High-order sliding-mode observer for linear time-varying systems with unknown inputs

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SUMMARY

In this article, an observer for linear time variant systems affected by unknown inputs is suggested. The proposed observer combines the deterministic least squares filter and the high-order sliding-mode differentiator to provide exact state reconstruction in spite of bounded unknown inputs and system instability. The cascade structure of the algorithm provides a correct state reconstruction for the class of linear time variant systems that satisfy the structural property of strong observability. Simulations illustrate the performance of the proposed algorithm. Copyright © 2016 John Wiley & Sons, Ltd.

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1. INTRODUCTION

1.1. State of the art

The state estimation of linear time-varying (LTV) systems is a problem that has an evident increase in complexity with respect to the linear time-invariant (LTI) case. Even when the literature related to this field is not as wide as for the LTI case, several authors have studied the conditions to reconstruct the states and to design the observers, see for example [1] and the references therein. In [1], besides the study of the conditions for the state estimation, an observer for the LTV systems is presented based on the application of a dynamic gain that provides asymptotic convergence of the estimated states. An adaptive estimation approach for LTV systems providing asymptotic convergence of the estimations have been proposed in [2]. A subsequent extension of this work, presented in [3], considers the asymptotic joint state and parameter estimations problem. The design of a finite-time convergent observer is presented in [4]. Nevertheless, the proposed observers and the system analysis developed in the mentioned works were developed for systems without unknown inputs.

The state-estimation of linear systems under uncertain conditions, and in particular, in the presence of unknown inputs, has attracted the attention of the control community since the decade of the 60s. The conditions for state reconstruction were studied by Silverman [5], Molinari [6], and Hautus [7]; in these works, the important concept of strong observability was introduced. In particular, in [5] and [6], the existence of a linear relation between the state and the output and its derivatives was suggested, and an algebraic methodology to obtain the linear transformation was presented. In [8], Kratz and Liebscher extended the concept of strong observability for LTV systems, and addition-
ally, they proved the existence of a linear relation between state and the output, and its derivatives, and provided an algebraic methodology to obtain the transformation.

The deterministic least square filter (DLSF) proposed by Willems [9] allows the asymptotic reconstruction of the states by means of a conventional linear structure for a linear system with unknown inputs. The DLSF provides the optimal estimation of the state in the least square sense. Another important feature of this work is that it presents a deterministic interpretation of the Kalman–Bucy filter [10], but using the principle of least squares estimation. However, the DLSF does not provide an exact reconstruction of the state; it only provides the asymptotic convergence of the estimation error to a small region, minimal in the least square sense, around the origin.

A useful approach to deal with uncertain conditions is the use of robust control techniques, such as sliding modes (for example [11], [12], [13]). Sliding modes have been successfully applied for control and observation; the interested reader can be referred to the tutorials [14], [15], and [16]. The high robustness of this control technique together with its characteristic of finite time convergence to the sliding surface and high-robustness against external disturbances have motivated its recent application to solve the problem of state estimation for systems with unknown inputs. In [17], the conventional sliding modes technique is applied for the state estimation of LTI plants in the presence of unknown inputs. A variable structure system observer is proposed in [18]; in this work, the problem of state estimation of LTV systems with unknown inputs is solved; however, the study is restricted to the case of systems with relative degree one with respect to the unknown input. Furthermore, the injection of the signum function to the estimated state gives rise to a smoothness problem that, in turn, gives rise a degradation of the estimation performance.

To avoid high-frequency signals, and the necessity of filtering, the discontinuous output injection is replaced by second-order sliding mode algorithms (for example [19], [20], [21]). The second order sliding mode technique is applied for the state reconstruction of hybrid linear systems in [22]. In [23], a second-order sliding mode observer is applied to a 3-DOF helicopter obtaining states and estimation of the disturbances and uncertainties. The observer-based output feedback control of a proton exchange membrane fuel cell system was studied in [24]. The observers obtained from this technique were restricted to systems when the relative degree of the system outputs with respect to the unknown inputs must be equal to two (e.g., [25]). In [26], a second-order sliding mode technique was applied for identification of LTV systems. It considers that the system matrix is bounded, and the robustness of the sliding mode term is used to compensate the unknown input. To overcome the relative degree restriction, the robust exact sliding-mode differentiator [27] was applied in combination with linear transformation techniques. In [28], the differentiator-based observer was applied for state-based fault estimation. However, this class of observers continue to be restricted, but now to the class of stable linear systems.

The aforementioned restriction was solved by the introduction of the high-order sliding modes observers (e.g., [29–32]). The exact reconstruction provided by these algorithms allows their application to complex nonlinear behaviors. In [33], the high-order sliding mode observers are applied for the state reconstruction of an electric car. The main features of these class of observers have been exploited for robust control purposes of systems with unmatched uncertainties in [34]. In these works, the state estimation problem was solved for linear systems with unknown inputs using the concept of strong observability. Thus, the strong observability concept has been used in the LTI context; however, its application to LTV systems is limited to a few works.

