based on the ISM and SM techniques. Performance of the obtained algorithms, as well as quality of disturbance identification, are verified for the inverted pendulum system in section 4. The simulation results are discussed in section 5.

2. Problem statement

Let (Ω, F, P) be a complete probability space with an increasing right-continuous family of σ-algebras $F_t, t \geq t_0$, and let $(W_1(t), F_t, t \geq t_0)$ and $(W_2(t), F_t, t \geq t_0)$ be independent Wiener processes. The F_t-measurable random process $(x(t), y(t))$ is described by a perturbed Ito differential equation for the system state

$$\mathrm{d}x(t) = (Ax(t) + Bu(t) + g(t))\mathrm{d}t + G\mathrm{d}W_1(t), \quad x(t_0) = x_0, \tag{1}$$

and a linear Ito differential equation for the observation process

$$\mathrm{d}y(t) = Cx(t)\mathrm{d}t + H\mathrm{d}W_2(t). \tag{2}$$

Here, $x(t) \in R^n$ is the system state, $u(t) \in R^m$, $m \leq n$, is the control input, $y(t) \in R^p$ is the observation vector. The initial condition $x_0 \in R^n$ is a Gaussian vector such that x_0, $W_1(t)$, and $W_2(t)$ are independent. It is assumed that $HH^T(t)$ is a positive definite matrix. All coefficients in (1)–(2) are deterministic time-invariant matrices of appropriate dimensions. The pair $\{A, B\}$ is controllable, the pair $\{A, C\}$ is observable, and $y(t) \in R^p$ is the output observation process. The plant operates under deterministic uncertainties $g(t)$ and stochastic noises $\mathrm{d}W_1(t)$ and $\mathrm{d}W_2(t)$ are represented as weak mean square derivatives (Pugachev 2001) of the Wiener processes, that is, white Gaussian noises.

The following condition is assumed

- The function $g(t)$ represents matched uncertainties such that $g(t) = B\gamma(t)$ and the norm $\|\gamma(t)\|$ is bounded by

$$\|\gamma(t)\| \leq q_a(t), \quad q_a(t) > 0, \tag{3}$$

where $q_a(t)$ is a finite time-dependent function; $\|x\| = \sqrt{(\sum_{i=1}^{n} x_i^2)}$ denotes the Euclidean 2-norm of a vector $x \in R^n$.

Note that the state and observation equations can also be written in the form of differential equations with white noises (Anderson and Moore 1971), which is commonly accepted in engineering applications

$$\dot{x}(t) = Ax(t) + Bu(t) + g(t) + G\upsilon(t), \tag{4}$$

$$y(t) = Cx(t) + H\omega(t), \tag{5}$$

where white Gaussian noises $\upsilon(t)$ and $\omega(t)$ are the weak mean square derivatives (Pugachev 2001) of the Wiener processes $W_1(t)$ and $W_2(t)$, respectively. The problem statements (1), (2) and (4), (5) are equivalent (Astrom 1970).

2.1 *Linear quadratic Gaussian (LQG) problem*

2.1.1 Optimal LQG filtering. Consider a linear stochastic system given by unperturbed linear state and observation equations

$$\mathrm{d}x(t) = (Ax(t) + Bu(t))\mathrm{d}t + G\mathrm{d}W_1(t), \quad x(t_0) = x_0, \tag{6}$$

$$\mathrm{d}y(t) = Cx(t)\mathrm{d}t + H\mathrm{d}W_2(t), \tag{7}$$

where the initial condition vector $x(t_0)$ is a Gaussian random variable with mean η and variance P_{e0}

$$E\{x(t_0)\} = \eta, \tag{8}$$

$$E\{[x(t_0) - \eta][x(t_0) - \eta]^T\} = P_{e0}. \tag{9}$$

Furthermore, $x(t_0)$ is independent of the Wiener processes $W_1(t)$ and $W_2(t)$.

