A Hybrid Robust Non-Homogeneous Finite-Time Differentiator

Denis V. Efimov, Member, IEEE, and Leonid Fridman, Member, IEEE

Abstract—A variant of super-twisting differentiator is proposed. Lyapunov function is designed and an estimate on finite time of derivatives estimation is given. The differentiator is equipped with hybrid adaptation algorithm that ensures global differentiation ability independently on amplitude of the differentiated signal and measurement noise.

Index Terms—PID control.

I. INTRODUCTION

The problem of a differentiator design is very important and challenging [3], [5], [13]. Numerical differentiation finds many application in control theory [14]. For example, many kinds of systems can be transformed to a canonical form with the state vector representation as a column of output function derivatives, in this case the problem of unmeasured state estimation is reduced to the derivatives computation of available for measurements output signal [1]. Another example is the class of flat systems, the state and the input of such nonlinear systems are functions of the output and its derivatives [8], their computation provides an access to the system internal dynamics and the input evaluation. Finally, despite of the great success achieved in nonlinear control theory, the PID control is still the most popular tool used in practical applications [20]. Realization of this control strategy requires estimation of the regulation error derivative.

In this work we are going for on-line or real time differentiation and by a differentiator we mean an algorithm or a dynamical system that derivatives of the given signal derivative. There exist many approaches to differentiators design providing similar performance in applications [19], [21]. One of the most popular is super-twisting differentiator [12]. This differentiator ensures finite-time robust differentiation of noisy signals. A shortage of the algorithm consists in complexity of Lyapunov functions design (to prove explicitly its stability and performance) and time of the estimation error convergence evaluation [17].

In this work we are going to develop the results from [12] proposing a variant of super-twisting differentiator with simple estimates on the time of convergence and accuracy of derivative calculation. Another design goal consists in robustness of the differentiator against a non-differentiable noise of any amplitude. Contrarily [12], the Lyapunov approach is chosen in this work to achieve these goals (as an alternative, in the book [2] the frequency domain framework for discontinuous systems investigation is presented).

The following sliding mode differentiator is designed:

\[
\begin{align*}
\dot{x}_1 &= -\alpha \sqrt{|x_1|} \mathrm{sign}(x_1 - f(t)) + x_2, \quad \alpha > 0 \\
\dot{x}_2 &= -\beta \mathrm{sign}(x_2 - f(t)) - \chi \mathrm{sign}(x_2 - x_1), \quad \beta > \chi \geq 0
\end{align*}
\]

(1a) where \(x_1 \in \mathbb{R}, x_2 \in \mathbb{R}\) are the state variables of the system (1), the function \(f: \mathbb{R} \to \mathbb{R}\) has two continuous derivatives (we assume that the constants \(L_i \in \mathbb{R}^+, i = 1, 2\) are given such that \(|f'(t)| \leq L_1\) and \(|f''(t)| \leq L_2\) for all \(t \in \mathbb{R}\)). The variable \(x_1(t)\) serves as an estimate of the function \(f(t)\) and \(x_2(t)\) converges to \(f'(t)\). Therefore, (1) has the input \(f(t)\) and the output \(x_2(t)\).

Comparing with conventional super-twisting differentiator [12], or other sliding-mode differentiators [16], the system (1) has two additional negative feedbacks in the (1b) causes the homogeneity property loss. As we will show, introduction of these feedbacks does not destroy excellent differentiation abilities of the differentiator [12] providing a hint to Lyapunov function design and time of convergence evaluation improving robustness.

As in [12] the requirement on existence of the second derivative can be replaced with Lipschitz continuity of the first derivative, then \(L_2\) is the corresponding Lipschitz constant. The system (1) is discontinuous, its solutions are understood in Filippov sense [4], [7], [18].

II. THE CASE OF EXACT MEASUREMENTS

Introducing variables \(e_1 = x_1 - f, e_2 = x_2 - f'\) we obtain

\[
\begin{align*}
\dot{e}_1 &= -\alpha \sqrt{|e_1|} \mathrm{sign}(e_1) + e_2 \\
\dot{e}_2 &= -\beta \mathrm{sign}(e_1) - \chi \mathrm{sign}(e_2 + f'(t)) - e_2 - f(t) - f''(t).
\end{align*}
\]

(2a) Define \(\gamma(t) = \beta + \{f'(t) + f''(t) - \chi \mathrm{sign}(e_2(t)) - \mathrm{sign}(e_2(t) + f''(t))\} \times \mathrm{sign}(e_1(t)),\) that is a piecewise continuous function (for \(\beta > L_1 + L_2 + 2\chi\) it is strictly positive and \(0 < \delta \leq \gamma(t) \leq \kappa,\) \(\delta = \beta - L_1 - L_2 - 2\chi,\) \(\kappa = \beta + L_1 + L_2 + 2\chi\)).

\[
\begin{align*}
\dot{e}_1 &= -\alpha \sqrt{|e_1|} \mathrm{sign}(e_1) + e_2 \\
\dot{e}_2 &= -\gamma(t) \mathrm{sign}(e_1) - \chi \mathrm{sign}(e_2) - e_2.
\end{align*}
\]

(3a) All solutions of (2) are captured by the corresponding solutions of (3). For the system (3) the origin \(e = 0\) contains an invariant solution. We are going to show that the origin is finite time stable.

