Chapter 1

Higher Order Sliding Modes as a Natural Phenomenon in Control Theory

Introduction

Sliding modes are used in order to keep a dynamic system to given constraints with utmost precision. They are also insensitive to external and internal disturbances. These features are provided due to a theoretically infinite frequency of control switching. At the same time, any presence of real actuators and measuring devices may cause considerable change in the real system behavior. There are also many arguments for the use of continuous controllers as approximations for discontinuous ones. Both reasons lead to the replacement of regular sliding modes in real systems by some special ones. Consider the phenomenon more explicitly.

A. Presence of fast actuators

Let the constraint be given by the equality of some constraint function σ to zero and the sliding mode $\sigma \equiv 0$ be provided by a relay control. Taking into account an actuator conducting a control signal to the process controlled, we achieve a more complicated dynamics. In this case a relay control enters the actuator and continuous output variables of the actuator

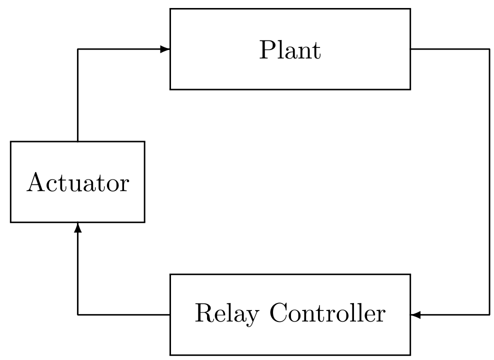

Figure 1.1: Control system with actuator

are transmitted to the plant input (Fig. 1.1). As a result discontinuous switching is hidden now in the higher derivatives of the constraint function [36, 37, 15, 16, 17, 18, 19, 20].

B. Artificial actuator-like dynamics

One of the main known drawbacks of regular sliding modes is the so-called chattering effect which is exhibited by high frequency vibration of the plant. This vibration features some definite vibration magnitude of the plant itself (in the state space) and of the plant velocity. While the first magnitude is infinitesimally small when switching imperfections (like switching delay) tend to zero, the second is approximately constant and eventually even large. Therefore the high frequency vibration energy is also finite or even large which may cause a system disaster. To avoid chattering several approaches have been proposed. The main idea is to change the dynamics in a small vicinity of the discontinuity surface in order to avoid real discontinuity and at the same time to preserve the main properties of the whole system. A transition to the new dynamics defined near the switching surface has to be sufficiently smooth. The idea is realized by insertion of some artificial actuator (Fig. 1.2). This actuator may be a functional [33] or may have its own dynamics [8, 7]. We are interested here in the latter case.

The actuator installed (artificial or real) has mainly some fast dynamics.

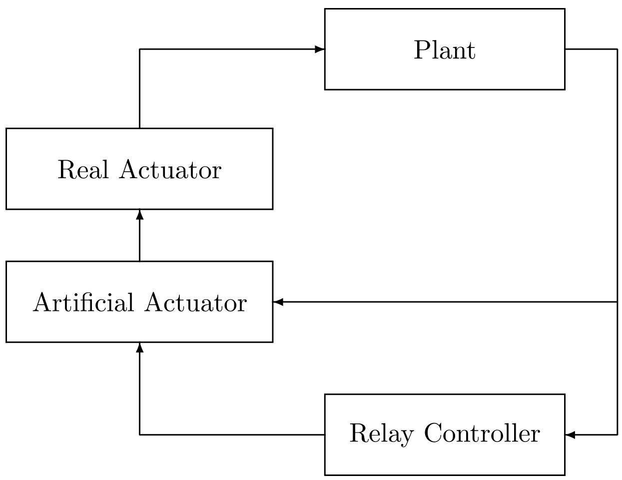

Figure 1.2: Control system with artificial actuator

The faster the dynamics, the more accurate is the modeling of the original discontinuous dynamics with all its advantages and disadvantages. However, from time to time a different mode, contiguous to the ordinary sliding mode, appears. The corresponding state and velocity vibration magnitudes both tend to zero when switching imperfections vanish. Moreover, the actuator dynamics does not need to be really fast for the existence of such a mode. Such modes were called higher order sliding modes [27, 9, 10, 11, 12, 2, 30, 31, 32, 24]. Convergence to this special mode may be asymptotic [15, 16, 17, 18, 19, 20, 27, 12, 2, 5, 30, 31, 32] or may feature a finite time as well [26, 9, 10, 11, 12, 24].

A higher order sliding mode (HOSM) is a movement on a special type integral manifold of a discontinuous dynamic system. It appears every time when some fast actuator-like dynamics is implanted instead of an ordinary relay in a variable structure system. Even if this mode is not stable, it plays the same role as integral manifold in ordinary differential equations. Thus, HOSM is an important natural phenomenon in control theory.

In Section 1 the definitions of HOSM are presented. The difference between the definitions is discussed. The place of HOSM in control theory and connection between the sliding order and sliding accuracy are discussed in Section 2. It is shown in Section 3 that HOSM emerges every time when we have dynamic actuators in sliding mode control systems. Section 4 is devoted to study of second order sliding mode stability in systems with actuators. Various examples of second order sliding modes are presented in Section 5. Section 6 deals with examples of higher order sliding modes, an example of third order sliding algorithm with finite convergence time is presented.

1.1 Definitions of higher order sliding modes

Regular sliding mode features few special properties. It is reached in finite time which means that the shift operator along the phase trajectory exists, but is not invertible in time at any sliding point. Other important features are that the manifold of sliding motions has a nonzero codimension and that any sliding motion is performed on a system discontinuity surface and may be understood only as a limit of motions when switching imperfections vanish and switching frequency tends to infinity. Any generalization of the sliding mode notion has to inherit some of these properties. A definition developed in [4] deals with the first property and allows, thus, extension of the definition to dynamic systems of even completely different nature. The definitions developed below utilize the other properties mentioned. In many cases both definition systems are satisfied (not only in the case of the regular sliding mode).

Let us remind first what are Filippov's solutions [13, 14] of a discontinuous differential equation

$$\dot{x} = v(x),$$

where $x \in R^n$, v is a locally bounded measurable (Lebesgue) vector function. In this case, the equation is replaced by an equivalent differential inclusion

$$\dot{x} \in V(x).$$

In the particular case when the vector-field v is continuous almost everywhere, the set-valued function $V(x)$ is the convex closure of the set of all possible limits of $v(y)$ as $y \to x$, while $\{y\}$ are continuity points of v. Any solution of the equation is defined as an absolutely continuous function $x(t)$, satisfying the differential inclusion almost everywhere.

In the following Definitions we follow the works by Levantovsky [26], Emelyanov et. al. [9, 10, 12], Levant [24].

1.1.1 Sliding modes on manifolds

Definition 1 *Let L be a smooth manifold. Set L itself is called the first order sliding set with respect to L. The second order sliding set is defined as the set of points $x \in L$, where $V(x)$ lies entirely in tangential space T_L to manifold L at point x.*

Definition 2 *It is said that there exists a first (or second) order sliding mode on manifold L in a vicinity of a first (or second) order sliding point x, if in this vicinity of point x the first (or second) order sliding set is an integral set, i.e. it consists of Filippov's sense trajectories.*

Denote by L_1 the set of second order sliding points with respect to manifold L. Assume that L_1 may itself be considered as a sufficiently smooth manifold. Then the same construction may be considered with respect to L_1. Denote by L_2 the corresponding second order sliding set with respect to L_1. L_2 is called *the* 3*-rd order sliding set with respect to manifold L* . Continuing the process, achieve sliding sets of any order.

