Discrete-time non-linear state observer based on a super twisting-like algorithm

Iván Salgado¹, I. Chairez², Bijnan Bandyopadhyay³, Leonid Fridman⁴ and Oscar Camacho¹

¹CIC, Instituto Politecnico Nacional, Mexico City, Mexico
²Department of Bioprocesses, UPIBI, Instituto Politecnico Nacional, Mexico City, Mexico
³SYSCON, Indian Institute of Technology, Bombay, Mumbai, India
⁴Engineering Universidad Nacional Autonoma de Mexico, Mexico City, Mexico
E-mail: j.chairez@ctrl.cinvestow.mx

Abstract: The properties of robustness and finite-time convergence provided by sliding mode (SM) theory have motivated several researches to deal with the problems of control and state estimation. In the SM theory, the super-twisting algorithm (STA), a second-order SM scheme, has demonstrated remarkable characteristics when it is implemented as a controller, observer or robust signal differentiator although the presence of noise and parametric uncertainties. However, the design of this algorithm was originally developed for continuous-time systems. The growth of microcomputers technology has attracted the attention of researchers inside the SM discrete-time domain. Recently, discretisation schemes for the STA were studied using majorant curves. In this study, the stability analysis in terms of Lyapunov theory is proposed to study a discrete-time super twisting-like algorithm (DSTA) for non-linear discrete-time systems. The objective is to preserve the STA characteristics of robustness in a quasi-sliding mode regime that was proved in terms of practical Lyapunov stability. An adequate combination of gains obtained by the same Lyapunov analysis forces the convergence for the DSTA. The problem of state estimation is also analysed for second-order mechanical systems of n degrees of freedom. Simulation results regarding the design of a second-order observer using the DSTA for a simple pendulum and a biped model of seven degrees of freedom are presented.

1 Introduction

Nowadays, the advance in the development of microcontrollers with capabilities to solve advanced mathematical algorithms has allowed the implementation of complex control schemes without the use of personal computers. In this way, the requirement of control and state estimation strategies downloadable in these microcontrollers has oriented the research of control strategies in two principal frameworks. The first one analyses the discretisation problem of several control techniques and its sampled time dependences in the discrete-time domain. On the other hand, as the second framework, new control strategies have been presented to study the case of pure discrete-time systems. These techniques include the well-known sliding mode (SM) theory [1]. In addition, the implementation of integration schemes could be an option to keep working with continuous strategies. However, the computational requirements are increased when this option is implemented.

The SM theory has been widely applied to solve the state estimation problem. The main features obtained by SM are robustness against parametric uncertainties and finite-time convergence. These advantages are evident in contrast with the conditions required for non-linear systems when a classical structure for observation is used, like a Luenberger one. Indeed, the necessity of the exact mathematical description constitutes a drawback in this kind of algorithms. Following the idea of Luenberger structures, the so-called high-gain observers have been developed to deal with complex non-linear systems [2].

Sliding motion is obtained by including a discontinuity term in the algorithm structure. The discontinuous injection must be designed such that the trajectories of the system are forced to remain on a prescribed submanifold (sliding surface) on the state space. The resulting motion in the surface is named as a sliding motion [3, 4]. If the system has relative degree greater than one with respect to the sliding surface, high-order sliding modes (HOSM) such as the second-order sliding modes (SOSM) could be implemented preserving the main characteristics of classical SM [5, 6]. Moreover, SOSM reduce the undesired chattering effect. In continuous-time, the super-twisting algorithm (STA) has been successfully analysed and applied like a robust exact differentiator [7], state estimator [8] or controller [9]. Although, there exist several researches in the continuous domain, the concept of high-order discrete-time sliding modes (DSM) has not deeply studied.

DSM have not received much attention as the continuous counterpart. The first ideas of DSM were introduced in [10, 11], where a quasi-sliding mode (QSM) is established for systems with relative degree one. In [12], the study of discrete-time single-input single-output (SISO) non-linear
systems with relative degree bigger than one was considered. A new definition of the QSM regime was addressed in [13]. In this paper, the motion of the system was restricted to a certain band around the sliding surface. In [14, 15] some researches have been reported for several classes of discrete-time linear systems. The idea of SOSM control in discrete-time systems was introduced in terms of a certain classes of discretisation-like Euler discretisation or finite differences [8, 16, 17]. However, a complete stability study has not been deeply analysed in terms of Lyapunov stability.

