The Problem of Chattering: an Averaging Approach

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Abstract. The singularly perturbed relay control systems (SPRCS) are examined. The mathematical apparatus for investigation of the fast periodic oscillations of SPRCS is developed. The theorem about existence of fast periodic solution of SPRCS is proved. The theorem about averaging is given. It is proved that the slow motions in SPRCS with fast periodic solutions are approximately described by equations obtained from the equations for the slow variables of SPRCS by averaging along fast periodic motions. The algorithm of asymptotic representation for the fast periodic solution of SPRCS is suggested. The algorithm for correction of the averaged equation is given. The stability of the fast periodic solution is investigated.
It is shown that in the case when the original SPRCS contains the relay control linearly the averaged equations and equations which describe the motions of the reduced system in the sliding mode are coincide. The example is given which shows that in the general case when the original SPCSC contains the relay control nonlinearly, the averaging equations do not coincide with the equivalent control equations or the Filippov extension definition which describe the motions in the sliding mode in the reduced system. The algorithm is proposed which allows to solve the problem of eigenvalue assignment for averaged equations using the additional dynamics of fast actuator.

1 Introduction

The chattering phenomena is one of the actual problems in modern sliding mode control theory. The presence of actuators and measuring devices is one of the basic reasons for chattering in sliding mode control systems ([17],[4],[2]). The behaviour of such systems is described by singularly perturbed relay control systems (SPRCS). Moreover, for such systems the conditions of dynamic uncertainty hold, which means that for the original SPRCS there are no stable first order sliding modes, but for the reduced system the sufficient conditions for existence of a stable first order sliding mode hold ([4],[3],[9]).

If the original SPRCS contains either a sliding mode of 3rd order and greater or positive feedback, then the sliding modes are unstable ([1],[10]). In such systems fast periodic oscillations can occur ([17],[9]).

The general model of sliding mode control systems with fast actuators and measuring devices has the following form (see [4])

$$\mu dz/dt = g(z, s, x, u(s)), \qquad (I.1)$$

$$ds/dt = h_1(z, s, x, u(s)), \quad dx/dt = h_2(z, s, x, u(s)),$$

where $z \in \mathbf{R}^m$, $s \in \mathbf{R}$, $x \in \mathbf{R}^n$, $u(s) = sign(s)$ and g, h_1, h_2 are smooth functions of their arguments. System (I.1) under such assumptions can describe for instance the behaviour of control systems in which variables x, s describe plant behaviour, and the vector z describes the behaviour of the fast actuator.

Conditions of dynamic uncertainty for system (I.1) mean that letting $\mu = 0$ and expressing $z_0 = \varphi(s, x, u(s))$ from the equation

$$g(z_0, s, x, u(s)) = 0$$

according to the formula $z_0 = \varphi(s, x, u(s))$, we obtain the reduced system

$$ds/dt = h_1(\varphi(s, x, u(s)), s, x, u(s)) = H_1(s, x, u(s)), \qquad (I.2)$$

$$dx/dt = h_2(\varphi(s, x, u(s)), s, x, u(s)) = H_2(s, x, u(s)).$$

It is assumed that

(i) almost everywhere on $s = 0$

$$h_1(z, 0, x, 1)h_1(z, 0, x, -1) > 0 \qquad (I.3)$$

or

$$h_1(z, 0, x, 1) > 0, \; h_1(z, 0, x, -1) < 0;$$

(ii) the measure of the domain

$$S = \{x : H_1(0, x, 1) < 0, \; H_1(0, x, -1) > 0\} \in \mathbf{R}^n$$

is nonzero and consequently S is the domain of stable first order sliding for system (I.2).

This chapter is devoted to the investigation of chattering problem in (I.2) from the viewpoint of averaging and specific features of fast periodic solutions of system (I.1). The chapter consists of four sections. Section 2 is devoted to the development of mathematical tools for the investigation of periodic solutions of SPRCS. In section 3 these tools are used for the investigation of the behaviour of sliding mode control systems with fast actuators. The proposed new approach is used in section 4 for the design of desired averaged equation in sliding mode control system with fast actuators.

2 Mathematical Tools

2.1 Problem Formulation

In this section we will consider the existence and stability of the fast periodic solutions for singularly perturbed relay control systems (SPRCS) of the form

$$\mu dz/dt = g(z, \xi, x, u(\xi)), \tag{1}$$

$$\mu d\xi/dt = h_1(z, \xi, x, u(\xi)),\ dx/dt = h_2(z, \xi, x, u(\xi)),$$

where $z \in \mathbf{R}^m$, $\xi \in \mathbf{R}$, $x \in \mathbf{R}^n$, $u(\xi) = sign\,(\xi)$ and g, h_1, h_2 are smooth functions of their arguments. Introducing the 'fast time' $\tau = t/\mu$ into (1), we obtain

$$dz/d\tau = g(z, \xi, x, u(\xi)), \tag{2}$$

$$d\xi/d\tau = h_1(z, \xi, x, u(\xi)), \quad dx/d\tau = \mu h_2(z, \xi, x, u(\xi)).$$

For smoothly singularly perturbed system the existence and stability of the first approximation of the fast periodic solution was investigated by [14]. The existence and stability of the first approximation of fast periodic solution of (1) was investigated in [9].

It turns out that for the investigation of the fast periodic solutions of the singularly perturbed system (2) it's impossible to use standard small parameter methods [7] for autonomous systems because setting $\mu = 0$ in (2) we will obtain a degenerate equation for the slow variables x.

In this section we develop the mathematical tools for investigation of the fast periodic oscillations (1),(2). To this end we employ the point mapping method (see [12],[13]).

2.2 Some Properties of the Point Mapping which Made by SPRCS

Let us denote the variation domain as Z, X with variables (z, s, x) and x.

Definition 1. We shall call the surface $\xi = 0$ the surface without stable sliding towards trajectories of the system

$$dz/d\tau = g(z, \xi, x, u(\xi)), \tag{3}$$

$$d\xi/d\tau = h_1(z, \xi, x, u(\xi)),$$

if all the trajectories of (3) which start outside the surface $\xi = 0$ cross it at the point $(z, 0, x)$ where the conditions

$$h_1(z, 0, x, 1)h_1(z, 0, x, -1) > 0$$

are fulfilled.

Suppose that the following conditions are true:

1^0 $h_1, h_2, g \in \mathbf{C}^2[\bar{Z} \times [-1,1]]$;

2^0 the surface $\xi = 0$ for all $x \in \bar{X}$ is a surface without stable sliding towards trajectories of system (3);

3^0 system (3) for all $x \in \bar{X}$ has an isolated orbitally asymptotically stable solution $(z_0(\tau, x), \xi_0(\tau, x))$ with period $T(x)$;

4^0 let $R(z, x)$ be a point mapping of the set $V = \{(z, x) : h_1(z, 0, x, 1) > 0\}$ on the surface $\xi = 0$ into itself performed by system (3), which has a fixed point $z^*(x)$ corresponding to $(z_0(\tau, x), \xi_0(\tau, x))$;

5^0 suppose that for all $x \in \bar{X}$ $\lambda_i(x)\,(i = 1, ..., m)$, the eigenvalues of the matrix

$$\frac{\partial R}{\partial z}(z^*(x), x),$$

the inequalities $|\lambda_i(x)| \neq 1$ are true;

6^0 the averaged system $dx/dt = p(x)$, where

$$p(x) = \frac{1}{T(x)} \int_0^{T(x)} h_2(z_0(\tau, x), \xi_0(\tau, x), x, u(\xi_0(\tau, x))d\tau, \tag{4}$$

has an isolated equilibrium point x_0 such that

$$p(x_0) = 0, \quad det|\frac{dp}{dx}(x_0)| \neq 0.$$

Let us denote as $z^{\pm}(\tau, z, x, \mu)$ and $\xi^{\pm}(\tau, z, x, \mu)$, the solutions of system (2) with the initial conditions $z^{\pm}(0, z, x, \mu) = z, \xi^{\pm}(0, z, x, \mu) = 0$ for $\xi > 0$ and $\xi < 0$.