Definition 3 *It is said that there exists an r-th order sliding mode on manifold L in a vicinity of an r-th order sliding point $x \in L_{r-1}$, if in this vicinity of point x the r-th order sliding set L_{r-1} is an integral set, i.e. it consists of Filippov's sense trajectories.*

1.1.2 Sliding modes with respect to constraint functions

Let a constraint be given by an equation $\sigma(x) = 0$, where $\sigma : R^n \to R$ is a sufficiently smooth constraint function. It is also supposed that total time derivatives along the trajectories $\sigma, \dot{\sigma}, \ddot{\sigma}, \ldots, \sigma^{(r-1)}$ exist and are single-valued functions of x, which is not trivial for discontinuous dynamic systems. In other words, this means that discontinuity does not appear in the first $r-1$ total time derivatives of the constraint function σ. Then the r-th order sliding set is determined by the equalities

$$\sigma = \dot{\sigma} = \ddot{\sigma} = \ldots = \sigma^{(r-1)} = 0. \qquad (1)$$

Here (1) is an r-dimensional condition on the state of the dynamic system.

Definition 4 *Let the r-th order sliding set* (1) *be non-empty and assume that it is locally an integral set in Filippov's sense (i.e. it consists of Filippov's trajectories of the discontinuous dynamic system). Then the corresponding motion satisfying* (1) *is called an r-th order sliding mode with respect to the constraint function σ.*

To exhibit the relation with the previous Definitions, consider a manifold L given by the equation $\sigma(x) = 0$. Suppose that $\sigma, \dot{\sigma}, \ddot{\sigma}, \ldots, \sigma^{(r-2)}$ are differentiable functions of x and that

$$\mathrm{rank}[\nabla\sigma, \nabla\dot{\sigma}, \ldots, \nabla\sigma^{(r-2)}] = r - 1 \qquad (2)$$

holds locally. Then L_{r-1} is determined by (1) and all $L_i, i = 1, \ldots, r-2$ are smooth manifolds. If in its turn L_{r-1} is required to be a differentiable manifold, then the latter condition is extended to

$$\text{rank}[\nabla\sigma, \nabla\dot{\sigma}, \ldots, \nabla\sigma^{(r-1)}] = r \tag{3}$$

Equality (3) together with the requirement for the corresponding derivatives of σ to be differentiable functions of x will be referred to as *the sliding regularity condition*, whereas condition (2) will be called *the weak sliding regularity condition*.

With the weak regularity condition satisfied and L given by the equation $\sigma = 0$ Definition 4 is equivalent to Definition 3. If the regularity condition (3) holds, then new local coordinates may be taken. In these coordinates the system will take the form

$$y_1 = \sigma, \;\; \dot{y}_1 = y_2; \;\; \ldots; \;\; \dot{y}_{r-1} = y_r;$$
$$\sigma^{(r)} = \dot{y}_r = \Phi(y, \xi);$$
$$\dot{\xi} = \Psi(y, \xi), \;\; \xi \in R^{n-r}.$$

Proposition 1 *Let regularity condition* (3) *be fulfilled and r-th order sliding manifold* (1) *be non-empty. Then an r-th order sliding mode with respect to the constraint function σ exists if and only if the intersection of the Filippov vector-set field with the tangential space to manifold* (1) *is not empty for any r-th order sliding point.*

Proof. The intersection of the Filippov set of admissible velocities with the tangential space to the sliding manifold (1), mentioned in the Proposition, induces a differential inclusion on this manifold. This inclusion satisfies all the conditions by Filippov [13, 14] for solution existence. Therefore manifold (1) is an integral one.□

Let now σ be a smooth vector function, $\sigma : R^n \to R^m, \sigma = (\sigma_1, \ldots, \sigma_m)$, and also $r = (r_1, \ldots, r_m)$, where r_i are natural numbers.

Definition 5 *Assume that the first r_i successive full time derivatives of σ_i are smooth functions, and a set given by the equalities*

$$\sigma_i = \dot{\sigma}_i = \ddot{\sigma}_i = \ldots = \sigma_i^{(r_i-1)} = 0, \;\; i = 1, \ldots, m,$$

is locally an integral set in Filippov's sense. Then the movement mode existing on this set is called a sliding mode with vector sliding order r with respect to the vector constraint function σ.

The corresponding sliding regularity condition has the form

$$\mathrm{rank}\{\nabla\sigma_i, \ldots, \nabla\sigma_i^{(r_i-1)} | i = 1, \ldots, m\} = r_1 + \ldots + r_m.$$

Definition 5 corresponds to Definition 3 in the case when $r_1 = \ldots = r_m$ and the appropriate weak regularity condition holds.

A sliding mode is called *stable* if the corresponding integral sliding set is stable.

Remarks.

1. These definitions also include trivial cases of an integral manifold in a smooth system. To exclude them we may, for example, call a sliding mode "not trivial" if the corresponding Filippov set of admissible velocities $V(x)$ consists of more than one vector.

2. The above definitions are easily extended to include non-autonomous differential equations by introduction of the fictitious equation $\dot{t} = 1$

1.2 Higher order sliding modes in control systems

All the previous considerations are translated literally to the case of a process controlled

$$\dot{x} = f(t, x, u), \quad \sigma = \sigma(t, x) \in R, \; u = U(t, x) \in R,$$

where $x \in R^n$, t is time, u is control, and f, σ are smooth functions. Control u is determined here by feedback $u = U(t, x)$, where U is a discontinuous function. For simplicity we restrict ourselves to the case when σ and u are scalars. Nevertheless, all statements below may also be formulated for the case of vector sliding order.

Regular sliding modes satisfy the condition that the set of possible velocities V does not lie in tangential vector space T to manifold $\sigma = 0$, but intersects with it, and therefore a trajectory exists on the manifold with velocity vector lying in T. Such modes are the main operation modes in variable structure systems [6, 35, 36, 37, 23, 3, 39] and according to the above definitions they are of the first order. When a switching error is present the trajectory leaves the manifold at a certain angle. On the other hand, in the case of second order sliding all possible velocities lie in the tangential space to the manifold and even when a switching error is present, the state trajectory is tangential to the manifold at the time of leaving.

To see connections with some well-known results of control theory, consider at first the case when

$$\dot{x} = a(x) + b(x)u, \quad \sigma = \sigma(x) \in R, \ u \in R,$$

where a, b, σ are smooth vector functions. Let the system have a relative degree r with respect to the output variable σ (Isidory [22]). This means that Lie derivatives $L_b\sigma, L_bL_a\sigma, \ldots, L_bL_a^{r-2}\sigma$ equal zero identically in a vicinity of a given point and $L_bL_a^{r-1}\sigma$ is not zero at the point. The equality of the relative degree to r means, in a simplified way, that u first appears explicitly only in the r-th full time derivative of σ. It is known that in this case $\sigma^{(i)} = L_a^i\sigma$ for $i = 1, \ldots, r-1$, regularity condition (3) is satisfied automatically and also $\partial_u\sigma^{(r)} = L_bL_a^{r-1}\sigma \neq 0$. There is a direct analogy between the relative degree notion and the sliding regularity condition. Loosely speaking, it may be said that the sliding regularity condition (3) means that the "relative degree with respect to discontinuity" is not less than r. Similarly, the r-th order sliding mode notion is analogous to the zero-dynamics notion.

The relative degree notion was originally introduced for the autonomous case only. Nevertheless, we will apply this notion to the non-autonomous case as well. As was already done above, we introduce for this purpose a fictitious variable $x_{n+1} = t, \dot{x}_{n+1} = 1$. It has to be mentioned that some results by Isidory will not be correct in this case, but the facts listed in the previous paragraph will still be true.