Sliding modes-based techniques have been applied to solve the state estimation problem of LTV systems, in particular, for the class of linear parameter varying (LPV) systems, for example, in [35], the sliding mode observer for fault reconstruction. In [36], an interval observer is proposed using higher-order sliding-mode (HOSM) for a class of LPV systems. In [37], the super-twisting algorithm is applied for state estimation, and, using the equivalent output injection property, the sensor and actuator faults reconstruction is obtained for LPV systems. The equivalent output injection is also exploited for the sensor fault reconstruction of LPV systems in [38]. However, none of these articles considered the systemic properties of strong observability; therefore, the system structure is not utilized for the construction of observers.

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1.2. Main contribution

In this work, a novel observer structure is proposed for the LTV systems providing the exact reconstruction of the state in spite of the presence of bounded unknown inputs. The proposed observer is designed for the class of LTV systems that possess the strong observability property, and the structure of the system is exploited for the design of the observer. With this aim, the following points are presented:

- The deterministic least squares filter [9] is applied to the LTV systems with unknown inputs to provide a globally stable estimation error. It is proven that the estimation error converges to a bounded region around the origin, in the infinity norm sense.
- The algebraic methodology, presented in [8], is used to obtain the linear transformation between the state estimation error and the output estimation error and its derivatives.
- The HOSM differentiator is applied to estimate the discrepancies between the system states and the estimated states provided by the deterministic least squares filter.

As far as the authors’ knowledge, the presented here is the first methodology that provides, in a finite-time and under the structural property of strong observability, an exact estimation of the real values of the state for LTV systems affected by bounded unknown inputs, even for unstable systems.

1.3. Structure of the article

The problem of state estimation for LTV systems with unknown inputs is described in Section 2. The preliminary concepts, required to design the observer, are presented in Section 3. A motivating example is presented in Section 4; in this section, it is shown how the known techniques are not able to reconstruct the states for a general class of LTV systems with unknown inputs. A cascade observer capable to reconstruct the exact value of the states in spite of the unknown inputs is presented in Section 5. In Section 6, the proposed observer is applied in the motivating example illustrating the exact reconstruction of the states in the presence of unknown inputs. Finally, in Section 7, the concluding remarks are presented.

2. PROBLEM STATEMENT

Let consider the LTV system with unknown inputs defined on the time interval $\mathcal{T} \subset \mathbb{R}_+$:

\[
\begin{align*}
\dot{x}(t) &= A(t)x(t) + B(t)u(t) + D(t)\xi(t), \\
y(t) &= C(t)x(t), \quad x(t_0) = x_0,
\end{align*}
\]

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^q$ is the control input vector, $y(t) \in \mathbb{R}^p$ is the output vector, $\xi(t) \in \mathbb{R}^m$ is the unknown inputs vector with $m \leq p$, and $A(t), B(t), C(t)$ and $D(t)$ are conformable known time-varying matrices.

Along this paper, the following assumptions are considered:

1. $A(t), D(t)$, and $C(t)$ are matrices of $n-2, n-2$ and $n-1$ continuously differentiable functions, respectively, and are assumed bounded, as well as their derivatives, that is,

   \[
   \|A^{(i)}(t)\| \leq k_{i1}; \quad \|D^{(i)}(t)\| \leq k_{i2}; \quad \|C^{(i)}(t)\| \leq k_{j}, \quad \forall i = 0, \ldots, n-2; \quad j = 0, \ldots, n-1,
   \]

   where $\| \cdot \|$ is any induced matrix norm.

2. The unknown input $\xi(t)$ is bounded, and its $(n-2)$-th derivatives are Lipschitz constant no greater than $\xi_i^+$, that is,

   \[
   \|\xi^{(i)}(t)\| \leq \xi_i^+, \quad \forall i = 0, \ldots, n-1
   \]

3. The triple $(A(t), D(t), C(t))$ is strongly observable on $\mathcal{T}$.

The main objective of this paper is to design an observer for the LTV systems affected by unknown inputs that allow to estimate the exact value of the systems state, preferably in finite time.
3. PRELIMINARY CONCEPTS

In this section, the concepts needed in the design of observers for LTV systems are recalled.

**Theorem 3.1 ([8, 39, 40])**

Let the given matrix functions \( A(t) \) and \( C(t) \) of the system (1) be \( n - 2 \) and \( n - 1 \) times continuously differentiable, respectively, on the non-degenerate time interval \( \mathcal{T} \). The observability matrix is defined by

\[
\Omega_{(A,C),n}(t) = \begin{bmatrix}
N_0(t) \\
N_1(t) \\
\vdots \\
N_{n-1}(t)
\end{bmatrix} \in \mathbb{R}^{pn \times n},
\]

where \( N_0(t) = C(t) \) and \( N_i(t) = N_{i-1}(t)A(t) + \frac{dN_{i-1}(t)}{dt} \) for \( i = 1, \ldots, n - 1 \). Then, the pair \((A(t), C(t))\) is observable on the time interval \( \mathcal{T} \) if and only if \( \operatorname{rank}(\Omega_{(A,C),n}(t)) = n \), for all \( t \in \mathcal{T} \).