The point mapping of the domain V of the surface $\xi = 0$, associated with system (2), has the following form

$$\Phi(z, x, \mu) = (\Phi_1(z, x, \mu), \Phi_2(z, x, \mu)) =$$

$$(z^-(\Theta, z^+(\theta, z, x, \mu), x^+(\theta, z, x, \mu), \mu), x^-(\Theta, z^+(\theta, z, x, \mu), x^+(\theta, z, x, \mu), \mu)),$$

where the functions $\theta(z, x, \mu), \Theta(z, x, \mu)$ are determined by equations

$$\xi^+(\theta, z, x, \mu) = 0,$$

$$\xi^-(\Theta, z^+(\theta, z, x, \mu), x^+(\theta, z, x, \mu), \mu) = 0.$$

This means that $\Phi_1(z, x, 0) = R(z, x)$.

The surface $\xi = 0$ is the surface without stable sliding for system (3). This means that there exists a neighbourhood of the point $(z^*(x_0), x_0)$ on the surface $\xi = 0$ for which

$$max\{|d\xi^+/d\theta|,\ |d\xi^-/d\Theta|\} > 0.$$

It follows from condition 1^0 and the implicit function theorem that for some small μ_0, the functions Φ, θ, Θ have continuous derivatives in some set $U \times [0, \mu_0]$ on the surface $\xi = 0$. This means that we can consider the function Φ as the point mapping of the set $U \times [0, \mu_0]$ on the surface $\xi = 0$ into itself. Moreover we can rewrite $\Phi(z, x, \mu)$ in the form

$$\Phi(z, x, \mu) = (\bar{R}(z, x, \mu), x + \mu\bar{Q}(z, x, \mu)),$$

where $\bar{R}(z, x, \mu), \bar{Q}(z, x, \mu)$ are sufficiently smooth functions and

$$\bar{Q}(z^*(x_0), x_0, 0) = 0, \bar{R}(z^*(x), x, 0) = z^*(x).$$

Let's introduce into the function Φ new variables using the formula $\eta = z - z^*(x)$. Then the point mapping Φ takes the form

$$\Psi(\eta, x, \mu) = (\Psi_1(\eta, x, \mu), \Psi_2(\eta, x, \mu)) =$$

$$= (\bar{R}(\eta + z^*(x), x, \mu) - z^*(x), x + \mu\bar{Q}(\eta + z^*(x), x, \mu)), \tag{5}$$

and consequently $\Psi(0, x, 0) = (0, x)$.

2.3 Existence of the Fast Periodic Solution

Theorem 2. *Under conditions $1^0 - 6^0$, system (1) has an isolated periodic solution with the period $\mu(T(x_0) + O(\mu))$ near to the circle*

$$(z_0(t/\mu, x_0), \xi_0(t/\mu, x_0), x_0).$$

Proof. We will prove the existence of the periodic solution as the existence of the fixed point $(\eta^*(\mu), x^*(\mu))$ of the point mapping Ψ. Let's rewrite the conditions of existence of this fixed point in the form

$$G(\eta^*, x^*, \mu) = \begin{pmatrix} G_1(\eta^*, x^*, \mu) \\ G_2(\eta^*, x^*, \mu) \end{pmatrix} = \begin{pmatrix} \eta^* - \Psi_1(\eta^*, x^*, \mu) \\ \frac{1}{\mu}[x^* - \Psi_2(\eta^*, x^*, \mu)] \end{pmatrix} = 0. \tag{6}$$

It is necessary to take into account that for $\mu = 0$ $\eta^*(0) = 0, x^*(0) = x_0$ and $G_2(0, x_0, 0) = -T(x_0)p(x_0) = 0$ and consequently for $\mu = 0$ conditions (6) are fulfilled. Moreover, taking into account that $G_1(0, x, 0) = 0$ for all $x \in \bar{X}$ we can conclude that $\frac{\partial G_1}{\partial x}(0, x_0, 0) = 0$. Let us compute the Jacobian of function G with respect to variables η, x at $\mu = 0$.

$$\frac{\partial G}{\partial(\eta, x)}(0, x_0, 0) =$$

$$= \begin{vmatrix} I_m - \frac{\partial R}{\partial z}(z^*(x_0), x_0) & 0 \\ \frac{\partial G_2}{\partial \eta}(0, x_0, 0) & -T(x_0)\frac{\partial p}{\partial x}(x_0) \end{vmatrix} \neq 0.$$

This means that there exists an isolated fixed point $(z^*(\mu), x^*(\mu))$ of the point mapping G which corresponds to the periodic solution of systems (1) and (2) and in this case $z^*(\mu) = z^*(x_0) + O(\mu), x^*(\mu) = x_0 + O(\mu)$. □

2.4 Stability in the First Approximation

Assume that

7^0 the eigenvalues $\lambda_i(x_0)$ of the matrix $\frac{\partial R}{\partial z}(z(x_0), x_0)$ satisfy the inequalities $|\lambda_i(x_0)| < 1\,(i = 1, ..., m)$;

8^0 the eigenvalues $\nu_j(x_0)$, $j = 1, ..., n$ of the matrix $\frac{dp}{dx}(x_0)$ satisfy the inequalities

$$\mathbf{Re}\,\nu_j(x_0) < 0.$$

Theorem 3. *Under conditions* 1^0-8^0 *the periodic solution of* (1), (2) *is orbitally asymptotically stable.*

Proof. Let's find the derivatives Ψ by variables η, x

$$\frac{\partial \Psi}{\partial(\eta, x, \mu)} = \Gamma(\eta, x, \mu) = \begin{bmatrix} I_m - \frac{\partial R}{\partial z}(x_0) + O(\mu) & O(\mu) \\ \frac{\partial \Psi_2}{\partial \eta}(0, x_0, 0) + O(\mu) & I_m + \mu T(x_0)\frac{\partial p}{\partial x}(x_0) + O(\mu) \end{bmatrix}.$$

Consequently the matrix $\Gamma(\eta, x, \mu)$ has at the small vicinity of $(0, x_0, 0)$ two groups of eigenvalues

$$\lambda_i(x_0) + O(\mu),\ i = 1, ..., m,$$
$$1 + \mu T(x_0)\nu_j(x_0) + o(\mu),\ j = 1, ..., n.$$

This means that under conditions of theorem 3 there exists some neighbourhood of $(0, x_0, 0)$ for which Ψ is contraction mapping and corresponding fast periodic solution of systems (1),(2) is orbitally asymptotically stable. □

2.5 Some Auxiliary Theorems about Decomposition of Two - Speed Point Mappings

It is obvious that the problem of stability of fast periodic solution of system (1) is equivalent to the problem of stability of the fixed point $\eta^*(\mu), x^*(\mu)$ of $\Psi(\mu)$. Let's introduce into Ψ new variables according to the formulae $\kappa = \eta - \eta^*(\mu), \chi = x - x^*(\mu)$. Then, taking into account that $\partial\Psi(0, x_0, 0)/\partial x = 0$, we have

$$\Lambda_1(\kappa, \chi, \mu) = P\kappa + Q(\kappa, \chi, \mu),$$
$$\Lambda_2(\kappa, \chi, \mu) = \chi + \mu R(\kappa, \chi, \mu), \qquad (D.1)$$

where Q and R are smooth functions and under conditions $1^0 - 7^0$

$$P = \partial\bar{R}/\partial z(z(x_0), x_0),\ ||P|| < 1;$$
$$Q(\kappa, \chi, \mu) = O(\mu)O(|\kappa| + |\chi|) + O(|\kappa|^2 + |\chi|^2),$$
$$R(\kappa, \chi, \mu) = O(|\kappa| + |\chi|).$$

Thus we can reduce Cauchy's problem for system (1) with initial conditions

$$z(0, \mu) = z^0,\ s(0, \mu) = 0,\ x(0, \mu) = x^0. \qquad (IC)$$

to the investigation of the two-speed discrete system

$$\kappa_{k+1} = P\kappa_k + Q(\kappa_k, \chi_k, \mu),\ \chi_{k+1} = \chi_k + \mu R(\kappa_k, \mu), \qquad (D.2)$$
$$\kappa_0 = z^0 - z^*(x^*(\mu)),\ \chi_0 = x^0 - x^*(\mu).$$

In the sequel we will use the following theorems about decomposition of point mappings (D.1),(D.2) (see [16]).

