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Quasi-continuous HOSM control for systems with unmatched perturbations ☆

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ABSTRACT

The control of nonlinear systems subject to unmatched perturbations is studied. A new design algorithm is proposed based on the block control and quasi-continuous higher order sliding modes techniques. The proposed method provides for the finite time exact tracking of a smooth desired signal in spite of unmatched perturbations.

1. Introduction

Traditionally sliding mode (SM) control is used in many applications; in nonlinear plants it enables high accuracy tracking and insensitivity to disturbances and plant parameter variations, e.g. (Utkin, 1992). Nevertheless, classical SM are not able to compensate unmatched perturbations, e.g. (Utkin, 1992).

Combinations of different robust techniques and SM have been applied to deal with systems with unmatched uncertainties. A design method is developed in Choi (2003) where the LMI-based switching surface is used. For uncertain nonlinear systems in strict-feedback form, Krstic, Kanellakopoulous, and Kokotovic (1995) develop the backstepping approach in a step-by-step design algorithm. The structure of the system allows considering, in each step, some states as a virtual control. Thus a virtual control based on Lyapunov methods is constructed in each step. The combination of the backstepping design and SM mode control is studied in Bartolini, Ferrara, Giacomini, and Usai (1996) for systems in strict-feedback form with parameter uncertainties and extended to the multi-input case in Ferrara and Giacomini (1998). The procedure proposed in Bartolini et al. (1996) and Ferrara and Giacomini (1998) reduces the computational load, as compared to the standard backstepping strategy, because it only retains $n-2$ steps of the original backstepping technique, coupling them with an auxiliary second order subsystem to which a second order SM control is applied. In Scarratt et al. (2000) the combination of dynamical adaptive backstepping and first and second order SM controls is applied to both triangular and nontriangular observable minimum phase nonlinear systems with parameter uncertainties, achieving asymptotic stability.

Another approach to the problem of unmatched uncertainty compensation is based on the Nonlinear Block Controllable form (NBC-form) (Loukianov, 2002). In Loukianov (2002) the SM technique is applied to compensate the matched perturbations. A high gain approach is used to achieve compensation of unmatched uncertainties and stabilization of the sliding mode dynamics. In Huerta-Avila, Loukianov, and Cañedo (2007) a SM controller is designed using the combination of block control (Loukianov, 1998), a sigmoid approximation to the integral SM control (Utkin, Guldner, & Shi, 1999) and nested SM control (Adhami-Mirhosseini & Yazdanpanah, 2005). A coordinate transformation is applied to design a nonlinear sliding manifold. This transformation requires smoothness of each virtual control; that is why sigmoid, instead of signum functions are used. In the proposals in Adhami-Mirhosseini and Yazdanpanah (2005), Huerta-Avila et al. (2007) and Loukianov (2002) they prove the convergence to a zone that depends on the high gain.

In this paper a new design algorithm for systems in strict-feedback form, a special case of the NBC-form, is proposed. *Finite time exact tracking* of the desired output is achieved with this algorithm in spite of the presence of unmatched perturbations i.e. parameter uncertainties and external disturbances. In the first step, the desired dynamics for the first state is defined by the desired tracking signal. After the first step, the desired dynamics for each state is defined by the previous one. Each virtual control is divided into two parts, the first one is intended to compensate the nominal nonlinear part of the system and the second one is

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aimed to achieve the desired dynamics in spite of perturbations. Since the reference signal and the perturbations are supposed to be bounded, the virtual controls are bounded too. The states acting as virtual controls are bounded. In the second part the quasi-continuous (QC) higher order sliding mode (HOSM) (Levant, 2005; Shtessel, Shkolnikov, & Levant, 2007) is used for unmatched uncertainty compensation. A key feature of the design algorithm proposed in this paper is the use of information on the nominal system used in the virtual controllers in order to reduce the control effort in comparison with the direct application of HOSM (Levant, 2005). In addition, the QC-HOSM controllers were chosen because, even when they are theoretically discontinuous, in the presence of switching delays, measurement noises and singular perturbations, they produce continuous control and produces less chattering than other HOSM controllers.