There are several generic methods for designing sliding mode observers (SMO). For continuous-time linear and non-linear systems, most of these observers are based on the equivalent control concept for handling disturbances and modelling uncertainties. In the same context, discrete-time sliding mode control (DSMC) methods have also been developed for linear and non-linear systems [12, 18, 19]. However, discrete-time sliding mode-based observer (DSMO) design has not received much attention, especially for non-linear systems. DSMO design for linear systems was given in [18]. In [20], the concept of sliding lattice for discrete-time systems was introduced and DSMO was designed for SISO linear systems using the Lyapunov min–max method. Besides, in several works the accuracy of the discretisation in SOSM controllers depending on the sampling period has been analysed [21, 22].

In this paper, a discrete-time SOSM algorithm is proposed based on the well known STA. Stability of discrete-time super twisting-like algorithm (DSTA) is analysed with a quadratic Lyapunov function. Stability conditions for the DSTA were obtained with the solution of a linear matrix inequality (LMI) and the trajectories were confined into a ball Br characterised with a radius ρmax.

2 Discrete-time super-twisting-like algorithm

The DSTA is composed by the following two equations in differences

\[
\begin{align*}
x_1(k+1) &= \rho_1 x_1(k) + \tau x_2(k) - \tau k_1 |x_1(k)|^{1/2} \text{sign}(x_1(k)) \\
x_2(k+1) &= \rho_2 x_2(k) - \tau k_2 \text{sign}(x_1(k))
\end{align*}
\]

(1)

where \(x_i(k) \in \mathbb{R}\) are the states, \(\rho_i \in \mathbb{R}^+\), \(|\rho_i| < 1\), \(k_i \in \mathbb{R}\) for \(i = 1, 2\), are the gains to be designed for ensuring the convergence of (1) to the origin and \(\tau > 0\) is the sampling period.

The sign(\(\epsilon\)) function is defined as

\[
\text{sign}(\epsilon) := \begin{cases} 
-1 & \text{if } \epsilon < 0 \\
0 & \text{if } \epsilon = 0 \\
1 & \text{if } \epsilon > 0
\end{cases}
\]

(2)

where \(\epsilon \in \mathbb{R}\).

Equation in (1) is designed as a slightly modification of an Euler discretisation from the continuous SOSM. The system (1) can be represented as

\[
x(k+1) = Ax(k) + B(k) \text{sign}(x_1(k))
\]

(3)

where \(A \in \mathbb{R}^{2 \times 2}\) and \(B(k) \in \mathbb{R}^2\) are

\[
A := \begin{bmatrix} \rho_1 & \tau \\ 0 & \rho_2 \end{bmatrix}, \quad B(k) := \left[ -\tau k_1 |x_1(k)|^{1/2} \right]
\]

(4)

2.1 Main result

The result obtained in this paper is stated in the following theorem.

Theorem 1: Consider the non-linear system given in (1), with gains selected as \(k_1 > 0\) and \(k_2 > 0\) and if the following

\[
A^T (P + PA) A - (1 - \varrho) P + Q \leq 0
\]

has a positive-definite solution \(P = P^T > 0\) for a given \(Q > Q^T > 0\) and \(\Lambda = \Lambda^T > 0\), then, the trajectories of the dynamic system given in (1) converge asymptotically to a ball Br centred at the origin \(B_r := \{x : \|x\|^2 < r\}\) characterised with a radius

\[
r = \frac{c}{1 - \varrho}
\]

(6)

where

\[
0 < \varrho < 1, \quad c := \delta_1 + \frac{1}{4} \delta_2^2 \|Q^{-1}\|_F^2
\]

(7)

\[
\delta_1 := \delta_1 + \tau^4 k_1^2 k_2^2 z_{11} \omega_1, \quad \delta_2 := k_2^2 r (\tau^4 k_1^2 z_{11}^2 \omega_1 + z_{22})
\]

(8)

\[
\omega_1 \in \mathbb{R}^+, \quad Z = [\Lambda^{-1} + P]^\top, \quad Z, \Lambda, P \in \mathbb{R}^{2 \times 2}
\]

\[
Z := \begin{bmatrix} z_{11} & z_{12} \\ z_{12} & z_{22} \end{bmatrix}
\]

Proof: The proof of the previous theorem is given in the appendix.

