Unmatched uncertainties compensation based on high-order sliding mode observation

Alejandra Ferreira de Loza1,*,†, Francisco J. Bejarano2 and Leonid Fridman3

1UVHC, LAMIH, F-59313 Valenciennes, France
2IPN ESIME Azcapotzalco, Sección de Estudios de Posgrado e Investigación, México
3Departamento de Control Automático, CINVESTAV-IPN, A.P. 14-740, C.P. 07000, México on leave from Facultad de Ingeniería, UNAM

SUMMARY

The problem of compensation of the effects of unmatched uncertainties/perturbations is considered. High-order sliding mode observers are employed for exact state and uncertainties/perturbations reconstruction. A sliding mode control design is proposed ensuring theoretically exact compensation of the uncertainties/perturbations for the corresponding unmatched states based on the identified perturbation values. An inverted pendulum simulation example is considered illustrating the feasibility of the proposed approach. Copyright © 2012 John Wiley & Sons, Ltd.

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1. INTRODUCTION

Motivation. Control under heavy uncertainties is one of the main problems of modern control theory. One of the most prospering control strategies insensitive with respect to uncertainties is sliding mode control (SMC) (see, e.g., [1]). This robust technique is well-known for its ability to withstand external perturbations and model uncertainties, which satisfy the matching condition [2]). This condition is met when the perturbation or parameters variation are implicit at the input channels, for example, in the case of completely actuated systems.

The SMC design methodology involves two stages: the design of a switching function which provides desirable system performance in the sliding mode and the design of the control law ensuring that the system states are driven to the sliding manifold and thus the desired performance is attained and maintained in spite of the matched uncertainties. Nevertheless, the resulting controller has some disadvantages: the necessity to measure the whole state and the lack of robustness against unmatched uncertainties.

In order to address the issue of robustness against unmatched perturbations, the main solution has been the combination of sliding mode technique with other robust strategies. In order to reduce the effects of unmatched uncertainties, a method that combines $H_\infty$ and integral SMC is proposed in [3]. The main idea is to choose such a projection matrix, ensuring not only that unmatched perturbations are not amplified, but even more, that their effects are minimized. In [4], a linear time-varying system with unmatched disturbances is replaced by a finite set of dynamic models such that each one describes a particular uncertain case; then, applying a min-max SMC, they develop an optimal robust sliding-surface design. A control scheme based on block control

*Correspondence to: Alejandra Ferreira de Loza, UVHC, LAMIH, F-59313 Valenciennes, France.
†E-mail: da_ferreira@yahoo.com

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and quasi-continuous high-order sliding modes (HOSM) techniques is proposed in [5] for control of nonlinear systems with unmatched perturbations; that method assures theoretically exact finite time tracking.

The sliding surface design for systems with unmatched uncertainties considering only output information has been considered in [6–8]. In [7], a linear matrix inequalities (LMI)-based method for designing an output feedback variable structure control system is presented. The author proposes an LMI-based sliding surface design considering $H_2$ performance. Another possible solution to overcome the full state requirement is to use an observer to estimate the state. In [6], an output robust stabilization problem for a class of systems with matched and mismatched uncertainties using sliding mode techniques is considered. The idea is to use an asymptotic nonlinear observer to estimate system states, then a variable structure controller is proposed to stabilize the system. In [9], an observer for estimating the state and the unknown inputs was proposed. In [8], an integral sliding surface is designed; once the system is steered to the sliding surface, a full order compensator is designed for the unmatched disturbance attenuation.

Contribution. In this paper, the problem of compensation of the effects of unmatched uncertainties/perturbations is considered.

In order to achieve this,

- A high-order sliding mode observer [10] is employed to reconstruct the state and identify the unknown inputs theoretically exact, and moreover, allowing to achieve the best possible observer precision [11] with respect to the sampling step or/and measurement noises.
- A sliding manifold is designed such that the system’s motion along the manifold meets the specified performance: the regulation of the nonactuated states and the theoretically exact unmatched uncertainties compensation.
- An SMC law is designed such that the system’s state is driven towards the manifold and stays there for all future time, regardless of disturbances and uncertainties.