Theorem 1: Let \(\beta > L_1 + L_2 + 2\chi\) and \(\alpha = 2\sqrt{2\kappa\chi + \chi^2 + \kappa^2} \times (\kappa - \delta)/(1.5\delta + 0.5\kappa),\) then the system (3) is finite-time stable for any initial conditions \(e(0) \in \Omega_0\)

\[
\Omega_0 = \{e \in \mathbb{R}^2 : \kappa|e_1(0)| + 0.5e_2^2(0) \leq 4\sqrt{2\kappa(\chi + \chi^2)}(\kappa - \delta)^{-1}\} \leq T_0 \leq \frac{4\sqrt{2\kappa(\chi + \chi^2)}(\kappa - \delta)^{-1}}{2\mu}
\]

with time of convergence to zero

\[
T_0 \leq 2\mu^{-1}\sqrt{\kappa|e_1(0)| + 0.5e_2^2(0)}. \quad \mu = \min(\delta/\sqrt{\kappa, \sqrt{2}\chi}.
\]

All proofs are presented in the Appendix.

0018-9286/$26.00 © 2011 IEEE
as $t \to T_0$. It is a semi-global result since the size of the set $\Omega_0$ can be arbitrary assigned by proper choice of $\alpha, \beta, \gamma$. The proof of global finite-time stability for the differentiator (1) is presented in [6], it is based on another Lyapunov function analysis. The shortage of the estimate on $T_0$ consists in its dependence on unavailable for measurements value $\varepsilon_2(0)$. Fortunately, choosing initial conditions and parameters $\alpha, \beta$ and $\gamma$ in particular way it is possible to compensate this problem.

**Corollary 1:** Let $x_1(0) = f(0)$, $x_2(0) = 0$ and (4), shown at the bottom of the page, for any $v \geq 0$, then $T_0 \leq L_1/(0.25 \sqrt{2} L_1 + v)$.  

According to the corollary result, taking $v$ large enough it is possible to ensure any desired rate of the estimation.

**Example 1:** Let $f(t) = \sin(\omega_1 t) + b \sin(\omega_2 t)$, $\omega_1 = 0.5$, $\omega_2 = 2$, $b = 0.3$, $L_1 = \omega_1 + b \omega_2$, $i = \frac{1}{2}$. Take $v = 1$, and from (4) $\gamma = 1.32$, $\beta = 7.53$, $\alpha = 10.63$, $T_0 = 0.82$. The results of simulation are shown in Fig. 1 (step is $10^{-4}$ for the Euler method).

**Remark 1:** Coefficients $\alpha$, $\beta$ and $\gamma$ can be chosen independently on amplitude of $f(t)$ ($f(t)$ can be unbounded).

### III. THE CASE OF NOISY MEASUREMENTS

Let the signal $f(t) = f(t) + \varphi(t)$ be available for measurements, where $f : R \to R$ is a useful signal and $\varphi : R \to R$ is a noise. In this case the system (1) takes form

$$\dot{x}_1 = -\alpha \sqrt{|x_1 - f(t)|} \cdot \text{sign} \left[ x_1 - f(t) \right] + x_2,$$
$$\dot{x}_2 = -\beta \cdot \text{sign} \left[ x_1 - f(t) \right] - \chi \cdot \text{sign} \left[ x_2 \right] - x_2.$$  

(5a)

(5b)

The system (5) is discontinuous and affected by the disturbance $\varphi$. First, we would like to prove that the system has bounded trajectories. Second, we would like to show that the accuracy of derivatives estimation depends continuously on the noise amplitude $\varphi$ (at least for small measurement errors), that is not true in general case [15].

Introducing variables $e_1 = x_1 - f$, $e_2 = x_2 - f \cdot$ the system (5) can be rewritten as follows:

$$\dot{e}_1 = -\alpha \sqrt{|e_1|} \cdot \text{sign} \left[ e_1 \right] + E_{1} + \delta_1(t),$$

$$\dot{e}_2 = -\gamma \cdot \text{sign} \left[ e_1 \right] - \chi \cdot \text{sign} \left[ e_2 \right] - E_{2} + \delta_2(t),$$

$$\delta_1(t) = \alpha \left\{ \sqrt{|e_1|} \cdot \text{sign} \left[ e_1 \right] - \sqrt{|e_1 - \varphi(t)|} \cdot \text{sign} \left[ e_1 - \varphi(t) \right] \right\};$$

$$\delta_2(t) = \beta \left\{ \text{sign} \left[ e_1 \right] - \text{sign} \left[ e_1 - \varphi(t) \right] \right\}.$$  

(6a)

(6b)

where $\delta_1$, $\delta_2$ are the disturbances originated by the noise $\varphi$ presence, $\gamma(t) = \beta + (f(t) + f'(t) - \chi \cdot \text{sign} \left[ e_2(t) \right] - \text{sign} \left[ e_2(t) + f(t) \right]) \cdot \text{sign} \left[ e_1(t) \right]$ is the same as in (3).