The optimal estimate $\hat{x}(t)$ for system state (6) is designed to *minimize* the error covariance

$$E\Big\{[x(t) - \hat{x}(t)][x(t) - \hat{x}(t)]^T\Big\}. \tag{10}$$

It is well-known that the optimal estimate $\hat{x}(t)$ is given by the Kalman–Bucy filtering equations (see, for example, Anderson and Moore 1971)

$$\mathrm{d}\hat{x}(t) = (A\hat{x}(t) + Bu(t))\mathrm{d}t + K_f[\mathrm{d}y(t) - C\hat{x}(t)\mathrm{d}t], \tag{11}$$

with the initial condition $\hat{x}(t_0) = \eta$, where

$$K_f = P_f(t)C^T\big(HH^T\big)^{-1} \tag{12}$$

is the optimal filter gain, and $P_f(t)$ is the solution to the differential Riccati equation

$$\dot{P}_f(t) = AP_f(t) + P_f(t)A^T + GG^T - P_f(t)C^T(HH^T)^{-1}CP_f(t), \qquad (13)$$

with the initial condition $P_f(t_0) = P_{e0}$.

2.1.2 Optimal LQG control. The quadratic functional to be minimized over the state and control vectors of the system (5) is defined by

$$J = E\left[\int_{t_0}^{\infty} \left(x^T Vx + u^T Wu\right)\mathrm{d}t\right]. \qquad (14)$$

Note that the inputs to the Kalman–Bucy filter are the plant input and output vectors, $u(t)$ and $y(t)$, and its output is the state estimate vector $\hat{x}(t)$. The separation principle claims that the optimal control law can be found taking the estimate $\hat{x}(t)$ instead of the state vector of a system without noises (see, for example, Anderson and Moore 1971, Maciejowski 1994):

$$u(\hat{x}(t), t) = -K_c\hat{x}(t), \qquad (15)$$

where

$$K_c = W^{-1}BP_C, \qquad (16)$$

and P_C satisfies the algebraic Riccati equation

$$AP_c + P_cA^T - P_cBW^{-1}B^TP_c + V = 0. \qquad (17)$$

2.1.3 Robust LQG control. The objective is to find the solution to the LQG control/estimation problem for the system (1) that coincides with the solution of this problem for the system (6) on a sliding mode manifold, i.e., is optimal with respect to the criterion (14) there. The robustness of the obtained solution against deterministic uncertainties is assured using the conventional SM and integral ISM control techniques.

2.2 *ISM and SM approaches to robust control problem*

The disturbance compensation, i.e., robustification of the optimal LQG control can be conducted via the sliding mode technique. Consider a linear perturbed system where stochastic noises are not yet included

$$\dot{x}(t) = Ax(t) + Bu(t) + g(t). \qquad (18)$$

If only matched uncertainties are present, the following robustification schemes are applicable.

Remark: The results of the next subsections are given for a deterministic system (18) and not applied directly to a stochastic system (1). The main article result, asymptotical convergence of the optimally controlled Kalman–Bucy estimate to the sliding mode motion, where the matched deterministic disturbance $g(t)$ is compensated, is proved using the general convergence properties of the Kalman–Bucy filter in section 3.

2.2.1 Integral sliding mode control. Consider a control input in the form

$$u(x, t) = u_0(x, t) + u_1(x, t), \qquad (19)$$

where $u_0(x, t)$ is the nominal (optimal LQ) control for the system without deterministic uncertainties

$$\dot{x}_0(t) = Ax_0(t) + Bu_0(t). \qquad (20)$$

Let the nominal control $u_0(x, t) = -K_cx(t)$ be assigned in accordance with the formula (16) for the gain matrix K_c. The compensator $u_1(x, t)$ should be designed to reject the disturbance $g(t)$ in the sliding mode $s(x, t) = 0$. The switching function $s(x, t)$ is defined as

$$s(x, t) = s_0(x, t) + \sigma(x, t) \quad s, s_0, \sigma \in R^m \qquad (21)$$

where $s_0(x, t)$ may be designed as a linear combination of the system state components $s_0(x, t) = B^+x(t)$, where $B^+ = (B^TB)^{-1}B^T$ is the left-inverse matrix of B, and the integral part σ is selected to achieve $x(t) = x_0(t)$ for all $t \in (t_0, \infty)$. In other words, from the initial time moment, the system state maintains the sliding mode motion, where the equivalent sliding mode control $u_{eq}(t)$ should compensate for the disturbance term

$$u_{eq}(t) = -g(t). \qquad (22)$$

To achieve this purpose, σ is determined from the equation

$$\dot{\sigma}(t) = -\frac{\partial s_0}{\partial x}\left\{Ax(t) + Bu_0(t)\right\}, \quad \sigma(t_0) = -B^+s(x(t_0)), \qquad (23)$$

where $\partial s_0/\partial x = \nabla \cdot s_0 = B^+$. Finally, the switching condition takes the form

$$s(x, t) = B^+\left[x(t) - x(t_0) - \int_{t_0}^{t}[Ax(\tau) + Bu_0]\mathrm{d}\tau\right]. \qquad (24)$$