**Definition 3.1 ([39])**

Let the observability index \( l_o \) be the minimum integer such that

\[ \operatorname{rank}(\Omega_{(A,C),l_o}(t)) = n, \text{ for all } t \in \mathcal{T} \]

**Definition 3.2 ([8])**

The triplet \((A(t), C(t), D(t))\) is called strongly observable in the non-degenerate interval \( \mathcal{T} \), if \( x(t) = A(t)x + D(t)\xi(t), C(t)x(t) \equiv 0 \), for some unknown input \( \xi(t) \), with \( D(t)\xi(t) \) being a continuous function, implies that \( x(t) \equiv 0 \), for all \( t \in \mathcal{T} \).

**Theorem 3.2 ([8])**

Let the elements of matrices \( A(t), D(t), \) and \( C(t) \) be \( l_o - 2, l_o - 2, l_o - 1 \) times continuously differentiable, respectively, in the time interval \( t \in \mathcal{T} \), and define the matrices \( \mathcal{D}_{\mu,\nu} = \mathcal{D}_{(A,C,D),\mu,\nu}(t) \), recursively by

\[
\mathcal{D}_{\mu,\mu-1} := C(t)D(t) \quad \text{for } 2 \leq \mu \leq l_o,
\]
\[
\mathcal{D}_{\mu,1} := N_{l_o-2}D(t) + \frac{d\mathcal{D}_{\mu-1,1}}{dt} \quad \text{for } 3 \leq \mu \leq l_o,
\]
\[
\mathcal{D}_{\mu,\nu} := \mathcal{D}_{\mu-1,\nu-1} + \frac{d\mathcal{D}_{\mu-1,\nu}}{dt} \quad \text{for } 3 \leq \nu < \mu \leq l_o.
\]

where \( N_1(t) = N_{l_o-1}(t)A(t) + \frac{dN_{l_o-2}(t)}{dt} \) and \( l_o \) are the observability index from Definition 3.1. Define the matrix functions \( S : \mathcal{T} \to \mathbb{R}^{p l_o \times (l_o - 1) n} \) and \( S^* : \mathcal{T} \to \mathbb{R}^{(p l_o - n) \times (n - (l_o - 1)n)} \) as

\[
S(t) := \begin{bmatrix}
\Omega_{(A,C),l_o}(t) & \mathcal{J}_{(A,C,D),l_o}(t)
\end{bmatrix},
\quad S^*(t) := \begin{bmatrix}
I_n \\
\Omega_{(A,C),l_o}(t) & \mathcal{J}_{(A,C,D),l_o}(t)
\end{bmatrix},
\]

with

\[
\mathcal{J}_{(A,C,D),l_o}(t) := \begin{bmatrix}
0 & 0 & \cdots & 0 \\
0 & \mathcal{D}_{2,1} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & \mathcal{D}_{l_o,1} & \cdots & \mathcal{D}_{l_o,l_o-1}
\end{bmatrix},
\]

and \( I_n \) the \( n \times n \) identity matrix, where the matrix \( \Omega_{(A,C),l_o} \) is defined in Equation (2) with \( n = l_o \).

Then the triplet \((A(t), C(t), D(t))\) is strongly observable on \( \mathcal{T} \) if and only if

\[ \operatorname{rank}(S(t)) = \operatorname{rank}(S^*(t)) \]

for all \( t \in \mathcal{T} \).
Corollary 3.1 ([8])
Assume that the matrices $A(t)$, $D(t)$, and $C(t)$ are $l_o - 2$, $l_o - 2$, and $l_o - 1$ times continuously differentiable, respectively, on the time interval $\mathcal{T}$; suppose that $D(t)\xi(t)$ is continuously differentiable and $y(t)$ is $l_o - 1$ continuously differentiable on $\mathcal{T}$.
Let $K(t) \in \mathbb{R}^{p_l \times p_l}$ such that $\ker K(A,C,D)(t) = \text{Im} J(A,C,D)(t)$. Define

$$
\mathcal{H}(A,C,D)(t) = \Phi^T(A,C),l_o(t)K^T(A,C,D)(t)K(A,C,D)(t)\Phi(A,C),l_o(t).
$$

Then $\mathcal{H}(A,C,D)(t)$ is invertible, and

$$
x(t) = \mathcal{H}^{-1}(A,C,D)(t)\Phi^T(A,C),l_o(t)K^T(A,C,D)(t)K(A,C,D)(t)\hat{y}(t)
$$

with $\hat{y}(t) = [y^T(t), \ldots, y^{(l_o-1)}T(t)]^T$ for all $t \in \mathcal{T}$.