By definition $|\delta_1(t)| \leq \alpha \sqrt{2\alpha}$, $\delta_2(t) = 0$ for $|e_1(t)| \geq \lambda_0$, $|\delta_2(t)| \leq 2\beta$ and $\delta_2 e_1(t) \geq 0$ for all $t \in R$.

### A. Global Boundedness of Solutions

**Lemma 1:** Let the signal $\varphi : R \to R$ be Lebesgue measurable and $|f(t)| \leq L_1$, $|f'(t)| \leq L_2$, $|\varphi(t)| \leq \lambda_0$ for all $t \in R$; $\alpha > 0$, $\beta > 0$ and $0 < \chi < \beta$. Then in (5) for all $t_0 \in R$ and initial conditions $x_1(t_0) \in R$, $x_2(t_0) \in R$ the solutions are bounded

$$|x_1(t) - f(t)| \leq \max \{|x_1(t_0) - f(t_0)|, 4\alpha^{-2} \left\{|x_2(t_0) - f(t_0)| + 3\beta + L_1 + L_2 + \chi \right\}^2 \},$$

$$|x_2(t) - f(t)| \leq |x_2(t_0) - f(t_0)| e^{-0.5\alpha} + |3\beta + L_1 + L_2 + \chi|.$$  

### B. Case of Differentiable Noise

Further for simplicity assume that $\varphi(t)$ also has two continuous derivatives and the constants $\lambda_i \in R$, $i \in \{1, 2\}$ are given such that $|\varphi(t)| \leq \lambda_0$, $|\varphi'(t)| \leq \lambda_1$, $|\varphi''(t)| \leq \lambda_2$ for all $t \in R$ (define the corresponding constants for the function $f(t)$ as $L_1 = L_1 + \lambda_i$, $i = \{1, 2\}$). It is required to estimate the signal $f(t)$ from measured $f(t)$. Introducing the variables $e_1 = x_1 - f$, $e_2 = x_2 - f$, $\gamma(t) = \beta + (f(t) + f'(t) - \chi \cdot \text{sign} \left[ e_2(t) \right] - \text{sign} \left[ e_2(t) + f(t) \right]) \cdot \text{sign} \left[ e_1(t) \right]$, the system (5) can be reduced to (3). The function $\gamma(t)$ is piecewise continuously and for $\beta > L_1 + L_2 + 2\chi$ it is strictly positive and $0 < \delta \leq \gamma(t) \leq \kappa$, where $\delta = \beta - L_1 - L_2 - 2\chi$, $\kappa = \beta + L_1 + L_2 + 2\chi$. The results of theorem 1 can be trivially extended to the system (5).

**Theorem 2:** Let $x_1(0) = f(0)$, $x_2(0) = 0$ and

$$\chi = 0.25 \sqrt{2} L_1 + v, \quad \beta > L_1 + L_2 + 3\chi,$$

$$\alpha = 4 \sqrt{2(\beta + L_1 + L_2 + 2\chi)} \chi + \sqrt{\beta + L_1 + L_2 + 3\chi} \chi$$

$$\chi \left( L_1 + L_2 + 2\chi \right) \left( 2\beta - L_1 - L_2 - 2\chi \right)$$  

(4)

for any $v > 0$, then the corresponding solutions of the system (5) are bounded and for all $t \geq T_0$

$$|x_1(t) - f(t)| \leq \lambda_0,$$

$$|x_2(t) - f(t)| \leq \lambda_1$$  

and the finite time $T_0$ of convergence possesses the estimate $T_0 \leq L_1/(0.25 \sqrt{2} L_1 + v)$.

This result means insensitivity of (5) to any constant noise.

**Example 2:** Let $f(t)$ be as in the first example and $\varphi(t) = v \sin(\omega_1 t)$, $v = 0.01$, $\omega = 5\omega_2$, then $\lambda_0 = v \omega$, $\chi = v \omega$, $i = \frac{1}{3}$. Let $v = 1$, $\chi = 1.357$, $\beta = 8.72$ and $\alpha = 12.008$, then $T_0 = 0.884$, step of simulation was chosen $5 \times 10^{-4}$ for the Euler method.
The results of the system (5) simulation presented in Fig. 2, where $e_{1}(t) = x_{1}(t) - f(t)$ and $e_{2}(t) = x_{2}(t) - f(t)$. The simulation results confirm theoretical findings of the work.

C. Non-Differentiable Noise

Let the signal $\varphi : R \to R$ be Lebesgue measurable and $|\varphi(t)| \leq \lambda_{0}$ for all $t \in R$.

