Uniform Robust Exact Differentiator

Emmanuel Cruz-Zavala, Jaime A. Moreno, Member, IEEE, Leonid Fridman Member, IEEE.

Abstract—The differentiators based on the Super-Twisting Algorithm (STA) yield finite-time and theoretically exact convergence to the derivative of the input signal, whenever this derivative is Lipschitz. However, the convergence time grows unboundedly when the initial conditions of the differentiation error grow. In this paper a Uniform Robust Exact Differentiator (URED) is introduced. The URED is based on a STA modification and includes high-degree terms providing finite-time, and exact convergence to the derivative of the input signal, with a convergence time that is bounded by some constant independent of the initial conditions of the differentiation error. Strong Lyapunov functions are used to prove the convergence of the URED.

Index Terms—Differentiation, Second Order Sliding Modes, Finite-Time Observers, Lyapunov functions, Discontinuous Observers.

I. INTRODUCTION

Motivation. Real-time differentiation is an old and important problem, with different applications: they are an important part of the classical PD and PID-controllers; more generally, they can be instrumental in the output-feedback control [1], [2]; and they can also be used as observers [3], [4]. The main issue in a real-time differentiator design is the trade-off between exactness and robustness with respect to noise and input signal sampling [5], [2]. Linear systems, approximating the transfer function of a differentiator in a frequency band, have been extensively used [6], [7]. High-Gain observers have also been frequently used as real-time differentiators, and applied in output-feedback control, providing for some separation results in nonlinear control schemes [1], [8]. However, the noise sensitivity is strongly amplified with the use of high gains, and the presence of the peaking effect deteriorates further their performance [1], [8].

In contrast, discontinuous differentiators can be theoretically exact for a wide class of signals [5], [9], [4], [10]. In particular, it has been shown that the super-twisting algorithm (STA) is very well suited for differentiators [5], [2], since it provides the best possible asymptotic precision in presence of deterministic Lebesgue-measurable bounded noises and discrete sample of the input signal, when its second time derivative is bounded [2]. This kind of differentiators can also be used for the construction of robust and exact observers with finite time convergence [3], [4], [11]. Prescribed time convergence is a crucial property for state estimation of hybrid systems, or for separation-like properties for nonlinear systems. However, a drawback of all previous results is that the convergence time of these observers/differentiators tends to infinity when the norm of the initial conditions of the differentiation error grows unboundedly. This means, for example, for the observation of a hybrid plant with a positive Dwell Time that despite using STA based observers, we cannot guarantee the state estimation convergence before the next plant’s switching takes place, if a bound for the initial conditions is not known a priori. For the output-feedback control of a nonlinear system based on an state observer this means that one cannot be sure that the observer converges before the plant’s trajectory has escaped to infinity. This motivates the importance of having differentiators/observers that converge exactly and robustly in finite time, with prescribed convergence time, independently of the initial differentiator/observer error and despite of bounded perturbations.

Methodology. Recently, strong Lyapunov functions have been developed for the Super-Twisting Algorithm (STA) [12], [13], [14], allowing to analyze the robustness of the algorithm for a wide class of uncertainties and perturbations. Besides, new algorithms have been proposed based on the STA (see [13], [14]), ensuring the finite time convergence and robustness characteristics by means of strong Lyapunov functions, and providing a formula to estimate the convergence time.

Contribution. In this paper the notion of uniform exact convergence is introduced for the first time, and a uniform robust exact differentiator (URED) w.r.t. initial conditions is proposed. It is based on a modification of the differentiator presented in [5], [2]. High-degree correction terms are added to have the uniform convergence property in the differentiator and convergence time bounded by a constant, independent of the initial differentiator error. Two Lyapunov-Like functions are used to guarantee the uniform exact convergence of the differentiator. One of them proves the uniform (in the initial condition) convergence of every trajectory to a compact set containing the origin. The other shows the exact convergence (in finite time and robust to bounded perturbations) to the origin of every trajectory starting in a compact set. Notice that the properties of uniform, exact and robust convergence cannot be obtained by continuous differentiators having globally Lipschitz injection terms.

The rest of the paper is organized as follows. In Section II the URED is introduced and the main results of the paper are presented. Section III presents the propositions used to prove the main result. In section IV the algorithm to estimate the convergence time is developed and in section V the convergence times of the URED and robust exact differentiator by Levant [5] are compared in simulation.