The compensator $u_1(x, t)$ is designed as a discontinuous unitary control

$$u_1 = -\rho(x, t)\frac{s(x, t)}{\|s(x, t)\|}.$$

For the Lyapunov function $\mathbf{V} = (1/2)(s^T s)$, its functional derivative along the trajectory is calculated

$$\begin{aligned}
\dot{\mathbf{V}} &= s^T \dot{s},\\
\dot{\mathbf{V}} &= s^T B^+(\dot{x}(t) - [Ax(t) + Bu_0]),\\
\dot{\mathbf{V}} &= s^T B^+\{Ax(t) + B(u_0 + u_1) + g(t)\}\\
&\quad - s^T B^+(Ax(t) + Bu_0),\\
\dot{\mathbf{V}} &\le -\|s\|(\rho(x, t) - \|\gamma(t)\|).
\end{aligned} \tag{25}$$

Thus, the sliding mode $s(x, t) = 0$ is attractive from the initial time moment, if

$$\rho(x, t) > \|\gamma(t)\|. \tag{26}$$

2.2.2 Conventional sliding mode control. Let us first transform the system (18) into the regular form using the nonsingular transformation matrix Castaños and Fridman (2006)

$$T = \begin{bmatrix} (B^\perp)^+ \\ B^+ \end{bmatrix}, \tag{27}$$

where the columns of $B^\perp \in R^{(n-m)\times n}$ span the null space of B^{T}, $B^+ \in R^{m\times n}$. Setting $z = Tx$, the following system equations are obtained in terms of the new variables

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix}u(t), \tag{28}$$

where $A_{11} \in R^{(n-m)\times(n-m)}$, $A_{12} \in R^{(n-m)\times m}$, $A_{13} \in R^{m\times(n-m)}$, and $A_{22} \in R^{m\times m}$. Define the switching condition as

$$s = s_0(z_1) + z_2, \quad s \in R^m. \tag{29}$$

Since the pair $\{A, B\}$ is controllable, the pair $\{A_{11}, A_{12}\}$ is controllable too. Moreover, the pair $\{TAT^{-1}, CT^{-1}\}$ is observable in view of observability of the pair $\{A, C\}$, since T is a nonsingular quadratic matrix. To stabilize the reduced-order system $\dot{z}_1 = A_{11}z_1 + A_{12}z_2$, where $z_2 = -Fz_1$ is employed as a virtual control, the optimal sliding mode (OSM) control technique (Uktin 1992) is used. The OSM control is associated with the following criterion taking its minimal value in the sliding mode

$$J = \int_{t_0}^{\infty} x^T Q x \, \mathrm{d}t. \tag{30}$$

The switching condition is then calculated as

$$s(x, t) = \begin{bmatrix} F & I \end{bmatrix} Tx = 0, \tag{31}$$

where $F = -(Q_{22}^{-1}A_{12}P + Q_{22}^{-1}Q_{12}^T)$ and

$$(T^{-1})^T Q T^{-1} = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}, \quad Q_{21}^T = Q_{12},$$

where P is the unique solution to the matrix Riccati equation

$$\begin{aligned}
&P(A_{11} - A_{12}Q_{22}^{-1}Q_{12}^T) + (A_{11} - A_{12}Q_{22}^{-1}Q_{12}^T)^T P\\
&\quad - PA_{12}Q_{22}^{-1}A_{12}^T P + (Q_{11} - Q_{12}Q_{22}^{-1}Q_{12}^T) = 0.
\end{aligned}$$

Finally, the control takes the form

$$u(x, t) = -\rho(x, t)\frac{s(x, t)}{\|s(x, t)\|}. \tag{32}$$

In this case, the derivative of the Lyapunov function $\mathbf{V} = (1/2)(s^T s)$ is estimated as

$$\begin{aligned}
\dot{\mathbf{V}} &\le \|(FA_{11} + A_{21})z_1 + (FA_{12} + A_{22})\|\,\|s\|\\
&\quad - (\rho(x, t) - \|\gamma(t)\|)\|s\|.
\end{aligned} \tag{33}$$

It is necessary to choose $\rho(x, t) > 0$ sufficiently large in order to yield $\dot{\mathbf{V}} < 0$ and, thereby, assure the attractiveness of the sliding mode $s(x, t) = 0$.