### 3.1. High-order sliding-mode differentiator

The HOSM differentiator [27] provides exact convergence of the estimated derivatives to its real values after a finite-time transient. In this section, $z_i$, $v_i$ are used to denote the scalar differentiator variables. Let $g_0$ be the function to be differentiated, under the assumption that a constant $\Gamma$ exists such that $|g_0^{(r+1)}(t)| \leq \Gamma$, the $r$-th order differentiator can be expressed in the following form:

$$
\dot{z}_0 = v_0 = z_1 - \kappa_1|z_0 - g_0(t)|^{\frac{r-1}{r}} \text{sign}(z_0 - g_0(t)),
$$

$$
\dot{z}_1 = v_1 = z_2 - \kappa_2|z_1 - v_0|^{\frac{r-1}{r}} \text{sign}(z_1 - v_0),
$$

$$
\vdots
$$

$$
\dot{z}_i = v_i = z_{i+1} - \kappa_{r-1}|z_i - v_{i-1}|^{\frac{r-1}{r}} \text{sign}(z_i - v_{i-1}),
$$

$$
\vdots
$$

$$
\dot{z}_r = -\kappa_0 \text{sign}(z_r - v_{r-1});
$$

for suitable positive constant coefficients $\kappa_i$ to be chosen recursively large in the given order.

The solutions of the differentiator (4) and the proposed observer are understood in Filippov’s sense [41], and this assumption is made in order to allow for discontinuous signals in the system and observer. It is important to remark that Filippov’s solutions are equivalent to the usual ones, when the right-hand side of (1) is continuous. It is also assumed that all the inputs allow the existence and solution’s extension for all the semi-axis $t \geq 0$.

A possible selection of the differentiator parameters is $\kappa_0 = 1.1\Gamma$, $\kappa_1 = 1.5\Gamma^{1/2}$, $\kappa_2 = 2\Gamma^{1/3}$, $\kappa_3 = 3\Gamma^{1/4}$, $\kappa_4 = 4\Gamma^{1/5}$, $\kappa_5 = 5\Gamma^{1/6}$ that are valid for $r \leq 5$. The following equalities are true after a finite time transient process in the absence of noise ([27]):

$$
|z_i - g_0^{(i)}(t)| = 0, \quad i = 0, \ldots, r.
$$

The notation $D^i_z[\cdot]g_0$ is used to represent the signal $z_i$ correspondent to the application of a differentiator of order $r$ to the signal $g_0$.

### 4. MOTIVATING EXAMPLE

In this section, a motivating example is included with the aim of illustrate the problems related to the separately application of the least square filter and the HOSM differentiator to the state observation problem of LTV systems with unknown inputs.
Consider the LTV model of an aircraft proposed by [42]

\[
\begin{align*}
\dot{x}(t) &= A(t)x(t) + B(t)u(t) + D(t)\zeta(t), \\
y(t) &= Cx(t),
\end{align*}
\]

(6)

where \(x_1, x_2, x_3, x_4\) are the \(x\)-axis velocity, the angle of attack, the pitch angle, and the pitch rate of the aircraft, respectively. The input \(u(t)\) is the elevator angle. For simulation purposes, the initial conditions are assumed as

\[
x(0) = \begin{bmatrix} 0.1 & -0.5 & -0.1 & 0.2 \end{bmatrix}^T,
\]

a null control input is considered, that is, \(u(t) = 0\), and the unknown input is taken as

\[
\zeta(t) = \sin(6\pi \cos(t)) + 0.5.
\]

The time-varying matrices in the model are given by

\[
A(t) = E^{-1}_c(t)Q_c(t), \quad B(t) = E^{-1}_c(t)R_c(t), \quad D(t) = B(t),
\]

\[
E_c(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & V(t) & 0 & 0 \\
0 & -M_\alpha(t) & 1 & \end{bmatrix}, \quad R_c(t) = \begin{bmatrix} 0 \\
0 \\
Z_\delta(t) \\
M_\delta(t) \end{bmatrix},
\]

\[
Q_c(t) = \begin{bmatrix} X_u(t) & -g \cos(\Theta_0) & X_a(t) & 0 \\
0 & 0 & 0 & 1 \\
Z_u(t) & -g \sin(\Theta_0) & Z_a(t) & V(t) + Z_q(t) \\
-M_\alpha(t) & 0 & M_\alpha(t) & M_q(t) \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \end{bmatrix}
\]

With the time-varying parameters defined as

\[
X_u(t) = \frac{bV(t)S(C_{xu} + 2C_L \tan(\Theta_0))}{2m},
\]

\[
Z_u(t) = \frac{bV(t)S(C_{zu} + 2C_L)}{2m},
\]

\[
M_u(t) = \frac{bV(t)Sc_{Cmu}}{2I_{yy}}, \quad X_a(t) = \frac{bV^2(t)SC_{xa}}{2m},
\]

\[
Z_a(t) = \frac{bV^2(t)SC_{za}}{2m}, \quad M_a(t) = \frac{bV^2(t)Sc_{Cma}}{2I_{yy}},
\]

\[
M_q(t) = \frac{bV(t)Sc_{Cm\dot{a}}}{4I_{yy}},
\]

\[
Z_q(t) = \frac{bV(t)Sc_{C\dot{z}q}}{4m}, \quad M_q(t) = \frac{bV(t)Sc_{Cmq}}{4I_{yy}},
\]

\[
Z_\delta(t) = \frac{bV^2(t)SC_{z\delta e}}{2m}, \quad M_\delta(t) = \frac{bV^2(t)Sc_{Cm\delta e}}{2m},
\]

where \(V(t) = 3(\sin(t) + 49)\) is the flight velocity, \(m\) is the aircraft’s mass, \(S\) is the main wing area, \(c\) is the main wing chord, \(b\) is the main wing span, and \(C_L\) is the lift coefficient. The other parameters are the non-dimensional stability and control derivatives. For more information about these parameters, consult [42]. The numerical values were taken from [43] for an A-7A aircraft.