This paper proceeds as follows. In Section 2 the class of systems studied and the problem formulation are introduced. Section 3 begins by presenting the QC controllers as proposed in Levant (2005). The section continues by introducing the hierarchical design algorithm proposed in this work. Section 3 ends with the convergence proof of the proposed algorithm. In Section 4 the algorithm is applied and simulations results are presented. The note then concludes with a brief comment on the proposed algorithm.

2. Problem statement

Consider a class of nonlinear systems, with constant relative degree, presented in the special NBC-form (Loukianov, 2002):

$$\left.\begin{array}{l}\dot{x}_1 = f_1(x_1,t) + B_1(x_1,t)x_2 + \omega_1(x_1,t)\\ \dot{x}_i = f_i(\bar{x}_i,t) + B_i(\bar{x}_i,t)x_{i+1} + \omega_i(\bar{x}_i,t)\\ \dot{x}_n = f_n(x,t) + B_n(x,t)u + \omega_n(x,t) \qquad i = 2,\ldots,n-1\end{array}\right\} \tag{1}$$

where $x \in R^n$ is the state vector, $x_i \in R, \bar{x}_i = [x_1 \cdots x_i]^T$; $u \in R$ is the control. Moreover f_i and B_i are smooth scalar functions, ω_i is a bounded unknown perturbation term due to parameter variations and external disturbances with at least $n-i$ bounded derivatives w.r.t system (1), $B_i \neq 0\ \forall x \in X \subset R^n, t \in [0,\infty)$. The output is $y = x_1$. The control problem is to design a controller such that the output y tracks a smooth desired reference y_d with bounded derivatives, in spite of the presence of unknown bounded perturbations. The whole state vector x is assumed to be known. In this paper, the solutions of all differential equations and inclusions are understood in the Filippov sense (Filippov, 1988).

3. Hierarchical quasi-continuous controller design

In this section the QC homogeneous controllers reported in Levant (2005) are presented; after that the hierarchical controller design algorithm is introduced. Finally, the convergence proof is included.

3.1. Quasi-continuous controller (Levant, 2005)

Consider a Single-Input–Single-Output system

$$\begin{array}{l}\dot{\xi} = a(t,\xi) + b(t,\xi)u, \quad \xi \in R^m,\ u \in R\\ \sigma : (t,\xi) \mapsto \sigma(t,\xi) \in R\end{array} \tag{2}$$

where σ is the measured output of the system, u is the control. The smooth functions a, b, σ are assumed to be unknown, the dimension m can also be uncertain. The task is to make σ vanish in finite time by means of a possibly discontinuous feedback and to keep $\sigma \equiv 0$. It is assumed that system (2) has relative degree r, constant and known. As follows from (Isidori, 1989) the equation

$$\sigma^{(r)} = h(t,\xi) + g(t,\xi)u, \quad g(t,\xi) \neq 0 \tag{3}$$

holds, where $h(t,\xi) = \sigma^{(r)}|_{u=0},\ g(t,\xi) = \frac{\partial}{\partial u}\sigma^{(r)}$. The uncertainty prevents immediate reduction of (2) to (3). Suppose that the inequalities

$$0 < K_m \leq \frac{\partial}{\partial u}\sigma^{(r)} \leq K_M, \quad |\sigma^{(r)}|_{u=0}| \leq C \tag{4}$$

hold globally for some $K_m, K_M, C > 0$. Then (3), (4) imply the differential inclusion

$$\sigma^{(r)} \in [-C, C] + [K_m, K_M]u. \tag{5}$$

The bounded feedback control, u, is constructed such that the r-sliding mode $\sigma = \dot{\sigma} = \cdots = \sigma^{(r-1)} = 0$ is established in finite time. In order to reduce chattering, a controller is designed which is continuous everywhere except in this set. Such a controller is naturally called quasi-continuous.