Consider a dynamic system of the form

$$\dot{x} = a(t,x) + b(t,x)u, \ \ \sigma = \sigma(t,x), \ u = U(t,x) \in R.$$

Theorem 1 *Let the system have relative degree r with respect to the output function σ at some r-th order sliding point (t_0, x_0). Let, also, the discontinuous function U take on values from sets $[K, \infty)$ and $(-\infty, -K]$ on some sets of non-zero measure in any vicinity of any r-th order sliding point near point (t_0, x_0). Then this provides, with sufficiently large K, for the existence of r-th order sliding mode in some vicinity of point (t_0, x_0).*

Proof. This Theorem is an immediate consequence of Proposition 1, nevertheless, we will detail the proof. Consider some new local coordinates $y = (y_1, \ldots, y_n)$, where $y_1 = \sigma, y_2 = \dot{\sigma}, \ldots, y_r = \sigma^{(r-1)}$. In these coordinates manifold L_{r-1} is given by the equalities $y_1 = y_2 = \ldots = y_r = 0$ and the dynamics of the system is as follows:

$$\begin{array}{l} \dot{y}_1 = y_2, \ \ \ldots, \ \ \dot{y}_{r-1} = y_r, \\ \dot{y}_r = h(t,y) + g(t,y)u, \quad g(t,y) \neq 0, \\ \dot{\xi} = \Psi_1(t,y) + \Psi_2(t,y)u, \ \ \xi = (y_{r+1}, \ldots, y_n) \end{array} \tag{4}$$

Denote $U_{eq} = -h(t,y)/g(t,y)$. It is obvious that with initial conditions being on the r-th order sliding manifold L_{r-1} control $u = U_{eq}(t,y)$ provides for keeping the system within manifold L_{r-1}. It is also easy to see that the substitution of all possible values from $[-K, K]$ for u gives us a subset of values from Filippov's set of the possible velocities. Let $|U_{eq}|$ be less than K_0, then with $K > K_0$ the substitution $u = U_{eq}$ determines a Filippov's solution of the discontinuous system which proves the Theorem. □

The trivial control algorithm $u = -K \operatorname{sign} \sigma$ satisfies Theorem 1. Usually, however, such a mode will not be stable.

It follows from the proof above that the movement in the r-th order sliding mode is described by the equivalent control method (Utkin [35]), on the other hand this dynamics coincides with the zero-dynamics [22] for corresponding systems.

There are some recent papers devoted to the higher order sliding mode technique. The sliding mode order notion, which appeared in 1990 [2, 5], seems to be understood in a very close sense (the authors had no possibility to acquaint themselves with the work by Chang [2]). The same idea is developed in a very general way from the differential-algebraic point of view in the papers by Sira-Ramirez [30, 31, 32]. In his papers sliding modes are not distinguished from the algorithms generating them. Consider this approach.

Let the following equality be fulfilled identically as a consequence of the dynamic system equations [32]:

$$P(\sigma^{(r)}, \ldots, \dot{\sigma}, \sigma, x, u^{(k)}, \ldots, \dot{u}, u) = 0. \tag{5}$$

Equation (5) is supposed to be solvable with respect to $\sigma^{(r)}$ and $u^{(k)}$. Function σ may itself depend on u. The r-th order sliding mode is considered as a steady state $\sigma \equiv 0$ to be achieved by a controller satisfying (5). In order to achieve for σ some stable dynamics

$$\Sigma = \sigma^{(r-1)} + a_1 \sigma^{(r-2)} + \ldots + a_{r-1}\sigma = 0$$

the discontinuous dynamics

$$\dot{\Sigma} = -\operatorname{sign} \Sigma \tag{6}$$

is provided. For this purpose the corresponding value of $\sigma^{(r)}$ is evaluated from (6) and substituted into (5). The obtained equation is solved for $u^{(k)}$.

Thus, a dynamic controller is constituted by the obtained differential equation for u which has a discontinuous right hand side. With this controller successive derivatives $\sigma, \ldots, \sigma^{(r-1)}$ will be smooth functions of closed system state space variables. The steady state of the resulting system will satisfy at least (1) and under some relevant conditions also the regularity requirement (3), and therefore Definition 4 will hold.

Hence, it may be said that the relation between our approach and the approach by Sira-Ramirez is a classical relation between geometric and algebraic approaches in mathematics. Note that there are two different sliding modes in system (5), (6): a regular sliding mode of the first order which is kept on the manifold $\Sigma = 0$, and an asymptotically stable r-th order sliding mode with respect to the constraint $\sigma = 0$ which is kept in the points of the r-th order sliding manifold $\sigma = \dot{\sigma} = \ddot{\sigma} = \ldots = \sigma^{(r-1)} = 0$.

Real sliding and finite time convergence

Remind that the objective is synthesis of such a control u that the constraint $\sigma(t, x) = 0$ holds. The quality of the control design is closely related to the sliding accuracy. In reality, no approaches to this design problem may provide for ideal keeping of the prescribed constraint. Therefore, there is a need to introduce some means in order to provide a capability for comparison of different controllers.

Any ideal sliding mode should be understood as a limit of motions when switching imperfections vanish and the switching frequency tends to infinity [13, 14]. Let ϵ be some measure of these switching imperfections. Then sliding precision of any sliding mode technique may be featured by a sliding precision asymptotics with $\epsilon \to 0$.

Definition 6 *Let $(t, x(t, e))$ be a family of trajectories, indexed by $\epsilon \in R^{\mu}$, with common initial condition $(t_0, x(t_0))$, and let $t \geq t_0$ (or $t \in [t_0, T]$). Assume that there exists $t_1 \geq t_0$ (or $t_1 \in [t_0, T]$) such that on every segment $[t|prime, t|prime|prime]$, where $t|prime * t_1$, (or on $[t_1, T]$) the function $\sigma(t, x(t, \epsilon))$ tends uniformly to zero with ϵ tending to zero. In this case we call such a family a real sliding family on the constraint $\sigma = 0$. We call the motion on the interval $[t_0, t_1]$ a transient process, and the motion on the interval $[t_1, \infty)$ (or $[t_1, T]$) a steady state process.*

Definition 7 *A control algorithm, dependent on a parameter $\epsilon \in R^{\mu}$, is called a real sliding algorithm on the constraint $\sigma = 0$ if, with $\epsilon \to 0$, it forms a real sliding family for any initial condition.*

Definition 8 *Let $\gamma(\epsilon)$ be a real-valued function such that $\gamma(\epsilon) \to 0$ as $\epsilon \to 0$. A real sliding algorithm on the constraint $\sigma = 0$ is said to be of order r ($r > 0$) with respect to $\gamma(\epsilon)$ if for any compact set of initial conditions and for any time interval $[T_1, T_2]$ there exists a constant C, such that the steady state process for $t \in [T_1, T_2]$ satisfies*

$$|\sigma(t, x(t, \epsilon))| \leq C|\gamma(\epsilon)|^r$$

.

In the particular case when $\gamma(\epsilon)$ is the smallest time interval of control smoothness, the words "with respect to γ" may be omitted. This is the case when real sliding appears as a result of switching discretization.

As follows from [24], with the r-th order sliding regularity condition satisfied, in order to get the r-th order of real sliding with discrete switching it is necessary to get at least the r-th order in ideal sliding (provided by infinite switching frequency). Thus, the real sliding order does not exceed the corresponding sliding mode order. The regular sliding modes provide, therefore, for the first order real sliding only. The second order of the real sliding was really achieved by discrete switching modifications of the second order sliding algorithms [26, 9, 10, 11, 12, 24]. A special discrete switching algorithm providing for the second order real sliding was constructed in [34]. Real sliding of the third order is demonstrated later in this paper.

