Fault Detection and Isolation for Nonlinear Non-Affine Uncertain Systems via Sliding-Mode Techniques

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This paper considers the fault detection and isolation problem for nonlinear uncertain non-affine systems. The proposed approach is based on an output-feedback stabilization strategy, which exploits a Luenberger-like nonlinear observer. As a main result a residual-based fault detection and isolation approach is developed, which relies on the measurable output, some of its derivatives, which are provided exactly by a uniform high-order sliding-mode differentiator, and the observer’s state. The proposed methodology is able to detect some possible components and actuators faults, and to isolate, under some mild conditions, the actuator faults from those in components. Simulation results illustrate the feasibility of the proposed approach.

Keywords: —Fault detection and isolation, Nonlinear systems, Non-affine systems, Sliding-Modes.

1. INTRODUCTION

1.1 State of the Art and Motivation

In the last decades, the fault detection and isolation (FDI) problem has stood out due to the increasing demands for efficiency, product quality and progressing integration of automatic control systems in highcost and safetycritical process. Economy and everyday life depend on the function of large power distributed networks and transportation systems, where the faults in a single component have major effects on the availability and performance of the system as a whole. Mainly, there exist two approaches: data-based and model-based methods, a comprehensive review on the FDI process can be seen in the series of papers: Venkatasubramanian et al. (2003a), Venkatasubramanian et al. (2003b) and Venkatasubramanian et al. (2003c). However, due to the improvements in modelling and control techniques based on mathematical models the second one turns out to be more appropriated (see, e.g. Isermann (1984), Patton et al. (1989) and Chen and Patton (1999)).

In this context, the analytical redundancy schemes for FDI are basically signal processing techniques using state estimation, parameter identification, adaptive filtering and so on; all these methods based on state-space and input-output models. The residual generation is of fundamental importance in these analytical redundancy schemes; these diagnostic signals can be exploited to determine the occurrence of faults in the considered processes (see, e.g. Ding (2008)). The observer-based techniques have been widely considered for the design of residual generators (Blanke et al.,

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Moreover, different kinds of observers have been exploited for FDI, based on several and different approaches. For example, sliding-mode observers (see Edwards et al. (2000), Yan and Edwards (2007), Chen and Saif (2008), Tan and Edwards (2010), Chen and Saif (2011) and Alwi et al. (2011)), where the main idea is to use the robust intrinsic features of the sliding-mode observers against matched disturbances, and the equivalent output injection to achieve FDI. In De-Persis and Isidori (2001), a differential geometric FDI approach is presented for affine nonlinear systems under disturbances; a residual generator based on the input and output information is proposed. An adaptive FDI approach is provided by Zhang et al. (2002) for nonlinear non-affine systems based on fault approximation and a bank of fault isolation estimators. Nevertheless, the information of the whole system state is required which is not very realistic. In Kabore and Wang (2001) an FDI scheme for affine nonlinear systems is proposed based on a fault diagnosis filter. Such an approach is performed using nonlinear observer techniques (nonlinear transformation and observer design), to generate a residual signal; but disturbances are not considered. It is worth noting that, to the best of our knowledge, most of the proposed works in the literature have been devoted for affine nonlinear systems. In this sense, the consideration of nonlinear non-affine systems can be regarded as a challenge itself. In this direction, one interesting work is presented by Shields (2005) for non-affine nonlinear polynomial systems. However, to achieve the FDI task a set of 22 different conditions must be satisfied. A further work is given in Ríos et al. (2014) based on sliding-mode theory but just the fault detection problem is tackled there for non-affine nonlinear systems.

1.2 Main Contribution

Motivated in the fact that most of the FDI literature has been devoted for affine nonlinear systems, this paper contributes with:

- An intuitive FDI approach for nonlinear uncertain non-affine systems. This approach is able to detect some possible component and actuator faults, and to isolate, under some mild conditions, the actuator faults from those in components.

The proposed approach is based on the following tools:

1. A nonlinear observer-based output feedback which is applied to estimate the state of the nonlinear system in the presence of parameter uncertainties and unknown inputs (Bartolini and Punta, 2012). Such an observer does not require any transformation to be designed.

2. A residual-based FDI scheme which is developed using the available information, i.e. the output, some of its derivatives which are provided exactly by a Uniform-High Order Sliding-Mode (U-HOSM) differentiator (Angulo et al., 2013), and the estimated state.

To the best of our knowledge, the FDI problem has not been fully tackled for this type of nonlinear non-affine systems under parameter uncertainties and unknown inputs.

1.3 Structure of the Paper

In Section 2, the considered nonlinear uncertain non-affine system is introduced together with the assumptions about the uncertainties and the faults possibly affecting the system. Some previous results are briefly recalled in Section 3 for the state estimation problem. In Section 4 the proposed approach to FDI is presented. A residual-based FDI approach is developed, which exploits the available output, some of its high order time derivatives from the U-HOSM differentiators and the observer’s state. A simulation example is presented in Section 5. Finally, the paper is concluded by Section 6.

**Notation:** Let $\mathbb{R}_+$ denote the set of all real positive numbers, i.e. $\mathbb{R}_+ = [0, +\infty)$. The symbol $|\cdot|$ denotes the Euclidian norm. Denotes an $n \times n$ identity matrix with the symbol $I_n$ while an $n \times m$ zeros matrix with $0_{n \times m}$. The symbol $\xi_x(x)$ denotes the Jacobian matrix, i.e. $\frac{\partial \xi(x)}{\partial x} \in \mathbb{R}^{m \times n}$
where $\xi: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a smooth function and $x \in \mathbb{R}^n$ is a vector. Denote a sequence of integers $1, \ldots, m$ as $1, m$.

2. PROBLEM STATEMENT

Consider the following uncertain nonlinear non-affine control system affected by faults:

$$\dot{\eta} = \varphi(t, \eta, u) + \Delta \varphi(t, \eta) + \Xi(t, \eta, u),$$
$$\zeta = \rho(\eta),$$

where $\eta \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $\rho \in \mathbb{R}^p$ are the state, control and measurable output vectors, respectively; with $m \leq n$. The smooth vector field $\varphi: \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ represents the nominal part of the system which is perfectly known, $\rho: \mathbb{R}^{n} \rightarrow \mathbb{R}^{p}$ is a smooth function defined on an open set of $\mathbb{R}^n$, while the term $\Delta \varphi: \mathbb{R}_+ \times \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ collects the uncertainties and unknown inputs acting on the system. The possible faults affecting the system are described by the term $\Xi: \mathbb{R}_+ \times \mathbb{R}^{n} \times \mathbb{R}^{m} \rightarrow \mathbb{R}^{n}$.

Let us consider that it is desirable to design continuous control $u$ that is robust against some bounded uncertainties and unknown inputs. The idea is to design Sliding-Mode Control in terms of the control function derivative. Thus, the following augmented control system is introduced

$$\begin{pmatrix} \dot{\eta} \\ \dot{u} \end{pmatrix} = \begin{pmatrix} \varphi(t, \eta, u) + \Delta \varphi(t, \eta) + \Xi(t, \eta, u) \\ v \end{pmatrix},$$
$$\begin{pmatrix} \zeta \\ \omega \end{pmatrix} = \begin{pmatrix} \rho(\eta) \\ \phi(u) \end{pmatrix},$$

where $v \in \mathbb{R}^m$ is regarded as a virtual control, $\omega \in \mathbb{R}^m$ is an auxiliary measurable output in terms on the smooth function $\phi: \mathbb{R}^m \rightarrow \mathbb{R}^m$ which only depends on the known input $u$. Note that the actual control $u$, which is the integral of the high-frequency virtual control $v$, is continuous.