In practice, in the presence of switching delays, measurement noises and singular perturbations, the motion will take place in some vicinity of the r-sliding set $\sigma = \dot{\sigma} = \cdots = \sigma^{(r-1)} = 0$ never hitting it, with $r > 1$. Denote

$$\begin{array}{l}\varphi_{0,r} = \sigma, \qquad N_{0,r} = |\sigma|, \qquad \Psi_{0,r} = \varphi_{0,r}/N_{0,r}\\ \varphi_{i,r} = \sigma^{(i)} + \beta_i N_{i-1,r}^{(r-i)/(r-i+1)}\Psi_{i-1,r}\\ N_{i,r} = |\sigma^{(i)}| + \beta_i N_{i-1,r}^{(r-i)/(r-i+1)}\\ \Psi_{i,r}(\cdot) = \varphi_{i,r}/N_{i,r}; \quad i = 0,\ldots,r-1.\end{array} \tag{6}$$

Theorem 1 (Levant, 2005)**.** *Provided that $\beta_1,\ldots,\beta_{r-1},\alpha > 0$ are chosen sufficiently large in the listed order, the above design result in the r-sliding homogeneous controller*

$$u = -\alpha\Psi_{r-1,r}(\sigma,\dot{\sigma},\ldots,\sigma^{(r-1)}) \tag{7}$$

providing for the finite time stability of (5), (7). The finite time stable r-sliding mode $\sigma \equiv 0$ is established in system (2), (7).

3.2. Design algorithm

At each step i the constraint $\sigma_i = 0$ is established and kept by means of the virtual control $x_{i+1} = \phi_i$, which forms the constraint $\sigma_{i+1} = x_{i+1} - \phi_i$ for the next step.

Step 1. Defining $x_2 = \phi_1$, the next virtual controller is constructed

$$\begin{array}{l}\phi_1(x_1,t,u_1) = B_1(x_1,t)^{-1}\{-f_1(x_1,t) + u_1\}\\ u_1^{(n-1)} = -\alpha_1\Psi_{n-1,n}(\sigma_1,\dot{\sigma}_1,\ldots,\sigma_1^{(n-1)})\end{array} \tag{8}$$

where $\sigma_1 = x_1 - y_d$ i.e. the nth order QC-HOSM, as defined in (6), is introduced in ϕ_1 through $n-1$ integrators. The derivatives $\sigma_1, \dot{\sigma}_1, \ldots, \sigma_1^{(n-1)}$ are calculated by means of robust differentiators with finite time convergence (Levant, 2003).

Remark 1. The values for each β_i are taken from Levant (2005); only α_i, for each QC controller, and the L parameter of the corresponding differentiator (Levant, 2003), have to be adjusted. The adjustment is made through simulation step-by-step.

Step i. The remaining virtual controls are constructed as follows.

$$\begin{array}{l}\phi_i(\bar{x}_i,t,u_i) = B_i(\bar{x}_i,t)^{-1}\{-f_i(\bar{x}_i,t) + u_i\}\\ u_i^{(n-i)} = -\alpha_i\Psi_{n-i,n-i+1}(\sigma_i,\dot{\sigma}_i,\ldots,\sigma_i^{(n-i)})\\ \sigma_i = x_i - \phi_{i-1}; \quad i = 2,\ldots,n.\end{array} \tag{9}$$

$\Psi_{n-i,n-i+1}$ is defined according to (6), with $\sigma = \sigma_i$. Notice that in *step n*, the real control is calculated i.e. $\phi_n = u$.

$$u = B_n(x,t)^{-1}\{-f_n(x,t) + u_n\}; \quad u_n = -\alpha_n \mathrm{sign}(\sigma_n).$$

It is possible to smooth the control signal by raising the order of the QC controller in each ϕ. If it is done, the super-twisting algorithm can also be used in u_n.