II. PROBLEM STATEMENT AND MAIN RESULTS

Let the input signal $f(t)$ to the differentiator be a Lebesgue-measurable function defined on $[0, \infty)$. $f(t)$ is assumed to be decomposed as $f(t) = f_0(t) + v(t)$, where the first term is a twice differentiable unknown base signal $f_0(t)$, to be differentiated, and having a derivative with known Lipschitz constant $L > 0$, and the second term $v(t)$ corresponds to a uniformly bounded noise signal. The discrete measurement of the input signal $f_0(t)$ can be also interpreted as the noise affecting the input signal. If $\sigma_0 = f_0(t)$ and $\dot{\sigma}_1 = f_0(t)$, a state representation of the base signal is given by

$$\dot{\sigma}_0 = \dot{\sigma}_1, \dot{\sigma}_1 = \ddot{f}_0.$$  

(1)

Our main aim is to construct a differentiator, that converges to $\ddot{f}_0(t)$, exactly, after a finite time, that can be prescribed to be a constant independent of the initial condition of the differentiator (uniform), using only the measurement of $f_0(t)$, and the knowledge of the bound $L$ of the second derivative of $f_0(t)$, $L = \left| \dddot{f}_0(t) \right| \leq L$.

Based on the Generalized Super-Twisting Algorithm (GSTA) [14], we propose as URED

$$\dot{z}_0 = -k_1 \phi_1(\sigma_0) + z_1, \dot{z}_1 = -k_2 \phi_2(\sigma_0),$$  

(2)

where $\sigma_0 = z_0 - \sigma_0$, $k_1$ and $k_2$ are positive gains to be designed, $\phi_1(\sigma_0) = |\sigma_0|^\frac{1}{2} \text{sign}(\sigma_0) + \mu |\sigma_0|^{\frac{3}{2}} \text{sign}(\sigma_0)$, $\phi_2(\sigma_0) = \frac{1}{2} |\sigma_0|^{\frac{3}{2}} \text{sign}(\sigma_0) + 2\mu \sigma_0 + \frac{3}{2} |\sigma_0|^{\frac{3}{2}} \text{sign}(\sigma_0)$, and $\mu \geq 0$ is a scalar. When $\mu = 0$ the standard robust exact differentiator proposed in [5] is recovered. It will be shown that the high-degree terms, $|\sigma_0|^{\frac{3}{2}} \text{sign}(\sigma_0)$ and $|\sigma_0|^{\frac{3}{2}} \text{sign}(\sigma_0)$, provide for uniform convergence of the differentiator, that is, the convergence...
time will be bounded by a constant independent of the initial conditions of the differentiator. $z_0$ and $z_1$ are the estimations of $f_0(t)$ and $\tilde{f}_0(t)$, respectively.

With $\sigma_1 = z_1 - z_1$ being the estimation error between the differentiator output $z_1$ and the first base signal derivative $\tilde{f}_0(t)$, the estimation error dynamics take the form

$$\dot{\sigma}_0 = -k_1 \phi_1(\sigma_0) + \sigma_1, \quad \dot{\sigma}_1 = -k_2 \phi_2(\sigma_0) - \tilde{f}_0(t),$$  \hspace{1cm} (3)

where $\tilde{f}_0(t)$ is supposed to be bounded $|\tilde{f}_0(t)| \leq L$. Solutions of (3) are understood in the Filippov’s sense [15].

Definition 1: System (3) is uniformly exact convergent if any trajectory of the system converges to the origin for every bounded perturbation $|\tilde{f}_0(t)| \leq L$, and its convergence time is upper bounded by some constant, independent of the initial condition $\sigma(0) = [\sigma_0(0), \sigma_1(0)]^T$.

The goal of this paper is to show that the differentiator (2) is uniformly exact convergent, i.e. that for every initial condition of the differentiator $z(0) = [z_0(0), z_1(0)]^T$, its output $z_1$ satisfies $z_1(t) = f_0(t)$, and also $z_0(t) = \tilde{f}_0(t), \forall t > T$, where $T$ is some constant independent of the initial conditions, whenever $|\tilde{f}_0(t)| \leq L$. Furthermore, an estimation of the convergence time and of the accuracy of the differentiator will be also provided.

A. Main result

1) Uniform exact convergence: Theorem 2: Suppose that $|\tilde{f}_0(t)| \leq L$, with $L > 0$ a known positive constant, and that $\mu > 0$. Then, the differentiator (2) is uniformly exact convergent if the gains $k_1, k_2$ are in the set

$$K = \left\{ (k_1, k_2) \in \mathbb{R}^2 | 0 < k_1 \leq 2\sqrt{T}, \quad k_2 > \frac{\epsilon \sqrt{T}}{4} \frac{2 + \lambda^2 T}{\lambda^2} \right\}, \quad \left\{ (k_1, k_2) \in \mathbb{R}^2 | k_1 > 2\sqrt{T}, \quad k_2 < 2L \right\}. \hspace{1cm} (4)

2) Asymptotic Accuracy of the Differentiator: Theorem 2 assures that the differentiator (2) estimates the derivative of $f(t) = f_0(t) + v(t)$ exactly after a finite time, in the absence of noise, i.e. $v(t) = 0$. This means that after a finite time the estimation error will vanish, i.e. $\sigma(t) = 0$, for system (3). From the Lyapunov function for (3) (see next Section III), and using standard arguments [8], it is possible to show that the error system (3) is Input-to-State Stable (ISS), considering the noise $v(t)$ as input, i.e. a bounded noise signal $|v(t)| \leq \zeta$ causes an ultimately bounded differentiation error vector $\sigma(t)$. For small values of $\zeta$ the norm of the differentiation error $\sigma$ is also small. In this case the differentiator error dynamics (3) is approximately equal to the error dynamics of Levant’s robust exact differentiator [5], since the lower order terms in $\phi_1$ and $\phi_2$ dominate the higher order ones for small values of $\sigma_0$.