Real sliding may also be achieved in a way different from the discrete switching realization of sliding mode. For example, high gain feedback systems [29, 38] constitute real sliding algorithms of the first order with respect to k^{-1}, where k is a large gain. Another example is adduced in Section 5 (Example 2).

It is right that in practice the final sliding accuracy is always achieved in finite time. Nevertheless, besides the pure theoretical interest there are also some practical reasons to search for sliding modes attracting in finite time. Consider a system with an r-th order sliding mode. Assume that with minimal switching interval τ the maximal r-th order of real sliding is provided. This means that the corresponding sliding precision $|\sigma| \sim \tau^r$ is kept, if the r-th order sliding condition holds at the initial moment. Suppose that the r-th order sliding mode in the continuous switching system was asymptotically stable and does not attract the trajectories in finite time. It is reasonable to conclude in this case that with $\tau \to 0$ the transient process time for fixed general case initial conditions will tend to infinity. If, for example, the sliding mode were exponentially stable, the transient process time would be proportional to $r\ln\tau^{-1}$. Therefore, it is impossible to observe such an accuracy in practice, if the sliding mode is only asymptotically stable. At the same time, the time of the transient process will not change drastically, if it was finite from the very beginning.

1.3 Higher order sliding modes and systems with dynamic actuators

Suppose that the plant has relative degree r with respect to the output function σ. That means that we can describe the behavior of the first r coordinates of the control system in form (4). Assume that relay control u is transmitted to the input of the plant (Fig. 1.1) by a dynamic actuator which itself has an l-th order dynamics. The behavior of the first l actuator coordinates and of the first r plant coordinates is described by the equations

$$\dot{y}_1 = y_2, \dots, \dot{y}_{r-1} = y_r,$$

$$\dot{y}_r = h(t,y) + g(t,y)z_1, \quad g(t,y) \neq 0,$$

$$\dot{z}_1 = z_2, \dots, \dot{z}_{l-1} = z_l \tag{7}$$

$$\dot{z}_l = p(t,y,z) + q(t,y,z)u, \quad q(t,y,z) \neq 0 \tag{8}$$

where $y_1 = \sigma$. This means that the complete model of the sliding mode control system has relative degree $r + l$, and, therefore, the $(r + l)$-sliding regularity condition holds. According to Theorem 1, a sliding mode with respect to σ has to appear here, which has sliding order $r+l$. If the controller itself is chosen in an actuator-like form (Fig. 1.2), the corresponding sliding order will be still larger.

Thus, higher order sliding modes emerge every time when we have to take into account dynamic actuators in a sliding mode control system.

Consider a special case when the actuator is fast. In this case equation (8) has the form

$$\mu \dot{z}_l = p(t,y,z,\mu) + q(t,y,z,\mu)u \tag{9}$$

where μ is a small actuator time constant. In fact all motions in system (7), (9) have fast velocities in such a case.

There are two approaches to investigation of systems with such actuators: consideration of a small neighborhood of HOSM set [16, 18], and transformation to a basis of eigenvectors. In both cases the complete model of the control system will be a singularly perturbed discontinuous control system. Fast motions in such systems are described by a system with an $(r + l)$-th order sliding mode.

Adduce some simple informal reasoning valid under sufficiently general conditions.

Let an actuator be called *precise* if its output is used by the process controlled exactly as a substitution for the control signal. This means, in particular, that the dimensions of the actuator output and control coincide. Also require for such an actuator that for any admissible constant input the output of the actuator be set at the input value after some time and that this transient time be small if the actuator is fast.

No chattering is generally observed in a system with a precise actuator, if the corresponding higher order sliding mode is stable. Indeed, let common conditions on regular sliding mode implementation be satisfied for the process controlled. This means, in particular, that $\dot{\sigma} = 0$ implies $u = u_{eq}$ where equivalent control u_{eq} is a sufficiently smooth function of the state variables. Therefore, the output of the actuator inevitably tends to this smooth function while the process enters the higher order sliding mode $\dot{\sigma} \equiv 0$ and the chattering is removed. However, if a fast actuator is not precise, fast motion stability in the higher order sliding mode is also to be required in order to avoid chattering (see remark at the end of the next section).

On the other hand instability of the sliding mode of corresponding order $r \geq 2$ leads to appearance of a real sliding mode which is usually accompanied by a chattering effect, if the system with a fast stable precise actuator is considered (Fig. 1.1). Indeed, suppose that the actuator output is always stabilized at some slow function value. This is possible only if the relay output is constant or an infinite frequency switching of the relay output takes place. The latter means that such a value may be achieved only if $\sigma \equiv 0$. Thus, also total derivatives of σ of orders up to $r-1$ equal zero in this mode and the higher order sliding mode is stable in contradiction to our conditions. On the other hand the actuator output will be set at the relay output value before the system leaves some small vicinity of the manifold $\sigma = 0$. This prescribes the needed sign to $\dot{\sigma}$ and prevents leaving this small vicinity of the manifold. Hence, the actuator output performs fast vibrations while the trajectory does not leave a small manifold vicinity.

1.4 Stability of second order sliding modes in systems with fast actuators

Consider a simple example of a dynamic system

$$\begin{array}{l} \dot{y}_1 = y_2, \ \ \dot{y}_2 = a y_1 + b y_2 + c y_3 + k \,\mathrm{sign}\, y_1 \\ \dot{y}_i = \sum_{j=1}^{n} a_{i,j} y_j, \ \ i = 3, \ldots, n \end{array} \tag{10}$$

The second order sliding set is given here by equalities $y_1 = y_2 = 0$. Following [1], we single out the exponentially stable and unstable cases.

- *Exponentially stable case.* Under the conditions

$$b < 0, \ \ k < 0 \tag{ES}$$

the set $y_1 = y_2 = 0, |cy_3| < k$ is an exponentially stable integral manifold for system (10).

- *Unstable case.* Under the condition

$$k > 0 \text{ or } b > 0 \tag{US}$$

the second order sliding set of system (10) is an unstable integral manifold.

- *Critical case.*

$$k \leq 0, \quad b \leq 0, \quad bk = 0. \tag{C}$$

With $c = b = 0, k < 0$ the second order sliding set of system (10) is stable but not asymptotically stable, with $c \neq 0$ stability is determined by the properties of the whole system.

Condition (ES) is used in works by Fridman [18, 19, 20] for analysis of sliding mode systems with fast dynamic actuators. It is not required that actuators be precise. Adduce a simple outline of these reasonings. Let the system under consideration be rewritten in the following form:

$$\begin{aligned} &\mu\dot{z} = Az + B\eta + D_1 x, \\ &\mu\dot{\eta} = Cz + b\eta + D_2 x + k \operatorname{sign} \sigma, \\ &\dot{\sigma} = \eta, \\ &\dot{x} = F(z, \eta, \sigma, x), \end{aligned} \tag{11}$$

where $z \in R^m, x \in R^n, \eta, \sigma \in R$.

With (ES) fulfilled and $\operatorname{Re}\operatorname{Spec} A < 0$ system (11) has an exponentially stable integral manifold of slow motions being a subset of the second order sliding manifold and given by equations

$$z = H(\mu, x) = -A^{-1}D_1 x + O(\mu), \ \sigma = \eta = 0.$$

Function H may be evaluated with any desired precision with respect to the small parameter μ .

Therefore, according to [18, 19, 20], under the conditions

$$\operatorname{Re}\operatorname{Spec} A < 0, \quad b < 0, \quad k < 0 \tag{CHA}$$

the motions in a system with a fast actuator of relative degree 1 consist of fast oscillations, vanishing exponentially, and slow motions on a submanifold of the second order sliding manifold.