Define the extended state vector $x = (\eta^T, u^T)^T \in \mathbb{R}^{n+m}$, and the extended output vector $y = (\zeta^T, \omega^T)^T \in \mathbb{R}^{p+m}$. Then, the corresponding dynamics are described as follows

$$\dot{x} = A(t, x) + Bu + g(t, \eta) + d(t, x),$$
$$y = h(x),$$

where

$$A(t, x) = \begin{pmatrix} \varphi(t, \eta, u) \\ 0_{m \times m} \end{pmatrix} \in \mathbb{R}^{n+m},$$
$$B = \begin{pmatrix} 0_{n \times m} \\ I_m \end{pmatrix} \in \mathbb{R}^{(n+m) \times m},$$
$$g(t, \eta) = \begin{pmatrix} \Delta \varphi(t, \eta) \\ 0_{m \times m} \end{pmatrix} \in \mathbb{R}^{n+m},$$
$$d(t, x) = \begin{pmatrix} \Xi(t, \eta, u) \\ 0_m \end{pmatrix} \in \mathbb{R}^{n+m},$$
$$h(x) = \begin{pmatrix} \rho(\eta) \\ \phi(u) \end{pmatrix} \in \mathbb{R}^{p+m}.$$

Then, considering the uncertain nonlinear control system (1)-(2), the aim of this paper is to detect and isolate possible faults acting on the system. For fulfilling such a goal, a nonlinear observer-based output feedback (Bartolini and Punta, 2012) is applied to estimate the state of the system.

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1 The uncertainty term $g$ only depends on $t$ and $x$ since the part that corresponds to the control $u$ is perfectly known.

2 It is worth saying that the main difference between the terms $g$ and $d$ consists in the fact that the uncertainties are always present in the system model, in some sense they represent something expected, while the faults represent something unexpected that may act in the system.
(1). Then, a residual-based fault detection and isolation approach is developed using the extended 
output (2), its derivatives (provided exactly by the U-HOSM differentiator (Angulo et al., 2013)), 
and the estimated state.

To deal with this problem the following assumption is imposed on the structure of the system:

**Assumption 1:** The Jacobian matrix $h_x(x)$ has maximum rank everywhere, i.e. $\text{rank}(h_x) = p+m$, 
$\forall x \in \mathbb{R}^{n+m}$.

The previous assumption implies that both Jacobian matrices $\rho_\eta(\eta)$ and $\phi_u(u)$ have maximum 
rank everywhere, i.e. $\text{rank}(\rho_\eta) = p$, $\forall \eta \in \mathbb{R}^n$, and $\text{rank}(\phi_u) = m$, $\forall u \in \mathbb{R}^m$, respectively. Such an 
assumption allows to design an observer for system (1).

### 2.1 Description of Uncertainties and Faults

Let us consider that the term $g(t, \eta)$ may be split into two parts, i.e.

$$g(t, \eta) = g_1(t, \eta) + g_2(t), \quad (3)$$

where $g_1(t, \eta)$ describes the parameter uncertainties due to imperfections on the model, and $g_2(t)$ 
plays the role of external unknown inputs. On the other way, assume that $d(t, x)$ is described in a 
similar way$^3$, i.e.

$$d(t, x) = d_1(t, \eta) + d_2(t, x), \quad (4)$$

where $d_1(t, \eta)$ represents component faults, and $d_2(t, x)$ describes actuator faults. In order to describe 
the type of actuator faults let us consider that the vector field $A(t, x)$ may be written as $A(t, x) = 
A_1(t, \eta) + A_2(t, \eta, u)$, where $A_1$ depicts the corresponding dynamics which are totally independent 
of the control input $u$ whilst $A_2$ is the actuated dynamics.

Then, the actuator fault may be represented as $d_2(t, x) = -A_2(t, \eta, K(t) u + \psi(t))$ where $\psi(t)$ may 
describe an abrupt, intermittent or oscillatory actuator fault, and $K(t) = \text{diag}(k_j(t))$, $j = 1, m$ is 
a diagonal weighting matrix modelling the effectiveness level of the actuators, i.e. if $1 > k_j(t) > 0$, 
an actuator fault is active, and if $k_j(t) = 0$, the corresponding $j$th actuator is fault free.

In this paper, the following assumption over the uncertainties and faults is introduced.

**Assumption 2:** The uncertainties and faults satisfy the following inequalities:

$$|g_1(t, \eta)| \leq \gamma_1 |\eta|, \quad |g_2(t)| \leq \gamma_2, \quad \forall t \geq 0, \quad \forall \eta \in \mathbb{R}^n, \quad (5)$$

$$\delta_1^{-} |\eta| \leq |d_1(t, \eta)| \leq \delta_1^{+} |\eta|, \quad \delta_2^{-} \leq |d_2(t, x)| \leq \delta_2^{+}, \quad \forall t \geq 0, \quad \forall x \in \mathbb{R}^{n+m}, \quad (6)$$

where $\gamma_1$, $\gamma_2$, $\delta_1^{-}$, $\delta_1^{+}$ and $\delta_2^{-}$ are positive constants; $\gamma_1$ and $\gamma_2$ known.

Note that Assumption 2 implies that $g_1(t, \eta)$ and $d_1(t, \eta)$ are parameter uncertainties and component 
faults vanishing at the origin, i.e. $g_1(t, 0) = 0$ and $d_1(t, 0) = 0$, respectively; while $g_2(t, \eta)$ and 
$d_2(t, x)$ are unknown inputs and actuator faults that do not vanish at the origin but are bounded, 
respectively.

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$^3$Note that the fault term $d(t, x)$ is totally unknown; however, the purpose of this description is to illustrate the type of 
considered faults and how to interpret them.
3. NONLINEAR OBSERVER-BASED OUTPUT FEEDBACK

Consider the following nonlinear Luenberger-like observer (Bartolini and Punta, 2012)

$$\dot{x} = A(t, x) + Bv + P(y - h(\hat{x})),$$  \hspace{2cm} (7)

where $\hat{x} = (\hat{y}^T, \hat{u})^T \in \mathbb{R}^{n+m}$ is the estimated state, and $P \in \mathbb{R}^{(n+m) \times (k+m)}$ is a design constant matrix. Then, define the following sliding surface

$$s(t, x) = \xi(\eta) \varphi(t, \eta, u) + \Lambda \xi(\eta),$$  \hspace{2cm} (8)

where $\xi(\eta) \in \mathbb{R}^m$ is a suitable sliding output such that $\text{rank} (\xi(\eta)) = m$, $\forall \eta \in \mathbb{R}^n$, and $\Lambda = \text{diag}(\lambda_j)$, $\lambda_j > 0$, $j = 1, m$ is a constant diagonal matrix. The idea for the state estimation is to use an output-feedback stabilization strategy, exploiting the nonlinear Luenberger-like observer (7) and the fact that it is possible to design suitably control $v$ and sliding output $\xi$ such that $s(t, x) = 0$.