Proposition D.1. *Assume that for system* $(D.1)$ *conditions* $(D.2)$ *hold. Then system* $(D.1)$ *has a slow motions integral manifold of the form* $\kappa = V(\chi, \mu)$ *for small* μ*. Then there exist* C_1, C_2 *such that*

$$||V(\chi, \mu)|| < C_1,$$

$$||V(\chi, \mu) - V(\bar{\chi}, \mu)|| < C_2 ||\chi - \bar{\chi}||.$$

The motion on the manifold $\kappa = V(\chi, \mu)$ *is described by the equation*

$$\Lambda_1(V(\chi, \mu), \chi, \mu) = \chi + \mu R(V(\chi, \mu), \chi, \mu). \tag{D.3}$$

For the slow coordinate solution of $(D.2)$ $\chi_k(\chi_0)$, *and* $\bar{\chi}_k(\tilde{\chi})$ *the solution of system* $(D.3)$ *with initial condition* $\bar{\chi}_0 = \tilde{\chi}$, *there exist* $c > 0$, $0 < q < 1$ *and* $\tilde{\chi} \in \mathbf{R}$ *for which the inequality*

$$|\chi(\chi_0) - \bar{\chi}(\tilde{\chi})| < cq^k$$

is true.

Proposition D.2 (reduction principle). *If*

$$Q(0, 0, \mu) = 0; \quad R(0, 0, \mu) = 0,$$

then the problem of stability of the zero solutions of systems $(D.1)$ *and* $(D.3)$ *are equivalent. This means that the zero solution of* $(D.1)$ *is stable (asymptotically stable, unstable) if and only if the zero solution of* $(D.3)$ *is stable (asymptotically stable, unstable).*

The function $V(\chi, \mu)$ may be found from the equation

$$PV(\chi, \mu) + Q(V(\chi, \mu), \chi, \mu) = V(\chi + \mu R(V(\chi, \mu), \chi, \mu), \mu)$$

with any level of precision in form

$$V(\chi, \mu) = V_0(\chi) + \mu V_1(\chi) + \mu^2 V_2(\chi) + \cdots.$$

The function $V_0(\chi)$ is a solution of the equation

$$PV_0(\chi) + Q(V_0(\chi), \chi, 0) = V_0(\chi).$$

Function $V_1(\chi)$ can be found from equation

$$PV_1(\chi) + Q'_\mu(V_0(\chi), \chi, 0) = V_1(\chi).$$

An equation describing the flow on the slow motions integral manifold has the form

$$\Lambda_2(V(\chi, \mu), \chi, \mu) = \chi + \mu R(V_0(\chi), \chi, \mu) + \tag{D.4}$$
$$+\mu^2 (R'_\kappa(V_0(\chi), \chi, 0) V_1(\chi) + R'_\mu(V_0(\chi), \chi, 0)) + O(\mu^3).$$

2.6 Theorem about Averaging

Assume that

9^0 The solution $\bar{x}(t)$ of the averaged system (3) with initial conditions $\bar{x}(0) = x^0$ for $t \in [0, L]$ is situated in the closed subdomain $\bar{X} \in X$.

Theorem 4. *Under conditions $1^0 - 7^0$ and 9^0, the slow coordinate $x(t, \mu)$ of solution* (1), (IC) *and $\bar{x}(t)$ satisfy the inequality*

$$\sup_{t\in[0,L]} |x(t,\mu) - \bar{x}(t)| = O(\mu).$$

2.7 Searching for the Periodic Solution

Assume now that

$1A^0 \quad h_1, h_2, g \in \mathbf{C}^{k+2}[\bar{Z} \times [-1, 1]]$.

We will find the period of the desired periodic solution of (2) in the form

$$T(\mu) = T_0 + \mu T_1 + \mu^2 T_2 + ..., \tag{7}$$

where $T_0 = T(x_0)$ and the time interval for which $u = 1$ and $u = -1$ in form

$$\theta^{\pm}(\mu) = \theta_0^{\pm} + \mu\theta_1^{\pm} + \mu^2\theta_2^{\pm} + \cdots + \mu^k\theta_k^{\pm} + \cdots,$$

where $\theta_0^{\pm} = \theta^{\pm}(x_0)$. Then the asymptotic representation of the desired periodic solution on $[0, T(\mu)]$ takes the form

$$z(\tau,\mu) = z_0(\tau) + \mu z_1(\tau) + \mu^2 z_2(\tau) + ... + \mu^k z_k(\tau) + ...,$$

$$x(\tau,\mu) = \xi_0(\tau) + \mu\xi_1(\tau) + \mu^2\xi_2(\tau) + ... + \mu^k\xi_k(\tau) + ...,$$

$$x(\tau,\mu) = x_0 + \mu x_1(\tau) + \mu^2 x_2(\tau) + ... + \mu^k x_k(\tau) + ... \quad .$$

Denote

$$\tilde{T}_k(\mu) = T_0 + \mu T_1 + \mu^2 T_2 + ... + \mu^k T_k,$$

$$\tilde{\theta}_k^{\pm}(\mu) = \theta_0^{\pm} + \mu\theta_0^{\pm} + \mu^2\theta_2^{\pm} + ... + \mu^k\theta_k^{\pm}.$$

Let's find the $k - th$ approximation of the asymptotic representation for the desired periodic solution for $\tau \in [0, \tilde{T}_k(\mu)]$ in the form

$$Z_k(\tau,\mu) = z_0(\tau) + \mu z_1(\tau) + \mu^2 z_2(\tau) + ... + \mu^k z_k(\tau),$$

$$\Xi_k(\tau,\mu) = \xi_0(\tau) + \mu\xi_1(\tau) + \mu^2\xi_2(\tau) + ... + \mu^k\xi_k(\tau),$$

$$X_k(\tau,\mu) = x_0 + \mu x_1(\tau) + \mu^2 x_2(\tau) + ... + \mu^k x_k(\tau),$$

where the continuous functions z_i, ξ_i, x_i are smooth on $[0, \tilde{\theta}_k^+(\mu)) \cup (\theta_k^+(\mu), T_k(\mu)]$ but have jumps in the derivative at $\tau = \tilde{\theta}_k^+(\mu)$. Let's show that under the conditions of theorem 2 the functions $z_i^{\pm}, \xi_i^{\pm}, x_i^{\pm}$ and the constants θ_i, Θ_i for every $i = 1, ..., k$ can be uniquely found.

Let's introduce in system (2) two 'new times' according to the formulae

$$\tau^+ = \tau/\tilde{\theta}_k^+(\mu); \tau^- = (\tau - \tilde{\theta}_k^+(\mu))/\tilde{\theta}_k^-(\mu), \tau^\pm \in [0,1],$$

and the auxiliary functions $z_0^\pm(\tau^\pm), \xi_0^\pm(\tau^\pm)$ as the solutions to

$$dz_0^\pm/d\tau^\pm = \theta_0^\pm g(z_0^\pm, \xi_0^\pm, x_0, \pm 1), \tag{8}$$

$$d\xi_0^\pm/d\tau = \theta_0^\pm h_1(z_0^\pm, \xi_0^\pm, x_0, \pm 1)$$

with initial and periodicity conditions

$$z_0^+(0) = z^*(x_0) = z_0^*, \quad \xi_0^+(0) = 0; \tag{9}$$

$$z_0^-(0) = z_0^+(1), \quad \xi_0^-(0) = \xi_0^+(1) = 0;$$

$$z_0^-(1) = z^*(x_0), \quad \xi_0^-(1) = 0.$$

From the periodicity of functions $z_0(\tau), \xi_0(\tau)$ it follows that the system (8),(9) has a unique solution.