Thus, if conditions (CHA) of chattering absence hold, the presence of a fast actuator of relative degree 1 does not lead to chattering in sliding mode control systems. For any chattering simulation in this case it is necessary to take into account some other factors like positive feedbacks [18] and time delays [21] or to consider systems with relative degree of actuator more than 1.

Remark

Stability of the fast actuator and of the second order sliding mode still do not guarantee absence of chattering if the actuator is not precise. Indeed, stability of a fast actuator corresponds to the stability of the fast actuator matrix

$$\operatorname{Re}\operatorname{Spec}\begin{pmatrix} A & B \\ C & b \end{pmatrix} < 0. \qquad (SA)$$

Consider the system

$$\begin{aligned}
&\mu\dot{z}_1 = z_1 + z_2 + \eta + D_1 x,\\
&\mu\dot{z}_2 = 2z_2 + z_3 + D_2 x;\\
&\mu\dot{\eta} = 24z_1 - 60z_2 - 9\eta + D_3 x + k\operatorname{sign}\sigma,\\
&\dot{\sigma} = \eta,\\
&\dot{x} = F(z_1, z_2, \eta, \sigma, x),
\end{aligned}$$

where z_1, z_2, η, σ are scalars. The dimensions of the actuator output z_1, z_2, η and relay output $\operatorname{sign}\sigma$ are not equal, so the actuator is not precise. It is easy to check that the spectrum of the matrix is $\{-1, -2, -3\}$ and condition (ES) holds for this system. On the other hand the motions in the second order sliding mode are described by the system

$$\begin{aligned}
&\mu\dot{z}_1 = z_1 + z_2 + D_1 x;\\
&\mu\dot{z}_2 = 2z_2 + D_2 x;\\
&\dot{x} = F(z_1, z_2, 0, 0, x).
\end{aligned}$$

The fast motions in this system are unstable and the absence of chattering in the original system cannot be guaranteed.

1.5 Examples of second order sliding modes

Without loss of generality we shall illustrate the approach by some simple examples. Consider, for instance, sliding mode usage for the tracking purpose. Let the process be described by the equation

$$\dot{x} = u, \quad x, u \in R,$$

and the problem is to track a signal $f(t)$ given in real time, where $|f|, |\dot{f}|, |\ddot{f}| < 0.5$. Only values of x, f, u are available. The problem is successfully solved by the controller

$$u = -\operatorname{sign}\sigma, \quad \sigma = x - f(t),$$

keeping $\sigma = 0$ in the sliding mode of the first order. In practice, however, there is always some actuator between the plant and the controller, which inserts some additional dynamics and removes the discontinuity from the real system. With respect to Fig. 1.1 let the system be described by the following scheme:

$$\dot{x} = z, \quad u \Rightarrow \text{some dynamics} \Rightarrow z, u = -\operatorname{sign} \sigma,$$

$$\dot{\sigma} = -\dot{f}(t) + z.$$

Example 1. Assume that the actuator has some fast first order dynamics. For example

$$\mu \dot{z} = u - z$$

where μ is a small positive number. The second order sliding manifold L_1 is given here by the equations

$$\sigma = x - f(t) = 0, \ \dot{\sigma} = z - \dot{f}(t) = 0. \quad (12)$$

Equality

$$\ddot{\sigma} = \frac{1}{\mu}(u - z) - \ddot{f}(t)$$

shows that the relative degree here equals 2 and, according to Theorem 1, a second order sliding mode exists in the system, provided $\mu < 1$. The motion in this mode is described by the equivalent control method or by zero-dynamics, which is the same: from $\sigma = \dot{\sigma} = \ddot{\sigma} = 0$ achieve $u = \mu \ddot{f}(t) + z, z = \dot{f}(t)$ and therefore

$$x = f(t), \ z = \dot{f}(t), \ u = \mu \ddot{f}(t) + z.$$

According to Section 4, the second order sliding mode is stable here with μ small enough. Note that the last equality describes the equivalent control [35, 36, 37] and is kept actually only in the average, while the first and the second are kept accurately in this sliding mode.

Here and further the examples are accompanied by simulation results with

$$f(t) = 0.08 \sin t + 0.12 \cos 0.3t \ , \quad x(0) = 0, \ z(0) = 0.$$

Here z is the actuator output. The plots of $x(t)$ and $f(t)$ with $\mu = 0.2$ are shown in Fig. 1.3, whereas the plot of $z(t)$ is demonstrated in Fig. 1.4.

Example 2. One of the main ideas of the binary system theory [8, 7] is to insert some artificial fast dynamics in the switching process. This may

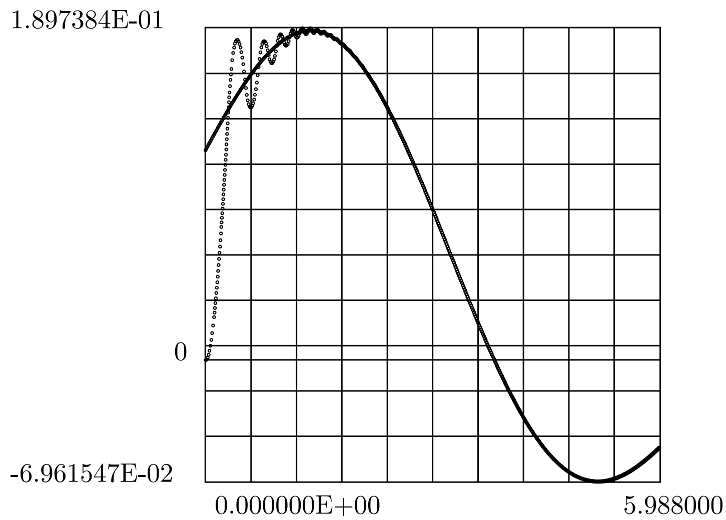

Figure 1.3: Asymptotically stable second order sliding mode in a system with a fast actuator. Tracking: $x(t)$ and $f(t)$.

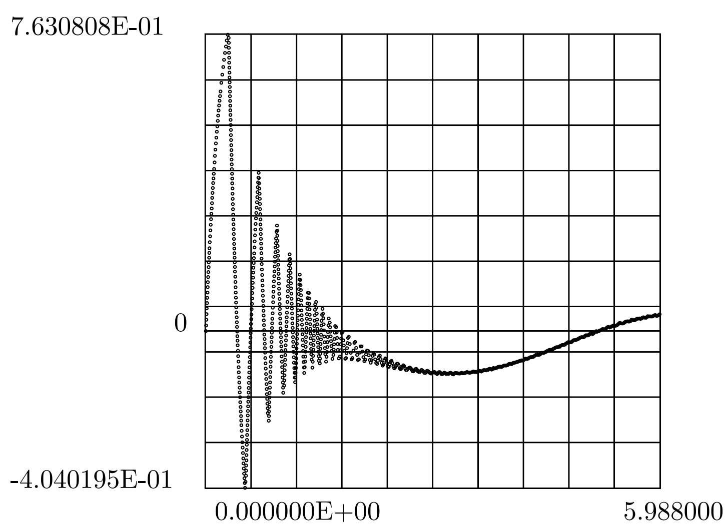

Figure 1.4: Asymptotically stable second order sliding mode in a system with a fast actuator: actuator output $z(t)$.