Paper Structure. In Section 2, the problem formulation and control challenge are presented. The high-order sliding mode observer is introduced in Section 3 as well as the perturbations identification algorithm. In Section 4, an output sliding mode controller rejecting the unmatched uncertainty is presented. An inverted pendulum simulation example illustrates the performance of the robust exact unmatched uncertainties compensation controller in Section 5.

2. PROBLEM STATEMENT

Let us consider a linear time invariant system with unknown inputs

$$\dot{x}(t) = Ax(t) + Bu(t) + Dw(t),$$

$$y(t) = Cx(t),$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, and $y(t) \in \mathbb{R}^p$ ($1 \leq p < n$) are the state vector, the control, and the system output, respectively. The disturbances and system uncertainties are represented by the unknown inputs function vector $w(t) \in \mathbb{R}^q$ and $\text{rank} C = p$, $\text{rank} B = m$, and $\text{rank} D = q$.

Thus, throughout the paper, the following conditions are assumed to be fulfilled.

A1. The $(A, B)$ pair is assumed to be controllable.
A2. For $u = 0$, the system is strongly observable, or equivalently $(A, C, D)$ has no zeros.
A3. $w(t)$ and its successive derivatives up to order $\alpha + 1$ are bounded by the same constant $w^+$, that is, $\|w(t)\| \leq w^+$ and $\|w^{(i+1)}(t)\| \leq w^+$, where $i = 0, 1, \ldots, \alpha$, $\alpha \geq 0$.

Here, $\| \|$ is understood as the vector Euclidean norm.

Now, let us transform the system into a suitable regular form [12]. In this form, the system is decomposed into two connected subsystems. Consider an invertible matrix of elementary row
operations\(^*\), \(\mathcal{T} \in \mathbb{R}^{n \times n}\)
\[
\mathcal{T} = \begin{bmatrix}
    B^+ \\
    B^+ B = 0, B^+ = (B^T B)^{-1} B^T.
\end{bmatrix}
\]

Applying the coordinate transformation \([ \begin{array}{c} x_1 \\
    x_2 \end{array} \] \(\mapsto \mathcal{T} x\) to system (1) yields
\[
\dot{x}_1(t) = A_{11} x_1(t) + A_{12} x_2(t) + D_1 w(t) \tag{3}
\]
\[
\dot{x}_2(t) = A_{21} x_1(t) + A_{22} x_2(t) + D_2 w(t) + u(t), \tag{4}
\]
where \(x_1 \in \mathbb{R}^{n-m}, x_2 \in \mathbb{R}^m, D_1 \in \mathbb{R}^{(n-m) \times q}, D_2 \in \mathbb{R}^{m \times q}\).

Control Goal. The objective of this work is to design a controller to regulate the perturbed non-actuated subsystem (3) based on the measurements of the output state and the identification of the unmatched uncertainties/perturbations.

Before designing the control law, the reconstruction of the state and the unknown inputs is needed. A brief description of observation procedure is given next.

### 3. OBSERVER DESIGN

Now, to realize the reconstruction of the state, let us introduce an HOSM observer. The HOSM observer provides the theoretically exact value of the state vector and the unknown inputs identification in a finite time. Basically, the HOSM observer design consists of two stages: firstly, a Luenberger observer is used to maintain the norm of the estimation error bounded; then, by means of a differentiation scheme, the state vector is reconstructed. For further details, see [10].

Before introducing the observer, let us define the following notation: let \(f(t)\) be a vector function, \(f^{[i]}(t)\) represents the \(i\)-th anti-differentiator of \(f(t)\), that is, \(f^{[0]}(t) = f(t)\).