Thus, for small noise signals the URED (2) has the same accuracy as the robust exact differentiator (5) with respect to noises and discrete sampling of $f_0(t)$. This means that if $|v(t)| \leq \zeta$, the inequalities $|z_0 - f_0(t)| < \eta_0 \zeta, \quad |z_1 - \tilde{f}_0(t)| < \eta_1 \zeta^{1/2}$ are established with some positive constants $\eta_0, \eta_1$. If $f_0(t)$ is sampled with a constant (and small) sampling interval $\tau > 0$, the inequalities $|z_0 - f_0(t)| < \eta_0 \tau^2, \quad |z_1 - \tilde{f}_0(t)| < \sigma_1 \tau$ are established with some positive constants $\eta_0, \eta_1$ [5].

III. PROOF OF THEOREM 2

To show the uniform exact convergence of (3), we use two Lyapunov-Like functions. The first Lyapunov function shows the global and exact convergence of (3), but not the uniformity w.r.t. the initial conditions. The second Lyapunov-Like function ensures the robust uniform convergence of all trajectories of (3).

A. Exact convergence via a global strong Lyapunov function

Inspired by [14] we propose the following global strong Lyapunov function for (3)

$$V_1(\sigma) = \zeta^2 P \zeta, \quad \sigma = [\sigma_0, \sigma_1]^T,$$  \hspace{1cm} (5)

where the vector $\zeta^T = [\phi_1(\sigma_0), \sigma_1]$ and $P$ is a symmetric and positive definite matrix. Suitable matrices $P = P^T > 0$ are the solutions of the Linear Matrix Inequality (LMI)

$$\begin{bmatrix} A^T P + P A + \epsilon I + 4 L^2 C^T C & PB \\ B^T P & -1 \end{bmatrix} \leq 0, \quad (6)
$$

where

$$A = \begin{bmatrix} -k_1 & 1 \\ -k_2 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad (7)
$$

with $(k_1, k_2) \in K$ defined in (4), and for some $\epsilon > 0$.

Proposition 3: Consider the error system (3), and assume that $f_0(t)$ satisfies $|\tilde{f}_0(t)| < L$.

1) If the gains satisfy $(k_1, k_2) \in K$, defined in (4), then the LMI

(6) is feasible, i.e. there exist $P = P^T > 0$ and a scalar $\epsilon > 0$ that fulfill the LMI.

2) For every $P$ satisfying (6) for some $\epsilon > 0$, the quadratic form (5) is a strong, robust Lyapunov function for system (3), that guarantees the exact convergence of $\sigma$ to the origin. Moreover, the derivative $V_1$ of the Lyapunov function along the trajectories of the system satisfies the inequality

$$V_1 \leq -\kappa_1 (P, \epsilon) V_1^1 (\sigma) - \kappa_2 (P, \epsilon) \mu |\sigma_0|^{3/4} V_1 (\sigma), \quad (8)
$$

where $\kappa_1 (P, \epsilon), \kappa_2 (P, \epsilon)$ are positive scalars given by

$$\kappa_1 (P, \epsilon) \leq \frac{4 \lambda_{\max} (P)}{2 \lambda_{\max}^2 (P)}, \quad \kappa_2 (P, \epsilon) \leq \frac{3 \lambda_{\max} (P)}{2 \lambda_{\max} (P)} V_1 (\sigma).$$

This implies that a trajectory of (3) starting at the initial condition $\sigma(0) = \sigma_0 \in \mathbb{R}^2$ reaches the origin in finite time, and despite the perturbations bounded by $L$, in a time $T_1 (\sigma_0)$ with

$$T_1 (\sigma_0) \leq \frac{4 \lambda_{\max} (P)}{\epsilon} V_1^{1/2} (\sigma_0). \hspace{1cm} (9)
$$

Proof: Appendix A.

B. Uniform convergence to a compact set containing the origin

The Lyapunov function (5) is not able to assure the uniform convergence of the URED, but the next function shows that every trajectory converges to a compact set, containing the origin, uniformly in the initial condition.