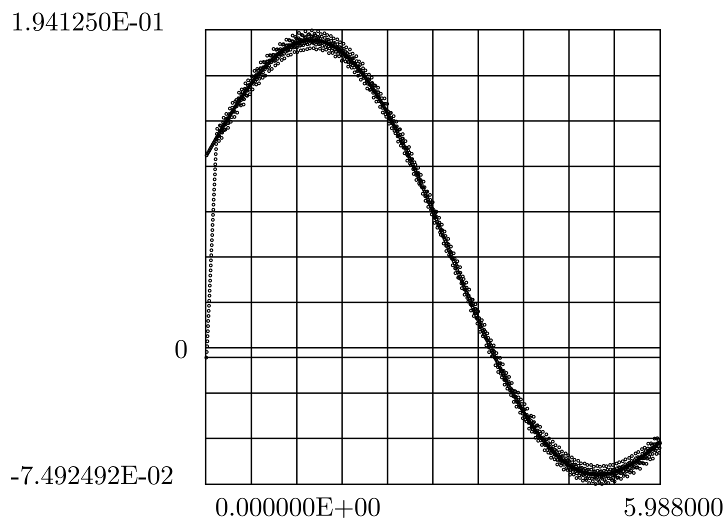

Figure 1.5: Unstable second order sliding mode in a system with A_μ-controller. Tracking: $x(t)$ and $f(t)$.

be regarded as an installation of a fast actuator. For example, let

$$\mu \dot{z} = \begin{cases} -z, & |z| > |u|, \\ -\operatorname{sign} u, & |z| \leq |u|, \end{cases}$$

where μ is a small positive number. This is a slightly modified A_μ-algorithm by Emelyanov and Korovin [8]. Having substituted $u = -\operatorname{sign} \sigma$ achieve the classical form of the algorithm with z being considered as a control.

$$\dot{z} = \begin{cases} -\frac{1}{\mu} z, & |z| > 1, \\ -\frac{1}{\mu} \operatorname{sign} \sigma, & |z| \leq 1, \end{cases} \tag{13}$$

Similarly to the previous example, achieve here a second order sliding mode provided $\mu < 2$. Simple calculation shows that the trajectories revolve around the second order sliding manifold in coordinates t, x, z. Algorithm (13) provides for appearance of a real sliding mode of the first order with respect to μ^{-1}. Simulation results for $\mu = 0.01$ are shown in Fig. 1.5, 1.6. The chattering is obvious here.

Example 3. The last actuator-like algorithm (13) may be modified in order to receive finite time convergence to the second order sliding mode.

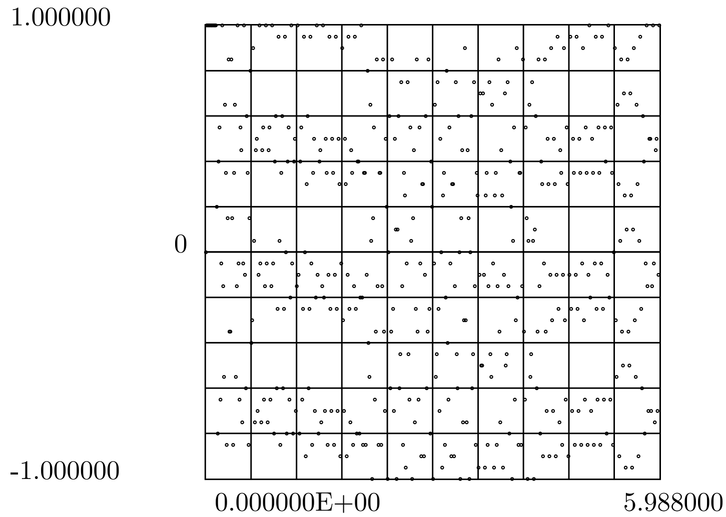

Figure 1.6: Unstable second order sliding mode in a system with A_μ-controller: control $z(t)$ (values are taken at discrete times).

The aim is gained by twisting algorithms [26, 28, 9, 10, 24]

$$\dot{z} = \begin{cases} -z, & & |z| > 1, \\ -5\,\mathrm{sign}\,\sigma, & \sigma\dot{\sigma} > 0, & |z| \leq 1, \\ -\,\mathrm{sign}\,\sigma, & \sigma\dot{\sigma} \leq 0, & |z| \leq 1. \end{cases}$$

Derivative $\dot{\sigma}$ has to be calculated here in real time. Having substituted the first difference of σ for $\dot{\sigma}$, achieve another version of the algorithm adapted for implementation:

$$\dot{z} = \begin{cases} -z(t_i), & & |z(t_i)| > 1, \\ -5\,\mathrm{sign}\,\sigma(t_i), & \sigma(t_i)\Delta\sigma_i > 0, & |z(t_i)| \leq 1, \\ -\,\mathrm{sign}\,\sigma(t_i), & \sigma(t_i)\Delta\sigma_i \leq 0, & |z(t_i)| \leq 1. \end{cases}$$

Here $t_i \leq t_{i+1}$. This algorithm modification constitutes a second order real sliding algorithm, which means that the sliding accuracy is proportional to the measurement time interval squared. The corresponding simulation results are shown in Fig. 1.7, 1.8.

Example 4. Another algorithm [12, 24] serving the same goal is the

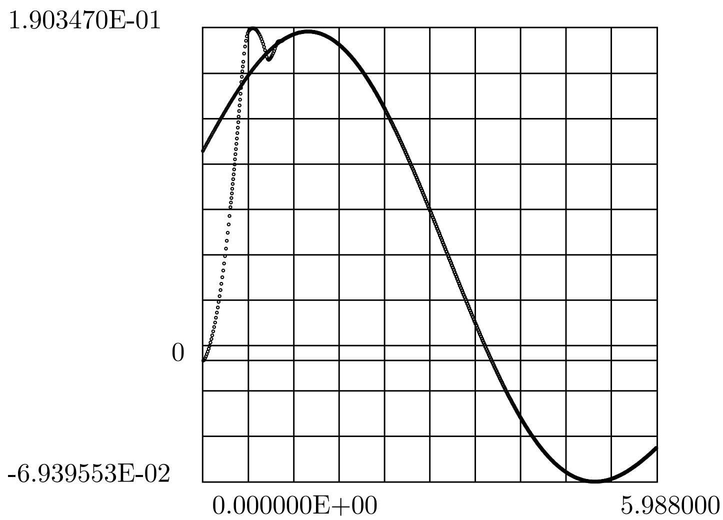

Figure 1.7: Second order sliding mode attracting in finite time: twisting algorithm. Tracking: $x(t)$ and $f(t)$.

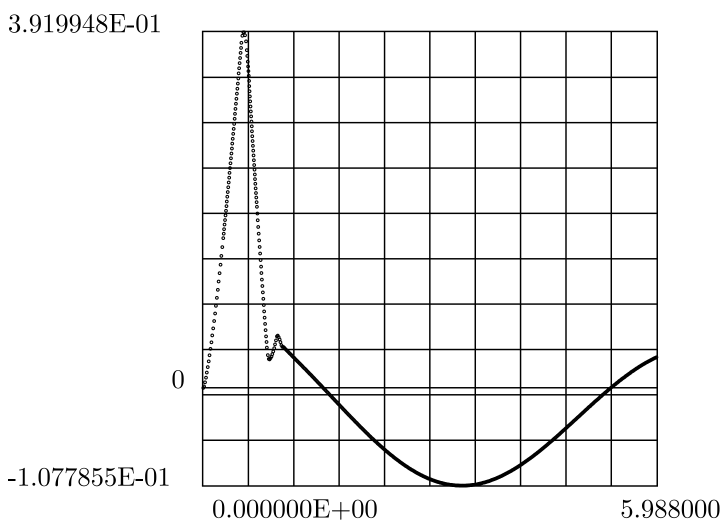

Figure 1.8: Second order sliding mode attracting in finite time: twisting algorithm. Control $z(t)$.