2.3 *Disturbance estimation*

In the sliding mode $s(x, t) = 0$, the control input takes its equivalent form

$$u = u_0 + u_{1eq}, \tag{34}$$

where the equivalent control value is determined from the equation $\mathrm{d}s(x, t)/\mathrm{d}t = 0$, following the equivalent control method (Uktin 1992), and, therefore, depends on the unknown disturbance $g(t)$. However, as shown in Uktin (1992), the equivalent control value (22) can be calculated as the asymptotic value of a first-order linear filter, with the discontinuous control as its input

$$\mu\dot{u}_{1av} + u_{1av} = u_1. \tag{35}$$

The constant μ should be selected small but such that the first-order filter does not eliminate the slow oscillation component, that is equal to u_{1eq}.

In addition, the use of the equivalent control estimate produced by the first-order filter alleviates the high frequency chattering arising due to the discontinuous nature of SM control schemes. Note that this approach does not lead to the loss of robustness or accuracy, which occurs if a continuous approximation of the sign function is used.

3. Problem solution

Consider a linear stochastic system (1) and the observations (2), assuming the conditions of section 2. Consider the Kalman–Bucy filtering equations (11) and (13), which are optimal if the disturbance is absent, $g(t) = 0$,

$$\mathrm{d}\hat{x}(t) = (A\hat{x}(t) + Bu(t))\mathrm{d}t + P(t)C^T(HH^T)^{-1}[\mathrm{d}y(t) - C\hat{x}(t)\mathrm{d}t], \quad (36)$$

$$\dot{P}(t) = AP(t) + P(t)A^T + GG^T - P(t)C^T(HH^T)^{-1}CP(t), \quad (37)$$

with the initial conditions $\hat{x}(t_0) = \eta$ and $P(t_0) = P_{e0}$. We can calculate the estimate $\hat{x}(t)$ of the state vector $x(t)$ as the solution of (36) and use it in the control law (19)

$$u(\hat{x}(t), t) = u_0(\hat{x}(t), t) + u_1(\hat{x}(t), t), \quad (38)$$

where $u_0(\hat{x}(t), t)$ is the LQG control for system without disturbances, $g(t) = 0$, and $u_1(\hat{x}(t), t)$ is the SM or ISM discontinuous control based on the estimate $\hat{x}(t)$. As follows from Jazwinski (1970), if the pair $\{A, B\}$ is controllable and the pair $\{A, C\}$ is observable, $\hat{x}(t)$ asymptotically converges almost surely (with probability one, w.p.1) to the state $\zeta(t)$ that satisfies the equation

$$\mathrm{d}\zeta(t) = (A\zeta(t) + Bu_0(\zeta(t), t))\mathrm{d}t + G\mathrm{d}W_1(t), \quad (39)$$

taking into account that $u_1(\hat{x}(t), t)$ is selected to reject $g(t)$ (equation 22). Consequently, the observation equation converges (w.p.1) to

$$\mathrm{d}y(t) = C\zeta(t)\mathrm{d}t + H\mathrm{d}W_2(t). \quad (40)$$

For this filtering system, $\hat{x}(t)$ serves as the optimal estimate in the Kalman sense, as follows from the results of subsection 2.1.1. Furthermore, $x(t)$ asymptotically converges almost surely (w.p.1) to $\zeta(t)$, taking into account that u_1 is selected to reject $g(t)$ and $\hat{x}(t)$ asymptotically converges almost surely (w.p.1) to the state $\zeta(t)$. Thus, there is an asymptotic compensation of matched disturbances almost surely (w.p.1).