**Stage 1**

In order to realize the differentiation process, we need to be sure that the observation error will be bounded. Firstly, design an auxiliary dynamic system
\[
\dot{x}(t) = \bar{A} \hat{x}(t) + \bar{B} u(t) + L (y(t) - \hat{y}(t)),
\]
where \(\hat{x} \in \mathbb{R}^m\) is an auxiliary state vector and \(\hat{y}(t) = C \hat{x}(t)\) and the gain \(L\) must be designed such that the matrix \(\bar{A} := (A - LC)\) is Hurwitz (notice that Assumption A2 implies that \((A,C)\) pair is observable). Let \(e(t) := x(t) - \hat{x}(t)\), whose dynamic equations are
\[
\dot{e}(t) = \bar{A} e(t) + D w(t). \tag{5}
\]

Thus, in view of Assumption A3, \(e(t)\) has a bounded norm, that is, there exists a known constant \(e^+\) and a finite time \(t_e\), such that
\[
\|e(t)\| \leq e^+, \text{ for all } t > t_e. \tag{6}
\]

**Stage 2**

This part of the state reconstruction is based on an algorithm that allows decoupling the unknown inputs from the successive derivatives of the output of the linear estimation error system \(y_e(t) := y(t) - C \hat{x}(t)\).

\[\text{By } B^+, \text{ we mean a full row rank matrix so that } B^+ B = 0; \text{ such definition is applicable to any matrix.}\]
0. Define $M_1 := C$.

1. Derive a linear combination of the output $y_e(t)$, ensuring that the derivative of this combination is unaffected by the uncertainties, that is,

$$
\frac{d}{dt} (M_1 D)^\top y_e(t) = (M_1 D)^\top C \tilde{A} e(t).
$$

Now, we form the extended vector

$$
\begin{bmatrix}
\frac{d}{dt} (M_1 D)^\top y_e(t) \\
\end{bmatrix}
\begin{bmatrix}
M_2 \\
\end{bmatrix}
\begin{bmatrix}
(M_1 D)^\top C \\
\end{bmatrix}
\begin{bmatrix}
e(t).
\end{bmatrix}
\tag{7}
$$

Then, moving the differentiation operator outside the parenthesis and defining $J_1 = (M_1 D)^\top$, the following equation is obtained

$$
\frac{d}{dt} \begin{bmatrix}
J_1 & 0 \\
0 & I_p \\
\end{bmatrix}
\begin{bmatrix}
y_e(t) \\
y_e^{[1]}(t) \\
\end{bmatrix}
= M_2 e(t),
$$
\tag{8}

where $I_p \in \mathbb{R}^{p \times p}$ is an identity matrix.

2. Derive a linear combination of $M_2 e(t)$, ensuring that the derivative of this combination is unaffected by uncertainties, that is, $\frac{d}{dt} (M_2 D)^\top M_2 e(t)$. Then, form the extended vector

$$
\begin{bmatrix}
\frac{d}{dt} (M_2 D)^\top M_2 e(t) \\
y_e(t) \\
\end{bmatrix}
\begin{bmatrix}
M_3 \\
\end{bmatrix}
\begin{bmatrix}
(M_2 D)^\top C \\
\end{bmatrix}
\begin{bmatrix}
\tilde{A} \\
\end{bmatrix}
\begin{bmatrix}
e(t).
\end{bmatrix}
\tag{9}
$$

Moving the differentiation operator outside the parenthesis from (9), we have that

$$
\frac{d}{dt} \left( (M_2 D)^\top M_2 e(t) \right) = M_3 e(t).
$$

From the aforementioned expression and from (8), and by moving the differentiation operator outside the parenthesis, we obtain

$$
\frac{d^2}{dt^2} \left( J_2 \begin{bmatrix}
J_1 & 0 \\
0 & I_p \\
\end{bmatrix}
\begin{bmatrix}
y_e(t) \\
y_e^{[1]}(t) \\
y_e^{[2]}(t) \\
\end{bmatrix}
\right) = M_3 e(t),
$$
\tag{10}

where $J_2 = (M_2 D)^\top \begin{bmatrix}
J_1 & 0 \\
0 & I_p \\
\end{bmatrix}$.

j. A general step $j (j \geq 1)$ can be summarized as follows. Derive $(M_{j-1} D)^\top M_{j-1} e(t)$. Then, from the identity

$$
\frac{d}{dt} \left( (M_{j-1} D)^\top M_{j-1} e(t) \right) = (M_{j-1} D)^\top M_{j-1} \tilde{A} e(t),
$$
\tag{11}

we obtain the expression

$$
\frac{d^{j-1}}{dt^{j-1}} \left( J_{j-1} \begin{bmatrix}
J_{j-2} & 0 \\
0 & I_p \\
\end{bmatrix}
\begin{bmatrix}
y_e(t) \\
y_e^{[1]}(t) \\
\vdots \\
y_e^{[j-1]}(t) \\
\end{bmatrix}
\right) = M_j e(t),
$$
\tag{12}

where $J_{j-1} = (M_{j-1} D)^\top \begin{bmatrix}
J_{j-2} & 0 \\
0 & I_p \\
\end{bmatrix}$. 