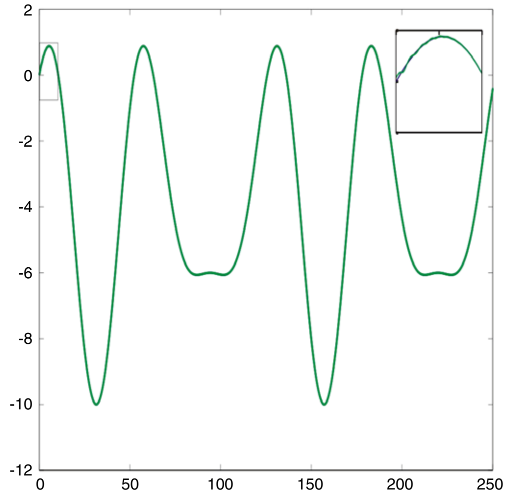

Fig. 1. Signals y_d, x_1.

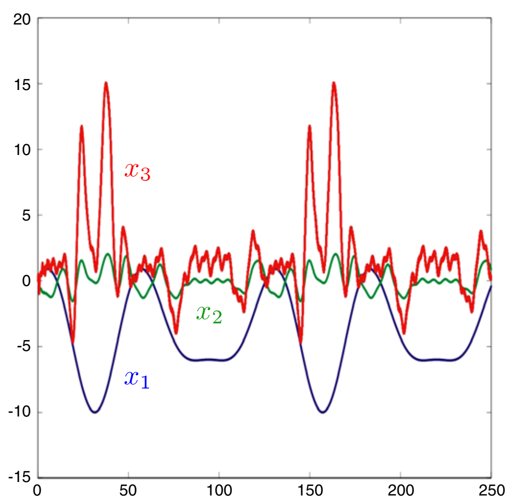

Fig. 2. States x_1, x_2, x_3.

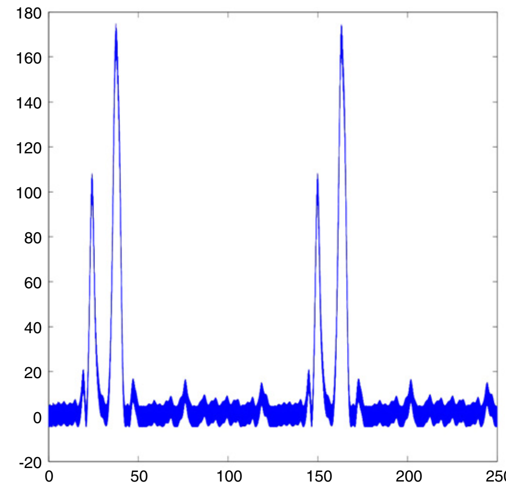

Fig. 3. Control signal u.

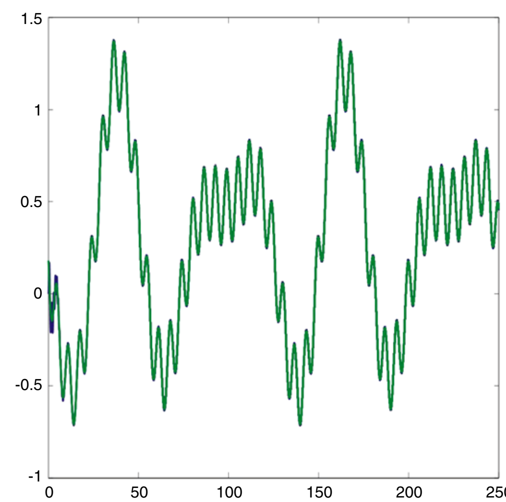

Fig. 4. $B_1 u_{11}, \dot{y}_d - \omega_1(\cdot)$.