The functions $x_1^\pm(\tau)$ are described by the equations

$$dx_1^\pm/d\tau^\pm = \theta_0^\pm h_2(z_0^\pm(\tau^\pm, x_0), \xi_0^\pm(\tau^\pm, x_0), x_0, \pm 1), \tag{10}$$

with initial conditions and periodicity conditions given by

$$x_1^+(0) = x_1^*, \quad x_0^-(0) = x_1^+(1), \quad x_1^-(1) = x_1^*, \tag{11}$$

Moreover

$$[h_{20}](x_0) = \int_0^1 h_2(z_0(\tau^+, x_0), \xi_0^+(\tau^+, x_0), x_0, 1)d\tau^+ +$$

$$+ \int_0^1 h_2(z_0(\tau^-, x_0), \xi_0^-(\tau^-, x_0), x_0, -1)d\tau^- = 0, \tag{12}$$

$$det|\frac{d[h_{20}]}{dx}(x_0)| = T(x_0)\frac{dp}{dx}(x_0) \neq 0.$$

This means that for every x_1^* there exists a unique solution of (10) and (11) for which $\int_0^1 \tilde{x}_1^+(\tau^+)d\tau^+ + \int_0^1 \tilde{x}_1^-(\tau^-)d\tau^- = 0$ and we can define the function $x_1(\tau)$ in the form

$$x_1(\tau) = x_1^* + \tilde{x}_1(\tau) =$$

$$= \begin{cases} x_1^* + \tilde{x}_1^+(\tau/\tilde{\theta}_k^+(\mu)) & \text{for} \quad \tau \in [0, \tilde{\theta}_k^+(\mu)] \\ x_1^* + \tilde{x}_1^-((\tau - \tilde{\theta}_k^+(\mu))/\tilde{\theta}_k^-(\mu)) & \text{for} \quad \tau \in [\tilde{\theta}_k^+(\mu), \tilde{T}_k(\mu)]. \end{cases}$$

The functions $z_1^\pm(\tau^\pm, x_1^*), \xi_1^\pm(\tau^\pm, x_1^*)$ are defined by equations

$$dz_1^\pm/d\tau = \theta_0^\pm(g_z'^\pm z_1^\pm + g_\xi'^\pm \xi_1^\pm + g_x'^\pm x_1^\pm) + \theta_1^\pm g^\pm; \tag{13}$$

$$d\xi_1/d\tau = \theta_0^\pm(h_{1z}'^\pm z_1^\pm + h_{1\xi}'^\pm \xi_1^\pm + h_{1x}'^\pm x_1^\pm) + \theta_1^\pm h_1^\pm,$$

where the values of $g^{\pm}, h_1^{\pm}$ and their derivatives are calculated at the points

$$(z_0^{\pm}(\tau^{\pm}, x_0), \xi_0^{\pm}(\tau^{\pm}, x_0), x_0, \pm 1).$$

Initial and periodicity conditions for system (13) are defined by equations

$$z_1^{+}(0, x_1^*) = z_1^{-}(1, x_1^*) = z_1^*, z_1^{-}(0, x_1^*) = z_1^{+}(1, x_1^*); \tag{14}$$

$$\xi_1^{+}(0, x_1^*) = \xi_1^{+}(1, x_1^*) = \xi_1^{-}(0, x_1^*) = \xi_1^{-}(1, x_1^*) = 0.$$

Equations (13) depend linearly on $z_1^{\pm}, \xi_1^{\pm}, \theta_1^{\pm}$ and consequently their solutions $z_1^{\pm}(\tau, x_1^*)$, $\xi_1^{\pm}(\tau, x_1^*)$, $\theta_1^{\pm}(x_1^*)$ are linearly dependent on the initial conditions $z_1^{\pm}(0, x_1^*)$. Expressing $z_1^{\pm}(\tau, x_1^*), \xi_1^{\pm}(\tau, x_1^*), \theta_1^{\pm}(x_1^*)$ through $z_1^{+}(0, x_1^*)$ and substituting the results in the first equation of (14) we have a system of algebraic equations linear in $z_i^{+}(0, x_1^*)$ whose determinant coincides with $det|I_m - \partial R(z^*(x_0), x_0)/\partial z| \neq 0$.

The functions $x_2^{\pm}(\tau)$ are described by the equations

$$dx_2^{\pm}/d\tau = \theta_0^{\pm}(h'_{2\,z} z_1^{\pm} +$$

$$+ h'_{2\,\xi}\xi_1^{\pm} + h'_{2\,x} x_1^{\pm}) + \theta_1^{\pm} h_2, \tag{15}$$

where the values of $h_2^{\pm}$ are calculated at the points

$$(z_0^{\pm}(\tau^{\pm}, x_0), \xi_0^{\pm}(\tau^{\pm}, x_0), x_0, \pm 1).$$

Initial and periodicity conditions are

$$x_2^{+}(0) = x_2^*, \quad x_2^{-}(0) = x_2^{+}(1), \quad x_2^{-}(1) = x_2^*. \tag{16}$$

The condition under which system (16) has periodic solutions for every x_2^* takes the form

$$\int_0^1 [\theta_0^{+}(h'^{+}_{2\,z} z_1^{+}(\tau^{+}, x_1^*) + h'^{+}_{2\,\xi}\xi_1^{+}(\tau^{+}, x_1^*) +$$

$$+ h'^{+}_{2\,x} x_1^{+}(\tau^{+}, x_1^*)) + \theta_1^{+}(\tau^{+}, x_1^*) h_2^{+}] d\tau^{+} +$$

$$+ \int_0^1 [\theta_0^{-}(h'^{-}_{2\,z} z_1^{-}(\tau^{-}, x_1^*) + h'^{-}_{2\,\xi}\xi_1^{-}(\tau^{-}, x_1^*) +$$

$$+ h'^{-}_{2\,x} x_1^{-}(\tau^{-}, x_1^*)) + +\theta_1^{-}(x_1^*) h_2^{-}] d\tau^{-} = 0. \tag{17}$$

Conditions (17) are a system of linear equations for obtaining x_1^*, whose determinant coincides with $\frac{dp}{dx}(x_0) \neq 0$. This means that we can find uniquely the function $x_2(\tau)$ in form $x_2(\tau) = \tilde{x}_2^* + \bar{x}_2(\tau)$, where $\bar{x}_2(\tau)$ is the function with zero averaged value.

Now suppose that the functions $z_j(\tau), \xi_j(\tau), x_j(\tau)$ and the constants $x_j^*, \theta_j^{\pm}$, $j = 1, ..., i-1$ have been found. Moreover, the periodic function $x_i(\tau)$ for every x_i^* can be represented in form of the sum of x_i^* and the function $\tilde{x}_i(\tau)$ with zero averaged value.

Then the functions $z_i^{\pm}(\tau^{\pm}, x_i^*), \xi_i^{\pm}(\tau^{\pm}, x_i^*), x_i^{\pm}(\tau^{\pm}, x_i^*)$ are defined by equations

$$dz_i/d\tau^{\pm} = \theta_0^{\pm}(g_z'^{\pm} z_i^{\pm} + g_\xi'^{\pm}\xi_i^{\pm} + g_x'^{\pm} x_i^{\pm})+$$

$$+\theta_i^{\pm}(x_i^*)g^{\pm} + \Pi_{1i}^{\pm}(\tau^{\pm}); \tag{18}$$

$$d\xi_i/d\tau^{\pm} = \theta_0^{\pm}(h_{1z}'^{\pm} z_i^{\pm} + h_{1\xi}'^{\pm}\xi_i^{\pm} + h_{1x}'^{\pm} x_i^{\pm})+$$

$$\theta_i^{\pm}(x_i^*)h_1^{\pm} + \Pi_{2i}^{\pm}(\tau^{\pm}),$$

where the values of $g^{\pm}, h_1^{\pm}$ and their derivatives are calculated at the points

$$(z_0^{\pm}(\tau^{\pm}, x_0), \xi_0^{\pm}(\tau^{\pm}, x_0), x_0, \pm 1),$$

and $\Pi_{ji}^{\pm}, j = 1,2$ are uniquely defined functions containing the terms of order μ^i in the asymptotic representations of $g^{\pm}, h_1^{\pm}$ depending on $z_j^{\pm}, \xi^{\pm}, x_j^{\pm}, x_j^*$, $j \leq i-1$. Initial and periodicity conditions for system (18) are defined by the equations

$$z_i^+(0, x_1^*) = z_i^-(1, x_i^*) = z_i^*,\ z_i^-(0, x_i^*) = z_i^+(1, x_i^*) \tag{19}$$

$$\xi_i^+(0, x_i^*) = \xi_i^+(1, x_i^*) = \xi_i^-(0, x_i^*) = \xi_i^-(1, x_i^*) = 0.$$

Equations (18) depend linearly on $z_i^{\pm}, \xi_i^{\pm}, \theta_i^{\pm}$ and consequently their solutions $z_i^{\pm}(\tau, x_i^*)$, $\xi_i^{\pm}(\tau, x_i^*)$, $\theta_i^{\pm}(x_i^*)$ are linearly dependent on the initial conditions $z_i^{\pm}(0, x_i^*)$. Expressing $z_i^{\pm}(\tau, x_i^*), \xi_i^{\pm}(\tau, x_i^*), \theta_i^{\pm}(x_i^*)$ through $z_i^+(0, x_i^*)$ and substituting the results in the first equation of (19) we have a system of algebraic equations linear in $z_i^+(0, x_i^*)$ whose determinant coincides with $det|I_m - \partial R(z^*(x_0), x_0)/\partial z| \neq 0$.