Proposition 4: For $\delta > 0$ sufficiently large the smooth function

$$V_2 (\sigma) = \frac{\delta}{2} k_2 |\sigma_0|^{3} - \sigma_0 |\sigma_1|^2 \text{sign} (|\sigma_1|) + \frac{\delta}{2} |\sigma_1|^2,$$  \hspace{1cm} (10)

is a robust Lyapunov-Like function for (3) and

$$V_2 (\sigma) \leq -\frac{1}{2} \left( \frac{V_2 (\sigma)}{2 C_2} \right)^2, \quad \forall \sigma \in \Gamma_e = \{ \sigma | V_2 (\sigma) \leq \epsilon \}, \hspace{1cm} (10)
$$

for $\epsilon$ sufficiently large. Moreover, any trajectory of system (3) starting in the set $\Gamma_e$ converges to its complement $\Gamma_e^c = \{ \sigma | V_2 (\sigma) \leq \epsilon \}$, a compact set containing the origin, robustly (i.e. despite of the perturbation) and uniformly, that is, the convergence time to $\Gamma_e^c$ is upper bounded by a time $T_2 (\epsilon)$ independent of the initial condition, that can be estimated as

$$T_2 (\epsilon) = 12 (2 C_2)^3 \left( \frac{1}{\epsilon} \right)^{3/2} \text{ for } \epsilon \geq C_1 \left( 2 C_3 + 2 \sqrt{C_4 + C_5^2} \right)^3, \hspace{1cm} (11)
$$
and $C_1$, $C_2$, $C_3$, and $C_4$ are positive constants independent of the initial conditions.

Proof: Appendix B.

Since the high-degree terms of the URED are stronger than the low-degree ones far from the origin, they are responsible for a faster convergence, and provide for the uniform convergence to a compact set $\Gamma^c$ containing the origin, as shown in Proposition 4. On the other hand, Proposition 3 ensures the exact convergence of system (3) to the origin, despite of the bounded perturbation $|f_0(t)| < L$, a fact that is due to the low-degree terms, which are stronger than the high-degree ones in a neighborhood of the origin. Both Propositions 3 and 4 together guarantee that the differentiator error (3) is uniformly exact convergent, for $f_0$ bounded, so that the convergence time of every trajectory can be bounded by the same constant.

IV. CONVERGENCE TIME ESTIMATION

Once $\mu > 0$ and the gains $(k_1, k_2)$ have been selected, according to Theorem 2, (9) provides a convergence time estimation for the differentiator, that grows unboundedly with the norm of the initial differentiator error. However, (11) shows that any trajectory of the same algorithm converges uniformly to the level set $\Gamma^c$, i.e. in a time bounded by the same constant, independent of the initial condition. Combining both time estimations, an upper bound $T$ for the convergence time of any trajectory of (3) is given by

$$T \leq \frac{4A_{2\max}^2}{\varepsilon} \eta^\frac{7}{2} \left( \frac{1}{\varepsilon} \right)^\frac{3}{2},$$

where $\varepsilon \geq C_1 \left( 2C_2 + 2\sqrt{C_2 + C_2^2} \right)^3$, and the values of $C_1, \ldots, C_4, \varepsilon$ and $P$ are calculated as described in Appendices A and B. Moreover, the value of $\eta$ is selected, such that $A_{2,\varepsilon} \subset \Omega_{1,\eta}$, where $A_{2,\varepsilon} = \{ \sigma \mid V_2(\sigma) = \varepsilon \}$ is a level surface of $V_2(\sigma)$, and $\Omega_{1,\eta} = \{ \sigma \mid V_1(\sigma) \leq \eta \}$ is a level set of $V_1(\sigma)$. $A_{2,\varepsilon} \subset \Omega_{1,\eta}$ can always be satisfied choosing $\eta$ large enough.

An appropriate value of $\eta$ can be calculated in the following form:

Choose $\omega > \left( \frac{2}{\sqrt{1}} \right)^\frac{3}{2}$, so that the homogeneous ball $B_{h,\omega} = \{ \sigma \mid ||\sigma||_{r,p} \leq \omega \} \supset A_{2,\varepsilon}$, where $||\sigma||_{r,p}$ is a homogeneous norm defined in Appendix B. Now, a value $\nu$ has to be found, such that $B_{\nu} = \{ \sigma \mid ||\sigma||_{r,p} \leq \nu \} \supset B_{h,\omega}$, where $||\sigma||_2^2 = ||\sigma_0|| + 2\mu ||\sigma_0||^2 + \mu^2 ||\sigma_0||^2 + \sigma_2^2$. This can be calculated by finding the maximum value of $||\sigma||_2$ on the boundary of $B_{h,\omega}$. By simple calculus for this maximum, $||\sigma||_{2\max} = \max \{ \rho(\sigma_{01}) \}, \rho(\sigma_{02}) \}$, where

$$\rho(\sigma_{0}) = (\mu ||\sigma_0|| + 1)^2, \sigma_{01} = \omega^\frac{2}{3}, \sigma_{02} = \omega^\frac{2}{7},$$

Finally, selecting $\nu > ||\sigma||_{2\max}$, the required value of $\eta$ is given by $\eta = \lambda_{2\max}^2 ||\sigma||_2$.  