Remark 1: The strategy to find the suitable sliding output $\xi$ and the control $v$ for the considered system is out of the scope of this paper. However, the sliding output $\xi$ must be chosen to guarantee that the motion of system (1) on the manifold $s(t, x) = 0$, even if it is achieved asymptotically, is characterized by properties such as stability, robustness, approximability and stable zero-dynamics (see, e.g., Utkin (1992), Kravaris et al. (2004) and Ebenbauer and Allgöwer (2007)).

Thus, let us consider that both the sliding output $\xi$ and the virtual control $v$ have been already designed. Note that systems (1) and (7) are affine with respect to the control $v$. Assuming that the Jacobian matrix $\xi(\eta) \varphi_u(t, \eta, u)$ is nonsingular everywhere for (1), and since this is also satisfied for (7), then there exists a unique solution $v_*(t, y, \hat{x})$, $\forall t \geq 0$, $\forall y \in \mathbb{R}^{p+m}$, and $\forall \hat{x} \in \mathbb{R}^{n+m}$, to the following equation

$$s_t(t, \hat{x}) + s_x(t, \hat{x}) [A(t, \hat{x}) + Bw(t, y, \hat{x}) + P(y - h(\hat{x}))] = 0.$$  \hspace{2cm} (9)

Definition 1: (Bartolini and Punta, 2012) The mapping $v_*(t, y, \hat{x})$, solution to (9), is defined as the observer’s equivalent control relevant to the output $y = h(x)$.

According to Definition 1, the solutions of (1) and (7) are understood in the Filippov or a.e. sense (Filippov, 1988) for the whole semi-axis $t \geq 0$, i.e.

$$\dot{x} = A(t, x) + Bv_*(t, h(x), \hat{x}) + g(t, \eta) + d(t, x),$$  \hspace{2cm} (10)

$$\dot{\hat{x}} = A(t, \hat{x}) + Bw_*(t, h(x), \hat{x}) + P(y - h(\hat{x})).$$  \hspace{2cm} (11)

Note that the observer (7) is a perfectly known nonlinear system with available state vector $\hat{x}$ and control vector $v$. Now, the following assumptions over the nonlinear observer (7) are introduced.

Assumption 3: The control vector $v$ can be designed to reach in finite-time the observer’s sliding manifold $s(t, \hat{x}) = 0$.

Assumption 4: For every solution $\hat{x}(t)$ to (11), on $s(t, \hat{x}) = 0$, $\forall t \geq 0$, there exist a matrix $M \in \mathbb{R}^{(n+m) \times (n+m)}$, two positive scalars $\alpha_1, \alpha_2 > 0$ such that the eigenvalues of $M$ are between $\alpha_1$ and $\alpha_2$, and three positive scalars $\beta_1, \beta_2, \beta_3 > 0$ such that the first time derivative of the Lyapunov function $V_1(\hat{x}) = \hat{x}^T M \hat{x}$ satisfies

$$\dot{V}_1 \leq -\beta_1 V_1 + \beta_2 |h(x) - h(\hat{x})|^2$$

$$\leq -\beta_1 \alpha_1 |\hat{x}|^2 + \beta_3 |x - \hat{x}|^2.$$  \hspace{2cm} (12)
The previous assumptions imply that the virtual control \( v \) is able to stabilize the nonlinear observer (11), and moreover, there exists a quadratic Lyapunov function which shows that on the sliding surface \( s(t, \hat{x}) = 0 \) the nonlinear observer (11) is Input to State Stable (see, e.g. Dashkovskiy et al. (2011)) with respect to the input \( (x - \hat{x}) \).

In the following section some results given in Bartolini and Punta (2012) are established to the state estimation problem.

### 3.1 Nonlinear Observation Results

Let us consider the fault-free case, i.e. \( d(t, x) = 0 \). Taking into account that both uncertainties \( g_1(t, \eta) \) and unknown inputs \( g_2(t) \) are affecting system (1). Then, the observation error and control objective are steered to a bounded neighborhood of the origin in a finite-time, according to the following theorem.

**Theorem 1:** (Bartolini and Punta, 2012) Consider system (1) and observer (7), for which Assumptions 1 and 2, 3 and 4 hold, respectively. Assume that it is possible to find a symmetric matrix \( Q \in \mathbb{R}^{(n+m) \times (n+m)} \), such that its eigenvalues are between two positive scalars \( \mu_1, \mu_2 > 0 \); and a positive scalar \( \kappa_1 > 0 \), satisfying the following matrix inequality

\[
Q (A_x - P h_x) + (A_x - P h_x)^T Q \leq -2 \kappa_1 I. 
\]

Assume moreover that \( |s_x(t, x)| \leq D \) everywhere, for some positive constant \( D \). Provided \( \mu_1, \mu_2, \) and \( \kappa_1 \) are such that \( b_1 = \frac{\mu_1}{\mu_2} - 4 \frac{\mu_2^2}{\kappa_1 \mu_1} \gamma_1^2 - \frac{\beta_1}{\mu_1} > 0 \) and \( b_2 = \beta_1 - 4 \frac{\mu_2^2}{\kappa_1 \alpha_1} \gamma_2^2 > 0 \) with \( \gamma_1, \gamma_2, \beta_1, \beta_2, \alpha_1 \) and \( \alpha_2 \) defined by (5), (12) and Assumption 4, respectively; then \( \forall t \geq 0 \)

\[
|s(t, x(t))| \leq D \left[ \frac{\mu_2}{\mu_1} |x(0) - \hat{x}(0)|^2 + \frac{\alpha_2}{\mu_1} |\hat{x}(0)|^2 \right] \exp(-2bt) + \frac{\beta_2 \gamma_2}{b \kappa_1} (1 - \exp(-2bt)) \right]^{1/2},
\]

where \( 2b = \min (b_1, b_2) \), for every solution \((x^T, \hat{x})^T\) to (10) and (11), such that \( s(0, \hat{x}(0)) = 0 \).

**Assumption 5:** The unknown initial conditions \( x(0) \) belong to a known bounded region of the state space.

Therefore, due to Assumption 5 it is always possible to fix the initial conditions \( \hat{x}(0) \) of the observer (7) such that \( |x(0) - \hat{x}(0)| \leq \rho^* \), where \( \rho^* \) is a known constant.

**Remark 2:** Theorem 1 implies that \( |s(t, x(t))| \leq D \varepsilon^*, \forall t \geq 0 \), where \( \varepsilon^* \) is a known constant given by \( \varepsilon^* = [(\mu_2 \rho^*/\mu_1 + \alpha_2 |\hat{x}(0)|^2 / \mu_1 + \mu_2^2 \gamma_2^2 / \mu_1 b \kappa_1)]^{1/2} \), which depends on \( \gamma_1, \gamma_2, \beta_1, \beta_2, \alpha_1, \alpha_2, \mu_1, \mu_2, \kappa_1, \rho^* \), and on the initial conditions \( \hat{x}(0) \). Then, the observation error \( (x - \hat{x}) \) and the sliding output \( s(t, x) \) for every \( t \geq 0 \) are in a bounded neighborhood of the origin. Particularly, the observation error \( (x - \hat{x}) \) is in a bounded neighborhood of the origin such that \( |x - \hat{x}| \leq \varepsilon^* \).