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Because of assumption A2, there exists a unique positive integer \( k \) such that after \( k \), steps of the algorithm \( 0 \leq k \leq n \), the matrix \( M_k \) generated recursively by (12), satisfies the conditions \( \text{rank} M_i < n \) for all \( i < k \) and \( \text{rank} M_i = n \) for all \( i \geq k \) (see, e.g., [13]). This means that the algebraic equation

\[
M_k e(t) = \frac{d^{k-1}}{dt^{k-1}} \begin{bmatrix} J_{k-1} & 0 \\ 0 & I_p \end{bmatrix} \begin{bmatrix} y_e(t) \\ y_{e}^{[k-1]}(t) \end{bmatrix}
\]

has a unique solution for \( e(t) \). Such solution may be found by premultiplying both sides of the previous equation by \( M_k^+ := (M_k^T M_k)^{-1} M_k^T \). That is

\[
e(t) = \frac{d^{k-1}}{dt^{k-1}} M_k^+ \begin{bmatrix} J_{k-1} & 0 \\ 0 & I_p \end{bmatrix} Y^{[k-1]},
\]

where \( Y^{[k-1]} = \left( (y_e(t))^T \ldots \left( y_{e}^{[k-1]}(t) \right)^T \right)^T \).

**Remark 1**

From the aforementioned expression, it is clear that \( e(t) \) can be reconstructed not in an iterative manner but in just one step using a high-order differentiation; the only matrices that should be obtained in an iterative manner are \( M_k \) and \( J_{k-1} \), which can be obtained using (11) with \( M_1 = C \).

From (13), the reconstruction of \( x(t) \) is equivalent to the reconstruction of \( e(t) \), which can be carried out by a linear combination of the output \( y_e \) and its \( (k-1) \)-th derivatives. Hence, a real time high-order sliding mode differentiator will be used in order to provide the theoretically exact observation and unknown inputs identification.

The assumption A3 allows realizing an \((\alpha + k - 1) - th\) order sliding mode differentiator, which is the highest order we can construct for this case. Beforehand, let us define

\[
H(t) := M_k^+ \begin{bmatrix} J_{k-1} & 0 \\ 0 & I_p \end{bmatrix} Y^{[k-1]},
\]

That is, from (13) and (14),

\[
e(t) = \frac{d^{k-1}}{dt^{k-1}} H(t).
\]

The HOSM differentiator is given by

\[
\begin{align*}
\dot{z}_0(t) &= -\lambda_0 \Gamma^{-\frac{1}{\alpha+1}} \Psi^\frac{1}{\alpha+1} (z_0(t) - H(t)) + z_1(t) \\
\dot{z}_1(t) &= -\lambda_1 \Gamma^{-\frac{1}{\alpha+1}} \Psi^\frac{1}{\alpha+1} (z_1(t) - \dot{z}_0(t)) + z_2(t) \\
&\vdots \\
\dot{z}_{i-1}(t) &= -\lambda_{i-1} \Gamma \Psi^0 (z_{i-1}(t) - \dot{z}_{i-2}(t)) + z_i(t) \\
\dot{z}_i(t) &= -\lambda_i \Gamma \Psi^0 (z_i(t) - \dot{z}_{i-1}(t)),
\end{align*}
\]

where \( i = \alpha + k - 1 \) is the differentiator order and \( z_i(t) \), \( H(t) \in \mathbb{R}^n \), \( \lambda_i \), \( \Gamma \in \mathbb{R} \). Consider \( \sigma = \begin{bmatrix} \sigma_1 & \ldots & \sigma_n \end{bmatrix}^T \), \( \beta \in \mathbb{R} \), the function vector \( \Psi^\beta (\sigma) \in \mathbb{R}^n \) is defined as

\[
\Psi^\beta (\sigma) = \begin{bmatrix} \sigma_1^\beta \text{sign} (\sigma_1) & \ldots & \sigma_n^\beta \text{sign} (\sigma_n) \end{bmatrix}^T.
\]