Remark 5: Note from (12) that the prescribed value of the URED is a constant, and it can be made arbitrarily small selecting the gains $k_1$ and $k_2$ properly.

V. SIMULATION EXAMPLE

We compare the URED with Levant’s robust differentiator [5]. For the simulation a value of $\mu = 1$ has been set for the URED and $\mu = 0$ for Levant’s differentiator. The base signal to be differentiated is $f_0(t) = 5t + \sin t$ with two different noise terms: $v_1(t) = 0.01 \cos 10t$, and $v_2(t) = 0.001 \cos 30t$. With $L = 2.5$ appropriate values for the gains are $k_1 = 2\sqrt{3}$, $k_2 = 6$ (see (4)) and two initial conditions for the output signal $z(0) = 0$ and $z(0) = [10, 0]^T$ are taken. The results are shown in the Fig. 1. Both differentiators have robust and exact convergence. However, as shown in Fig. 2, the convergence time of Levant’s differentiator [5] grows unboundedly with the norm of the initial condition, while the convergence time of the URED is asymptotically bounded by a constant for growing initial condition’s norm. This prescribed convergence time can be estimated by the expression (12).  

1) Selecting (see Appendix) $\delta = 1.22228$, we have $\tilde{\gamma}_m = 2.14039$, 

\[ \tilde{\gamma}_M = 0.361173, \xi = 2.74119, C_1 = 0.398254, C_2 = 3.68254, C_3 = 22.1792, C_4 = 20.4383 \text{ and, consequently, } \varepsilon = 28.6752 \text{ and } T_2(\epsilon) = 15.1811. \]

2) Find $P = P^T > 0$ and $\epsilon > 0$ such that (6) is satisfied. This happens for

$$A = \begin{bmatrix} -3.4641 & 1 \\ 0 & -6 \end{bmatrix}, P = \begin{bmatrix} 10.4315 & -2.7068 \\ -2.7068 & 2.0680 \end{bmatrix},$$

Preprint submitted to IEEE Transactions on Automatic Control. Received: June 14, 2011 07:49:19 PST
\( \epsilon = 0.6042 \) and \( \lambda_{\text{min}} \{ P \} = 1.2684, \lambda_{\text{max}} \{ P \} = 11.2311. \) Taking \( \omega = \sqrt{\frac{3}{\epsilon_{1}}} + 0.001 = 89.63 \), we can select \( \nu^{2} = 8.64 > \| \zeta \|_{2}^{2} \max = \max \{ 8.6173, 8.6367 \} \) and \( \eta = \lambda_{\text{max}} \{ P \} \nu^{2} = 97.0367. \) An upper bound for the prescribed time can be then estimated by

\[
T_{p} = \frac{4A_{\text{max}} \{ P \} }{\epsilon} \eta \frac{1}{2} + T_{2}(\epsilon) = 218.5538 + 15.1811 = 233.7349.
\]

From the simulations (see Fig. 2) it is clear that \( T_{p} \) is a very crude estimation of the true prescribed convergence time of the URED, so that in practice it is better to obtain its value by means of simulations.

VI. CONCLUSIONS

In this paper, a uniform robust exact differentiator is proposed for the first time. It is based on a modification of the robust exact differentiator [5] by adding high-degree correction terms, that improve the convergence properties. These high-degree terms ensure the uniform convergence, and the lower-degree terms guarantee the exact convergence to the origin. In this form the differentiator converges in finite time and exactly to the derivative of the input signal, if its second derivative is bounded, and with a prescribed convergence time that is independent of the initial condition of the differentiator. Two Lyapunov-Like functions are used to establish the properties of the algorithm: a first Lyapunov function is proposed to prove robust exact convergence, and a second Lyapunov-Like function was proposed to prove the uniform convergence to a compact set containing the origin.

VII. ACKNOWLEDGMENTS

The authors gratefully acknowledge the financial support from CONACYT (Consejo Nacional de Ciencia y Tecnología), grants 132125 and 51244, Programa de Apoyo a Proyectos de Investigación e Innovación Tecnológica (PAPIIT) UNAM, grants IN117610 and IN17211-2, and FONCyT project 93302.

REFERENCES


APPENDIX A

Part 1: Here it is shown that the LMI (6) is feasible when the gains are in \( K \) defined in (4). Using the classical frequency interpretation of the LMI (6), for example through the circle criterion [8], (6) will be satisfied if and only if the Nyquist diagram of the transfer function \( G(s) = C(sI - A)^{-1}B = 1/(s + k_{1}s + k_{2}) \) is contained in the circle of radius \( 1/(2L) \) and centered at the origin, that is, if and only if

\[
\max_{\omega} \left| G(j\omega) \right|^{2} = \max_{\omega} \left( \frac{1}{(k_{2} - \omega^{2})} + \frac{\omega^{2}}{1} \right) < \left( \frac{1}{2L} \right)^{2}.
\]