Note that the previous results can be perfectly applied in the faulty case, i.e. \( d(t, x) \neq 0 \), if the fault \( d(t, x) \) can be bounded as in (6), and the corresponding bounds are taken into account in the Theorem 1.

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4Inequalities (12)-(13) are sufficient conditions to guarantee the stability of the estimation error in absence of measurable noise (Bartolini and Punta, 2012). The measurement noise case is not tackled in this work but an extension for Theorem 1 is possible if bounded measurement noise is considered.
4. FAULT DETECTION AND ISOLATION SCHEME

In the previous section the nonlinear observer (7) has been proposed such that, provided no faults have occurred in the system, a suitably designed controller based on the estimated state \( \hat{x} \) is able to guarantee that the sliding output \( s(t,x) \) remains bounded.

Now, let us consider the output vector \( y = h(x) \) of the system given by (2) and some structural assumptions imposed to its derivatives in order to establish later on the residual generator and the corresponding FDI approach.

**Assumption 6:** Consider the first \( k \) elements of the vector \( y = h(x) \) defined by (2), i.e. \( y_i = h_i(x) = \zeta_i = \rho_i(\eta), i = 0, \ldots, k \). Assume that \( \zeta = \rho(\eta) \) has a finite and known vector relative degree \( r = (r_1, \ldots, r_k) \) with respect to \( g(t, \eta) \) and \( d(t,x) \) (Isidori, 1996), such that for \( i = 0, k \), \( y_i \) is differentiable up to the order \( r_i \), and it holds

\[
\dot{y}_i(t) = H_{i,1}(t,x), \\
\ddot{y}_i(t) = H_{i,2}(t,x), \\
\vdots \\
y_i^{(r_i)}(t) = H_{i,r_i}(t,x,g(t,\eta),d(t,x)), \quad \forall r_i \leq n, \ i = 0, k.
\]

It is important to note that, according to Assumption 6, the effect of the uncertainties \( g(t,\eta) \) and faults \( d(t,x) \) only appears explicitly in \( y_i^{(r_i)}(t) \), \( i = 0, k \), and that each \( H_{i,r_i}, i = 0, k \), can be expressed in the following way

\[
H_{i,r_i}(t,x,g(t,\eta),d(t,x)) = H_{i,r_i}^1(t,x) + H_{i,r_i}^2(t,x) (g(t,\eta) + d(t,x)),
\]

where \( H_{1,r_i} : \mathbb{R}_+ \times \mathbb{R}^{(n+m)} \rightarrow \mathbb{R} \), and \( H_{2,r_i} : \mathbb{R}_+ \times \mathbb{R}^{(n+m)} \rightarrow \mathbb{R}^{1 \times (n+m)} \) are completely known. Then, the following two vectors are defined

\[
H^1(t,x) = \begin{pmatrix} H_{1,r_1}^1(t,x), H_{2,r_2}^1(t,x), \ldots, H_{k,r_k}^1(t,x) \end{pmatrix}^T, 
\]

\[
H^2(t,x) = \begin{pmatrix} H_{1,r_1}^2(t,x), H_{2,r_2}^2(t,x), \ldots, H_{k,r_k}^2(t,x) \end{pmatrix}^T.
\]

Define the vector \( Y(t) \) as follows

\[
Y(t) = \begin{pmatrix} y_1^{(r_1)}(t), y_2^{(r_2)}(t), \ldots, y_k^{(r_k)}(t) \end{pmatrix}^T,
\]

Note that the vector \( Y(t) \) contains the \( r_i - \)th derivatives of \( y_i(t), i = 1, k \). To make use of this information it is necessary to estimate somehow the corresponding derivatives. For this purpose, the following U-HOSM differentiator is introduced.

4.1 **U-HOSM Differentiator**

Each element of the vector \( Y(t) \) will be estimated using the following U-HOSM differentiator (Angulo et al., 2013), in the component-wise sense, i.e.

\[
\dot{\hat{\theta_j}} = -\lambda_j \theta \left[ e_y \right]^{\frac{\gamma_j}{\gamma_j+1}} - \kappa_j (1 - \theta) \left[ e_y \right]^{\frac{\gamma_j+1+\delta_j}{\gamma_j+1}} + \hat{\vartheta}_{j+1}, \quad j = 1, r,
\]

\[
\dot{\hat{\vartheta}_{r+1}} = -\lambda_{r+1} \theta \text{sign} (e_y) - \kappa_{r+1} (1 - \theta) \left[ e_y \right]^{1+\alpha},
\]

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where \( e_y = \dot{\vartheta}_1 - y \), the constants \( \lambda_j, j = \Gamma, r+1 \), are chosen according to Levant (2003), i.e. for the case that \( r+1 \leq 6 \), \( \lambda_0 = 1.1M, \lambda_5 = 1.5M^{1/2}, \lambda_4 = 2M^{1/3}, \lambda_3 = 3M^{1/4}, \lambda_2 = 5M^{1/5}, \lambda_1 = 8M^{1/6} \), with \( M \geq |y^{(r)}(t)|, \forall t \geq 0 \). The parameters \( \kappa_j, j = \Gamma, r+1 \), are chosen such that the following matrix is Hurwitz

\[
K = \begin{pmatrix}
-\kappa_1 & 1 & 0 & \cdots & 0 \\
-\kappa_2 & 0 & 1 & \cdots & 0 \\
& \vdots & \ddots & \ddots & \vdots \\
-\kappa_{r+1} & 0 & 0 & \cdots & 0
\end{pmatrix},
\]

the parameter \( \alpha > 0 \) is chosen small enough, and the function \( \theta : \mathbb{R}_+ \to \{0, 1\} \) is chosen as follows

\[
\theta(t) = \begin{cases}
0, & \text{if } t \leq T_u, \\
1, & \text{otherwise},
\end{cases}
\]

with some arbitrarily chosen \( T_u > 0 \). It has been shown by Angulo et al. (2013) that the U-HOSM differentiator provides an exact estimation of \( y^{(j)}(t), j = 0, r, \) by means of \( \vartheta_j(t), j = \Gamma, r+1 \), after a finite-time transient \( t^* \) that is upper bounded independently of the initial conditions. Note that, for simplicity, it was omitted the index \( i \) in the output \( y \), but it is clear that the U-HOSM differentiator (19)-(20) will be applied for each output \( y_i, i = \Gamma, k \), according to the corresponding relative degree \( (r_1, \ldots, r_k) \).

### 4.2 Residual-Based FDI

The fault detection approach is based on the analytical redundancy of the available information, that is the measured output \( y = h(x) \), its estimated derivatives contained in \( Y(t) \), and the estimated state of the system \( \hat{x} \).