The system state vector is shown in Figure 1, where it is important to remark the unstable behavior of the system. Given that the system is linear, several techniques for state reconstruction can be applied to solve the state observation problem. However, the existence of unknown inputs increases the complexity of the problem; hence, the performance of the existent techniques is affected, impeding the obtaining of the same results than in the nominal case. The main question that arises is: can the existent methods estimate the states of an LTV system with unknown inputs with the same accuracy than in the LTV nominal case?
In order to introduce one of the components of the proposed methodology, the deterministic least squares filter [9] is presented here. In Figure 2, the application of the DLSF for the system (6) is presented. Notice that the DLSF provided the stabilization of the estimation error to a bounded region around the origin. However, the exact value of the unknown states cannot be reached accurately, as it is shown in Figure 3.
On the other hand, the application of the HOSM differentiator is considered to reconstruct the states of the system. Notice that the observability index of this system is $l_o = 2$. The estimated states are shown in Figure 4. It is important to remark that given the unstable nature of the system, the estimation error diverges, which can be seen in Figure 5.

It is clear that the individual application of the proposed methods cannot provide an accurate reconstruction of the states in the presence of the unknown inputs. Therefore, a novel methodology is necessary to provide exact state reconstruction in the presence of unknown inputs even when the system is unstable. With this aim, in the forthcoming section, a methodology that provides and exact state estimation, in finite-time, for the LTV systems in the presence of unknown inputs is presented.

5. OBSERVER DESIGN

The observer is built in the following form:

\[ \dot{z}(t) = A(t)z(t) + B(t)u(t) + L(t)\left( y(t) - C(t)z(t) \right), \]

(7)

\[ \hat{z}(t) = z(t) + F(t)W_{o}(e_y(t)); \]

(8)

where $e_y(t) = y(t) - C(t)z(t)$ and matrix $L(t)$ is a correction factor designed as

\[ L(t) = P^{-1}(t)C^T(t), \]

(9)

the symmetric positive definite matrix $P(t) = P^T(t)$ is the solution of the differential Riccati equation

\[ \dot{P}(t) = -P(t)A(t) - A^T(t)P(t) + 2C^T(t)C(t) - Q(t), \]

(10)

with initial condition $P(t_0) = P_0$ and for some symmetric positive definite matrix $Q(t)$, the gain $F(t)$ is given by

\[ F(t) = \mathcal{H}^{-1}_{(A,C,D)}(t)e^{T}_{(A,C),l_o}(t)K^T_{(A,C,D)}(t)K_{(A,C,D)}(t) \]

(11)
Figure 4. Higher-order sliding-mode differentiator: state estimation.

Figure 5. Higher-order sliding-mode differentiator: estimation errors.
and, considering the differentiators gains as
\[
\Gamma > 2\zeta_0^+ ||N_{l_0-1}|| |D(t)||P(t)|| + \sum_{j=1}^{l_0-1} ||Q_{l_0,j}|| \zeta_{j-1}^+
\] (12)

The vector function \( W_{l_0-1}(e_y(t)) \) is computed as follows:

\[
W_{l_0-1}(e_y(t)) = \begin{bmatrix}
e_y(t)_1 \\
\vdots \\
D_{l_0-1}^1[e_y(t)_1] \\
\vdots \\
D_{l_0-1}^t[e_y(t)_1] \\
\vdots \\
D_{l_0-1}^{l_0-1}[e_y(t)_1] \\
\vdots \\
D_{l_0-1}^{l_0-1}[e_y(t)_p]
\end{bmatrix}
\] (13)

where \( e_y(t)_i \) denotes the \( i \)-th row of the vector function \( e_y(t) \). The following theorem summarizes the main contribution of this article.

**Theorem 5.1**

Let the system (1) be affected by the unknown inputs \( \zeta(t) \), satisfying Assumptions A1–A3. The observer (7)–(13) provides global exact convergence of the estimation error \( e = x - \hat{x} \) to zero after a finite-time transient, that is, \( e \to 0 \) after a finite-time transient; therefore, \( \hat{x} \to x \) after a finite-time transient.