Lemma 1: If \(\varrho\) is selected as \(0 < \varrho < 1\) such that

\[
(\lambda_{\text{max}}(A))/(\sqrt{(1 - \varrho)}) < 1,
\]

the LMI expressed in (5) always has positive definite solution \(P = P^T > 0\) given by

\[
P = \sum_{k=0}^{\infty} (A^\top)^k Q_k A^k
\]

with

\[
\tilde{A} := \sqrt{\gamma}^{-1} A, \quad \tilde{Q}_k := \tilde{\gamma}^{-1} Q_k
\]

This solution can be obtained because (5) can be represented as a classical Lyapunov discrete-time equation. Also, for a particular value of \(P\) and \(Q\) we can exactly get the corresponding value of \(\varrho\).
Suppose that the systems states are bounded, then the existence of a constant $f^+$ is ensured, such that the next inequality
\[
[f(x_1, \hat{x}_2, u)] \leq f^+
\]
with $f^+ \in \mathbb{R}^+$ holds for any possible $k, x_1, x_2$. The previous assumption is commonly used in the design of SM observers [8, 23].

Remark 1: For a physical second-order system, the constant $f^+$ can be found as the double maximal possible value of the derivative of $x_2$ (for example, the acceleration for a mechanical system or the flux for an electrical system). Moreover, the estimation constant $f^+$ does not depend on the control terms. Such assumption of the state boundedness is true too, if, for example system (11) is bounded-input bounded-output (BIBS) stable, and the control input $u(k)$ is bounded.

Equation (13) can be rewritten by
\[
e(k+1) = \Theta e(k) + \Phi(k)sign(e_1(k))
\]
\[
\Theta := \begin{bmatrix} 1 - k_1 \tau \\ -k_4 \end{bmatrix}, \quad \Phi(k) := \begin{bmatrix} -\tau k_1 |e_1(k)|^{1/2} \\ -\tau k_2 + \tau f \end{bmatrix}
\]
\[
(16)
\]
This equation has a similar form to the equation given in (3). According to the statement given in Theorem 1, the following result is obtained

Corollary 1: Consider the non-linear system in (11), where the output available is $x_1(k)$, if the assumption required for equation (14) is fulfilled and the observer gains are selected such that the LMI given by
\[
\Theta^T (R + R \Lambda R) \Theta - (1 - \varrho) R < -G
\]
with $\Theta$ defined as in (16) and $k_2 > f^+$, has a positive-definite solution $R = R^T > 0$, then, the observer trajectories of (12) converge to the real states values of (11) in a QSM regime.

Proof: Equations in (13) have the same structure as (3). Then, it is straightforward to follow the proof of Theorem 1 given in the appendix to obtain the LMI given in (17). The exact solution of (17) can be obtained with similar arguments to the ones described in lemma 1.

2.2.2 Vectorial case: Consider the case for systems with more than one degree of freedom but with relative degree two. The dynamics of these kind of systems is described by
\[
x(k+1) = \begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix}
\]
\[
= \begin{bmatrix} x_1(k) + \tau x_2(k) \\ x_2(k) + \tau f(x_1(k), u(k)) + \xi(k) \end{bmatrix}
\]
\[
(18)
\]
where \( x_a(k+1) \in \mathbb{R}^n \) and \( x_b(k+1) \in \mathbb{R}^n \) are defined as

\[
x_a(k+1) = \begin{bmatrix} x_1(k+1) \\ \vdots \\ x_n(k+1) \end{bmatrix} = \begin{bmatrix} x_1(k) + \tau x_{n+1}(k) \\ \vdots \\ x_n(k) + \tau x_{2n}(k) \end{bmatrix}
\]

\[
x_b(k+1) = \begin{bmatrix} x_{n+1}(k+1) \\ \vdots \\ x_{2n-1}(k+1) \\ x_{2n}(k+1) \end{bmatrix}
\]

\[
= \begin{bmatrix} x_{n+1}(k) + \tau (f_1(x(k), u(k)) + \xi_1(k)) \\ \vdots \\ x_{2n-1}(k) + \tau (f_{n-1}(x(k), u(k)) + \xi_{n-1}(k)) \\ x_{2n}(k) + \tau (f_n(x(k), u(k)) + \xi_n(k)) \end{bmatrix}
\]