Let \( \mathcal{F}_0 \) be the situation when no faults \( d(t, x) \) have occurred in the system (1) and \( \mathcal{F}_1 \) be the case when faults appeared in the system. Then, considering (1) and (2), the following logical statements are true

\[
\mathcal{F}_0 \iff \dot{x} = A(t, x) + Bv + g(t, \eta) \land y = h(x), \\
\mathcal{F}_1 \iff \dot{x} \neq A(t, x) + Bv + g(t, \eta) \lor y \neq h(x).
\]

Now, the measured output of the system \( y = h(x) \), given by (2), the vector \( Y(t) \) from (19) and (20), and the estimated state \( \hat{x} \) in (7), are used to construct residuals, the behaviour of which will be different under the different cases \( \mathcal{F}_0 \) and \( \mathcal{F}_1 \) allowing us to detect some faults on the system. The proposed residual is given by

\[
R(t) = Y(t) - \mathbf{H}^1(t, x) = \begin{pmatrix}
y_1^{(r_1)}(t) - H_{1, r_1}^1(t, x) \\
y_2^{(r_2)}(t) - H_{2, r_2}^1(t, x) \\
\vdots \\
y_k^{(r_k)}(t) - H_{k, r_k}^1(t, x)
\end{pmatrix},
\]

where \( Y(t) \) and \( \mathbf{H}^1(t, x) \) are defined by (18) and (16), respectively. In the following, some conditions in order to achieve fault detection under uncertainties and unknown inputs will be given considering two different cases, i.e. available and unavaiable state information, respectively.
4.2.1 Fault Detection: Available State

Let us take into account that both uncertainties \( g_1(t, \eta) \) and unknown inputs \( g_2(t) \) are affecting system (1), and they satisfy Assumption 2, i.e. \( |g_1(t, \eta)| \leq \gamma_1 |\eta| \) and \( |g_2(t)| \leq \gamma_2 \). Then, the conditions under which it is possible to ensure the fault detection are given in the following theorem.

**Theorem 2:** Consider the system (1) affected by the fault \( d(t, x) \) and the residual (23), for which Assumption 6 holds. Then, if the following condition is satisfied

\[
(\delta_1^- |\eta| + \delta_2^-) > 2 (\gamma_1 |\eta| + \gamma_2),
\]

where \( H^2(t, x) \) is defined by (17); the following logical statements are true \( \forall t \geq 0 \)

\[
|R(t)| \leq |H^2(t, x)| (\gamma_1 |\eta| + \gamma_2) \iff F_0,
\]

\[
|R(t)| > |H^2(t, x)| (\gamma_1 |\eta| + \gamma_2) \Rightarrow F_1.
\]

**Proof.** Until faults \( d(t, x) \) do not appear in the system (1), according to (23), the residual \( R(t) \) can be written as

\[
R(t) = H^2(t, x)g(t, \eta),
\]

from which, taking into account (5), it follows that \( \forall t \geq 0 \)

\[
|R(t)| \leq |H^2(t, x)| (\gamma_1 |\eta| + \gamma_2),
\]

and therefore it holds

\[
F_0 \Rightarrow |R(t)| \leq |H^2(t, x)| (\gamma_1 |\eta| + \gamma_2).
\]

When faults occur in the system (1), according to (23), the residual is given by

\[
R(t) = H^2(t, x)g(t, \eta) + H^2(t, x)d(t, x),
\]

which, taking into account (5) and (24), leads to

\[
|R(t)| > |H^2(t, x)| (\gamma_1 |\eta| + \gamma_2).
\]

Therefore, (25)-(26) hold and the theorem is proven.

It is important to highlight two points:

- The available state case is described here to show that even when the state is completely known, the residual \( R(t) \) has to overpass certain threshold, in this case \( |H^2(t, x)| (\gamma_1 |\eta| + \gamma_2) \), due to the uncertainties and unknown inputs in the system; in order to achieve fault detection.
- The condition (24) represents a fault detectability condition. Moreover, it is worth saying that this condition does not have to be tested since this one simply depicts the type of faults that can be detected, i.e. faults \( d(t, x) \) whose minimum effect satisfies (24).

4.2.2 Fault Detection: Unavailable State

Let us consider the case in which the state is not available. The information which can be used to construct the residual \( R(t) \) is the measured output \( y = h(x) \), its estimated derivatives contained in
Then, based on this information, the estimated residual is given by

\[ \hat{R}(t) = Y(t) - H^1(t, \hat{x}) = \begin{pmatrix} y_1^{(r_1)}(t) - H_{1, r_1}^1(t, \hat{x}) \\ y_2^{(r_2)}(t) - H_{2, r_2}^1(t, \hat{x}) \\ \vdots \\ y_k^{(r_k)}(t) - H_{k, r_k}^1(t, \hat{x}) \end{pmatrix}. \] (28)

Hence, it is necessary to find some conditions to guarantee the fault detection for the uncertain system with unavailable state. To fulfill this aim let us consider the following assumption over the vectors \( H^1(t, \cdot) \) and \( H^2(t, x) \) defined by (16) and (17), respectively.

**Assumption 7:** The vectors \( H^1(t, \cdot) = (H_{1, r_1}^1(t, \cdot), \ldots, H_{k, r_k}^1(t, \cdot))^T \), and \( H^2(t, x) = (H_{1, r_1}^2(t, x), \ldots, H_{k, r_k}^2(t, x))^T \) are such that

\[
\begin{align*}
|H^1(t, x) - H^1(t, \hat{x})| & \leq H_1^+ |x - \hat{x}|, \quad \forall t \geq 0, \forall x, \hat{x} \in \mathbb{R}^{n+m}, \\
|H^2(t, x)| & \leq H_2^+ |x|, \quad \forall t \geq 0, \forall x \in \mathbb{R}^{n+m},
\end{align*}
\]

where \( H_1^+ > 0 \) and \( H_2^+ > 0 \) are known constants.

Therefore, taking into account that both uncertainties \( g_1(t, \eta) \) and unknown inputs \( g_2(t) \) are affecting system (1), and they satisfy Assumption 2, i.e., \( |g_1(t, \eta)| \leq \gamma_1 |\eta| \) and \( |g_2(t)| \leq \gamma_2 \), the conditions under which it is possible to ensure the fault detection are given by the following theorem.

**Theorem 3:** Consider the system (1) affected by the fault \( d(t, x) \) and the residual (28), for which Assumptions 6 and 7 hold. Let us assume that the U-HOSM differentiator (19)-(20) has already converged in a finite-time \( t^* \). Then, if the following condition is satisfied

\[
(\delta_1^- |\eta| + \delta_2^-) > \frac{2}{H_2^+} \left[ H_2^+ (|\hat{\eta}| + \epsilon^*) \gamma_1 (|\hat{\eta}| + \epsilon^*) + \gamma_2 \right] + H_1^+ \epsilon^*,
\] (29)

where \( \epsilon^* \) is specified in Remark 2; the following logical statements are true \( \forall t \geq t^* \)

\[
\begin{align*}
|\hat{R}(t)| & \leq H_2^+ (|\hat{\eta}| + \epsilon^*) [\gamma_1 (|\hat{\eta}| + \epsilon^*) + \gamma_2] + H_1^+ \epsilon^* \Leftrightarrow \mathcal{F}_0, \\
|\hat{R}(t)| & > H_2^+ (|\hat{\eta}| + \epsilon^*) [\gamma_1 (|\hat{\eta}| + \epsilon^*) + \gamma_2] + H_1^+ \epsilon^* \Rightarrow \mathcal{F}_1.
\end{align*}
\] (30) (31)

**Proof.** Until faults \( d(t, x) \) do not appear in the system (1), according to (28), the residual \( \hat{R}(t) \) can be written as

\[ \hat{R}(t) = R(t) + [H^1(t, x) - H^1(t, \hat{x})] = H^2(t, x)g(t, \eta) + [H^1(t, x) - H^1(t, \hat{x})], \]

from which, taking into account (5), Remark 2, and Assumption 7, it is obtained that \( \forall t \geq t^* \)

\[ |\hat{R}(t)| \leq H_2^+ (|\hat{\eta}| + \epsilon^*) [\gamma_1 (|\hat{\eta}| + \epsilon^*) + \gamma_2] + H_1^+ \epsilon^*, \]