Theorem 3: Let $\beta > L_{1} + L_{2} + 2\chi, \chi > 0$ and $\alpha = 2\sqrt{2\lambda_{0} + \frac{\chi}{\delta} - \delta^{-1}}/1.5\delta + 0.5\chi$, then for any initial conditions $e(0) \in \Omega_{\alpha}, \Omega_{\beta, \alpha} = \{e \in R^{2} : |e| \leq \delta^{-1}\}$ the trajectories of the system (6) satisfy the estimate for all $t \geq T_{0}$

$$|e_{1}(t)| \leq \delta^{-1}e(0) + \frac{2\lambda_{0}}{\alpha e_{2}(0),}
\|e_{2}(t)\| \leq \sqrt{2\lambda_{0}} + \frac{\delta^{-1}}{\alpha}.

c_{1} = \max\left\{\frac{1}{\delta^{-1}}, \frac{1}{\alpha}, \frac{\delta^{-1}}{\alpha}, \frac{\delta^{-1}}{\alpha} \right\}.
\frac{c_{2}}{\alpha \sqrt{\lambda_{0}}}, \frac{c_{2}}{\alpha^{2}} \leq \frac{\delta^{-1}}{\alpha}.
\frac{c_{1} \lambda_{0} + c_{2} \sqrt{\lambda_{0}}}{\alpha \sqrt{\lambda_{0}}}, \frac{c_{2}}{\alpha \sqrt{\lambda_{0}}}, \frac{c_{2}}{\alpha^{2}} \leq \frac{\delta^{-1}}{\alpha}.
$$

where $T_{0} \geq 4 \mu^{-1} \sqrt{\lambda_{0} e_{2}(0)} + 0.5\delta^{-1}$. Theorem 3 proof is based on the observation that $\delta$ (the product $e_{2} \delta$) influences negatively on (6) onto the set $\Gamma = \{e(0) : c_{1} < \lambda_{0} \wedge 3\delta \sqrt{2\lambda_{0}} < c_{2} < 2\beta \wedge e_{1} e_{2} > 0\}$ (only see Fig. 3). The result of the theorem says that if the noise amplitude $\lambda_{0}$ is comparable with the chosen $\alpha, \beta, \chi$ (the constraint $c_{1} \lambda_{0} + c_{2} \sqrt{\lambda_{0}} \leq 2\lambda_{0} \sqrt{\chi + \delta^{-1}}$ holds), then the estimate on the derivative $f$ has the error proportional to $\lambda_{0}^{2}$ (theoretical limitations of this estimate improvement are established in [10]). If the noise amplitude is very high, then the result of lemma 1 is satisfied guaranteeing boundedness trajectories. It is worth to stress, that the value $2 \lambda_{0} \sqrt{\chi + \delta^{-1}}$ can be taken arbitrary high adjusting $\alpha, \beta, \chi$.

Remark 2: It is well known fact that the system with discontinuous feedback is robust with respect to sufficiently small measurement noise if and only if the system admits a continuously differentiable Lyapunov function [11]. The Lyapunov function used in theorem 1 is not continuously differentiable, therefore, the result of [11] cannot be applied here for robustness approving.

IV. HYBRID DIFFERENTIATOR

The shortage of theorem 2 conditions (namely $\beta > 3\chi + L_{1} + L_{2}$, $\chi = 0.25 \sqrt{2} L_{1} + \lambda_{0}$) consists in their dependence on the constants $L_{1}, L_{2}$, which can be unknown. Instead of these constants we can use guess values $L_{1}^{\mu}, L_{2}^{\mu}$, then their substitution into the conditions of theorem 2 can provide us with sample parameters for the system (5)

$$\chi_{0} = 0.25 \sqrt{2} L_{1}^{\mu} + v, \beta_{0} = L_{1}^{\mu} + L_{2}^{\mu} + 3\chi_{0} + \frac{v}{\mu}.
\alpha_{0} = 4 \left\{ \sqrt{2} \left( \beta_{0} + L_{1}^{\mu} + L_{2}^{\mu} + 2\lambda_{0} \right) \right\}.
\right.
\left. \sqrt{2} \left( \beta_{0} + L_{1}^{\mu} + L_{2}^{\mu} + 2\lambda_{0} \right) \right\}.
$$

where $v > 0$ is a design constant. Taking $x_{1}(t_{0}) = \tilde{f}(t_{0}), x_{2}(t_{0}) = 0, t_{0} \geq 0$ and simulating the system (5) with derived $\alpha_{0}, \beta_{0}, \chi_{0}$ we may expect that $x_{1}(t) = \tilde{f}(t)$ for all $t \geq t_{0}$. If there exists a time instant $t_{1} \geq t_{0} + T_{0}^{\mu}$ such that $x_{1}(t) \neq \tilde{f}(t)$, then the values of $L_{1}^{\mu}, L_{2}^{\mu}$ were guessed wrongly. Next, it is worth to increase the guess values for the constants $L_{1}, L_{2}$ (say $L_{1}^{\mu} > L_{1}, L_{2}^{\mu} > L_{2}$) and recalculate $\alpha_{1}, \beta_{1}, \chi_{1}, t_{0}$ for these $L_{1}^{\mu}, L_{2}^{\mu}$ repeating the simulation. Formally this algorithm can be written as follows:

$$L_{1}^{1} = L_{1}^{\mu}, L_{2}^{1} = L_{2}^{\mu}, \chi_{1} = 0.25 \sqrt{2} L_{1}^{1} + v, \beta_{1} = 3\chi_{1} + L_{1}^{1} + L_{2}^{1} + v
\alpha_{1} = \frac{4 \left\{ \sqrt{2} \left( \beta_{1} + L_{1}^{1} + L_{2}^{1} + 2\chi_{1} \right) \right\}}.
\left. \sqrt{2} \left( \beta_{1} + L_{1}^{1} + L_{2}^{1} + 2\chi_{1} \right) \right\}.
\right.
\left. \sqrt{2} \left( \beta_{1} + L_{1}^{1} + L_{2}^{1} + 2\chi_{1} \right) \right\}.
$$

where the functions $\Lambda_{i}, i = \frac{1}{\mu}$ guarantee strict increasing of the system (7) solutions for all $j \geq 1$ (the concrete choice of the functions $\Lambda_{j}$ depends on hypothesis available for $f, \varphi$ and their derivatives, $\Lambda_{1}(L_{1}, j) = L_{1} + 1$ for instance). The (7) defines dynamics of the guess estimates $L_{1}, i = \frac{1}{\mu}$. In (8) the parameters of (5) are derived, the equation (9) estimates the finite time of convergence if the sample parameters in (8) are chosen correctly. The simulation should be performed on the interval $[t_{j}, t_{j+1})$, where the instant of time $t_{j+1}$ is defined in (10),
it is the instant of the current values $\hat{L}_i^j$, $i = \frac{j}{2}$ falsification (if the constants $\hat{L}_i^j$, $i = \frac{j}{2}$ have been chosen correctly, then $t_{j+1} = +\infty$).

**Theorem 4:** Let $\hat{f}(t) = f(t) + \varphi(t)$, where $f : R \to R$, $\varphi : R \to R$ are two times continuously differentiable signals and there exist some $L_1 \in R_+$, $i = \frac{j}{2}$ and $\lambda_1 \in R_+$, $i = \frac{j}{2}$ such that $|f'(t)| \leq L_1$, $|f''(t)| \leq L_2$ for all $t \in R_+$, $i = \frac{j}{2}$, $|\varphi'(t)| \leq \lambda_1$, $|\varphi''(t)| \leq \lambda_2$ for all $t \in R_+$.

Then for the system (5) and the algorithm (7)-(10) for any $\nu > 0$ and $L_1 \geq 0$, $i = \frac{j}{2}$ there exists $T_0 > 0$ such that

$$|x_1(t) - f(t)| \leq \lambda_1, \quad |x_2(t) - f(t)| \leq \lambda_1$$

for all $t \geq T_0$ providing that the discrete systems (7) have strictly increasing solutions for any $j \geq 1$.

The result of theorem 4 implies that the algorithm (5), (7)-(10) for the parameters $\alpha, \beta, \chi$ calculation provides an estimate on the derivative $f'$ in finite time independently on $\hat{L}_i$, $i = \frac{j}{2}$. For the differentiator [12] a similar adaptation problem has been solved in [9], where a conventional continuous time tuning technique is used (finiteness of the adjusting parameters is not guaranteed).

**Example 3:** Let $f(t) = et + \sin(\omega_1 t) + b \sin(\omega_2 t)$, $e = 1$, $\omega_1 = 0.5$, $\omega_2 = 2$, $b = 0.3$, $\varphi(t) = r \sin(\omega_3 t)$, $r = 0.1$. Let $\alpha = 2$, $L_1 = 0.5$ and $\lambda_1(L, j) = L + 1$, $i = \frac{j}{2}$, the corresponding trajectories of the differentiator (5) with the hybrid adaptation algorithm (7)-(10) for the cases $\omega_3 = 5$ (dashed lines) and $\omega_3 = 15$ (solid lines) are shown in Fig. 4. Simulations were performed on the interval $t \in [0, 100]$, step of simulation was chosen to be $10^{-3}$ for the Euler method, evolution of the error variable $e_2(t)$ is plotted in logarithmic time scale in Fig. 4(a). Growth of the parameter $\alpha_3$ is shown in Fig. 4(b) and the case $\alpha_3 = 5$ the algorithm needs 1 step, for the case $\alpha_3 = 15$ the algorithm stops after 3 steps, the derivative estimation was ensured after $t_1 + T_0^1 = 0.439$ and $t_2 + T_0^2 = 1.258$ respectively.

The asymptotic chattering depicted in Fig. 4(b) is caused by Euler algorithm used for computation of the system (5) solutions with the step of discretization $10^{-3}$. Thus during simulation the condition (10) was replaced with $t_{j+1} = \arg \max_{t 
abla \hat{f}(t), \hat{f}(t) > \varepsilon}$ and $\varepsilon = 0.001$. Note, that $f(t)$ is not bounded in this example. \( \square \)

**V. CONCLUSION**

The proposed hybrid differentiator ensures finite-time exact observation of the derivative $f'$ for any signal $f$ (not necessarily bounded) with bounded derivatives $f$, $f''$. The solutions of the differentiator stay bounded even for wrongly chosen parameters and non differentiable noise. If the noise amplitude is comparable with the values of parameters $\alpha, \beta, \chi$, then the derivative estimation error stays proportional to $|\varphi|^{0.25}$. The hybrid algorithm adaptation is proposed for the differentiator parameters adjustment providing the derivative $f'$ estimation uniformly in the norms of $f'$ and $f''$.