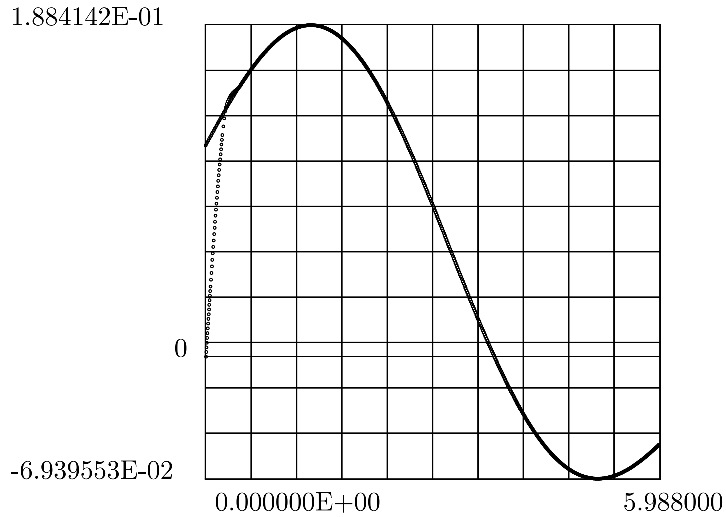

Figure 1.9: Example 4: second order sliding mode attracting in finite time. Tracking: $x(t)$ and $f(t)$.

algorithm

$$z = -2\sqrt{|\sigma|} + z_1, \quad \dot{z_1} = \begin{cases} -z, & |z| > 1, \\ -\mathrm{sign}\,\sigma, & |z| \le 1, \end{cases}$$

The discrete switching modification of this algorithm also constitutes a second order real sliding algorithm. Its simulation results are shown in Fig. 1.9, 1.10.

Note that in the latter example only the weak regularity condition (2) holds. Examples 3 and 4 are representatives of large algorithmic families. Details and a number of other examples for second order sliding modes attracting in finite or infinite time may be found in [9, 11, 12, 26, 27, 28, 2, 5, 31, 32, 24].

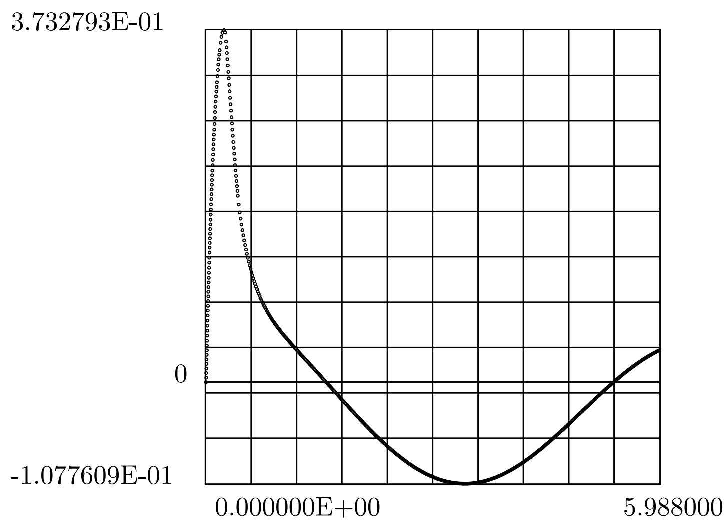

Figure 1.10: Example 4: second order sliding mode attracting in finite time. Control $z(t)$.

1.6 Sliding modes of order 3 and higher

Note, following [1], that for any $l \geq 3, a_{i,j}, k \neq 0$ the l-th order sliding sets in systems

$$\begin{aligned} &\dot{y}_1 = y_2, \ \ldots, \ \dot{y}_{l-1} = y_l, \\ &\dot{y}_l = \textstyle\sum_{j=1}^{n} a_{l,j} y_j + k \operatorname{sign} y_1, \\ &\dot{y}_i = \textstyle\sum_{j=1}^{n} a_{i,j} y_j, \ \ i = l+1, \ldots, n \end{aligned}$$

are always unstable with $k \neq 0$.

This leads to an important conclusion. Even a stable high order actuator may insert additional chattering into the closed dynamic system. Whenever a possibility of using actuators with r-th order dynamics ($r > 2$) for first order sliding mode control systems is concerned, one has to search for stable attractors of a corresponding $(r+1)$-dimensional fast dynamic system or use some special control algorithms.

Example 5. Continuing the example series, let now the actuator have second order dynamics:

$$\dot{z} = z_1, \ \ \dot{z}_l + 3\alpha z_1 + 2\alpha^2 z = 2\alpha^2 u,$$

where $\mu = 1/\alpha \to 0$. Let also $|f^{(3)}(t)| < 0.5$ be true. Calculation shows that

$$\sigma^{(3)} = -f^{(3)}(t) - 3\alpha z_1 - 2\alpha^2 z + 2\alpha^2 u.$$

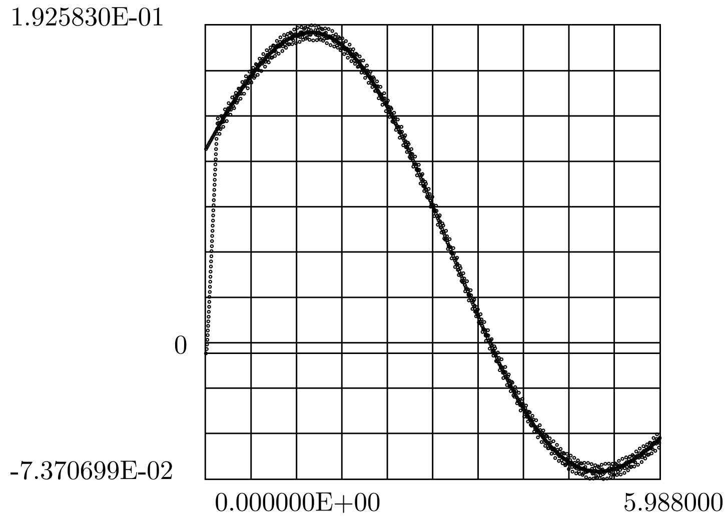

Figure 1.11: Unstable third order sliding mode in a system with actuator of relative degree 2: $x(t)$ and $f(t)$.

It is easy to check (see Theorem 1) that, with $\mu < \sqrt{3}/3$, there is a Filippov's solution lying on set $\sigma = \dot{\sigma} = \ddot{\sigma} = 0$, which corresponds to the third order sliding mode. However, it is unstable according to the classical result by Anosov [1]. Certainly, an approximation of ideal regular sliding mode is achieved with $\mu \to 0$ (Fig. 1.11). However, the actuator introduces here considerable chattering (Fig. 1.12).

The following is the first published example of a third order sliding algorithm with finite convergence time as well as of a third order sliding mode being attractive in finite time at all.