10
and therefore it holds

\[ F_0 \Rightarrow \left| \hat{R}(t) \right| \leq H_2^+ (|\hat{\eta}| + \varepsilon^*) \gamma_1 (|\hat{\eta}| + \varepsilon^*) + \gamma_2 + H_1^+ \varepsilon^* \] \hspace{1cm} (32)

When faults occur in the system (1), according to (28), the estimated residual is given by

\[ \hat{R}(t) = H^2(t,x)g(t,\eta) + H^2(t,x)d(t,x) + \left[ H^1(t,x) - H^1(t,\hat{x}) \right], \]

which, taking into account (5) and (29), leads to

\[ \left| \hat{R}(t) \right| > H_2^+ (|\hat{\eta}| + \varepsilon^*) \gamma_1 (|\hat{\eta}| + \varepsilon^*) + \gamma_2 + H_1^+ \varepsilon^*. \]

Therefore, (30)-(31) hold and the theorem is proven. \(\square\)

Remark 3: According to Theorem 3, the proposed fault detection procedure is able to detect faults satisfying the fault detectability condition (29) once the U-HOSM differentiator has converged in a finite-time \(t^*\). Note that due to the uncertainties and unknown inputs affecting the system, besides the state estimation error, the presented approach cannot discriminate between the fault-free case and the occurrence of faults \(d(t,x)\) whose minimum effect does not satisfy (29).

4.2.3 Fault Isolation: Actuator and/or Component Faults

In previous part of this section a strategy for fault detection, which is able to detect faults (either plant or actuator faults) such that the detectability condition (29) is satisfied, for the estimated state case. Clearly, the fault detection scheme is not able to distinguish between component and actuator faults.

For isolation purpose, consider the last \(m\) elements of the vector \(y = h(x)\) defined by (2), i.e. \(y_i = h_i(x) = \omega_i = \phi_i(u), i = p + 1, m\). Thus, let us define the following actuator residual

\[ R_a(t) = \begin{pmatrix} \phi_1(u) - \phi_1(\hat{u}) \\ \phi_2(u) - \phi_2(\hat{u}) \\ \vdots \\ \phi_m(u) - \phi_m(\hat{u}) \end{pmatrix}. \] \hspace{1cm} (33)

Note that \(R_a(t)\) is an available vector since for all \(t \geq 0\), the vector \(\phi(u)\) is measured and the vector \(\phi(\hat{u})\) is perfectly known. It is clear that due to the structure of the extended system (1), the uncertainties \(g_1(t,\eta)\) and unknown inputs \(g_2(t)\) do not affect the corresponding imposed dynamics for \(u\).

Thus, according to the results given by Theorem 1, the estimation error \((u - \hat{u})\), can be bounded by

\[ |u(t) - \hat{u}(t)| \leq \left[ \frac{\mu_2}{\mu_1} |u(0) - \hat{u}(0)|^2 + \frac{\alpha_2}{\mu_1} |\hat{u}(0)|^2 \right]^{1/2} \exp(-bt), \forall t \geq 0. \]

Therefore, since \(u(0)\) is known and \(\hat{u}(0)\) is a design parameter, it is always possible to fix \(\hat{u}(0)\) such that \(|u(t) - \hat{u}(t)| = 0\), for all \(t \geq 0\). Hence, the following logical statements hold

\[ |R_a(t)| = 0 \Leftrightarrow F_0 \text{ (Actuator Fault-Free)}, \] \hspace{1cm} (34)

\[ |R_a(t)| > 0 \Rightarrow F_1 \text{ (Actuator Fault)}. \] \hspace{1cm} (35)
Let us consider the results of Theorem 3 with the corresponding logical statements (30) and (31), and the logical statements (34) and (35), for the actuator residual (33). Then, it appears that four situations may take place, from which the threshold-based decision criterion in Table 1 can be derived.

<table>
<thead>
<tr>
<th>( \hat{R}(t) )(^{1,\dagger} )</th>
<th>( R_a(t) )(^{1,\dagger} )</th>
<th>Conclusion</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>( F_0 ) – Fault-free or Component Fault(^a)</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>( F_1 ) – Actuator Fault</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>( F_1 ) – Component Fault(^b)</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>( F_1 ) – Actuator Fault or Actuator and Component Fault(^b)</td>
</tr>
</tbody>
</table>

\(^{1}\)The logical 0, for \( \hat{R}(t) \) and \( R_a(t) \), means that condition (30) and (34) hold, respectively.

\(^{\dagger}\)The logical 1, for \( \hat{R}(t) \) and \( R_a(t) \), means that condition (31) and (35) hold, respectively.

\(^a\)Component fault not satisfying condition (29), see Remark 3.

\(^b\)Component fault satisfying condition (29).

Remark 4: Note also that the actuator residual \( R_a(t) \) is only sensitive to actuator faults. Due to this fact, the proposed FDI procedure can isolate three different cases (no faults, only actuator faults and only component faults). In the fourth case the proposed scheme is able to detect the actuator faults, yet it cannot assert or exclude the simultaneous occurrence of faults also in the components.

4.2.4 Residual Implementation

Theoretically, if conditions (31) and (35) are satisfied under statements given by Theorem 3 and the previous fault isolation section; the simple threshold-based decision criterion described in Table 1 could be applied to the signals \( \hat{R}(t) \) and \( R_a(t) \) in order to achieve FDI.

However, the residual \( \hat{R}(t) \) and \( R_a(t) \) may occasionally cross the zero value when there exist faults, which would produce a false alarm. Therefore, the average energy of the residual \( \hat{R}(t) \) and \( R_a(t) \) is taken over a suitable receding-horizon time interval of finite length (Ding, 2008), i.e.

\[
J_{RMS}(t) = \left( \frac{1}{\Delta T} \int_{t-\Delta T}^{t} |\hat{R}(\tau)|^2 d\tau \right)^{1/2},
\]

\[
J_{aRMS}(t) = \left( \frac{1}{\Delta_a T_a} \int_{t-\Delta_a T_a}^{t} |R_a(\tau)|^2 d\tau \right)^{1/2},
\]

where \( \Delta T \) and \( \Delta_a T_a \) are time window widths in which the corresponding signals are evaluated. Hence, according to Ding (2008), the corresponding thresholds may be defined as follows

\[
J_{th,RMS} = \sup_{d=0} \left| g \right| \leq \gamma_1 \eta + \gamma_2
\]

\[
\left| x - \hat{x} \right| \leq \epsilon^*, \quad \left| e_y \right| = 0,
\]

\[
J_{ath,RMS} = \sup_{d=0} J_{aRMS}.
\]

Note that the corresponding thresholds \( J_{th,RMS} \) and \( J_{ath,RMS} \) can be calculated since \( |\hat{R}(t)| \) and \( |R_a(t)| \) depend only on known variables, see (31) and (35). Thus, the new threshold-based decision
criterion for component and actuator fault is given as follows

\[
\begin{align*}
J_{\text{RMS}} & \leq J_{\text{th,RMS}} \quad \Leftrightarrow \quad \text{No Alarm, } \mathcal{F}_0, \\
J_{\text{aRMS}} & \leq J_{\text{ath,RMS}} \quad \Leftrightarrow \quad \text{No Alarm, } \mathcal{F}_0,
\end{align*}
\]

\[
\begin{align*}
J_{\text{RMS}} & \leq J_{\text{th,RMS}} \\
J_{\text{aRMS}} & > J_{\text{ath,RMS}} \quad \Rightarrow \quad \text{Alarm, } \mathcal{F}_1.
\end{align*}
\]

The fault isolation conclusions given in Table 1 remain the same now for \( J_{\text{RMS}} \) and \( J_{\text{aRMS}} \). Note that this implementation approach is, in some sense, robust against some measurable noises and false alarms.