The functions $x_{i+1}(\tau)$ are described by the equations

$$dx_{i+1}^{\pm}/d\tau = \theta_0^{\pm}(h_{2z}' z_1^{\pm} + h_{2\xi}'\xi_1^{\pm} + h_{2x}' x_1^{\pm})+$$

$$+\theta_1^{\pm} h_{i+1} + \pi_{3i}^{\pm}(\tau),$$

where the values of $h_2^{\pm}$ and its derivatives are calculated at the

$$(z_0^{\pm}(\tau^{\pm}, x_0), \xi_0^{\pm}(\tau^{\pm}, x_0), x_0, \pm 1)$$

and $\pi_{3i}^{\pm}$ are uniquely defined functions containing the terms of order μ^i in the asymptotic representations of $h_2^{\pm}$ depending on $z_j^{\pm}, \xi_j^{\pm}, x_j^{\pm}, x_j^*$, $j \leq i-1$. Initial and periodicity conditions are

$$x_{i+1}^+(0) = x_{i+1}^*,\ x_{i+1}^-(0) = x_{i+1}^+(1),\ x_{i+1}^-(1) = x_{i+1}^*. \tag{20}$$

The conditions under which system (20) has a periodic solution with zero averaged value for every x_{i+1}^* takes the form

$$\int_0^1 [\theta_0^+(h_{2z}'^+ z_i^+(\tau^+, x_i^*) + h_{2\xi}'^+\xi_i^+(\tau^+, x_i^*)+$$

$$+h_{2x}'^+ x_i^+(\tau^+, x_i^*)) + \theta_i^+(\tau^+, x_i^*)h_2^+]d\tau^+ + \tag{21}$$

$$+\int_0^1 [\theta_0^- (h'^-_{2\,z} z_i^- (\tau^-, x_i^*) + h'^-_{2\,\xi} \xi_i^- (\tau^-, x_i^*) +$$

$$+ h'^-_{2\,x} x_i^- (\tau^-, x_i^*)) + \theta_i^- (x_i^*) h_2^-] d\tau^- = 0.$$

Conditions (21) are a system of linear equations for obtaining of x_i^*, whose determinant coincides with $\frac{dp}{dx}(x_0) \neq 0$. This means that for every x_{i+1}^* we can uniquely find the function $x_{i+1}(\tau)$ in the form $x_{i+1}(\tau) = \tilde{x}_{i+1}^* + \bar{x}_{i+1}(\tau)$, where $\bar{x}_{i+1}(\tau)$ is a function with zero averaged value. To finish the algorithm for the design of the desired asymptotic representation, it is necessary to define

$$(z_i(\tau), \xi_i(\tau)) =$$

$$= \begin{cases} (z_i^+ (\tau/\tilde{\theta}_k^+(\mu), x_i^*), \xi_i^+ (\tau/\tilde{\theta}_k^+(\mu), x_i^*)) \text{ for } \quad \tau \in [0, \tilde{\theta}_k^+(\mu)], \\ (z_i^- ((\tau - \theta_k^+(\mu))/\tilde{\theta}_k^-(\mu), x_i^*), \xi_i^+ ((\tau - \theta_k^+(\mu))/\tilde{\theta}_k^-(\mu), x_i^*)) \\ \text{for} \quad \tau \in [\tilde{\theta}_k^+(\mu), \tilde{T}_k(\mu)], i = 0, ..., k. \end{cases}$$

$$x_j(\tau) = \begin{cases} x_j^* + \tilde{x}_j^+ (\tau/\tilde{\theta}_k^+(\mu)) \quad \text{for} \quad \tau \in [0, \tilde{\theta}_k^+(\mu)], \\ x_j^* + \tilde{x}_j^- ((\tau - \tilde{\theta}_k^+(\mu))/\tilde{\theta}_k^-(\mu)) \\ \text{for} \quad \tau \in [\tilde{\theta}_k^+(\mu), \tilde{T}_k(\mu)], j = 1, ..., k. \end{cases}$$

2.8 Correction of Averaged Equations

Let us now show how we can use knowledge about the fast periodic solution for the correction of averaged equations with any precision level according to the small parameter degrees. The knowledge of such equations is necessary the case when the linear part of the averaged equations (3) has spectral points on the imaginary axis.

Assume that we have found the functions

$$\theta^{\pm}(x, \mu) = \theta_0^{\pm}(x) + \sum_{i=1}^{\infty} \mu^i \theta_i^{\pm}(x),$$

$$T(x, \mu) = \theta^+(x, \mu) + \theta^-(x, \mu)$$

and

$$(z_i(\tau, x), \xi_i(\tau, x)) =$$

$$= \begin{cases} (z_i^+ (\tau/\theta^+(\mu, x), x), \xi_i^+ (\tau/\theta^+(\mu, x), x)) \\ \text{for} \quad [0, \theta^+(\mu, x)], \\ (z_i^- ((\tau - \bar{\theta}^+(\mu))/\theta^-(\mu, x), x), \xi_i^- ((\tau - \theta^+(\mu, x))/\theta^-(\mu, x), x)) \\ \text{for} \quad [\theta^+(\mu, x), T(\mu, x)], \end{cases}$$

then

$$z(\tau, x, \mu) = z_0(\tau, x) + \mu z_1(\tau, x) + ... + \mu^i z_i(\tau, x) + ...,$$

$$\xi(\tau, x, \mu) = \xi_0(\tau, x) + \mu \xi_1(\tau, x) + ... + \mu^i \xi_i(\tau, x) + ...,$$

$$x(\tau, x, \mu) = \mu x_1(\tau, x) + ... + \mu^i x_i(\tau, x) + ... \quad .$$

Then the precise averaged equation has the form

$$dx/dt = \frac{1}{T(x\mu)} \int_0^{T(x,\mu)} h_2(z(\tau,x,\mu),\xi(\tau,x,\mu),x+$$

$$+\tilde{x}(\tau,\mu),u(\xi(\tau,x,\mu)))d\tau. \qquad (PAE)$$

Equation (PAE) correspond to the system (D.4) which describes a flow on the slow motion manifold in system (D.1). In this case the first order approximation of (PAE) has the form

$$dx/dt = \frac{1}{T_0(x)} \Bigg\{ (1-\mu T_1(x)) \int_0^{T_0(x)} h_2 d\tau +$$

$$+\mu \Bigg[\int_0^{T_0(x)} \Bigg(h'_{2z} z_1(\tau,x) + h'_{2\xi}\xi_1(\tau,x) + h'_{2x}\tilde{x}_1(\tau) \Bigg) d\tau +$$

$$+\theta_1^+(x) h_2(z_0(\theta_0(x),x),\xi_0(\theta_0(x),x),x,1)+$$

$$+\theta_1^-(x) h_2(z_0(T_0(x),x),\xi_0(T_0(x),x),x,-1) \Bigg] \Bigg\}, \qquad (FAAE)$$

where the values of the functions h_2 and its derivatives in the integral terms are calculated at the points $(z_0(\tau,x),\xi_0(\tau,x),x,u(\xi_0(\tau,x)))$. Analogously we can obtain the averaged equations with any precision level expanding in powers of the small parameter.

2.9 Investigation of stability in critical Case

Theorem 5 Reduction Principle. *Under conditions $1^0 - 7^0$ the periodic solution for original system (1) is stable (asymptotically stable, unstable) if and only if the equilibrium point of system (PAE) is stable (asymptotically stable, unstable).*

Corollary 6. *Assume that for system (1) conditions $1^0 - 7^0$ are true. If the equilibrium point of system (FAAE) is asymptotically stable (unstable) in the first approximation then the periodic solution for original system (1) is asymptotically stable (unstable).*

3 Analysis of Averaged Equations in Sliding Mode with Fast Actuators

3.1 Averaged Equations of Systems which Linearly Depend on Relay Control

In this section we will consider the SPRCS which linearly depend on relay control. We will show that the averaged equations which describe the slow motions in such SPRCS and the equations which describe the sliding motion in the reduced systems coincide.