Since

\[
\max_{\omega} \left| G(j\omega) \right|^{2} = \left\{ \begin{array}{ll}
\frac{1}{k_{2}^{2}} & \text{if } k_{2} - \frac{k_{1}^{2}}{2} \\
\frac{1}{k_{1}^{2}(k_{2} - \frac{k_{1}^{2}}{4})} & \text{if } k_{2} - \frac{k_{1}^{2}}{2} > 0
\end{array} \right.,
\]

(6) is feasible if and only if

\[
\left\{ \begin{array}{l}
k_{2} - \frac{k_{1}^{2}}{2} \leq 0 \text{ and } \frac{1}{k_{2}} < \frac{1}{2L} \\
k_{2} - \frac{k_{1}^{2}}{2} > 0 \text{ and } \frac{1}{k_{2}^{2}(k_{2} - \frac{k_{1}^{2}}{4})} < \left( \frac{1}{2L} \right)^{2}
\end{array} \right.,
\]

that can be equivalently described by (4).

Part 2: Note that \( \phi_{2}(\sigma_{0}) = \phi_{1}'(\sigma_{0}) \phi_{1}(\sigma_{0}) \), where \( \phi_{1}'(\sigma_{0}) = \left( \frac{1}{2\sigma_{0}^{2}} \right) > \frac{3}{2} \mu |\sigma_{0}|^{2} > 0 \), and that, for \( \sigma_{0} \neq 0 \)

\[
\dot{\zeta} = \phi_{1}'(\sigma_{0}) \left( -k_{1}\phi_{1}(\sigma_{0}) + \sigma_{1} \right) = \phi_{1}'(\sigma_{0}) \left( A\zeta - Bp(t, \zeta_{1}) \right),
\]

where \( A, B \) are given in (7) and, since \( \phi_{1} = \phi_{1}(\sigma_{0}) = \left( 1 + \mu |\sigma_{0}| \right) |\sigma_{0}|^{2} \), sign \( \sigma_{0} \),

\[
\rho(t, \zeta_{1}) = \frac{\dot{\hat{f}}(t)}{\phi_{1}'(\sigma_{0})} = \frac{2\hat{f}(t) \text{ sign } (\sigma_{0})}{(1 + 3\mu |\sigma_{0}|)(1 + \mu |\sigma_{0}|)} \zeta_{1}.
\]

Moreover, \( \hat{f}(t) \leq L \) implies \( |\rho(t, \zeta_{1})| \leq 2L |\zeta_{1}| \), or equivalently

\[
\omega(\rho, \zeta) = -\rho^{2}(t, \zeta_{1}) + 4L^{2}\zeta_{1}^{3} = \left[ \begin{array}{c}
\frac{\sigma_{0}}{\rho}
\end{array} \right]^{T} \left[ \begin{array}{cc}
4L^{2}C^{T}C & 0 \\
0 & -1
\end{array} \right] \left[ \begin{array}{c}
\sigma_{0}/\rho
\end{array} \right] \geq 0,
\]

Preprint submitted to IEEE Transactions on Automatic Control. Received: June 14, 2011 07:49:19 PST
where $C$ is as in (7). The time derivative of $V_1(\sigma)$ for $\sigma_0 \neq 0$ satisfies
\[
\dot{V}_1 = \phi_1(\sigma_0) \begin{bmatrix} \zeta^T & \frac{A^T P + PA}{B^T P} & PB \end{bmatrix} \begin{bmatrix} \zeta \\ \rho \end{bmatrix} \\
\leq \phi_1 \left\{ \begin{bmatrix} \zeta \\ \rho \end{bmatrix} \begin{bmatrix} A^T P + PA & PB \end{bmatrix} \begin{bmatrix} \zeta \\ \rho \end{bmatrix} + \omega(\rho, \zeta) \right\} \\
= \phi_1 \begin{bmatrix} \zeta \\ \rho \end{bmatrix} \begin{bmatrix} A^T P + PA + 4L^2G^2C & PB \end{bmatrix} \begin{bmatrix} \zeta \\ \rho \end{bmatrix} -1 \left\{ \begin{bmatrix} \zeta \\ \rho \end{bmatrix} \begin{bmatrix} A^T P + PA + 4L^2G^2C & PB \end{bmatrix} \begin{bmatrix} \zeta \\ \rho \end{bmatrix} \right\} \\
\leq -\phi_1(\sigma_0) \epsilon \|\zeta\|^2 - \frac{\epsilon}{2|\sigma_0|^{1/2} \epsilon} \|\zeta\|^2 - \frac{3}{2} \mu |\sigma_0|^{1/2} \epsilon \|\zeta\|^2 \\
\leq -\frac{\epsilon}{2} \|\zeta\|^2 - \frac{3}{2} \mu |\sigma_0|^{1/2} \epsilon \|\zeta\|^2 \\
\leq -\frac{\epsilon}{2} \lambda_{\max}^2 \left\{ P \right\} \mu |\sigma_0|^{1/2} \|V_1(\sigma)\| = \frac{3}{2} \lambda_{\max}^2 \left\{ P \right\} \mu |\sigma_0|^{1/2} V_1(\sigma),
\]
where (6) has been assumed to be satisfied. The latter inequalities are derived applying the well-known inequality $\lambda_{\max} \{ P \} \|\zeta\|^2 \leq \zeta^T P \zeta \leq \lambda_{\max} \{ P \} \|\zeta\|^2$, where $\|\zeta\|^2 = \|\sigma_0\| + 2\mu |\sigma_0|^{1/2} \mu |\sigma_0|^2 + |\sigma_0|^2$ is the Euclidean norm of $\zeta$, and $-|\sigma_0|^2 - \frac{1}{2} \leq -\|\zeta\|^2$, that is immediate from the definition of $\|\zeta\|^2$. It follows that $V_1$ is negative definite. Note that the solution of the differential equation
\[
\dot{v} = -\gamma v^p, \quad v(0) = v_0 \geq 0, \quad p > 0, \quad p \neq 1
\]
is given by
\[
v(t) = \left( v_0^{1-p} - (1-p) \gamma t \right)^{\frac{1}{1-p}} \text{ if } \gamma > 0. \quad (13)
\]
Since for $p = 1/2$ it is $v(t) = \left( v_0^{1/2} - \frac{1}{2} \gamma t \right)^{2}$, it follows from (8), and the comparison principle [8], that $V(\sigma(t)) \leq v(t)$ when $V(\sigma(0)) \leq v_0$, and therefore, that $\sigma(t)$ converges in finite time to zero (when $\gamma > 0$) and reaches that value before a time given by (9).