An advantage of the differentiator (1) with respect to the standard super-twisting differentiator [12] consists in the Lyapunov function existence (that facilitates performance analysis in the presence of noise and allows us to evaluate the time of convergence). The disadvantages include the requirement on boundedness of the first two derivatives (in [12] only the second derivative has to be bounded) and the conservatism of differentiation accuracy, that is proportional to the noise magnitude in the power $1/4$ (in [12] this power is $1/2$).

**APPENDIX**

**Proof of theorem 1:** Consider for the system (3) the following Lyapunov function:

$$W(e) = \Gamma(e)|e_1| + 0.5\varepsilon_2$$

where $\varepsilon_2 = 0.5$ and $\epsilon \equiv 0.5$. Then

$$\varepsilon_2 \leq \frac{\gamma}{\sqrt{\epsilon}}$$

The function $W$ is continuous (an example of its contour plot is given in Fig. 5, but not continuously differentiable. For $|e_2| \geq \sqrt{\epsilon}/|e_1|$ we have $W(e) = \Gamma(e)|e_1| + 0.5\varepsilon_2$ and

$$W = \alpha \Lambda(e) \sqrt{|e_1|} + \Lambda(e) < \sqrt{|e_1|}$$

where $\theta > 0$ is the design parameter. From the function $\Gamma$ definition $\delta \leq \Gamma(e) \leq \kappa$ for all $e \in R^2$

$$\delta \leq \varepsilon_2 \leq \frac{\gamma}{\sqrt{\epsilon}}$$

The case $|e_2| < \sqrt{\epsilon}/|e_1|$ is more complicated

$$W(e) = \frac{1}{2} \epsilon_1^{\gamma - 1} - \epsilon_1^{\gamma - 1} \epsilon_2 + \epsilon_2^{\gamma - 1}$$

From $\sqrt{|e_1|} \sqrt{|e_1|} |e_2| \leq 0.5 \epsilon_1^{\gamma - 1} \epsilon_2 + 0.5 \epsilon_1^{\gamma - 1}$

$W \leq 0.25\epsilon_1^{\gamma - 1} - \epsilon_1^{\gamma - 1} \epsilon_2 + 0.5 \epsilon_1^{\gamma - 1}$

From some $\rho > 0$, to be specified later, choose $\alpha = \{\rho + (\kappa + \chi) \theta^{\gamma - 1} \theta^{-(\kappa + \chi) \theta^{\gamma - 1}} \} / (1.5 \delta + 0.5 \kappa)$, then $\gamma = \alpha \theta^{\gamma - 1} + (\theta + 0.5 \alpha) \theta^{-\gamma - 1} - \epsilon_2^{\gamma - 1} - 0.5 \epsilon_2^{\gamma - 1}$

Let the constant $|e_1| \leq \theta \rho \kappa - 1$ hold. Then

$$W \leq -0.25 \rho \theta \sqrt{|e_1|} \leq -0.25 \rho \theta \sqrt{|e_1|}$$

and combining it with the estimate computed for the case $|e_2| \geq \sqrt{\epsilon}/|e_1|$ we finally obtain:

$$W \leq -\mu \sqrt{|W|}, \quad \mu = \min(\alpha \epsilon^{\gamma - 1} \epsilon_2 + 0.5 \epsilon_1^{\gamma - 1})$$
that gives upper estimate for the time of convergence $T_0$. Since
\[ \delta |e_1(t)| \leq |e_1(t)| + 0.5 e_2^2(t) \leq W(e(t)) \leq W(e(0)) \leq \kappa |e_1(0)| + 0.5 e_2^2(0) \]
for the initial conditions $e(0) \in \Omega_0$, $\Omega_0 = \{ e \in R^2 : \kappa |e_1(0)| + 0.5 e_2^2(0) \leq \rho \theta \kappa (\kappa - \delta)^{-1} \}$ the constraint $\delta |e_1(t)| \leq \rho \theta \kappa (\kappa - \delta)^{-1}$ holds for all $t \geq 0$ and the derived estimates are valid.