In [14], it was shown that there is a finite time \( T \) such that the identity

\[
\dot{z}_j(t) = \frac{d^j}{dt^j} H(t)
\]

is achieved for every \( j = 0, \ldots, \alpha + k - 1 \).
The values of the λ’s can be calculated as it is shown in [14]; Γ is a Lipschitz constant of $H^{(\alpha+k)}(t)$, which for our case can be calculated in the following way: from (6) and (15) $\|H^{(k-1)}(t)\| \leq e^+\|\hat{A}\|e^+ + \|B\|w^+$. In general, $e^{\alpha+1}(t)$ can be represented as a linear combination of $\{e^{(k)}, e^{(k+1)}, \ldots, e^\alpha, w, \ldots, w^{(\alpha+1)}\}$ and it can be verified that

$$\Gamma \geq \|\hat{A}\|^{\alpha+1}e^+ + \sum_{j=0}^{\alpha} \|\hat{A}\|^j \|B\|w^+. \quad (18)$$

### 3.1. State variables reconstruction

The vector $e(t)$ can be reconstructed from the $(k-1)$-th order sliding dynamics. Thus, from (17), we achieve the identity $z_{k-1}(t) = e(t)$, and consequently,

$$\hat{x}(t) := z_{k-1}(t) + \hat{x}(t) \text{ for all } t \geq T, \quad (19)$$

where $\hat{x}$ represents the estimated value of $x$. Therefore, the identity

$$\hat{x}(t) \equiv x(t) \quad (20)$$

is achieved for all $t \geq T$.

### 3.2. Uncertainties identification

Now, from error dynamics (5), we can recover $\hat{e}(t)$ using the HOSM differentiator (16). From (17), the equality $z_k(t) = \hat{e}(t)$ is achieved for all $t \geq T$ and the next equation holds

$$\hat{w}(t) = D^+[z_k(t) - \hat{A}z_{k-1}(t)], \quad (21)$$

where $\hat{w}(t)$ is the identified exact value of the unknown input $w(t)$. Thus, after a finite time $T$, when the HOSM differentiator converges (see [14]), the identity $\hat{w}(t) \equiv w(t)$ holds.

### 3.3. Precision of the observation and identification processes

Theoretically, the HOSM differentiator provides for the exact value of the state and the identified unknown input. Nevertheless, suppose that we would like to realize the HOSM observer with a sampling step $\delta$ while considering that a deterministic noise signal $\nu(t)$ (a Lebesgue measurable function of time with a maximal magnitude $\eta$) is presented in the system output. Let

$$\tilde{H}(t) = H(t) + \nu(t), \quad \|H^{(\alpha+1)}(t)\| < \Gamma, \|\nu(t)\| \leq \eta, \quad (22)$$

where $\tilde{H}$ is the signal measured online.

Now, assuming $\delta \leq k_4\Delta$ and $\eta \leq k_5\Gamma\Delta^{i+1}$ with $k_4, k_5, \Delta$ some positive constants. Then, as follows from Theorem 3.1 of [15], the error for an $(\alpha + k - 1) - th$ order HOSM differentiator, is

$$\|H^{(j)}(t) - z_j(t)\| \leq O(\Delta^{\alpha+k-j}) \text{ for } j = 0, \ldots, \alpha + k - 1. \quad (23)$$

For recovering the estimated state, $(k - 1)$ differentiations are needed. From expression (23) follows that the observation error is $O(\Delta^{\alpha+1})$.

Now, Equation (21) shows that $k$ differentiations are needed in order to recover the estimated unknown input $\hat{w}(t)$. Therefore, the identification error will be $O(\Delta^\alpha)$.