It is possible to introduce another concepts like false alarm rate (FAR) and fault detection rate (FDR) in order to evaluate and improve the performance of the FD scheme in terms of the intensity of false alarms and from the fault detectability point of view, respectively. Also, in order to add robustness to the residual implementation, one can use probabilistic robustness theory. Nevertheless, these issues are out of the scope of this paper but a lot of work is available in the literature (see, e.g. Zhang et al. (2005), Ding (2008) and the references therein).

5. SIMULATION EXAMPLE

Let us consider a system quite frequently used in non-linear control literature: the double inverted pendulum connected by a spring depicted by Fig 1. This system is interesting since it may describe the behaviour of an inter-vehicle spacing regulation problem in a platoon of an automated highway system (see, e.g. Spooner and Passino (1999) and Hovakimyan et al. (2001)).

![Figure 1. Two inverted pendulums connected by a spring.](image)

The equations which describe the motion of the pendulums are the following

\[
\begin{align*}
\begin{pmatrix}
\eta_1 \\
\eta_2 \\
\eta_3 \\
\eta_4
\end{pmatrix}
& = 
\begin{pmatrix}
\alpha_1 \sin(\eta_1) + \epsilon_1 \sin(\eta_3) - \zeta_1 \eta_2 + \beta_1 + \xi_1 \tanh(u_1) \\
\alpha_2 \sin(\eta_3) + \epsilon_2 \sin(\eta_1) - \zeta_2 \eta_4 - \beta_2 + \xi_2 \tanh(u_2)
\end{pmatrix}
+ 
\begin{pmatrix}
0 \\
0
\end{pmatrix}
\Delta \varphi_1(t, \eta) \\
\Delta \varphi_2(t, \eta)
+ 
\begin{pmatrix}
\Xi_1(t, \eta, u) \\
\Xi_2(t, \eta, u)
\end{pmatrix},
\end{align*}
\]

\[
\begin{align*}
\begin{pmatrix}
\zeta_1 \\
\zeta_4
\end{pmatrix}
& = 
\begin{pmatrix}
\eta_1 \\
\eta_3
\end{pmatrix},
\end{align*}
\]

where \( \eta_1 \) and \( \eta_3 \) are the angular displacements of the pendulums 1 and 2, respectively; whilst \( \eta_2 \) and \( \eta_4 \) represent the corresponding angular velocities. The torque inputs \( u_1 \) and \( u_2 \) are applied in a non-affine fashion by a servomotor at its base, while \( \zeta_1 \) and \( \zeta_2 \) are the measurable outputs. The
terms $\Delta \varphi_i$ describe the uncertainties and $\Xi_i$ the possible faults. The parameters given in (42) are defined as:

$$\alpha_i = \frac{m_0 g}{l_i} - \beta_i \frac{k}{l_i^2}, \quad \epsilon_i = \beta_i \frac{k}{l_i^2}, \quad \varsigma_i = \frac{f_{p1}}{l_i}, \quad \beta_i = \frac{k}{l_i^2} (l - b), \quad \xi_i = \frac{u_{\text{max}}}{\gamma}, \quad i = 1, 2.$$  

The system parameters and their corresponding values are listed in Table 2.

Table 2. **Parameters of the Double Pendulum**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Description</th>
<th>Value/Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m_1$</td>
<td>Mass of the Pendulum 1</td>
<td>2[kg]</td>
</tr>
<tr>
<td>$m_2$</td>
<td>Mass of the Pendulum 2</td>
<td>2.5[kg]</td>
</tr>
<tr>
<td>$J_1$</td>
<td>Moment of Inertia 1</td>
<td>0.5[kgm$^2$]</td>
</tr>
<tr>
<td>$J_2$</td>
<td>Moment of Inertia 2</td>
<td>0.625[kgm$^2$]</td>
</tr>
<tr>
<td>$f_{p1}$</td>
<td>Rotational Friction 1</td>
<td>0.1354[Nms]</td>
</tr>
<tr>
<td>$f_{p2}$</td>
<td>Rotational Friction 2</td>
<td>0.0654[Nms]</td>
</tr>
<tr>
<td>$k$</td>
<td>Spring Constant</td>
<td>100[N/m]</td>
</tr>
<tr>
<td>$r$</td>
<td>Pendulum Height</td>
<td>0.5[m]</td>
</tr>
<tr>
<td>$l$</td>
<td>Natural Length of the Spring</td>
<td>0.5[m]</td>
</tr>
<tr>
<td>$g$</td>
<td>Acceleration</td>
<td>9.81[m/s$^2$]</td>
</tr>
<tr>
<td>$b$</td>
<td>Distance between Pendulum Hinges</td>
<td>0.4[m]</td>
</tr>
<tr>
<td>$u_{1\text{max}}$</td>
<td>Maximum Torque Input 1</td>
<td>20[Nm]</td>
</tr>
<tr>
<td>$u_{2\text{max}}$</td>
<td>Maximum Torque Input 2</td>
<td>20[Nm]</td>
</tr>
</tbody>
</table>

Note that system (42)-(43) may be easily represented in the extended form (1)-(2) with $x = (\eta_1, \eta_2, \eta_3, \eta_4, u_1, u_2)^T$ and $y = (\eta_1, \eta_3, u_1, u_2)^T$. Then, according to the output-feedback stabilization strategy, the sliding manifold is designed as

$$s(t, x) = \begin{pmatrix} \alpha_1 \sin(x_1) + \epsilon_1 c_1 x_1 + \varsigma_1 \sin(x_3) + (\epsilon_1 + \varsigma_1) x_2 + \beta_1 + \xi_1 \tanh(x_5) \\ \alpha_2 \sin(x_3) + \epsilon_2 c_2 x_3 + \varsigma_2 \sin(x_1) + (\epsilon_2 + \varsigma_2) x_4 - \beta_2 + \xi_2 \tanh(x_6) \end{pmatrix},$$

where $c_i$ and $\bar{c}_i$ are positive design gains. Then, the following control law is applied

$$v(t, y, \hat{x}) = \begin{pmatrix} \frac{\cosh^3(\hat{x}_2)}{\xi_1} (\Upsilon_1(t, y, \hat{x}) - \Gamma_1 \text{sign}(s_1(t, \hat{x}))) \\ \frac{\cosh^3(\hat{x}_2)}{\xi_2} (\Upsilon_2(t, y, \hat{x}) - \Gamma_2 \text{sign}(s_2(t, \hat{x}))) \end{pmatrix},$$

where $\Gamma_i$ are positive design gains, and the functions $\Upsilon_i$ have the following form