Figure 6. Cascade observer block diagram.
**Proof**

The proposed observer is composed by three elements (Figure 6):

- An auxiliary state \( z(t) \), obtained from equation (7), that provides an inaccurate reconstruction of the state \( x(t) \), but it provides a bounded estimation error. In this equation, the time-varying gain \( L(t) \) is designed such that \( \dot{A}(t) = A(t) - L(t)C(t) \) is stable for all \( t \), in particular the implementation of the deterministic least squares filter [9] is proposed in (9)–(10) providing the convergence of the estimation error to a bounded region around the origin.
- A differentiator, presented in (13), that provides after a finite-time transient the exact reconstruction of the derivatives up to \( l_o \) of difference between the output of the system \( y(t) \) and its estimated value \( C(t)z(t) \).
- Finally, an error compensation equation, given in (8), that using a projection matrix computed in (11), compensates for the discrepancies between \( x(t) \) and \( z(t) \) and provides the exact estimation of the real-state values \( x(t) \).

With this structure in mind, the proof is divided in three steps.

First, the convergence of the auxiliary estimation error \( e_z = x - z \) to a bounded region around the origin is proven. With this aim, the dynamics of \( e_z \) is given by

\[
\dot{e}_z(t) = \dot{A}e_z(t) + D(t)\xi(t),
\]

where \( \dot{A} = (A(t) - L(t)C(t)) \).

Let a Lyapunov candidate function be

\[
V(e_z(t)) = e_z^T(t)P(t)e_z(t)
\]

The following chain of equalities is satisfied:

\[
\frac{d}{dt}V(e_z(t)) = e_z^T(t)P(t)e_z(t) + e_z^T(t)\dot{P}(t)e_z(t) + e_z^T(t)P(t)\dot{e}_z(t) + e_z^T(t)P(t)e_z(t)
\]

\[
\frac{d}{dt}V(e_z(t)) = e_z^T(t)\left[P(t)(A(t) - L(t)C(t)) + (A(t) - L(t)C(t))^T P(t) + \dot{P}(t)\right]e_z(t) + 2e_z^T(t)P(t)D(t)\xi(t)
\]

If there exist symmetric positive definite matrices \( P(t) \) and \( Q(t) \) such that the following equation is satisfied

\[
\dot{P} + P(t)A(t) + A^T(t)P(t) - 2C^T(t)C(t) + Q(t) = 0
\]

Thus, the time derivative of (15) when the gain \( L(t) \) is chosen according to (9) satisfies:

\[
\frac{d}{dt}V(e_z(t)) = -e_z^T(t)Q(t)e_z(t) + 2e_z^T(t)P(t)D(t)\xi(t)
\]

Therefore, the foregoing derivative satisfies the following inequality:

\[
\frac{d}{dt}V(e_z(t)) \leq -e_z^T(t)Q(t)e_z(t), \quad \forall \|e_z\| \geq 2\zeta_0^+\|D(t)\|\|P(t)\|\|Q(t)\|\]

As a consequence, the proposed choice of the gain \( L(t) \) guarantees the global exponential convergence of the estimation error \( e_z \) to a bounded neighborhood of the origin dependent of the unknown-input bound (for a detailed analysis on this algorithm, the interesting readers can refer to [9]).

As a second step, let us prove that the estimated value of the output and its derivatives can be computed in finite time. Because of the Assumptions A1 and A2, the estimation error derivative \( \dot{e}_z(t) \) will be also uniformly bounded. Define the output error as

\[
e_y(t) = C(t)e_z(t) = y - Cz
\]

Let introduce the variable \( \tilde{e}_y \), a variable composed by the first \( l_o - 1 \) derivatives of \( e_y \), that is,

\[
\tilde{e}_y(t) = \left[ e_y^T(t), \ldots, e_y^{(l_o-1)}(t) \right]^T.
\]
A straightforward computation allows to compute the following relation:
\[
\hat{e}_y(t) = \mathcal{O}_{(\hat{A},C),I_o}(t)e_z(t) + \mathcal{J}_{(\hat{A},C),I_o}(t)\hat{\zeta}(t)
\]
where
\[
\hat{\zeta}(t) = \begin{bmatrix} \zeta^T(t), \ldots, \zeta^{(I_o-2)}T(t) \end{bmatrix}^T.
\]
Now, it is necessary to prove that the aforementioned \(I_o - 1\) derivatives of \(e_y(t)\) can be estimated using the differentiator described in Section 3.1. With this aim, let one study the \(I_o\)-th row of vector \(\hat{e}_y(t)\), it takes the form:
\[
e_{y}^{(I_o-1)}(t) = N_{I_o-1}e_z(t) + \mathcal{D}_{I_o,1}\hat{\zeta}(t) + \mathcal{D}_{I_o,2}\hat{\zeta}(t) + \ldots + \mathcal{D}_{I_o,I_o-1}\zeta^{(I_o-2)}(t)
\]
The norm of this term satisfies the following inequality:
\[
||e_{y}^{(I_o-1)}(t)|| \leq ||N_{I_o-1}|| ||e_z(t)|| + \sum_{j=1}^{I_o-1} ||\mathcal{D}_{I_o,j}|| ||\zeta^{j-1}_i||
\]
in view of the boundedness of the auxiliary estimation error \(e_z\). Then, with the proper selection of the differentiators gains according to (12), the \(I_o - 1\) derivatives of \(e_y(t)\) are computed as follows:
\[
\hat{\hat{e}}_y = W_{I_o}(e_y(t))
\]
where \(W_{I_o}(e_y(t))\) is computed according to (13). In view that Assumptions 1 and 2 are satisfied, the estimated values of \(\hat{\hat{e}}_y\) converge to the real values \(\hat{e}_y\) after a finite-time transient (the interested readers can consult [27] for more details in the prove of convergence).