### 4. CONTROL DESIGN

The sliding surface is designed considering the estimated values of the state and the identified unknown input signal, $[\hat{x}_1 \hat{x}_2] \leftrightarrow T\hat{x}$, as follows:

$$s(t) = K\hat{x}_1(t) + \hat{x}_2(t) + G\hat{w}(t). \quad (24)$$
The matrix $K \in \mathbb{R}^{m \times (n-m)}$ could be designed to prescribe the required performance of the reduced-order system. The term $G \hat{w}(t)$ is added to compensate unmatched uncertainties. The control law is

$$u(t) = -\rho(\hat{x}) \frac{s(t)}{\|s(t)\|}.$$  \hspace{1cm} (25)

First, it is guaranteed that the aforementioned control law induces a sliding motion despite the presence of uncertainties.

### 4.1. Ideal sliding mode design

Because of (20), the identities $\dot{x}_1 = x_1, \dot{x}_2 = x_2$ are certainly obtained. Then, the time derivative of $s(t)$ is given by

$$\dot{s}(t) = \Phi x(t) + (KD_1 + D_2)w(t) + G\dot{w}(t) + u(t),$$  \hspace{1cm} (26)

where matrix $\Phi \in \mathbb{R}^{m \times n}$ is defined as $\Phi := \left[ \begin{array}{cc} KA_{11} + A_{21} & KA_{12} + A_{22} \end{array} \right] ^T$.

Choosing a Lyapunov candidate function $V(s) = \frac{s^T(t)s(t)}{2}$ and taking its derivative along time yields

$$\dot{V}(s) = s^T(t) \left( \Phi x(t) + (KD_1 + D_2)w(t) + G\dot{w}(t) - \rho(x) \frac{s(t)}{\|s(t)\|} \right) \leq -\|s(t)\| (\rho(x) - \|\Phi\| \|x(t)\| - \theta),$$  \hspace{1cm} (27)

where $\theta := \|(KD_1 + D_2)\| w^+ + \|G\| w^+$. The scalar gain $\rho(x)$ satisfies the condition

$$\rho(x) - \|\Phi\| \|x(t)\| - \theta \geq \zeta > 0,$$

where $\zeta$ is a constant.

Combining inequalities (27) and (28), it follows that the derivative of the Lyapunov function satisfies

$$\dot{V}(s) \leq -\zeta V^{\frac{1}{2}},$$

and consequently, gain $\rho(x)$ will induce the sliding motion.

### 4.2. Sliding mode dynamics

When the system reaches the sliding surface $s = 0$, we have

$$x_2(t) = -Kx_1(t) - G\dot{w}(t)$$  \hspace{1cm} (29)

$$\dot{x}_1(t) = (A_{11} - A_{12}K)x_1(t) - A_{12}G\dot{w}(t) + D_1w(t).$$  \hspace{1cm} (30)

It is well-known that the $(A_{11}, A_{12})$ pair will be controllable because the $(A, B)$ pair is also controllable [16]. Hence, there exists a matrix $K$ such that matrix $A_s := (A_{11} - A_{12}K)$ has stable eigenvalues. The $G$ gain matrix should be selected in order to compensate the unmatched uncertainties. In order to compensate $w(t)$ from $x_1(t)$, matrix $D_1$ must be matched with respect to $A_{12}$; therefore, it will be assumed that

A4. $\text{im} \,(D_1) \subset \text{im} \,(A_{12}).$

Thus, matrix $G \in \mathbb{R}^{m \times p}$ may be chosen so that

$$A_{12}G = D_1.$$  \hspace{1cm} (31)

Then, Equation (30) yields

$$\dot{x}_1(t) = (A_{11} - A_{12}K)x_1(t) + D_1(w(t) - \dot{w}(t)).$$  \hspace{1cm} (32)
so, in the ideal case after a finite time $T$, $w(t) \equiv \dot{w}(t)$; and therefore,

$$\dot{x}_1(t) = A_s x_1(t). \tag{33}$$

In particular, when $\text{rank}(A_{12}) = n - m$, matrix $G = A_{12}^+, D_1$.