Let's consider the system

$$\mu dz/dt = A(s,x)z + f_1(s,x) + K_1(s,x)u(s),$$

$$ds/dt = B(s,x)z + f_2(s,x) + K_2(s,x)u(s),$$

$$dx/dt = D(s,x)z + f_3(s,x) + K_3(s,x)u(s), \qquad (22)$$

where $z \in \mathbf{R}^m$, $s \in \mathbf{R}$, $x \in \mathbf{R}^n, u(s) = sign\,(s)$ and $f_i, K_i\,(i = 1,2,3)$ are smooth functions of their arguments. Setting $\mu = 0$ and expressing z_0 from the first equation of system (22) according to the formula $z_0 = -A^{-1}(s,x)[f_1(s,x) + K_1(s,x)u(s)]$ we obtain the reduced system

$$ds/dt = -B(s,x)A^{-1}(s,x)f_1(s,x) + f_2(s,x) -$$

$$-[B(s,x)A^{-1}(s,x)K_1(s,x) - K_2(s,x)]u(s),$$

$$dx/dt = D(s,x)A^{-1}(s,x)f_1(s,x) + f_3(s,x) -$$

$$-[D(s,x)A^{-1}(s,x)K_1(s,x) - K_3(s,x)]u(s).$$

Suppose that for the original system (22) the conditions of dynamic uncertainty hold which means that

$$K_2(0,x) \geq 0, \quad B(0,x)A^{-1}(0,x)K_1(0,x) - K_2(0,x) > 0. \qquad (CDU)$$

The equations which describe the motion in sliding modes in the reduced system have the form

$$dx/dt = -D(0,x)A^{-1}(0,x)f_1(0,x) + f_3(0,x) -$$

$$-[D(0,x)A^{-1}(0,x)K_1(0,x) - K_3(0,x)](u(s) - u_{eq}(x)), \qquad (23)$$

$$u_{eq}(x) = [B(0,x)A^{-1}(0,x)K_1(0,x) - K_2(0,x)]^{-1}\times$$

$$\times[-B(0,x)A^{-1}(0,x)f_1(0,x) + f_2(0,x)].$$

Let's show that the averaged equations for the original system (22) coincide with system (23).

Suppose that for all $x \in \bar{X}$ the following conditions are true:

$(*)$ $\quad Re \quad Spec\,A(0,x) < 0;$

$(**)$ $\quad |u_{eq}(x)| < 1.$

It is obvious that if the conditions of dynamical uncertainty are true it is reasonable to consider only solutions of system (22) with initial conditions

$$z(0,\mu) = z^0,\ s(0,\mu) = \mu s^0,\ x(0,\mu) = x^0,$$

which are situated in the $O(\mu)$ vicinity of the switching surface. Following [3],[9] let us increase $1/\mu$ times the neighbourhood of the discontinuity surface $s = 0$ in system (22) and let the variable $\xi = s/\mu$. Then we will rewrite system (22) in the form

$$\begin{aligned} \mu dz/dt &= A(\mu\xi,x)z + f_1(\mu\xi,x) + K_1(\mu\xi,x)u(\xi), \\ \mu d\xi/dt &= B(\mu\xi,x)z + f_2(\mu\xi,x) + K_2(\mu\xi,x)u(\xi), \\ dx/dt &= D(\mu\xi,x)z + f_3(\mu\xi,x) + K_3(\mu\xi,x)u(\xi). \end{aligned} \tag{24}$$

In this case the system which describes the fast motions in (24) has, analogous to (3), the form

$$\begin{aligned} dz/d\tau &= A(0,x)z + f_1(0,x) + K_1(0,x)u(\xi), \\ d\xi/d\tau &= B(0,x)z + f_2(0,x) + K_2(0,x)u(\xi), \quad (x - parameter). \end{aligned} \tag{25}$$

Introducing into system (25) the new variables $\eta = z + A^{-1}(0,x)[f_1(0,x) + K_1(0,x)u_{eq}(x)]$ we will have

$$\begin{aligned} d\eta/d\tau &= A(0,x)\eta + K_1(0,x)\bar{u}(\xi,x), \\ d\xi/d\tau &= B(0,x)\eta + K_2(0,x)\bar{u}(\xi,x), \\ \bar{u}(\xi,x) &= u(\xi) - u_{eq}(x). \end{aligned} \tag{26}$$

Let's consider the point mapping of the surface $\xi = 0$ into itself which is made by system (26). The solution of system (26) with initial conditions

$$\eta^+(0,\mu) = \eta, \quad \xi^+ = 0;$$

$$\eta \in \Omega^+ = \{(\eta,\mu) : B(0,x)\eta + K_2^+(0,x)\bar{u}(\xi,x) > 0\},$$

$$K_i^+ = K_i(1 - u_{eq}), i = 1,2$$

has the form

$$\begin{aligned} \eta^+(\tau,\eta,x) &= e^{A\tau}(\eta + A^{-1}K_1^+) - A^{-1}K_1^+, \\ \xi^+(\tau,\eta,\mu) &= BA^{-1}(e^{A\tau} - I)(\eta + A^{-1}K_1^+) - \\ &\quad -(BA^{-1}K_1^+ - K_2^+)\tau. \end{aligned}$$

Here and in the sequel the functions A, B, K_1, K_2 are computed at the point $(0,x)$. For $\tau = 0$ $d\xi/d\tau = B(0,x)\eta + K_2^+(0,x)\bar{u}(\xi,x)$ and consequently

$$\xi^+(\tau,\eta,\mu) > 0$$

at least for small $\tau > 0$. On the other hand from condition (i) it follows that

$$\lim_{\tau\to\infty} \xi^+(\tau,\eta,\mu) = -\infty.$$

This means that there exists $\theta(\tau,\eta,\mu)$, the smallest root of equation

$$\xi^+(\theta(\tau,\eta,\mu),\eta,\mu)=0.$$

Let's rewrite this equation in form

$$BA^{-1}(e^{A\theta}-I)(\eta+A^{-1}K_1^+)=(BA^{-1}K_1^+-K_2^+)\theta.$$

It follows from the definition of θ that $d\xi^+/d\tau(\theta)\le 0$. This means that we can define the point mapping of the set Ω^+ into the set

$$\Omega^-=\{(\eta,\mu):\ B(0,x)\eta^+(\theta,\eta,x)-K_2^-(0,x)\bar{u}_{eq}(\xi,x)<0\},$$

where $K_i^-=K_i(1+u_{eq}), i=1,2$. Analogously the point mapping

$$\eta^-(\Theta,\eta,x)=e^{A\Theta}(\eta-A^{-1}K_1^-)+A^{-1}K_1^-,$$

$$BA^{-1}(e^{A\Theta}-I)(\eta-A^1K_1^-)=-(BA^{-1}K_1^--K_2^-)\Theta$$

transforms the set Ω^+ into the set

$$\Omega^*=\{(\eta,\mu):B(0,x)\eta-K_2^-(0,x)>0,\ x\in\bar{X}\}.$$

This means that the point mapping

$$\Phi(\eta,x)=\eta^-(\Theta,\eta^+(\theta,\eta,x),x)$$

given by formula

$$\Phi(\eta,x)=e^{A\Theta}(e^{A\theta}(\eta+A^{-1}K_1^+)-2A^{-1}K_1)+A^{-1}K_1^-$$

transforms the set Ω^* into itself. Let's denote $\eta^*(x)$ as the fixed point of the point mapping $\Phi(\eta,x)$ which corresponds to the periodic solution

$$(z_0(\tau,x),\xi_0(\tau,x)).$$

For $\eta^*(x)$ we have the formula

$$\eta^*(x)=\{2[I-e^{A(\theta+\Theta)}]^{-1}[I-e^{A\Theta}]-(1-u_{eq})\}A^{-1}K_1.$$

Let's study the properties of averaged values of the $T(x)=\theta(x)+\Theta(x)$ - periodic solutions $\eta_0(\tau,x)$ and $\xi_0(\tau,x)$.

$$I(x)=\int_0^{T(x)}\eta_0(\tau,x)d\tau=[(1+u_{eq})\Theta-(1-u_{eq})\theta]A^{-1}K_1.$$

Taking into account that $\xi_0(T(x),x)=0$ we have

$$\xi_0(T(x),x)=$$

$$=\int_0^{T(x)}B\eta_0(\tau,x)d\tau-K_2[(1+u_{eq})\Theta-(1-u_{eq})\theta]=0.$$

This means that $(1+u_{eq})\Theta=(1-u_{eq})\theta$. The following lemma is true.

Lemma 7. *If there exists a solution of system* (26) *of period* $T(x)$ *then*

$$\int_0^{T(x)} \eta_0(\tau, x) d\tau = 0,$$

$$\int_0^{T(x)} u(\eta_0(\tau, x)) d\tau = \frac{\Theta(x) - \theta(x)}{T(x)} = u_{eq}(x).$$

Remark. This lemma was obtained for the first time in [11] by using transfer function methods.