The last thing to do is optimize values of the parameters $\theta$ and $\rho$. The value of $\mu$ is not changing if $0.25 |x| \sqrt{\kappa} = \sqrt{2} \kappa$, then $\rho = 4 \sqrt{2} \kappa$. The function $\kappa$ reaches for its minimum $\alpha = 2 \sqrt{2} \kappa \kappa (\kappa - \delta)^{-1}$, this estimate is satisfied if $0.5 |L_1| \leq 4 \sqrt{2} \kappa$, that gives the proposed choice of $\alpha$ admissible for any nonnegative $\kappa$. In accordance with theorem 1 result the function
\[ \mu = \min \left\{ \kappa - L_1 - L_2 - 2 \kappa \right\} \]

The first term under the minimum sign is an increasing function of $\beta$. If we are able to show that for the minimum value $L_1 + L_2 + 3 \kappa$ of $\beta$ the first term is always bigger than $\sqrt{2} \kappa$, then the expression for $\mu$ can be simplified
\[ \alpha(\beta - L_1 - L_2 - 2 \kappa) / (\sqrt{\kappa + L_1 + L_2 + 2 \kappa}) \]
\[ \geq 4 \left\{ \sqrt{2} L_1 + 2 L_2 + 5 \kappa \right\} \]
\[ \geq 4 \left\{ \kappa - L_1 - L_2 - 2 \kappa \right\} / (\sqrt{\kappa + L_1 + L_2 + 2 \kappa}) \]
\[ \geq 4 \left\{ \kappa - L_1 - L_2 - 2 \kappa \right\} / (\sqrt{\kappa + L_1 + L_2 + 2 \kappa}) \]
The function $(L_1 + L_2 + (2 + \sqrt{2} \kappa)) / (L_1 + L_2 + 2 \kappa)$ is strictly decreasing in $\kappa$ and its minimum is $0.25 (2 + \sqrt{2})$, therefore
\[ \alpha(\beta - L_1 - L_2 - 2 \kappa) / (\sqrt{\kappa + L_1 + L_2 + 2 \kappa}) \geq 0.25 (2 + \sqrt{2}) \]
that gives $\mu = \sqrt{2} \kappa = \sqrt{2} (0.25 \sqrt{2} L_1 + e) \gamma$ and the required upper estimate for $T_0$ follows by theorem 1 result.

Proof of corollary 1: Let us start with the second equation in the system (6), considering the Lyapunov function $U(e_2) = 0.5 e_2^2$, with the derivative $U \leq -U + 0.5 (\gamma \kappa + L_1 + L_2 + \kappa)^2 \gamma$. That gives the desired estimate. Next consider $U(e_1) = 0.5 e_1^2$
\[ U \leq -\alpha \sqrt{|e_1|} + |e_1| (e_2 + \alpha \sqrt{2L_0}) \]
Since $|e_2(t)| \leq |e_2(t)| + (\gamma \kappa + L_1 + L_2 + \kappa) \gamma$ for $|e_2(t)| + (\gamma \kappa + L_1 + L_2 + \kappa) \gamma \leq 0.5 \alpha \sqrt{|e_1|}$, we have $U \leq -0.5 \alpha \sqrt{|e_1|} < 0$, that implies the result.

Proof of theorem 2: The proof follows from theorem 1 and corollary 1 under observation that in coordinates $e_1 = x_1 - f, e_2 = x_2 - f'$ the system (5) is reduced to (3).

Proof of theorem 3: Consider for the system (6) the same Lyapunov function $W(e)$ as in the proof of theorem 1. Several cases have to be analyzed.

1. The case $|e_2| \leq \theta |e_1|$
\[ W(e) = 0.5 [\theta^2 (e_1 + (\beta + \kappa)|e_1|] + 0.5 e_2^2 \]
\[ W \geq |e_2| \theta \kappa (\kappa - \delta)^{-1} \]
\[ \leq |e_2| \kappa \theta (\kappa - \delta)^{-1} \]
Taking in mind inequalities
\[ \theta \kappa (\kappa - \delta)^{-1} \]
\[ \leq \theta \kappa (\kappa - \delta)^{-1} \]
for some $\theta > 0$ (to be specified later) we obtain
\[ W \leq 0.5 \theta (\gamma \kappa |e_1| e_1^2 + 0.5 |e_1| \gamma^2) \]
\[ \leq 0.5 \theta (\gamma \kappa |e_1| e_1^2 + 0.5 |e_1| \gamma^2) \]
\[ \leq 0.5 \theta (\gamma \kappa |e_1| e_1^2 + 0.5 |e_1| \gamma^2) \]
Let the constraint $|e_1| \leq \theta \kappa (\kappa - \delta)^{-1}$ hold, then
\[ W \leq -\rho \theta \kappa \gamma |e_1| - e_2^2 + 0.5 \theta \kappa |e_1| + 0.25 \theta \kappa |e_1| \gamma^2 \]
where we used the series of relations $e_2 e_2 \leq 2 \kappa |e_2| \gamma \kappa |e_1| \gamma^2 \leq 2 \kappa |e_1| \gamma^2$ (the last step is based on that $e_2 \leq 0$ for $|e_1| > \lambda_0$).