**APPENDIX B**

We recall first some useful results. The homogeneous norm [16] is defined as
\[
\|\sigma\|_{r,p} = \left( \sum_{i=1}^{n} |r_i|^{\frac{p}{r}} \right)^{\frac{1}{p}}, \quad \forall \sigma \in \mathbb{R}^n,
\]
where $p \geq 1$ and $r_i$ are the weights of the $\sigma_i$. If $r_1 = 1$, $r_2 = \frac{3}{2}$ and $p = \frac{5}{2}$
\[
\|\sigma\|_{\frac{5}{2}} = \|\sigma_0\| + |\sigma_1|.
\]
By the fundamental (generalized) mean inequality [17, Thm. 16, Section 2.9]
\[
0 < \alpha < 1, \quad \forall x \in \mathbb{R}^2, \quad s < r. \quad \text{From (14) it follows that}
\]
\[
\frac{1}{2} \left( |\sigma_0|^3 + |\sigma_1|^2 \right)^{\frac{2}{3}} \leq \left( \frac{1}{2} \left( |\sigma_0|^2 \right)^{\frac{2}{3}} + \frac{1}{2} \left( |\sigma_1|^2 \right)^{\frac{1}{3}} \right)^{\frac{2}{3}} \\
= \left( \frac{1}{2} \right)^{\frac{2}{3}} \left( |\sigma_0|^2 + |\sigma_1|^2 \right)^{\frac{1}{3}} \left( \frac{1}{2} \right)^{\frac{2}{3}} \|\sigma\|_{\frac{5}{2}}^{\frac{2}{3}} (15)
\]
\[
\frac{1}{2} \left( |\sigma_0|^2 + |\sigma_1| \right) \leq \left( \frac{1}{2} \left( |\sigma_0|^2 \right)^{\frac{3}{5}} + \frac{1}{2} \left( |\sigma_1|^2 \right)^{\frac{1}{5}} \right)^{\frac{5}{2}} \\
= \left( \frac{1}{2} \right)^{\frac{5}{2}} \left( |\sigma_0|^2 + |\sigma_1|^2 \right)^{\frac{1}{5}} \left( \frac{1}{2} \right)^{\frac{5}{2}} \|\sigma\|_{\frac{5}{2}}^{\frac{1}{5}} (16)
\]
From (18) and (15) we conclude that
\[
C_1 \|\sigma\|_{r,p} \leq V_2(x) \leq C_2 \|\sigma\|_{r,p}.
\]
Finally, from the classical arithmetic and geometric mean inequality [17, Thm. 9, Section 2.5]: $ab \leq p^{-a} + q^{-b}$ for every real numbers $a > 0, b > 0, p > 1, q > 1$, with $p^{-1} + q^{-1} = 1$, it follows that for every real number $c > 0$ also the following inequality is satisfied
\[
ab \leq e^a b^p + e^{-a} b^{-p}.
\]
We will prove first, that $V_2(\sigma)$ is positive definite (p.d.). Using the inequality (derived from (19))
\[
|\sigma_0| |\sigma_1|^{\frac{2}{3}} \leq \frac{1}{3} \gamma_3 |\sigma_0|^3 + \frac{2}{3} \gamma_2 |\sigma_1|^2, \quad p = 3, \quad q = \frac{3}{2}, \quad \forall \gamma > 0,
\]
in $V_2(\sigma) = \frac{2}{2} \mu^2 k_2 |\sigma_0|^3 - \sigma_0 |\sigma_1|^{2} \text{ sign}(\sigma_1) + \frac{2}{7} |\sigma_1|^2$ (10) it follows that
\[
C_1 \left( |\sigma_0|^3 + |\sigma_1|^2 \right) \leq V_2(x) \leq C_2 \left( |\sigma_0|^3 + |\sigma_1|^2 \right),
\]
where
\[
C_1 = \left( \frac{\delta}{3} \mu^2 k_2 - \frac{\gamma_3}{3} \right), \quad C_2 = \left( \frac{\delta}{2} \mu^2 k_2 + \frac{\gamma_3}{3} \right),
\]
and $\gamma_3 = \frac{\gamma_3}{3}$, where $\gamma_1$ is the positive real root of $4 + 36 (\mu^2 k_2 - 1) \zeta = 2 \zeta^3$, and $\gamma_M = \frac{\gamma_3}{3}$, where $\gamma_2$ is the positive real root of $4 - 36 (\mu^2 k_2 - 1) \zeta = 2 \zeta^3$. $V_2(\sigma)$ is p.d. if $C_1 > 0$, that is, if
\[
\delta > \frac{2 \gamma_3}{3 \mu^2 k_2}.
\]
From (18) and (15) it will be shown that $V_2(\sigma)$ is negative out of a compact set containing the origin. Considering $\|\sigma\|_{r,p} \leq L$, the derivative of $V_2(\sigma)$ along the trajectories of (3) satisfies
\[
V_2(\sigma) \leq \frac{3}{2} \delta k_1 k_2 |\sigma_0|^2 - \frac{3}{2} \mu^2 k_1 k_2 \delta |\sigma_0|^2 + \mu k_1 |\sigma_0|^3 |\sigma_1|^2 \quad + 2 \delta k_2 |\sigma_0|^3 |\sigma_1|^2 + \mu k_1 |\sigma_0|^3 |\sigma_1|^2 \quad + 2 \mu k_2 |\sigma_0|^2 |\sigma_1|^2 + \frac{2}{3} (2L + k_2) |\sigma_0|^3 |\sigma_1|^2 \quad + \frac{1}{2} \delta k_2 + \delta L |\sigma_1| - |\sigma_1|^2.
\]
The use of the inequalities, derived from (19), valid for $\forall \gamma_i > 0$