The signals \( \xi_i(k) \) represent internal disturbances for each state \( x_i(k) \) in the system. The vector \( \xi(k) \in \mathbb{R}^n \) is assumed to be as follows: \( \xi(k) := [\xi_1(k), \ldots, \xi_n(k)]^T \) and by assumption bounded, that is, \( \|\xi(k)\|_{\mathcal{L}_1}^1 \leq \Upsilon, \Lambda_1 = \Lambda_1^T > 0, \forall k \in \mathbb{Z}^+ \cup \{0\} \). Every couple of states for the system (18) can be seen as a decoupled equations and they can be treated as a class of independent systems taking into account the relationship between them. In this way, the proposed observer showed in (12) can be extended as

\[
\hat{x}_a(k+1) = \hat{x}_a(k) - \tau \beta_1 \lambda(\hat{x}_a(k))S(\hat{x}_a(k)) - \beta_3 \hat{x}_a(k) \\
\hat{x}_b(k+1) = \hat{x}_b(k) + \tau f(\hat{x}_a(k), \hat{x}_b(k), u(k)) - \beta_5 S(\hat{x}_a(k)) - \beta_3 \hat{x}_a(k)
\]

where \( \hat{x}_a, \hat{x}_b \) are the estimates of the state vectors \( x_a, x_b \) respectively; the gain matrices \( \beta_1 \in \mathbb{R}^{n \times n}, \beta_2 \in \mathbb{R}^{n \times n}, \beta_3 \in \mathbb{R}^{n \times n}, \beta_4 \in \mathbb{R}^{n \times n}, \lambda(\hat{x}_a(k)) \in \mathbb{R}^{n \times n} \) and \( S(\hat{x}_a(k)) \in \mathbb{R}^{n \times n} \) are defined as

\[
\beta_1 = \text{diag}\{\beta_{11}, \beta_{12}, \ldots, \beta_{1n}\} \\
\beta_2 = \text{diag}\{\beta_{21}, \beta_{22}, \ldots, \beta_{2n}\} \\
\beta_3 = \text{diag}\{\beta_{31}, \beta_{32}, \ldots, \beta_{3n}\} \\
\beta_4 = \text{diag}\{\beta_{41}, \beta_{42}, \ldots, \beta_{4n}\} \\
\lambda(\hat{x}_a(k)) = \text{diag}\{|\tilde{x}_1(k)|^{1/2}, |\tilde{x}_2(k)|^{1/2}, \ldots, |\tilde{x}_n(k)|^{1/2}\} \\
S(\hat{x}_a(k)) = \text{sign}(\hat{x}_1(k)), \text{sign}(\hat{x}_2(k)), \ldots, \text{sign}(\hat{x}_n(k))\}
\]

The estimation error becomes

\[
\hat{x}(k+1) = \begin{bmatrix} (I_{n \times n} - \beta_3) \tilde{x}_a(k) + \tau \tilde{x}_b(k) \\ -\beta_2 \tilde{x}_a(k) + \tau \tilde{x}_b(k) + f(\hat{x}(k), u(k)) + \xi(k) \end{bmatrix}
\]

\[
- \tau \begin{bmatrix} \beta_1^T \\beta_2^T \end{bmatrix} \begin{bmatrix} \lambda(\tilde{x}_a(k)) \\ I_{n \times n} \end{bmatrix} S(\hat{x}_a(k))
\]

where \( \tilde{x}(k), \tilde{x}_a(k), \tilde{x}_b(k), (k+1) := f(\hat{x}(k), \hat{x}_b(k), u(k)) - f(x_a(k), x_b(k), u(k)), \) in the same way as in the scalar observer (15), \( \|\tilde{x}(k), \tilde{x}_a(k), \tilde{x}_b(k), u(k)\| \leq f^+ \) and \( \|\xi(k)\| \leq \xi, \forall k \in \mathbb{Z}^+ \cup \{0\} \). Let us denote \( \tilde{x}(\hat{x}_a(k), \hat{x}_b(k), (k), u(k)) \) and \( \tilde{x}_i(k) \) to the ith row of the functions \( f(\hat{x}_a(k), \hat{x}_b(k), (k), u(k)) \) and \( \xi(k) \) correspondingly. Owing to the boundedness assumption (15) an upperbound can be found for each couple of coordinates such that

\[
\|\tilde{x}(x(k), u(k)) + \xi(k)\| \leq f^+_i, f^+_i \in \mathbb{R}^n
\]

This bound is justified by the same facts used in the scalar case. If the new variables \( \Theta \) and \( \Phi \) are defined as

\[
\Theta := \begin{bmatrix} I_{n \times n} - \beta_3 & \tau I_{n \times n} \\ -\beta_2 & I_{n \times n} \end{bmatrix}, \quad \Phi(k) := \begin{bmatrix} \beta_1 \Lambda(\hat{x}_a(k)) \\ \beta_2 \end{bmatrix}
\]

with \( I_{n \times n} \) being the \( n \times n \) matrix identity, the dynamics of the observation error can be represented in the form of (16) and following the results presented in Theorem 1, the next Lemma is easily verified