The compensation algorithm functioning can be explained as follows. Both the random state $x(t)$ under influence of deterministic disturbances and the optimal estimate $\hat{x}(t)$ converge to the random state $\zeta(t)$, which is free of deterministic disturbances (see equation 39). The state $x(t)$ asymptotically converges to $\zeta(t)$ in view of the disturbance compensator u_1 dependent on the optimal estimate $\hat{x}(t)$ (equation 38), which asymptotically converges to the state $x(t)$, feeding up the current-time information on the disturbed state $x(t)$ through the observations $y(t)$. Note that availability of real-time information on the disturbed state $x(t)$ for the optimal estimate $\hat{x}(t)$ and, as a consequence, for the compensator $u_1(\hat{x}(t), t)$ is the decisive point of the algorithm. Thus, the algorithm is based on the system of two variables simultaneously converging to the same limit, as time goes to infinity, in such a way that one of them, $\hat{x}(t)$, collects information on and regulates the other one, $x(t)$.

Two sliding mode control approaches, ISM and conventional SM, are further used to form the compensating control u_1. Note that the sliding mode control $u_1(\hat{x}(t), t)$, based on the estimate $\hat{x}(t)$, is realized taking into account that the equation (36) for $\hat{x}(t)$ is a posteriori deterministic. Indeed, since $\hat{x}(t)$ is defined as an expectation conditional with respect to the observation process $y(t)$, the equation (36) is *a priori* stochastic, but becomes deterministic upon substituting a specific realization of $y(t)$ into its right-hand side.

3.1 *Solution using LQG/SM*

Define u_1 as

$$u_1(\hat{x}(t), t) = -K_c\hat{x}(t) - \rho(\hat{x}, t)\frac{s(\hat{x}(t), t)}{\|s(\hat{x}(t), t)\|},$$

where $s(\hat{x}(t), t)$ is designed using the optimal sliding mode control theory (31)

$$s(\hat{x}(t), t) = \begin{bmatrix} F & I \end{bmatrix} T\hat{x}(t), \quad (41)$$

and $K_c = W^{-1}BP_C(t)$ is the LQG gain (16).

3.2 *Solution using LQG/ISM*

Define u_1 as

$$u_1(\hat{x}(t), t) = -K_c\hat{x}(t) - \rho(\hat{x}, t)\frac{s(\hat{x}(t), t)}{\|s(\hat{x}(t), t)\|}, \quad (42)$$

where $s(\hat{x}(t), t)$ is designed using the ISM approach (24)

$$s(\hat{x}, t) = B^{+}\left[\hat{x}(t) - \hat{x}(t_0) - \int_{t_0}^{t}[A\hat{x}(\tau) + Bu_0(\hat{x}(\tau), \tau)]\mathrm{d}\tau\right], \tag{43}$$

and $K_c = W^{-1}BP_C(t)$ is the LQG gain (16).

A study of the LQG/ISM-SM control of a linear stochastic system describing an inverted pendulum system perturbed with deterministic uncertainties and stochastic Gaussian noises is presented in the next section.

4. LQG-optimal and robust control problem for inverted pendulum system

Consider an inverted pendulum system, which consists of a cart moving along a metal guiding bar. The corresponding Euler–Lagrange model is given by

$$\begin{bmatrix} M+m & m\ell \\ m\ell\cos\theta & m\ell^2 \end{bmatrix}\begin{bmatrix} \ddot{r} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} m\ell\dot{\theta}^2\sin\theta - f_c + F \\ mg\ell\sin\theta - f_\tau \end{bmatrix}.$$

The specific pendulum parameters are listed in table 1. The system equations linearized in state variables take the form

$$\mathrm{d}x(t) = (Ax(t) + Bu(t) + g(t))\mathrm{d}t + \mathrm{d}W_1(t), \tag{44}$$

$$\mathrm{d}y(t) = Cx(t)\mathrm{d}t + \mathrm{d}W_2(t), \tag{45}$$

where

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -0.8818 & -1.9148 & 0.0056 \\ 0 & 21.4964 & 3.8498 & -0.1362 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 \\ 0 \\ 0.3088 \\ -0.6209 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The control input $u(t)$ is applied to the cart in order to move and maintain the pendulum in the vertical position. The state variable $x(t)$ consists of four components: x_1 is the car position, that is the distance from a reference point to the trolley, x_2 is the angle between the vertical line and the pendulum, x_3 is the trolley velocity, and x_4 is the pendulum angular velocity.

Table 1. Inverted pendulum parameters.