Example 6. *An example of a third order sliding algorithm with finite convergence time.* Define, determining by continuity when necessary,

$$\Psi(\sigma, \dot{\sigma}) = \max\{0,\ \min[1,\ \frac{1}{2} + 6(\dot{\sigma} + \frac{5}{12}|\sigma|^{2/3}\mathrm{sign}\,\sigma)|\sigma|^{-2/3}]\},$$

$$\Phi(\sigma, \dot{\sigma}) = \frac{2}{3}(1 - 2\Psi(\sigma, \dot{\sigma}))(\frac{1}{2}|\dot{\sigma}|^3 + \frac{1}{2}|\sigma|^2)^{\frac{1}{6}}.$$

In order not to change the notation, variable z is used below as an actual control. Introduce also an auxiliary variable z_1. The following algorithm

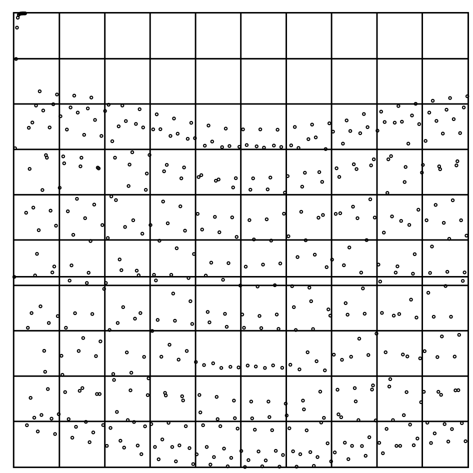

Figure 1.12: Unstable third order sliding mode in a system with actuator of relative degree 2: actuator output $z(t)$ (values are taken at discrete times).

provides for finite time convergence to the third order sliding mode

$$\dot{z} = z_1,$$

$$\dot{z_1} = -15\,\mathrm{sign}\,(\ddot{\sigma} - \Phi(\sigma, \dot{\sigma})).$$

The proof is provided by a sequence of simple calculations. It is necessary to check that

- $\Phi(\sigma, \dot{\sigma})$ is a continuous piece-wise smooth function;
- Functions $\sigma, \dot{\sigma}, \ddot{\sigma}$ may be taken as new coordinates. There is a first order sliding mode on the manifold

$$\ddot{\sigma} = \Phi(\sigma, \dot{\sigma}); \tag{14}$$

- The corresponding sliding motion is described by equation (14) which provides for finite time convergence to the origin $\sigma = \dot{\sigma} = 0$.

In a similar way a finite time convergence algorithm of an arbitrary sliding order may be constructed.

The discrete switching modification of this algorithm

$$\dot{z} = z_1(t_i),$$

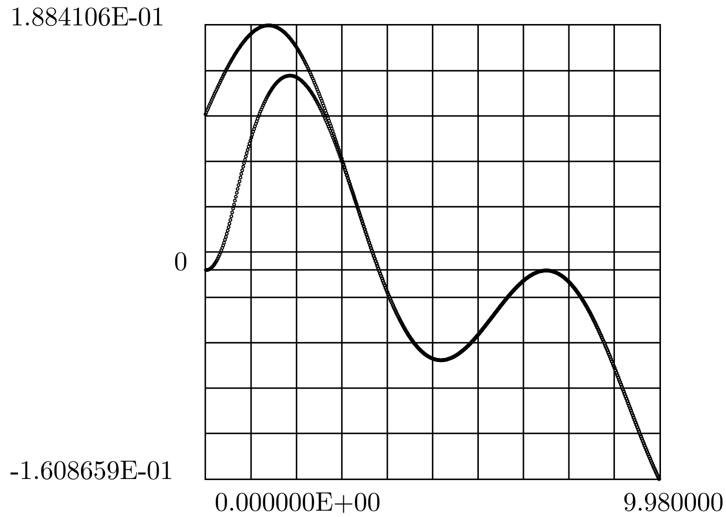

Figure 1.13: Third order sliding mode attracting in finite time. Tracking: $x(t)$ and $f(t)$.

$$\dot{z}_1 = -15\,\mathrm{sign}\,(\Delta\dot{\sigma}(t_i) - \tau\Phi(\sigma(t_i), \dot{\sigma}(t_i))).$$

constitutes a third order real sliding algorithm providing for the sliding accuracy being proportional to the third power of the discretization interval τ. The simulation results are shown in Fig. 1.13, 1.14, 1.15. It was taken that $\tau = 10^{-3}, 10^{-4}, 5 \cdot 10^{-5}$ and the sliding precision $\sup|\sigma| = 2.8 \cdot 10^{-6}, 1.9 \cdot 10^{-9}, 2.7 \cdot 10^{-10}$ was achieved.

Note that the sliding algorithms from examples 3, 4, 6 cannot be produced by the powerful differential-geometric methods by Sira-Ramirez. At the same time these algorithms are beyond any doubt of practical interest. One of the present authors has already successfully applied such algorithms in solving avionics problems and constructing robust differentiators [25]. It has to be mentioned that these algorithms also provide for much higher accuracy than the regular sliding modes [24].

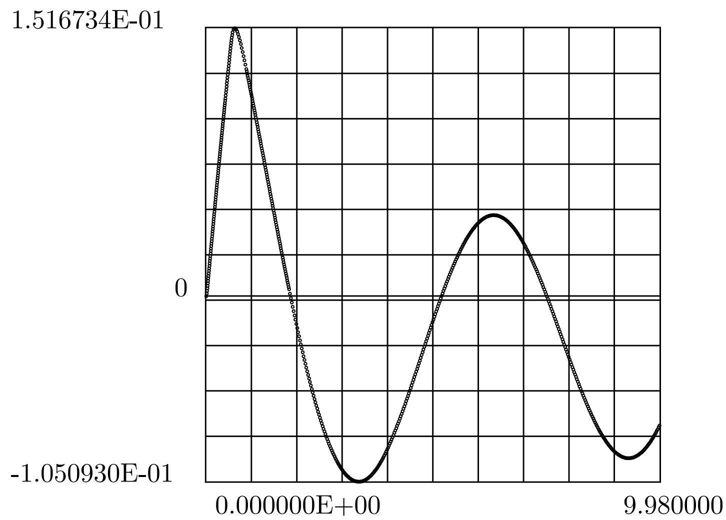

Figure 1.14: Third order sliding mode attracting in finite time. Control $z(t)$.

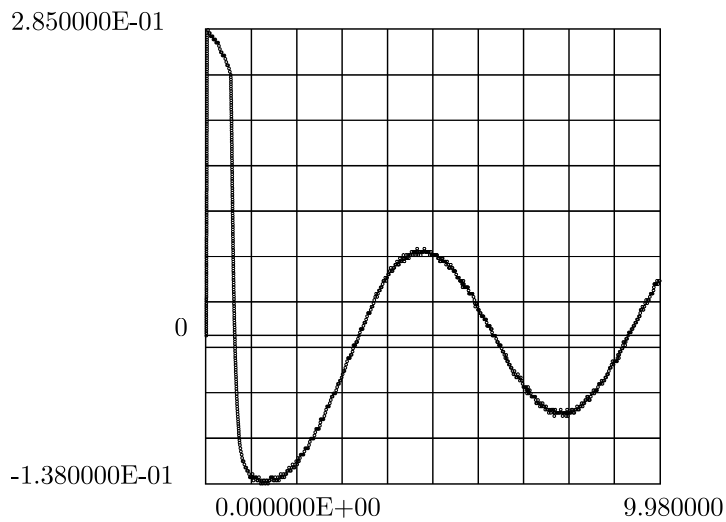

Figure 1.15: Third order sliding mode attracting in finite time. Control derivative $z_1(t) = \dot{z}(t)$.

Conclusions

- Higher order sliding mode definitions were formulated.
- It was shown that higher order sliding modes are natural phenomena for control systems with discontinuous controllers if the relative degree of the system is more than 1 or a dynamic actuator is present.
- A natural logic of actuator-like algorithm introduction was presented. Such algorithms also provide for the appearance of higher order sliding modes.
- Stability was studied of second order sliding modes in systems with fast stable dynamic actuators of relative degree 1.
- A number of examples of higher order sliding modes were listed. Among them the first example was presented of a third order sliding algorithm with finite time convergence. The discrete switching modification of this algorithm provides for the third order sliding precision with respect to the minimal switching time interval.

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