**Lemma 2:** Consider the system given in (18), using the extended version of (12) given by (21), selecting \( \beta_{3i} > 0, \beta_{3i} > f^+_i, \beta_{4i} > 2 \) and \( \beta_{4i} > 0 \) with \( i = 1 : n \), and if the following LMI

\[
(\Theta^T \{P + P \Upsilon P\} \Theta - (1 - \phi)P) < -Q
\]

has a positive-definite solution \( P = P^T > 0 \) with \( \Upsilon = \Upsilon^T > 0, P, \Upsilon \in \mathbb{R}^{2n \times 2n} \) then the observer states \( (\hat{x}_a, \hat{x}_b) \) converge to the real states \( x_a, x_b \) in a QSM regime.

**Proof:** With the definitions given in (25) the estimation error becomes

\[
\tilde{x}(k+1) = \Theta \tilde{x}(k) + \Phi(k)S(\hat{x}_a(k))
\]

following the Theorem 1, the Lemma is proven.

**3 Numerical results**

**3.1 Simulation of the DSTA**

For the simulation, the DSTA parameters were chosen as \( \rho_1 = 0.9, \rho_2 = 0.3 \), the free gains were \( k_1 = 10.1 \) and \( k_2 = 10 \), and the sampled period was selected as \( \tau = 0.01 \). Fig. 1 shows that the first state \( x_1 \) converged to the origin and remains in a band near to zero. High-frequency oscillations when the state \( x_2 \) evolved to zero around the origin are depicted in Fig. 2. At the same time, the Lyapunov function remained in a band for bounded values of the states as shown in Fig. 3. Selecting \( Q = 3 \times 10^{-3} \) and \( \phi = 0.81 \), the matrix \( \Lambda \) defined in Lemma 1 becomes

\[
\tilde{A} = \begin{bmatrix} 0.9994 & 0.001 \\ 0 & 0.3331 \end{bmatrix}
\]
The matrix $P$ solution for the LMI (10) is

$$P = \begin{bmatrix} 2.4337 & 1.87 \times 10^{-5} \\ * & 0.0034 \end{bmatrix}$$

with this solution selecting $\hat{A} = I_{2 \times 2}$, $\Lambda = \begin{bmatrix} 0.1688 & -0.6762 \\ * & 8.78 \times 10^{4} \end{bmatrix}$. The radius of the ball $R$, defined in (6) becomes $r = 0.0038$, this fact can easily verified in the second graph of Fig. 3. The steady-state value of $V(x)$ is $3 \times 10^{-5}$, that is less than the ratio $r$. With this fact, the result claimed in the main theorem was numerically proven. In addition, in Figs. 1 and 2, it could be appreciated that the DSTA states $x_1$ and $x_2$ presented small oscillations in steady state (chattering effect); therefore the Euclidean norm and the trajectories of the Lyapunov function presented in Fig. 3 have a value of 0.075 and $3 \times 10^{-5}$, respectively. Both graphs in Fig. 3 are quite similar because the Lyapunov function is the weighted Euclidean norm by the matrix $P$, that is $V(x) = \|x\|_P^2$.

### 3.2 Non-linear DSTO

A second illustration of the results presented in this paper was the design of a DSTO for a non-linear pendulum system. Consider a pendulum whose state model is given by

$$
x_1(k+1) = x_1(k) + \tau x_2(k) + \frac{1}{J} u(k) - \frac{mg}{2J} \sin(x_1(k))
$$

$$
x_2(k+1) = x_2(k) + \tau \psi(k)
$$

$$
y(k) = x_1(k)
$$

(30)

The previous equations represent an Euler discretisation of the continuous pendulum model, where $x_1(k)$ is the angle of oscillation, $x_2(k)$ is the angular velocity, $m$ is the pendulum mass, $g$ is the gravitational force, $l$ is the pendulum length, $J = ml^2$ is the inertia arm, $V_\psi$ is the pendulum viscous friction coefficient. For simulation the bounded perturbation was expressed as $\psi(k) = k_1 \sin(2\pi k) + k_2 \cos(5\pi k)$, with $x_1 = \dot{x}_1 = 0.5$, the initial conditions were chosen as $x_1(0) = 3$ and $x_2(0) = -1$ for the model and $\dot{x}_1(0) = 10$ and $\dot{x}_2(0) = -10$ for the observer. The following numeric values were applied to simulate the pendulum parameters $m = 1.1 \text{kg}$, $l = 1 \text{m}$, $g = 9.81 \text{m/s}^2$ and $V_\psi = 0.18 \text{kg} \cdot \text{m/s}^2$. The input applied into the system is $u(k) = \sin(2\pi k) \cos(5\pi k)$.