As the third step of the proof, one should prove that equation (8) provides a valid reconstruction of the state \(x(t)\). To do that, one should explore the relation between \(e_z\) and \(\hat{e}_y\). Let chose the matrix \(K(\hat{A},C,D)(t) = K(t) \in \mathbb{R}^{p_o \times p_o}\) such that \(\ker K(\hat{A},C,D)(t) = \text{Im} \mathcal{J}_{(\hat{A},C),I_o}(t)\). Therefore, the following equality is satisfied:
\[
K(t)\hat{e}_y(t) = K(t)\mathcal{O}_{(\hat{A},C),I_o}(t)e_z(t) + K(t)\mathcal{J}_{(\hat{A},C),I_o}(t)\hat{\zeta}(t)
\]
Introducing the Moore–Penrose pseudo-inverse of \(K(t)\mathcal{O}_{(\hat{A},C),I_o}\) as
\[
\left[K(t)\mathcal{O}_{(\hat{A},C),I_o}\right]^+ = \left[K(t)\mathcal{O}_{(\hat{A},C),I_o}\right]^T \left[K(t)\mathcal{O}_{(\hat{A},C),I_o}\right]^{-1} \left[K(t)\mathcal{O}_{(\hat{A},C),I_o}\right]^T
\]
This pseudo-inverse always exists by the definition of \(K(\hat{A},C,D)(t)\). It is important to remark the fact that the second term in the right-hand of equation (16) is equal to zero. Now, by pre-multiplying (16) by (17), the error can be reconstructed by
\[
e_z(t) = \mathcal{J}_{(A,C),D}^{-1}(t)K_{(\hat{A},C),I_o}^T(t)K_{(\hat{A},C),I_o}(t)e_y(t) = F_{(\hat{A},C),D}(t)e_y(t);
\]
Notice that the value of the gain \(F_{(\hat{A},C),D}(t)\) can be obtained analytically by the consecutive derivatives of the vectors and matrices \(e_y, A(t), L(t), C(t),\) and \(D(t)\) ([8]). The derivatives of the output estimation error can be obtained on-line by using the HOSM differentiator (4).

Finally, in view of the finite-time convergence of the differentiator, \(W_{I_o}(e_y(t)) \equiv \hat{e}_y\) after finite-time transient, then the estimation of the state can be deduced directly from the equation \(e_z = x - z\), now \(e_z\) and \(z\) are known values. Therefore, the state can be reconstructed exactly after a transient of finite duration by (8).

\[\square\]

\[\text{Here, we can use any left generalized inverse } \left[K(t)\mathcal{O}_{(\hat{A},C),I_o}\right]^R, \text{ such that } \]
\[
\left[K(t)\mathcal{O}_{(\hat{A},C),I_o}\right]^R K(t)\mathcal{O}_{(\hat{A},C),I_o} = I;
\] and it can be computed using any methodology (e.g., [44–47], for more details about generalized inverses and its computation).

Lemma 5.1
Let Assumptions A1–A3 be satisfied and the output be measured with an additive noise $\hat{y}$, being a Lebesgue-measurable function of time with the maximal magnitude $\varepsilon$, that is, $||\hat{y} - y|| \leq \varepsilon$. With sufficiently small $\varepsilon$, the state estimation is obtained with the accuracy of the order of $\varepsilon^{1/n}$.

Proof
Let follow the proof of Theorem 5.1, now the estimation error dynamics (14) becomes

$$
\dot{e}_x(t) = \tilde{A}e_x(t) + D(t)\xi(t) + L(t)e,
$$

(18)

Considering the Lyapunov candidate function

$$
V(e_x(t)) = e_x^T(t)P(t)e_x(t),
$$

its derivative satisfies the inequality

$$
V(e_x(t)) \leq -e_x^T(t)Q(t)e_x(t), \quad \forall ||e_x|| \geq 2(||D(t)||e_0^+ + ||L(t)||\varepsilon)||P(t)||
$$

Now, the proposed choice of the gain $L(t)$ guarantees the global exponential convergence of the estimation error to a bounded region around the origin, and this region depends on the unknown-input bound and the maximal noise magnitude.