Parameters	Value
M	Mass of the cart
m	Mass of the pendulum
θ	Angular position of the pendulum
r	Linear position of the car
f_τ	Friction proportional to the angular velocity
f_c	Friction proportional to the linear velocity
ℓ	Center of gravity position
F	Input force

The disturbance is matched: $g(t) = B\gamma(t)$, and $\gamma(t) = 7 + 6\sin t$; thus, $\gamma(t)$ satisfies the condition (3). The matrices in the criterion (30) are equal to

$$V = \begin{bmatrix} 10 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad W = 1.$$

To simulate the system, the state initial condition is chosen as

$$x(t_0) = \begin{bmatrix} 1 & \frac{9\pi}{180} & 0 & 0 \end{bmatrix}.$$

The pair $\{A, B\}$ is controllable. Moreover, the vector state is assumed completely measurable but corrupted by noise, therefore, $C \in R^{n\times n}$, and the pair $\{A, C\}$ is observable.

The first task is to implement the Kalman–Bucy filter (36)–(37), which yields the state estimate vector $\hat{x}(t)$. Then, the estimate-based control is defined according to (38):

$$u(\hat{x}(t), t) = u_0(\hat{x}(t), t) + u_1(\hat{x}(t), t),$$

where $u_0(\hat{x}(t), t) = -K_c\hat{x}(t)$ is the nominal LQG control (15), and

$$K_c = \begin{bmatrix} 3.1 & 106.6 & 15.4 & 23 \end{bmatrix}.$$

The discontinuous control $u_1(\hat{x}(t), t) = -\rho(\hat{x}, t) \times (s(\hat{x}(t), t)/\|s(\hat{x}(t), t)\|)$, which compensates for the matched disturbances, the gain $\rho(\hat{x}, t)$, and the switching function $s(\hat{x}(t), t)$ are further designed depending on a selected technique, conventional SM or ISM.

4.1 *LQGSM control*

Using LQGSM control yields

$$u(\hat{x}(t),t) = u_0(\hat{x}(t),t) + u_1(\hat{x}(t),t) = -K_c\hat{x}(t) - \rho(\hat{x},t)\frac{s(\hat{x}(t),t)}{\|s(\hat{x}(t),t)\|}.$$

The system without stochastic noises (18) should be transformed into the regular form using the transformation matrix

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0.4973 \\ 0 & 0 & 0.6422 & -1.2912 \end{bmatrix}.$$

The obtained regular form is given by

$$\begin{bmatrix} \dot{z}_{1,1} \\ \dot{z}_{1,2} \\ \dot{z}_{1,3} \\ \dot{z}_{2,1} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0.8017 & 0.3088 \\ 0 & 0 & 0.3987 & -0.6209 \\ 0 & 9.8093 & -0.0249 & 0.0385 \\ 0 & -28.3222 & -4.8993 & -2.0261 \end{bmatrix}\begin{bmatrix} \dot{z}_{1,1} \\ \dot{z}_{1,2} \\ \dot{z}_{1,3} \\ \dot{z}_{2,1} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}u + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 6\sin t + 7 \end{bmatrix}. \tag{46}$$

To stabilize the reduced order system

$$\begin{bmatrix} \dot{z}_{1,1} \\ \dot{z}_{1,2} \\ \dot{z}_{1,3} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0.8017 \\ 0 & 0 & 0.3987 \\ 0 & 9.8093 & -0.0249 \end{bmatrix}\begin{bmatrix} \dot{z}_{1,1} \\ \dot{z}_{1,2} \\ \dot{z}_{1,3} \end{bmatrix} + \begin{bmatrix} 0.3088 \\ -0.6209 \\ 0.0385 \end{bmatrix}\left[\dot{z}_{2,1}\right], \tag{47}$$

the optimal sliding mode (OSM) control is designed, and the functional (30) reaches its minimum value if the condition (31) holds

$$\dot{z}_2 = -\left[-4.5602 \quad -15.7301 \quad -4.7292\right]\dot{z}_1.$$

The corresponding switching condition is calculated as

$$s(\hat{x}(t),t) = \left[-4.56 \quad -15.73 \quad -4.72 \quad 1\right]\hat{z}(t) = \left[-4.56 \quad -15.73 \quad -4.08 \quad -3.64\right]\hat{x}(t), \tag{48}$$

and the estimate-based control takes the form

$$u(\hat{x}(t),t) = [-3.1 \;\; -106.6 \;\; -15.4 \;\; -23]\hat{x}(t) + \rho(\hat{x},t)\frac{s(\hat{x}(t),t)}{\|s(\hat{x}(t),t)\|}.$$