The observer parameters were chosen as $k_1 = 7.1$, $k_2 = 12$, $k_3 = 0.1$ and $k_4 = 0.01$ for the SM observer. A Luenberger observer was designed for comparison purposes. The linear gains for the Luenberger observer were selected as $l_1 = 20$ and $l_2 = 84$ with the following structure

$$
\dot{x}_1(k+1) = \dot{x}_1(k) + \tau l_1 e_1(k)
$$

$$
\dot{x}_2(k+1) = \dot{x}_2(k) + \frac{1}{J} u(k)
$$

$$
-\frac{mg}{2J} \sin(x_1(k)) - \tau \frac{V_\psi}{J} \dot{x}_2(k) + \tau l_2 e_1(k)
$$

(31)

The numerical simulation of the angular velocity estimation is shown in Fig. 4. This brings out the fact that the DSTO behaves as the STA observer for continuous-time systems with small sampled times. It is clear that even when the Luenberger high-gain observer reaches the trajectories of the measurable state before the DSTO, the steady-state error for the DSTO was smaller than the Luenberger error. This fact was explained by the gains selected for the Luenberger observer that were greater than the DSTO gains. When the...
error trajectories were faraway from the real ones, the linear gain has a more important effect than those including the signum function. The zone of convergence can be calculated with the result provided in Theorem 1. Selecting $Q = 1 \times 10^{-7}I_{2 \times 2}$, $\varrho = 0.9999$, the solution for the LMI (17) were

$$P = \begin{bmatrix} 5.0191 & 0.0075 \\ 0.0075 & 0.001 \end{bmatrix} \quad (32)$$

and the radius $r$ defined in (6) was $r = 0.0078$. In Fig. 5, the performance indexes for the observers were depicted for several values of the perturbation term $\psi(k)$. The term $\psi(k)$ was defined as

$$\psi(k) = \kappa_1 \sin(2\tau k) + \kappa_2 \cos(5\tau k)$$

The upperbound of this term was given by

$$|\psi(k)| \leq |\kappa_1 \sin(2\tau k) + \kappa_2 \cos(5\tau k)| \leq \kappa_1 + \kappa_2$$

To complete the numerical test, one can chose $\kappa_1 = \kappa_2$, and then $|\psi(k)| \leq 2\kappa = \kappa^*$. A measure of robustness was studied in means of the magnitude of this perturbation. The following cases were considered in the paper, $\kappa = 0.5, 1, 2, 5$, then $\kappa^* = 1, 2, 4, 10$. The following figure showed the Euclidean norm of the estimation. The solid line depicts the performance of the DSTO observer, while the dotted line does for the linear observer. One can note that the DSTO performance was unaffected by the increment of the perturbation norm ($\kappa^*$).

### 3.3 State estimation for 2n-dimensional systems

The model of a biped robot was formed with five links connected with frictionless joints. The identical legs have knee joints between the shank and thigh parts, and one rigid body forms the torso. Fig. 6a shows the model structure and variables that were taken from [24].

As the system can move freely in the $x$–$y$-plane and contains five links, it has seven degrees of freedom. The corresponding seven coordinates were selected according to Fig. 6a

$$q = \begin{bmatrix} x_0 & y_0 & \alpha & \beta_L & \beta_R & \gamma_L & \gamma_R \end{bmatrix} \quad (33)$$

The coordinates $(x_0, y_0)$ fix the position of the torso centre of mass, and the rest of the coordinates describe the joint angles. The link lengths were denoted as $(l_0, l_1, l_2)$ and masses as $(m_0, m_1, m_2)$. The centres of mass of the links were located at the distances $(r_0, r_1, r_2)$ from the corresponding...
Fig. 6  BipedSim

a  Coordinates and constants
b  External forces
c  Leg tip touches the ground in point \((x'_0, 0)\) (grey) and penetrates it; the current position of the leg tip is \((x'_G, y'_G)\) (black)

The model was actuated with four moments:

\[
M = \begin{bmatrix}
M_{L1} & M_{L2} & M_{R1} & M_{R2}
\end{bmatrix}
\]  \(34\)

two of them acting between the torso and both thighs and two at the knee joints (Fig. 6b). The walking surface was modelled using external forces:

\[
F = [F_{Lx}, F_{Ly}, F_{Rx}, F_{Ry}] \]

that affect the leg tips. When the leg should touch the ground, the corresponding forces are switched to support the leg. As the leg rises, the forces are zeroed (Fig. 6c).