$$|\sigma_0|^2 |\sigma_1|^2 \leq \frac{3}{7} \gamma_1^2 |\sigma_0|^2 + \frac{4}{7} \gamma_1^2 |\sigma_1|^2,$$

$$|\sigma_0|^2 |\sigma_1|^4 \leq \frac{6}{7} \gamma_1^2 |\sigma_0|^2 + \frac{5}{7} \gamma_1^2 |\sigma_1|^2,$$

$$|\sigma_0|^2 |\sigma_1|^4 \leq \frac{2}{5} \gamma_1^2 |\sigma_0|^2 + \frac{5}{3} \gamma_1^2 |\sigma_1|^2,$$

$$|\sigma_0|^2 |\sigma_1|^4 \leq \frac{5}{5} \gamma_1^2 |\sigma_0|^2 + \frac{5}{3} \gamma_1^2 |\sigma_1|^2,$$

leads to

$$V_2(\sigma) \leq -\eta_10 |\sigma_0|^2 \eta_1 |\sigma_1|^2 + \eta_2 (|\sigma_0|^2 + |\sigma_1|^2),$$

where

$$\eta_{10} (\gamma_1, \gamma_2) = \left( \frac{3}{7} \mu^2 \delta k_1 k_2 - \frac{3}{7} |\mu k_1|^2 - \frac{12}{7} \mu^2 k_2 \gamma_2 \right),$$

$$\eta_{11} (\gamma_1, \gamma_2) = \left( 1 - \frac{4}{7} \mu k_1 \gamma_1 \gamma_2 - \frac{2}{7} \mu k_2 \gamma_2 \right),$$

$$\eta_2 = \frac{2}{5} \left( \frac{k_1}{\xi} + \frac{1}{3} \frac{\gamma}{\mu} \right) \mu k_2 + \frac{4}{3} \mu k_2, \eta_3 = \frac{4}{9} (2L + k_2) \zeta_3,$$

$$\gamma_6 = \zeta_3^2,$$ where $\zeta_3$ is the positive real root of $2 + 9 \beta \zeta^2 = 4 \xi^2$, given explicitly by

$$\zeta_3 = \frac{\beta^2 + 3 \delta \beta + 9 \delta^2}{4 \beta}, \beta = \left( 16 + 27 \xi^2 + 4 \sqrt{2 \xi} \right)^{\frac{1}{2}}.$$