Let's turn to system (22). If for system (24) the conditions of theorem 3 are true there exists an isolated periodic solution $(z(\tau, \mu), \xi(\tau, \mu), x(\tau, \mu))$ which corresponds to the periodic solution $(\eta_0(\tau, x), \xi_0(\tau, x))$ of system (26). Moreover

$$\int_0^{T(x)} z_0(\tau, x) d\tau = A^{-1}(0, x)(f_1(0, x) + K_1(0, x) u_{eq}(x)).$$

This means that the averaged equations which approximately describe the behaviour of the slow motions in system (22) coincide with equations (23) for the sliding motions in the reduced system .

3.2 Example

Suppose that a mathematical model of a control system taking account of actuator behaviour has the following form

$$\mu dz/dt = -z - u, ds/dt = z + (\alpha + x)u, \ \alpha > 0 \tag{27}$$

$$dx/dt = -z + x - u, \tag{28}$$

$z, s, x \in \mathbf{R}, u(s) = sign(s), \mu$ - actuator time constant. The fast motions taking place in (27), (28) are described by the system

$$dz/d\tau = -z - u, \ d\xi/d\tau = z + (\alpha + x)u, \ u = sign(\xi). \tag{29}$$

System (29) is symmetric relative to the point $z = \xi = 0$ so we shall consider it as a point mapping $R(z, x)$ of the domain $z + \alpha > 0$ on the switching line $\xi = 0$ into the domain $z + \alpha < 0$ with $\xi > 0$. Then $\Psi(z) = -1 + e^{-T}(z + 1)$ where τ is the smallest root of equation

$$(1 - e^{-T})(z + 1) = (1 - \alpha - x)\tau.$$

The fixed point $z^* = \Psi(z^*(x), x)$ corresponding to the periodic solution (29) is determined by the equation $\Psi(z^*(x), x) = -z^*(x)$. Then the fixed point $z^*(x)$ (amplitude) and the semiperiod $T(x)$ of the periodic solution are determined by equations

$$2th(T/2) = (1 - \alpha - x)T, \ z^*(x) = th(T/2). \tag{30}$$

Equations (30) with $0 < \alpha + x < 1$ have positive solutions which corresponds to the existence of a $2T$ periodic solution in system (29). The slow motions averaged equation for system (28) assumes the form

$$dx/dt = -x.$$

This equation has the asymptotically stable equilibrium point $x = 0$. It follows from (30) that $T \approx 3.83, \lambda \approx -0.07$, and so system (27), (28) has an orbitally asymptotically stable periodic solution which lies in the $O(\mu)$ neighbourhood of the switching surface.

3.3 The Systems Containing The Relay Control Nonlinearly

Consider the control system which is described by the equations

$$\mu dz/dt = -z - u, ds/dt = z + \alpha u,$$

$$dx/dt = (z^4 - z^2 + \beta)x, \tag{31}$$

where $x, s, z \in \mathbf{R}$, $u(s) = sign(s)$, $0 < \alpha, \beta < 1$ and μ is the actuator time constant. If we take $\mu = 0$ in system (31) we will have

$$ds/dt = (\alpha - 1)u,\ dx/dt = (u^4 - u^2 + \beta)x. \tag{32}$$

In system (32) a stable sliding mode exists. Both the classical extension definition of solutions according Filippov [8] and the equivalent control method [17] coincide. These motions are described by the equation $dx/dt = \beta x$. The zero solution of this equation is unstable for $\beta > 0$.

At the same time if $0 < \alpha < 1$ in system (31) fast periodic solutions occur. Let us denote $z(\tau)$ as the first coordinate of the periodic solution (29) for $x = 0$. If α and β are selected so that

$$-\gamma = \int_0^{T(x_0)} [z^4(\tau) - z^2(\tau)]d\tau < -\beta < 0$$

the averaged equation has the form $dx/dt = -(\gamma - \beta)x$. The zero solution of this equation is asymptotically stable. This means that system (31) has an asymptotically orbitally stable periodic solution in the $O(\mu)$ neighbourhood of the point $s = x = 0$. The averaged equation does not coincide with the equations of the equivalent control method and Filippov determination of solution. Moreover the introduction of positive feedback was used for transition from one vector of convex closure of the right hand part to the other one and for giving the system desired dynamic properties.

4 Eigenvalue Assignment in Averaged Equations using Dynamics of Actuators

4.1 Problem Formulation

Let us suppose that the behaviour of control system is described by the state vector (s, x) $\quad (s \in R, x \in R^n)$ with equations

$$\dot{s} = A_1 s + A_2 x + b_1 u(s), \dot{x} = A_3 s + A_4 x + b_2 u(s), \tag{33}$$

where $s \in \mathbf{R}$, $x \in \mathbf{R}^n$ the discontinuous control law has been designed in the form $u(s) = sign(s)$. Let us suppose that this control law ensures a stable sliding mode on the surface $s = 0$.

Then the motions in the sliding mode in system (33) are described by the equations

$$\dot{x} = (A_4 - b_2 b_1^{-1} A_3)x. \tag{34}$$

In [17] two methods were proposed to solve the problem of eigenvalue assignment in (34):

(i)to extend the state space by using additional dynamics and to solve the problem of eigenvalue assignment in the extended state space;

(ii) to include the derivatives of the variable s into the equation of the switching surface.

In [15] the fast variable describing the behaviour of the fast actuator was introduced in the equation for the switching surface for motion control in singularly perturbed discontinuous control systems. This approach ensures the existence of a first order sliding mode in overall system. For such systems the composite control method (see [6]) was used [5].

We suggest using the dynamics of the actuators which are present in the original system to solve the problem of eigenvalue assignment in (34). The proposed algorithm is based on theorems 2,3. It is necessary to note that in this case we can use only the slow coordinates of the state-vector for control design and we can solve the eigenvalue assignment problem in the space of the sliding mode equations. On the other hand, the proposed algorithm is useful only in case when the actuator is a MIMO system.

4.2 The Eigenvalues Assignment in Averaged Equations

Let us suppose that the complete model of the control system taking into account the fast actuator dynamics has the form

$$\mu \dot{z} = B_1 z + B_2 s + B_3 x + d_1 v$$

$$\dot{s} = B_4 z + B_5 s + B_6 x + d_2 v, \quad \dot{x} = B_7 z + B_8 s + B_9 x + d_3 v, \tag{35}$$

where $z \in \mathbf{R}^m$, $v \in \mathbf{R}^l$ and μ is actuator time constant. Now we suppose that the conditions

$$rank \begin{pmatrix} B_4 \\ B_7 \end{pmatrix} \geq 2,$$

$$rank\, d_1 \geq\ 2, \quad m \geq\ l \geq\ 2 \tag{36}$$

are satisfied. The conditions (36) mean that the discontinuous control is transmitted to the plant through the actuators with the number of inputs and outputs no less than two. Ignoring fast dynamics, having accepted that $\mu = 0$ and expressing z according to the formula $z = -B_1^{-1}(B_2 s + B_3 x + d_1 v)$ we obtain

$$\dot{s} = (B_5 - B_4 B_1^{-1} B_2)s + (B_6 - B_4 B_1^{-1} B_3)x + (d_2 - B_4 B_1^{-1} d_1)v, \tag{37}$$

$$\dot{x} = (B_8 - B_7 B_1^{-1} B_2)s + (B_9 - B_7 B_1^{-1} B_3)x + (d_3 - B_7 B_1^{-1} d_1)v.$$

Let us suppose that in the case when the control law has been designed in the form $v = Ku(s)$ $(u(s) = sign\,(s), K$ is constant vector), systems (37) and (33) coincide.

The proposed algorithm uses theorem 3. We propose to use the control law in the form

$$v = Ku(s) + w. \tag{38}$$

From theorem 3 it follows that the slow motions in (35) are described by the equations

$$\dot{x} = (A_4 - b_2 b_1^{-1} A_2)x - [d_3 - B_7 b_1^{-1}(d_2 - B_4 B_1^{-1} d_1)]w \tag{39}$$

Assume that

D.1.*Matrices*

$$(A_4 - b_2 b_1^{-1} A_2)\, and\, [d_3 - B_7 b_1^{-1}(d_2 - B_4 B_1^{-1} d_1)]$$

are controllable.