Using the Lagrangian mechanics, the dynamic equations for the biped system can be derived:

\[
A(q) \ddot{q} = b(q, \dot{q}, M, F) \]

Here \(A(q) \in \mathbb{R}^{7 \times 7}\) is the inertia matrix and \(b(q, \dot{q}, M, F) \in \mathbb{R}^7\) is a vector containing the right-hand sides of the seven differential equations. Matrices \(A(q)\) and \(b(q, \dot{q}, M, F)\) are defined in [24]. Using the Euler discretisation, the system (36) is implemented as a discrete one using the physical parameters for the biped model summarised in the following table.

Initial conditions for the biped robot were given by (see (37))

Initial conditions for the observer were selected as (see (38))

\[
x_u = \begin{bmatrix} 5.730 & 1.37 & 0.011 & 0.299 & 0.033 & 0.206 & 0.240 \end{bmatrix}^T
\]

\[
x_{\dot{u}} = 0.8 * \begin{bmatrix} 0.545 & -0.029 & -0.159 & -1.130 & -0.970 & 0.064 & -0.387 \end{bmatrix}^T
\]

\[
\hat{x}_u = \begin{bmatrix} 4.011 & 0.959 & 0.008 & 0.209 & 0.023 & 0.144 & 0.168 \end{bmatrix}^T
\]

\[
\hat{x}_{\dot{u}} = \begin{bmatrix} 0.436 & -0.023 & -0.127 & -0.904 & -0.776 & 0.051 & -0.309 \end{bmatrix}^T
\]
The trajectories for the DSTO and the Luenberger observer are depicted in Fig. 7. It can be seen how the trajectories of the DSTO converged closer to the real system trajectories. The Luenberger state estimator did not converge to the real states. In the same way, a similar comparison with the velocities can be seen in Fig. 8. Again, a remark should be

\[ L_1 = 5 \times \text{diag} \{15, 15, 15, 25, 25, 25, 25\} \]
\[ L_2 = 5 \times \text{diag} \{25, 25, 25, 35, 35, 35, 35\} \]  

(40)
done in the gains selection. The gains for the linear observer were five times greater than the DSTO, this fact, implied a faster convergence into a zone for the Luenberger observer in similar way as it was claimed for the scalar case. However, the error was bigger in the linear case when the system was in the steady state. It is important to note that only four states were plotted with the objective of keeping the length of the paper small enough. A complete analysis including all the states of the biped model was done by means of the Euclidean norm of the error (Fig. 9). Moreover, following the results presented in [24], where an animation of a biped was performed, in Fig. (10) this animation was reprogrammed to present some clips of the animation with the observer trajectories. In this figure, the mesh line are used to differentiate the left leg from the right left of the biped model. This animation was modified from the one presented [24] in order to include a second biped that represented the trajectories of the state estimator. In Fig. 10, the first biped corresponded to the DSTO and the second one to the real system. In the instant $t = \tau k = 0$, the centre of mass of the biped represented by the observer is ubicac at the $x$-coordinates in $x = 4.011$ and the real system on $x = 5.730$. After some steps of time, the animation obtained with the DSTO reached the system. After $t = \tau k = 1.75$, the estimated biped reached the real biped animation. This confirmed the complete reconstruction of the biped model states by the observer ones (21).

4 Conclusions

The Euler-like discretisation of the super-twisting algorithm was successfully analysed in terms of Lyapunov stability. The behaviour of the DSTA has similar characteristics like its continuous-time counterpart for small sampled periods. However, finite-time convergence cannot be proved with this quadratic Lyapunov function. The convenient gains to ensure the convergence of the DSTA were obtained by means of an LMI. A direct application of this new discrete structure is the problem of state estimation. Numerical results showed how the observer can reach the real trajectories of a $n$-degrees of freedom non-linear mechanical system in a small period of time with better performance than the results obtained by a Luenberger observer.