Theorem 2. *Provided that $\omega_i(\bar{x}_i, t)$ in system (1) and y_d are smooth functions with $n-i$ and n bounded derivatives, respectively, the above hierarchic design results in an ultimate controller u, providing for the finite time stability of $\sigma_1 = x_1 - y_d = \dot{\sigma}_1 = \cdots = \sigma_1^{(n-1)} = 0$ in system (1).*

3.3. Convergence proof

• Consider the state x_n

$$\dot{x}_n = f_n(x, t) + B_n(x, t)u + \omega_n(x, t)$$

with $u = B_n(x, t)^{-1}\{-f_n(x, t) - \alpha_n \mathrm{sign}(\sigma_n)\}$;

$\sigma_n = x_n - \phi_{n-1}$; $\quad \phi_{n-1}$ sufficiently smooth.

Thus $\dot{\sigma}_n = -\alpha_n \mathrm{sign}(\sigma_n) + \omega_n(x, t) - \dot{\phi}_{n-1}$. Taking $\alpha_n \geq |\omega_n(x, t)| + |\dot{\phi}_{n-1}|$, provides for the appearance of a 1-sliding mode for the constraint σ_n.

• Now for the state x_{n-1}, we have

$$\begin{aligned}\dot{\sigma}_{n-1} &= \dot{x}_{n-1} - \dot{\phi}_{n-2} \\ &= f_{n-1}(\bar{x}_{n-1}, t) + B_{n-1}(\bar{x}_{n-1}, t)\phi_{n-1} \\ &\quad + \omega_{n-1}(\bar{x}_{n-1}, t) - \dot{\phi}_{n-2} \\ &= u_{n-1,1} + \omega_{n-1}(\bar{x}_{n-1}, t) - \dot{\phi}_{n-2}\end{aligned}$$

$$\ddot{\sigma}_{n-1} = \dot{u}_{n-1} + \dot{\omega}_{n-1}(\bar{x}_{n-1}, t) - \ddot{\phi}_{n-2} \tag{10}$$

and according to (9)

$$\dot{u}_{n-1} = -\alpha_{(n-1)} \Psi_{1,2}(\sigma_{n-1}, \dot{\sigma}_{n-1}). \tag{11}$$

That is (10) takes the form (3):

$$\ddot{\sigma}_{n-1} = h_{n-1}(t, x) + g_{n-1}(t, x)\dot{u}_{n-1} \tag{12}$$

with $h_{n-1}(t, x) = \ddot{\sigma}_{n-1}|_{\dot{u}_{n-1}=0} = \dot{\omega}_{n-1} - \ddot{\phi}_{n-2}$

$g_{n-1}(t, x) = \partial \ddot{\sigma}_{n-1} / \partial \dot{u}_{n-1}$.

If for some $K_{m_{n-1}}, K_{M_{n-1}}, C_{n-1} > 0$ the inequalities $0 < K_{m_{n-1}} \leq g_{n-1} \leq K_{M_{n-1}}$, $|h_{n-1}| \leq C_{n-1}$ are fulfilled, then the next differential inclusion is implied

$$\ddot{\sigma}_{n-1} \in [-C_{n-1}, C_{n-1}] + [K_{m_{n-1}}, K_{M_{n-1}}]\dot{u}_{n-1} \tag{13}$$

and controller (11) provides for the finite time stability of (13), (11). The finite time stable 2-sliding mode is established for the constraint σ_{n-1}.

• It is possible to obtain an analogous equation to (10) for each of the remaining states, thus for the state x_1

$$\sigma_1^{(n)} = h_1(t, x) + g_1(t, x)u_1^{(n-1)} \tag{14}$$

$$u_1^{(n-1)} = -\alpha_1 \Psi_{n-1,n}(\sigma_1, \dot{\sigma}_1, \ldots, \sigma_1^{(n-1)}) \tag{15}$$

$$\sigma_1^{(n)} \in [-C_1, C_1] + [K_{m_1}, K_{M_1}]u_1^{(n-1)} \tag{16}$$

the differential inclusion (16) is implied for some constants K_{m_1}, K_{M_1} and C_1. The controller (15) provides for the finite time stability of (16). The finite time stable n-sliding mode is established for the constraint σ_1.