By following the same order than in proof of Theorem 5.1. In the presence of noise, component by component, the estimation of the output error $e_y$ and its derivatives of the order up to $l_o - 1$, that are estimated by (13), satisfy the following inequalities according to [27]:

$$
\begin{align*}
|z_{0_j} - e_y(t)_j| & \leq \delta_{0_j}\varepsilon \\
|z_{1_j} - \dot{e}_y(t)_j| & \leq \delta_{1_j}\varepsilon^{(l_o - 1)/l_o} \\
& \vdots \\
|z_{(l_o-1)_j} - e_y^{(l_o-1)}(t)_j| & \leq \delta_{(l_o-1)_j}\varepsilon^{1/l_o}
\end{align*}
$$

for positive scalar values $\delta_{i_j}$, dependent of the observer and system parameters. Then, the following inequality is satisfied for some positive scalar constant $\gamma^+$ that depends on the system and observer parameters:

$$
||\tilde{e}_y(t) - W_{l_o-1}(e_y(t))|| \leq \gamma^+\varepsilon^{1/l_o}
$$

Notice that according to equation (8), the estimated values $\tilde{e}_y = W_{l_o}(e_y(t))$ are used to obtain the estimated states $\tilde{x}$. Therefore, the following inequality is satisfied $||x - \tilde{x}|| \leq \gamma^+\varepsilon^{1/l_o}$, where $\gamma^+$ depends on the norm of $F(t)$ and $\delta^+$.

Corollary 5.1
Let assumptions of Theorem 5.1 be satisfied, and let the observer be designed according to (7)–(10), (8) and under the assumption that estimation error (14) satisfies the following equality:

$$
\begin{bmatrix}
C(t)D(t) \\
C(t)\tilde{A}(t) \\
\vdots \\
C(t)\tilde{A}^{l_o-1}D(t)
\end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}
$$
Then, the matrix $F(t) = I_{l_{o-1}}$ can be used instead of (11).

**Remark 5.1**
The application of the HOSM differentiator (4) ensures the finite time convergence of the cascade observer. In order to guarantee fixed time convergence of the estimated state, the proposed HOSM
differentiator must be replaced by a suitable differentiator that guarantees this kind of convergence, for example, the uniform robust exact differentiator [48].
Figure 11. Accuracy comparison for the pitch rate estimation obtained by means of the hierarchical observer in the presence of different noises in the output measurements.

Remark 5.2
The usage of the DLSF assures the convergence of the estimation error to a ball defined by the bound of the measurement noises and the unknown inputs. For LTI systems, to attenuate the effect of the measurement noises, a descriptor observer has been proposed in [49] guaranteeing asymptotic convergence of the estimation error in the presence of constant input disturbances. In this case, the DLSF can be replaced by this descriptor observer, and Lemma 5.1 is still valid for higher measurement noises.

5.1. Computational algorithm

To implement the aforementioned observer, a computational algorithm is proposed

1. Compute $P(t)$, the positive definite solution of the differential Riccati equation (10)
2. By using the computed matrix $P(t)$, obtain the gain $L(t)$ using (9)
3. Define the matrix $\tilde{A}(t) = A(t) - L(t)C(t)$
4. Calculate the matrices $\Theta(\tilde{A},C), J_o(t)$ and $\mathcal{J}(\tilde{A},C,D), J_o(t)$ as in Theorem 3.2
5. Obtain the matrix $K(\tilde{A},C,D)(t)$ by using the null space basis of the matrix $\mathcal{J}(\tilde{A},C,D), J_o(t)$
6. Construct the matrix $\mathcal{H}(\tilde{A},C,D)(t)$
7. Calculate the $l_o - 1$ derivatives of $e_y$ with the HOSM differentiator
8. Obtain the observed value $\hat{x}$ with (8)
6. EXAMPLE

Let us recall the system presented in the motivating example, and as it was shown in Figure 1, the system is unstable and has observability index $l_0 = 2$. The proposed algorithm, which is a combination of a least squares filter and the HOSM differentiator, is able to perform an exact reconstruction of the states. Here is important to remark that this reconstruction cannot be achieved by the algorithms individually as it was shown in Section 4.

The cascade combination of both methodologies follows the computational algorithm and provides exact estimation of the states, and this estimation is shown in Figure 7. The matrix $P(t)$ is obtained numerically using a vectorization procedure with a numerical integration of the vectorized Equation (10). The proposed cascade observer provides an accurate estimation of the states. The corresponding estimation errors are presented in Figure 8.

Let us remark the finite time convergence of the proposed observer to the real value of the states in spite of the instability of the LTV system and the existence of the unknown inputs.

Now, considering additive noise in the output measurement such that $\|\hat{y} - y\| \leq \epsilon = 0.3$, the cascade observer obtains an estimated state given in Figure 9. Observe that the reconstructed states present small errors because of the effect of the noise (Figure 10). Moreover, the cascade observer satisfies Lemma 5.1 for small measurement noise as it is illustrated in Figure 11 with a precision of order $\epsilon^{1/2}$.

7. CONCLUSIONS

In this paper was presented an state observer capable to reconstruct the states of an LTV system with unknown inputs. A combination of a deterministic least squares filter and the HOSM differentiator is used in order to compensate the effects of instability and unknown inputs. The proposed observer provides a finite-time exact reconstruction of the real value of the states in the presence of bounded unknown inputs. The global finite-time exact convergence of the observer is illustrated by simulations.

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REFERENCES


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