Then, choosing the control vector in the form $w = Lx$, we can solve the eigenvalue assignment problem for (39). This means, that an algorithm for design of the control vector w is proposed which allows us to solve the eigenvalue assignment problem for system (39). If the conditions of lemma 7 are fulfilled, the equations (39) coincide with the averaged equations which approximately describe the slow motions in a small neighbourhood of the switching surface of system (35). To use this algorithm it is necessary to ensure existence and stability in the first approximation of periodic solutions of the system

$$dz/d\tau = B_1 z + d_1 K u(s),\; ds/d\tau = B_4 z + d_2 K u(s), \tag{40}$$

describing the fast motions in (35) in the small neighbourhood of the equilibrium point $x = 0$ of the averaged equation (39).

To formulate the sufficient conditions consider the point mapping $R(z)$ of the domain

$$\Omega^* = \{z\, :\, B_4 z - d_2 K > 0, z \in \mathbf{R}^m\}$$

on the surface $s = 0$ into itself, given by the formulae

$$R^+(z) = e^{B_1 \tau_1}(z + B_1^{-1} d_1 K) - B_1^{-1} d_1 K,$$

$$R(z) = e^{B_1 \tau_2}(R^+(z) - B_1^{-1} d_1 K) + B_1^{-1} d_1 K,$$

where $\tau_1, \tau_2$the smallest positive roots of the equations

$$B_4 B_1^{-1}(e^{B_1\tau_1} - I)(z + B_1^{-1} d_1 K) = (B_4 B_1^{-1} d_1 - d_2) K\tau_1,$$

$$B_4 B_1^{-1}(e^{B_1\tau_2} - I)[e^{B_1\tau_1}(z + B_1^{-1} d_1 K) -$$
$$-2B_1^{-1} d_1 K] = -(B_4 B_1^{-1} d_1 - d_2) K\tau_2.$$

Taking into account the symmetry of system (33) for $u(s) = sign\,(s)$ we can rewrite the conditions of existence of the fixed point in the form $R^+(z^*) = -z^*$. Then

$$z^* = [I + e^{B_1 T}]^{-1}(I - e^{B_1 T}) B_1^{-1} d_1 K,$$

where the semiperiod of the desired periodic solution $T > 0$ is the smallest root of the equation

$$B_4 B_1^{-1}(e^{B_1 T} - I)(z^* + B_1^{-1} d_1 K) = (B_4 B_1^{-1} d_1 - d_2) KT.$$

Then from theorems 2,3 it follows

Theorem 8. *Assume that condition D.1 is true and* B_1 *and* $R(z)$ *satisfy the conditions*

D.2 $\;Re\;\; Spec\, B_1 < 0$.

D.3 The point mapping R(z) has an isolated fixed point $z^* \in \Omega^*$.

D.4. For $\lambda_i(x_0)$ $(i = 1, ..., m)$, *the eigenvalues of the matrix* $\frac{\partial R}{\partial z}(z^*)$, *the inequalities* $|\lambda_i| < 1$ *hold.*

Then there exists a matrix L *which ensures that the characteristic polynomial of the matrix*

$$(A_4 - b_2 b_1^{-1} A_2) - [d_3 - B_7 b_1^{-1}(d_2 - B_4 B_1^{-1} d_1)]L$$

has the desired form and the averaged equations for system (35) *has the form* (39) *and for systems* (35) *and* (39) *theorems* 2, 3 *are true.*

4.3 Example

Let us suppose that the state vector of the control system is described by the equations

$$\dot{s} = u(s)/2,\; \dot{x}_1 = x_2,\; \dot{x}_2 = -x_1,\; s, x_1, x_2 \in \mathbf{R} \tag{41}$$

and the discontinuous control $u(s) = -sign(s)$ has been designed. The motions in the sliding mode in (41) are described by the equations

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1. \tag{42}$$

The spectrum of matrix in (42) is situated on the imaginary axis. Let us suppose that the discontinuous control $u(s)$ is transmitted to the plant with the help of actuators whose behaviour is described by the variables z_1, z_2 and the overall model of the system has the following form

$$\mu\dot{z}_1 = -z_1 + v_1 - x_1, \mu\dot{z}_2 = -z_2 + v_2,$$

$$\dot{s} = z_2 + v_2/2,\ \dot{x}_1 = x_2,\ \dot{x}_2 = z_1. \tag{43}$$

It can be easily seen that in the case where we suppose that $v_1 = v_2 = u(s) = -sign(s)$, system (43) takes the form

$$\mu\dot{z}_1 = -z_1 - sign\,(s) - x_1, \quad \mu\dot{z}_2 = -z_2 - sign\,(s),$$

$$\dot{s} = z_2 + 1/2sign\,(s),\ \dot{x}_1 = x_2,\ \dot{x}_2 = z_1$$

and the slow motions in it are described by system (42) with an accuracy of $O(\mu)$.

Let's show that for system (43) the conditions of theorem 8 are fulfilled. Denote $z = (z_1, z_2)$. Consider the point mapping $R^+(z)$ of the domain

$$\Omega^* = \{(z, x) \,:\, z_2 - 1/2 > 0\}$$

on the surface $s = 0$ into the domain $\Omega^- = \{(z, x) \,:\, z_2 + 1/2 < 0\}$ made by the system

$$dz_1/d\tau = -z_1 - sign\,(s), \quad dz_2/d\tau = -z_2 - sign\,(s),$$

$$d\xi/d\tau = z_2 + 1/2sign\,(s). \tag{44}$$

The point mapping $R^+(z)$ of the domain Ω^* into the domain $\Omega^- = \{z : z_2 + 1/2 < 0\}$ made by the system (44) has the form

$$R^+(z) = \{R_1^+(z), R_2^+(z)\} =$$

$$= \{-1 + e^{-\tau}(z_1 + 1), -1 + e^{-\tau}(z_2 + 1)\},$$

where $\tau > 0$ is the smallest root of the equation $(1 - e^{-\tau})(z_2 + 1) = \tau/2$. System (44) is symmetric with respect to the point $(0, 0)$ and consequently the condition of existence of a fixed point z^* corresponding to the desired periodic solution of (44) takes the form $R^+(z^*) = -z^*$. Then z^* and the semiperiod T satisfy the equations $z_2^* = th(T/2)$, $4th(T/2) = T$ with the solution $z_1^* = z_2^* = 0,95$, $T \approx 3,83$. Moreover

$$\frac{\partial R^+}{\partial(z_1, z_2)}(z^*) = \begin{pmatrix} -0,07 & 0 \\ 0 & -0,07 \end{pmatrix}.$$

This means that for system (44) the conditions of theorem 3 are fulfilled and the slow motions in (44) are described by the averaged equations (42). This means that for eigenvalue assignment in the system we can use a control law of the form

$$v_1 = -sign\,(s) + l_1 x_1 + l_2 x_2, v_2 = -sign\,(s).$$

Assume that for our goal the desired characteristic polynomial of averaged equations

$$\dot{x}_1 = x_2,\ \dot{x}_2 = (l_1 - 1)x_1 + l_2 x_2 \tag{45}$$

has the form

$$\lambda^2 + \alpha\lambda + \beta,\ \alpha, \beta \,\text{are constants.} \tag{46}$$

This means that choosing $l_1 = 1 - \beta, l_2 = -\alpha$, we can ensure that the characteristic polynomial of the averaged system (45) has the form (46).

Conclusion

As a mathematical model of chattering in the small neighbourhood of the switching surface in sliding mode systems the singularly perturbed relay control systems was examined.

The following mathematical apparatus for the investigation of the SPRCS has been designed

- the sufficient conditions for existence of fast periodic solutions;
- the averaging theorem;
- the algorithm for asymptotic representation of fast periodic solutions;
- the sufficient conditions and reduction principle theorem for the investigation of the stability of fast periodic solutions.

It was shown that the slow motions in such SPRCS are approximately described by equations obtained from the equations for the slow variables of SPRCS by averaging along fast periodic motions.

The analysis of oscillations of sliding mode systems in the small neighbourhood of the sliding mode control systems has shown that

- in the general case when the original SPCSC contains the relay control nonlinearly, the averaging equations do not coincide with the equivalent control equations or the Filippov extension definition which describe the motions in the sliding mode in the reduced system;
- in the case when the original SPCSC contain the relay control linearly, the averaging equations and equations which describe the motions in the sliding mode in the reduced system coincide.

The results obtained were used for eigenvalue assignment for slow motions using the dynamics of the actuators in the case when the actuator is a MIMO system.

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