5 References

2 Atassi, A. N., Khalil, H.: ‘Separation results for the stabilization of nonlinear systems using different high-gain observer design’, Syst.
6 Appendix

Proof: (Proof of Theorem 1) Consider the following function like a candidate Lyapunov one
\[ V(x) := \| x \|^2 \] \hspace{1cm} (41)
Let \( \Delta V(k) := V(k+1) - V(k) \) then
\[ \Delta V(k) = x^T(k+1) P x(k+1) - x^T(k) P x(k) \] \hspace{1cm} (42)
Substituting (4) into (42) the terms \( \Delta V(k) \) becomes
\[ \Delta V(k) = x^T(k) (A^T P A - P) x(k) + 2 x^T(k) A^T P B(k) \]
\[ \times \text{sign}(x_1(k)) + B^T(k) P B(k) \] \hspace{1cm} (43)
Using the so-called lambda inequality [25] \( X^T Y + Y^T X \leq X^T \Lambda^{-1} X + Y^T Y \) for \( 2 x^T(k) A^T P B(k) \text{sign}(x_1(k)) \) the following result was obtained
\[ 2 x^T(k) A^T P B(k) \text{sign}(x_1(k)) \leq x^T(k) A^T P A^{-1} P A x(k) + B^T(k) \Lambda B(k) \] \hspace{1cm} (44)
Equation (43) turns in
\[ \Delta V(k) \leq x^T(k) (A^T P + P A^T) A - (1 - \rho) P x(k) + B^T(k) (\Lambda^{-1} + P) B(k) - \varrho V(k) \] \hspace{1cm} (45)
Then, expanding the term \( B^T(k) Z B(k) \) with \( Z := (A^{-1} + P) \), one has
\[ \Delta V(k) = B^T(k) Z B(k) = \delta_1 x_1(k) + \delta_2 |x_1(k)|^{1/2} + \delta_3 \]
\[ \delta_1 := \tau^2 k_2^2 z_{11}, \quad \delta_2 := \tau^2 k_2 z_{12}, \quad \delta_3 := \tau^2 k_2^2 z_{22} \] \hspace{1cm} (46)
Using again the lambda inequality in the term containing \( \delta_2 \) and if there exists a matrix \( Q = Q^T > 0 \) solution for the LMI given by \( A^T (P + P A) A - (1 - \rho) P = -Q \), then \( \Delta V(k) \) becomes into (for any \( \alpha \in \mathbb{R}^n \))
\[ \Delta V(k) \leq - \| x(k) \|^2 + \delta_1 |x_1(k)| + \delta_2 - \varrho V(k) \]
\[ \delta_1 := \delta_1 + \tau^2 k_2^2 z_{11}, \quad \delta_2 := \tau^2 k_2 z_{12}, \quad \delta_3 := \tau^2 k_2^2 z_{22} \] \hspace{1cm} (47)
Following the previous result
\[ \Delta V(k) \leq - \| x(k) \|^2 + \delta_1 |x_1(k)| - \alpha V(k) + \tilde{\delta}_2 \]
\[ = - \left( \| Q^{1/2} x(k) \| - \frac{1}{2} \delta_1 \| Q^{-1/2} \| \right)^T \]
\[ \times \left( \| Q^{1/2} x(k) \| - \frac{1}{2} \delta_1 \| Q^{-1/2} \| \right) \]
\[ - \alpha V(k) + \tilde{\delta}_2 + \frac{1}{4} \delta_2^2 \| Q^{-1} \|^2 \] \hspace{1cm} (48)
then
\[ \Delta V(k) \leq - \varrho V(k) + c \] \hspace{1cm} (49)
with \( c := \delta_2 + \frac{1}{4} \delta_2^2 \| Q^{-1} \|^2 \).
\[ V(k+1) \leq (1 - \varrho) V(k) + c \] \hspace{1cm} (50)
The last equation is well defined as a discrete linear one and the solution is given by
\[ V(k+1) \leq (1 - \varrho)^i V(0) + \sum_{i=1}^{k} (1 - \gamma_s)^{i-1} c \] \hspace{1cm} (51)
If the upper limit when \( k \) goes to infinity is considered, one has
\[ \lim_{k \to \infty} V(k) \leq \frac{c}{1 - \varrho} \] \hspace{1cm} (52)
and the radius of the region of convergence of the DSTA is defined as
\[ r \leq \frac{c}{1 - \varrho} \] \hspace{1cm} (53)
This result completes the proof. \( \square \)
CTA20130568

Author Queries

Iván Salgado, I. Chairez, Bijnan Bandyopadhyay, Leonid Fridman and Oscar Camacho

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