The gain $\rho(\hat{x},t)$ should satisfy the condition (equation 33):

$$\rho(\hat{x},t) > \left\|\left[0 \quad -74.71 \quad -10.76 \quad -15.25\right]\right\|\|\hat{x}\| + \|\gamma(t)\|;$$

since

$$\|\gamma(t)\| < 20,$$

then

$$\rho(\hat{x},t) > 77\|\hat{x}\| + 20.$$

This presents the worst case, which was experimentally adjusted to $\rho(\hat{x},t) = 20\|\hat{x}\| + 20$. Finally,

$$u(\hat{x}(t),t) = \left[-3.1 \quad -106.6 \quad -15.4 \quad -23\right]\hat{x}(t) - (20\|\hat{x}\| + 20)\frac{s(\hat{x}(t),t)}{\|s(\hat{x}(t),t)\|}. \tag{49}$$

4.2 *LQGISM control*

In the integral sliding mode (LQGISM), the nominal system control u_0 is the LQG control (15), and the sliding mode is defined according to (24)

$$s(\hat{x}(t),t) = B^{+}\left[\hat{x}(t) - \hat{x}(t_0) - \int_{t_0}^{t}(A - BK_c)\hat{x}(\tau)\mathrm{d}\tau\right].$$

The gain $\rho(\hat{x},t)$ in (42) should satisfy the condition (see equation 26): $\rho(\hat{x},t) > \|\gamma(t)\|$, where $\|\gamma(t)\| \le 13$, and is set to $\rho = 20$. Finally, the entire control is given by

$$u(\hat{x}(t),t) = -K_c\hat{x}(t) - 20\frac{s(\hat{x}(t),t)}{\|s(\hat{x}(t),t)\|},$$

where $K_c = \left[3.1 \;\; 106.6 \;\; 15.4 \;\; 23\right]$.

4.3 *Disturbance estimation*

As described in section 2.3, passing $u_1(\hat{x}(t),t)$ through first-order filter yields an estimate of the disturbance term. The constant μ of the first-order filter (35) is experimentally set to $\mu = 0.2$.

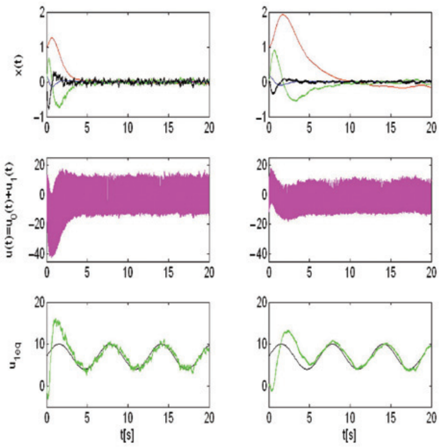

Figure 1. Inverted pendulum system perturbed with stochastic noises and deterministic uncertainty: unmeasured real states and their estimates (above), sum of the equivalent control and disturbance (middle), disturbance identification (below).

5. Simulations

Numerical simulations are conducted for both controllers, LQGSM and LQGISM, and the perturbed stochastic system. Figure 1 shows the perturbed system response in both cases, the LQGSM results in the left graph and the LQGISM results in the right one. One can observe more control input oscillations when the conventional SM is used, because there is no insensibility to perturbations until the state enters the sliding mode; due to this reason the control signal is larger during transient response. The LQGISM control also performs better, providing less oscillatory estimate trajectories and identifying the disturbance more precisely, than the LQGSM does. Finally, the LQGISM approach leads to a closer approximation of the real state by the estimate. Note that a good estimate for the disturbance $g(t)$ could be obtained setting $\hat{g}(x,t) = E\{u_1(t,\hat{x}(t))\}$.

6. Conclusions

- The disturbance identification problem is solved for a linear time-invariant stochastic uncertain system, applying the conventional SM and ISM based approaches.
- Although the real state value is unknown (unmeasured), the combination of the Kalman–Bucy estimate with the LQG optimal control and ISM technique yields a reliable state estimate, robust control minimizing the cost functional, and trustworthy disturbance identification.
- The robust controller problem is solved for a linear stochastic system under white Gaussian noises and deterministic uncertainties, using the conventional SM and ISM techniques.
- A better performance is obtained for the ISM-based approach, in comparison to the conventional SM.

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