2. The case $|e_2| \geq \theta \kappa |e_1|, W(e) = \Lambda(e)|e_1| + 0.5 e_2^2$
\[ W = \alpha \Lambda(e)|e_1| + \Lambda(e)|e_1| - e_2^2 + 0.5 \theta \kappa |e_1| \gamma^2 \]
\[ \gamma \leq -0.5 \theta \kappa |e_1| - e_2^2 + 0.5 \theta \kappa |e_1| \gamma^2 \]
\[ \leq -0.5 \theta \kappa |e_1| - e_2^2 + 0.5 \theta \kappa |e_1| \gamma^2 \]
The properties \(|b_1| \leq \alpha \sqrt{2} \lambda_0\) and \(|b_2| \leq 2 \beta\) hold. The main issue of the last estimate is how to treat the disturbance \(d_2\) computing the required bounds on the trajectories convergence dependent on \(\lambda_0\) only. Fortunately, the disturbance \(d_2\) affects negatively on the system dynamics in two compact sets only. In Fig. 3 the partition of the planar state space of the system (6) is shown, where \(d_2 = 0\) for \(|e_1| \geq \lambda_0\) (more precisely for \(|e_1| \geq \lambda_1|\)). Since always \(|b_2| \leq \alpha \sqrt{2} \lambda_0\) by construction of \(d_2\), then \(e_2 d_2 \leq 0\) in two quadrants with \(e_1 e_2 \leq 0\). Finally, for \(|e_2| \geq 2 \beta\) the inequality \(e_2 d_2 = e_2^2 \leq 0\) is satisfied provided that always \(|d_2| \leq 2 \beta\), and \(|e_2 d_2| \leq \alpha (\lambda_0 + 6 \beta |e_2|)^2 \lambda_0\) for \(|e_2| \leq 3 \alpha \sqrt{2} \lambda_0\). Thus, appearance of the constructive amplitude 2/3 of the disturbance \(d_2\) is possible into the compact set \(\mathcal{Y} = \{ |e_1| < \lambda_0 \land \alpha (\lambda_0 + 6 \beta |e_2|)^2 \lambda_0 < |e_2| < 2 \beta \land e_1 e_2 > 0\} \) only (see Fig. 3). Consider all these subsets separately.

A. For the cases \(|e_1| \geq \lambda_0\), \(|e_2| \geq 2 \beta\) or \(e_1 e_2 \leq 0\) we have \(W \leq -\alpha \delta \sqrt{|e_1|} - \chi |e_2| + \kappa \lambda_0^2\). B. For the case \(|e_2| < 3 \alpha \sqrt{2} \lambda_0\) we obtain \(W \leq -\delta \sqrt{|e_1|} - \chi |e_2| - e_2^2 + (\alpha (\lambda_0 + 6 \beta)^2 \lambda_0^2\). Combining these estimates with the one computed for the case \(|e_2| < \theta \sqrt{|e_1|}\) we finally get:

\[
W(t) \leq \max \left\{ \max \left\{ \sqrt{W(t_0)} - 0.25 \mu (t-t_0)^2, 0 \right\}, 8 \mu^{-2} \left( 0.25 \delta + \kappa \alpha + \max \{2 \delta \theta, - \theta \} \right)^2 \lambda_0 \right\}.
\]

Finally, taking maximum over all estimates obtained for the Lyapunov function \(W\) we obtain:

\[
W(t) \leq \max \left\{ \sqrt{W(0)} - 0.25 \mu (t-t_0)^2, 0 \right\} + c_1 \lambda_0 + c_2 \sqrt{\lambda_0},
\]

where \(c_1 = \max \left\{ \frac{8 \mu^{-2} \left( 0.25 \delta + \kappa \alpha + \max \{2 \delta \theta, - \theta \} \right)^2 \lambda_0}{\theta \sqrt{|e_1|}} \right\}\)

For the trajectories into the set \(\mathcal{Y}\) we know that \(t \leq t_0 + \sqrt{\lambda_0/(\alpha \sqrt{2})}\), consequently

\[
W(t) \leq \max \left\{ \sqrt{W(t_0)} - 0.25 \mu (t-t_0)^2, 0 \right\} + \kappa \lambda_0^2 + \beta^2 / (\alpha \sqrt{2}) \sqrt{\lambda_0}.
\]

Therefore, for any \(t \geq t_0 \geq 0\) we have

\[
S(t) \leq \max \left\{ \left[ \sqrt{S(t_0) - 0.5 \mu (t-t_0)^2} \right]^2, 0 \right\},
\]

\[
S(t) = S(t) + s(t) \leq \max \left\{ \left[ \sqrt{W(t_0)} - 0.5 \mu (t-t_0)^2 \right]^2, 0 \right\} + \int_{t_0}^{t} \kappa \lambda_0 + \beta^2 d\tau.
\]

For the trajectories into the set \(\mathcal{Y}\) we know that \(t \leq t_0 + \sqrt{\lambda_0/(\alpha \sqrt{2})}\), consequently

\[
W(t) \leq \max \left\{ \sqrt{W(t_0)} - 0.25 \mu (t-t_0)^2, 0 \right\} + \kappa \lambda_0 + \beta^2 / (\alpha \sqrt{2}) \sqrt{\lambda_0}.
\]

The authors would like to thank anonymous reviewers for their helpful suggestions for improving the technical note.

ACKNOWLEDGMENT

The authors would like to thank anonymous reviewers for their helpful suggestions for improving the technical note.

REFERENCES