Due to the dependence on states of functions in (14), the inclusion (16) may be ensured only locally. The same applies to the inclusion obtained for each virtual control.

4. Example

Consider the perturbed third order system

$$\begin{aligned}
\dot{x}_1 &= 2\sin(x_1) + 1.5x_2 + \omega_1(x_1, t)\\
\dot{x}_2 &= 0.8x_1x_2 + x_3 + \omega_2(\bar{x}_2, t)\\
\dot{x}_3 &= -x_3^2 + 2u + \omega_3(x, t)\\
\omega_1(x_1, t) &= 0.2\sin(t) + 0.1x_1 + 0.12\\
\omega_2(\bar{x}_2, t) &= 0.3\sin(2t) + 0.2x_1 + 0.2x_2 - 0.4\\
\omega_3(x, t) &= 0.2\sin(2t) + 0.2x_1 + 0.3x_2 + 0.2x_3 + 0.3.
\end{aligned}$$

Tracking of $y_d = 2\sin(0.15t) + 4\cos(0.1t) - 4$ by x_1 is desired.

$$\begin{aligned}
\phi_1 &= \frac{1}{1.5}\{-2\sin(x_1) + u_1\}; \qquad \dot{u}_1 = -\alpha_1\Psi_{2,3}(\sigma_1, \dot{\sigma}_1, \ddot{\sigma}_1)\\
\Psi_{2,3} &= \frac{\ddot{\sigma}_1 + 2(|\dot{\sigma}_1| + |\sigma_1|^{2/3})^{-1/2}(\dot{\sigma}_1 + |\sigma_1|^{2/3}\mathrm{sign}(\sigma_1))}{|\ddot{\sigma}_1| + 2(|\dot{\sigma}_1| + |\sigma_1|^{2/3})^{1/2}}\\
\phi_2 &= -0.8x_1x_2 + u_{21}; \qquad \ddot{u}_2 = -\alpha_2\Psi_{1,2}(\sigma_2, \dot{\sigma}_2)\\
\Psi_{1,2} &= \frac{\dot{\sigma}_2 + |\sigma_2|^{1/2}\mathrm{sign}(\sigma_2)}{|\dot{\sigma}_2| + |\sigma_2|^{1/2}}\\
u &= \frac{1}{2}\{x_3^2 + u_3\}; \quad u_3 = -\alpha_3\mathrm{sign}(\sigma_3)
\end{aligned}$$

where $\sigma_1 = x_1 - y_d, \sigma_2 = x_2 - \phi_1$ and $\sigma_3 = x_3 - \phi_2$. Results obtained by simulation are shown in Figs. 1–4, using $\alpha_1 = 4.5, \alpha_2 = 3.6$ and $\alpha_3 = 9$. Since $\sigma_1 = x_1 - y_d$, straightforward algebra reveals that $B_1u_{11} = \dot{y}_d - g_1(\cdot)$ has to be accomplished in order to achieve that x_1 tracks y_d, these signals are shown in Fig. 4.

5. Conclusion

The compensation of unmatched perturbations via NBC-form usually requires the use of a smooth approximation of SM control as virtual controllers, for example: high gain approximation (Loukianov, 2002), sigmoid functions instead of signum functions (Adhami-Mirhosseini & Yazdanpanah, 2005; Huerta-Avila et al., 2007). The approximative nature of such controllers cannot ensure the exact compensation of the unmatched perturbations. In this paper, we proposed the corresponding order integrated QC-HOSM controllers as the virtual control. It is shown that the proposed control law ensures exact tracking w.r.t. unmatched perturbations.

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