$$\Upsilon_1(t, y, \hat{x}) = -c_1 \bar{c}_2 x_2 - \alpha_1 \bar{x}_2 \cos(\hat{x}_1) - \epsilon_1 \bar{x}_4 \cos(\hat{x}_3)$$
$$+ (\varsigma_1 - c_1 - \bar{c}_1)(\alpha_1 \sin(\eta_1) + \epsilon_1 \sin(\eta_3) - \varsigma_1 \eta_2 + \beta_1 + \xi_1 \tanh(\hat{x}_5)), $$
$$\Upsilon_2(t, y, \hat{x}) = -c_2 \bar{c}_2 \bar{x}_4 - \alpha_2 \bar{x}_4 \cos(\hat{x}_3) - \epsilon_2 \bar{x}_2 \cos(\hat{x}_1)$$
$$+ (\varsigma_2 - c_2 - \bar{c}_2)(\alpha_2 \sin(\hat{x}_3) + \epsilon_2 \sin(\hat{x}_1) - \varsigma_2 \hat{x}_4 - \beta_2 + \xi_2 \tanh(\hat{x}_6)).$$

The standard first order sliding-mode technique guarantees $s(t, \hat{x}) = 0$ with the estimated state $\hat{x}$ taken from (7) with $Q = I_6$ and

$$P = \begin{pmatrix} 91.3502 & 8.9598 & 0 & 0 \\ 259.8379 & 162.8739 & 400 & 0 \\ 9.3207 & 84.8954 & 0 & 0 \\ 140.8835 & 263.9390 & 0 & 320 \\ 0 & 0 & 20 & 0 \\ 0 & 0 & 0 & 10 \end{pmatrix}.$$
which satisfy the conditions of Theorem 1. Consider the uncertainties $\Delta \varphi_1(t, \eta) = -0.2 \alpha_1 \sin(x_1) + 0.1 \sin(t)$ and $\Delta \varphi_2(t, \eta) = -0.2 \alpha_2 \sin(x_3) + 0.1 \cos(t)$, which describe a 20% of uncertainty in the parameters $\alpha_1$ and $\alpha_2$, respectively; and some oscillatory unknown inputs.

Note that the outputs $y_1 = x_1$ and $y_2 = x_3$ have a vector relative degree $r = (2, 2)$ with respect to $g(t, \eta)$ and $d(t, x)$. Thus, a U-HOSM differentiator is designed for each output with the following parameters: $\lambda_1 = 2M^{1/3}$, $\lambda_2 = 1.5M^{1/2}$, $\lambda_3 = 1.1M$, $\kappa_1 = 4$, $\kappa_2 = 1$, $\kappa_3 = 2$, $\alpha = 0.01$, $T_a = 1$ and $M = 3$. The following simulations have been done in Mat-Lab environment with the Euler discretization method and a sampling time equal to 0.001[sec]. The initial conditions are set on $x(0) = (0.1, 0, -0.1, 0, 0, 0)^T$ and $\dot{x}(0) = (0.1, 0.1, -0.1, 0.1, 0, 0)^T$ for the system and observer, respectively.

The observation and the differentiation errors are depicted in Fig. 2 for the fault-free case with uncertainties.

![Figure 2. Estimation Errors. In the left graph is depicted the observation error whose behaviour is according to the statements given by Theorem 1. The differentiation error given by the U-HOSM differentiator is described by the right graph. It is possible to see that the differentiator error converges to zero in a finite-time $t^* \approx 2.4[sec]$.](image)

Then, the residuals $R(t)$ and $R_a(t)$ are given by

$$R(t) = \begin{pmatrix} \dot{y}_1 - (\alpha_1 \sin(\hat{x}_1) + \epsilon_1 \sin(\hat{x}_3) - \xi_1 \hat{x}_2 + \beta_1 + \xi_1 \tanh(\hat{x}_5)) \\ \dot{y}_2 - (\alpha_2 \sin(\hat{x}_3) + \epsilon_2 \sin(\hat{x}_1) - \xi_2 \hat{x}_4 - \beta_2 + \xi_2 \tanh(\hat{x}_6)) \end{pmatrix},$$

$$R_a(t) = \begin{pmatrix} u_1 - \hat{x}_5 \\ u_2 - \hat{x}_6 \end{pmatrix}.$$  

The residual implementation described in subsection 4.2.4 is applied to the residuals $R(t)$ and $R_a(t)$ with $\Delta T = \Delta T_a = 1$. The fault-free case is depicted by Fig. 3a. Consider now that for some reason the spring suffers a fault at 5[sec], and the constant spring value reduces to $k = 50[N/m]$, i.e. a component fault. The results are described and depicted by Fig. 3b. Then, let us consider that an actuator fault representing a 10% loss of effectiveness appears at 6[sec]. The results are described and depicted by Fig. 3c. Afterwards, an actuator fault (10% loss of effectiveness) takes place at 4[sec] while the component fault in the spring (constant spring value reduces to $k = 50[N/m]$) appears at 15[sec]. The results are described and depicted by Fig. 3d.

Finally, the Fig. 4 shows the results obtained for different type of actuator faults, i.e. abrupt, intermittent, incipient and oscillatory, which are typically tested in the FDI community.
Figure 3. Residues - Fault-free and Fault Cases. Based on the statements given by Theorem 3, and some simulations, it is possible to obtain the thresholds for the corresponding residuals, see graph a) for the Fault-free case. Note that $|J_{ath,RMS}| = 1 \times 10^{-4}$ (due to sampling time is not possible to have the equality $|J_{ath,RMS}| = 0$) and that the residuals take their corresponding value after $t^* \approx 2.4$ when the U-HOSM differentiator has converged. The Component Fault case is depicted by graph b), and it is concluded by the residuals that a Component Fault has occurred (third case in the isolation Table 1). For the Actuator Fault, graph c), it is possible to see that, at the beginning of the fault occurrence, the residuals conclude that an Actuator Fault or an Actuator and Component Fault has occurred, fourth case described in the isolation Table 1; but after some seconds, the second case appears and it is concluded that an Actuator fault has occurred as it was expected. Finally, the graph d) shows the residuals for the Actuator and Component Fault (fourth case in the isolation Table 1).

6. CONCLUSIONS

This paper contributes with an FDI approach for nonlinear uncertain non-affine systems. The proposed approach is based on a nonlinear observer-based output feedback which does not require any transformation is applied to estimate the state of the nonlinear system in the presence of parameter uncertainties and unknown inputs. Then, a residual-based FDI approach is developed using the available information, i.e. the output, some of its derivatives which are provided by a U-HOSM differentiator, and the estimated state. This approach may detect some possible component and actuator faults, and to isolate them under some mild conditions. Some simulation results illustrate the feasibility of the proposed approach. Future research topics may be related to fault estimation with fault tolerant control purposes as well as the FDI problem with measurable uncertainties and/or sensor faults for nonlinear non-affine systems.
Figure 4. Residues - Actuator Faults. Four different types of actuator faults, i.e. abrupt, intermittent, incipient and oscillatory, were simulated. In each case, the actuator fault appears at 5[sec]. The results show that an Actuator Fault or an Actuator and Component Fault has occurred, and due to the behaviour of the residual $J_{RMS}$, which in this case is also sensitive to these faults, it is not possible to isolate the component fault.

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