Because the eigenvalues of $A_s$ have negative real part, Equation (33) is exponentially stable. Hence, the unmatched uncertainties are compensated, and coordinate $x_1$ is stabilized. The trajectories of the state $x_1$ will converge to a bounded region, that is, there exist some constants $a_1, a_2 > 0$ such that

$$\|x_1(t)\| \leq a_1 \|x_1(0)\| \exp^{-a_2 t} \quad \forall t > t_\sigma,$$

where $t_\sigma$ is the time taken to reach the sliding surface. Furthermore, $x_2(t)$ is bounded as well indeed during sliding motion. Taking the norm of Equation (29), we have

$$\|x_2(t)\| \leq \|K\| \|x_1(t)\| + \|G\| \|w^+\| \quad \forall t > t_\sigma. \tag{34}$$

From the aforementioned equation, it is clear that the trajectories of $x_2(t)$ are bounded.

4.3. Compensator realization error

Theoretically, perturbations are exactly compensated in finite time. Nevertheless, in the previous section, we discussed how the discretization affects the observation and identification accuracy. Furthermore, an additional error, because of the actuator time constant $\mu$ will cause an error of order $O(\mu)$ [17]. Now, the stabilization of the $x_1$ coordinate will be

$$\dot{x}_1(t) = A_s x_1(t) + \varepsilon, \tag{35}$$

where $A_s$ is Hurwitz and $\varepsilon = O(\mu) + O(\Delta^\alpha) + O(\Delta^\alpha+1)$. Thus, we do not stabilize the origin, but we can achieve an ultimate bound that depends on the system time constants. This bound can be improved by reducing $\mu, \Delta$, or whenever is possible, increasing the order of the differentiator.

5. SIMULATION EXAMPLE

Consider the inverted pendulum system shown in Figure 1. The system consists of a cart (e) moving along a metal guiding bar (d). A cylindrical weight (f) is fixed to the cart by an axis (g). The cart is connected by a transmission belt (c) to a drive wheel (b). The wheel is driven by a current controlled direct current motor (a), which transforms the voltage $u$ in torque such that the cart is accelerated.

Figure 1. Inverted cart pendulum system.
The state equations, considering the actuator dynamics are

\[
\dot{x} = \begin{bmatrix}
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & \frac{m_p \ell^2 g}{(m_p + m_c) I + m_p m_c \ell^2} & 0 & 0 & \frac{K_t (I + m_p \ell^2)}{(m_p + m_c) I + m_p m_c \ell^2} \\
0 & \frac{m_p \ell g}{(m_p + m_c) I + m_p m_c \ell^2} & 0 & 0 & -\frac{m_p \ell K_t}{m_p + m_c} \\
0 & 0 & 0 & 2\pi K_m \frac{I_m}{T_m} & 0 \\
\end{bmatrix} x + \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
\frac{1}{I_m} \\
\end{bmatrix}
\]

\[y = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
\end{bmatrix} x, \]

(36)

The state vector is given by \(x = [r \phi \dot{r} \dot{\phi} di dt]^T\), where \(r, \phi, i\) represent the longitudinal position of the cart, the angular position of the pendulum, and the motor current, respectively. The unknown input \(w\) is a perturbing force acting on the cart. The variable description and their corresponding values are given in Table I. For this example, we are considering an unknown input \(w = 2.5 \sin(2.2t) + 1.5\).

Observer design. First, a Luenberger-type observer is designed such that matrix \(A - LC\) has a set of eigenvalues given by \([-780, -9, -2.6, -2]\). Systems (38)–(39) is strongly observable. A way to check the system's strong observability property is to apply the unknown inputs decoupling algorithm introduced in the Observer Section. Thus, if the system is strongly observable, \(k\) iterations \((k \leq n)\) are needed to find a full column rank matrix \(M_k\). For systems (38)–(39), \(k = 2\). From (13), we need to differentiate once (i.e., \((k-1)\) times) in order to reconstruct the state. Additionally, for recovering the unknown inputs, a second differentiation is needed. The total order of the differentiator (16) is determined by the smoothness of the unknown input; we select an HOSM differentiator of the 2\(nd\) order. The input of the HOSM differentiator (14) is

\[
H(t) = \begin{bmatrix}
-0.07 & 0 & 0 & 0.45 & 0.05 & 0.48 \\
0 & 0 & 0 & 0.05 & 0.99 & -0.48 \\
-0.32 & 0 & 0 & -2.4 & 0.18 & -2.76 \\
0.76 & 1 & 0 & 4.22 & 9.16 & 3.83 \\
0.06 & 0 & 0 & 0.48 & -0.04 & 0.56 \\
\end{bmatrix} y^{[1]}.
\]

Table I. Inverted-cart pendulum system description.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>Value</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>(m_p)</td>
<td>Mass of the pendulum</td>
<td>0.36</td>
<td>(kg)</td>
</tr>
<tr>
<td>(m_c)</td>
<td>Mass of the cart</td>
<td>4</td>
<td>(kg)</td>
</tr>
<tr>
<td>(I)</td>
<td>Pendulum moment of inertia</td>
<td>0.084</td>
<td>(kg \cdot m^2)</td>
</tr>
<tr>
<td>(\ell)</td>
<td>Longitude</td>
<td>0.5</td>
<td>(m)</td>
</tr>
<tr>
<td>(g)</td>
<td>Gravitational acceleration constant</td>
<td>9.81</td>
<td>(m/s^2)</td>
</tr>
<tr>
<td>(K_t)</td>
<td>Motor torque constant</td>
<td>0.0295</td>
<td>(N \cdot m)</td>
</tr>
<tr>
<td>(I_m)</td>
<td>Motor inductance</td>
<td>0.0087</td>
<td>(H)</td>
</tr>
<tr>
<td>(K_m)</td>
<td>Motor back electromotive constant</td>
<td>0.212</td>
<td>(V/(rad/s))</td>
</tr>
<tr>
<td>(R)</td>
<td>Motor armature resistance</td>
<td>3.12</td>
<td>(\Omega)</td>
</tr>
</tbody>
</table>
Figure 2. Column (B) shows the compensated system and in column (A), the system without compensation. The underactuated states are $x_{11}, \ldots, x_{14}$, whereas the completely actuated state is $x_2$.

Following [14], we select $\lambda_0 = 1.1, \lambda_1 = 1.5, \lambda_2 = 2$. The observer gain is $\Gamma = 2.8e6$. The sampling step is $\delta = 10(\mu s)$.

Control design. Regularizing the system,

$$\dot{x}_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ -0.4377 & 0 & 0 & 0 \\ 10.6 & 10 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ 0.8345 \\ -0.8632 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} 0 \\ 0.2396 \\ -0.2479 \end{bmatrix} w$$

(38)

$$\dot{x}_2 = \begin{bmatrix} 0 & -1.332 & 0 & 0 \end{bmatrix} x_1 - 386.35u.$$  

(39)

From the aforementioned equation, it can be seen that condition $\text{im}(D_1) \subseteq \text{im}(A_{12})$ is satisfied. The compensator gain is selected as $G = 0.28$. The gain $K$ was designed by eigenvalues assignment, such that the reduced order system has eigenvalues set given by $\{-1 - 1.3 - 4.53 - 3.9\}$. From (28), the sliding mode gain is selected as $\rho(x) = 1.2e3 (\|v(t)\| + 1)$.

The simulation was carried comparing two approaches for sliding surface design. The approach in (A) was carried designing a conventional sliding mode surface (see [1]), that is, $s_A(t) = x_1(t) + Kx_2(t)$, whereas in (B), the surface was designed to cope with the unmatched perturbations (24), that is, $s_B(t) = x_1(t) + Kx_2(t) + G\dot{w}(t)$. Figure (2) shows the states of the regularized system, column (A) shows the results when no compensation is carried: the perturbation effects are present in all the states. Column (B) shows the states when the compensation of unmatched uncertainties is performed through the sliding surface; here, the stabilization of state $x_1$ is achieved, whereas the trajectories of state $x_2$ remain bounded.

6. CONCLUSIONS

An output SMC approach was designed for the systems with unmatched uncertainties/perturbations. A high-order sliding mode observer was used to reconstruct the states and perturbations. The sliding surface is designed to reject, theoretically exact, the unmatched uncertainties while stabilizing the underactuated dynamics. At the same time, the remaining state trajectories are maintained bounded.
A discontinuous control law was designed such that the state was steered to the sliding manifold and stayed there in spite of system unknown inputs. The proposed method rejects the uncertainties/perturbations presented in the underactuated system dynamics. This implies the satisfaction of assumption A4. Further analysis must be performed to relax this condition.

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