Interval estimation for LPV systems applying high order sliding mode techniques

Denis Efimov, Leonid Fridman, Tarek Raïssi, Ali Zolghadri, Ramatou Seydou

1. Introduction

The problem of state estimation for nonlinear systems is very challenging and has been extensively studied in the literature, see for instance Besançon (2007), Fossen and Nijmeijer (1999) and Meurer, Graichen, and Gilles (2005). Although a complete palette of solutions exists for linear systems, in the nonlinear case mainly particular approaches are available. Many solutions are based on the system representation in a canonical form (frequently, partially linear). For nonlinear systems, it has been shown that an LPV equivalent representation can be an appealing alternative to deal with the original nonlinear system (Marcos & Balas, 2004; Shamma & Cloutier, 1993; Tan, 1997). The basic idea is to replace the nonlinear complexity of the original system by an enlarged parametric variation in the LPV representation, which may simplify the observer design. There are several approaches to design observers for LPV systems (Bernard & Gouzé, 2004; Jaulin, 2002; Kieffer & Walter, 2004; Moisan, Bernard, & Gouzé, 2009). The work presented in this paper falls within the scope of interval observers (Bernard & Gouzé, 2004; Moisan et al., 2009). This approach has been recently extended in Raïssi, Videau, and Zolghadri (2010) to nonlinear systems using LPV representation with known minorant and majorant matrices, and in Raïssi, Efimov, and Zolghadri (2012) for observable nonlinear systems relaxing the requirement on cooperativity (monotonicity) of the original system dynamics. Basically, the interval observers propagate the parameter uncertainty in the width of the estimated interval for the state values. So, the length of the interval determines the estimation accuracy of the approach. This is why decreasing the uncertainty is very important for improvement of the interval (set-membership) estimation performance. This key feature is the subject studied in this paper.

On the other hand, the HOSM techniques have become very popular for design of observers for linear and nonlinear systems (Barbot, Boutat, & Floquet, 2009; Bejarano & Pisano, 2011; Efimov & Fridman, 2011; Moreno & Osorio, 2012; Pisano & Usai, 2011; Shtessel, Baev, Edwards, & Spurgeon, 2010). The sliding modes ensure a finite time of the estimation error convergence to zero and complete insensitivity to a matched uncertainty (Barbot, 2008; Edwards & Spurgeon, 1998; Perruquet & Barbot, 2002). Mainly these features can be achieved under the assumption that the system is strongly observable or strongly detectable (Bejarano & Pisano, 2011). The objective of this work is to combine both...
approaches (the interval observers and the HOSM techniques) in order to improve the accuracy of estimation achieved by interval observers. The idea is that under a linear transformation of coordinates, an LPV system always has a strongly observable subsystem. Applying the HOSM differentiation approach it is possible to estimate the state and the state derivative for this subsystem, which can be further used for improved evaluation of the input and the parameter uncertainty in the rest of the system. This combination leads to a significant decrease of the interval estimation conservatism. Moreover, a relaxation of some applicability constraints usually met in interval estimation can be obtained.

The paper is organized as follows. The next section is devoted to some preliminaries. The main result is described in Section 3. Finally, Section 4 provides some simulation results.

2. Preliminaries

The Euclidean norm for a vector $x \in \mathbb{R}^n$ will be denoted as $|x|$, and for a measurable and locally essentially bounded input $u : \mathbb{R}_+ \to \mathbb{R}$, $r \geq 0$ the symbol $\|u\|_{L_1, t_0} = \|u(t)|t \in \mathbb{R}_+, t_0\|$ is accessible for measurements, in the following this assumption is made. Applying the HOSM differentiation approach it is possible to improve the accuracy of estimation achieved by interval observers and the HOSM techniques (the interval observers and the HOSM techniques) for unknown input estimation and compensation in linear systems has been studied in [11], an extension to nonlinear systems is presented in [12].

2.2. Interval estimation

For two vectors $x_1, x_2 \in \mathbb{R}^n$ or matrices $A_1, A_2 \in \mathbb{R}^{n \times n}$, the relations $x_1 \leq x_2$ and $A_1 \leq A_2$ are understood elementwise. Given a matrix $A \in \mathbb{R}^{n \times n}$ or a vector $x \in \mathbb{R}^n$, define $A^+ = \max(A, 0)$, $A^- = A^+ - A$ to $x^+ = \max(0, x)$, $x^- = x^+ - x$ respectively.

Lemma 2. Let $x \in \mathbb{R}^n$ be a vector variable, $\underline{x} \leq x \leq \bar{x}$ for some $\underline{x}, \bar{x} \in \mathbb{R}^m$.

(1) If $A \in \mathbb{R}^{m \times n}$ is a constant matrix, then

$$A^+ \bar{x} - A^- \underline{x} \leq Ax \leq A^+ \bar{x} - A^- \underline{x}.$$  

(2) If $A \in \mathbb{R}^{m \times n}$ is a matrix variable, $\underline{A} \leq A \leq \bar{A}$ for some $\underline{A}, \bar{A} \in \mathbb{R}^{m \times n}$, then

$$A^+ \bar{x} - A^- \underline{x} \leq Ax \leq A^+ \bar{x} - A^- \underline{x} - \bar{A}^+ \bar{x} + \underline{A}^{-} \underline{x} - \bar{A}^+ \bar{x} + \underline{A}^{-} \underline{x}.$$  

Proof. To prove the first part note that $Ax = (A^+ - A^-)x$, which for $\underline{x} \leq x \leq \bar{x}$ gives the required inequalities. The proof of the second part is based on (3), where $A^+$ and $A^-$ are the functions with $\underline{A} \leq A \leq \bar{A}$. The relations (3) can be rewritten as follows:

$$A^+ (x^+ - x^-) - A^- (x^+ - x^-) \leq Ax \leq A^+ (x^+ - x^-) - A^- (x^+ - x^-),$$

which for $\underline{A} \leq A \leq \bar{A}$ gives the desired inequalities. □

A matrix $A \in \mathbb{R}^{n \times n}$ is called Hurwitz if all its eigenvalues have a negative real part, it is called Metzler if all its elements outside the main diagonal are non-negative. Any solution of the linear system $\dot{x} = Ax + \omega(t), \quad \omega : \mathbb{R}_+ \to \mathbb{R}_n^p$ with $x \in \mathbb{R}^n$ and a Metzler matrix $A$, is elementwise non-negative for all $t \geq 0$ provided that $\omega(0) \geq 0$ (Smith, 1995). Such dynamical systems are called cooperative (monotone) (Smith, 1995).

3. Main result

For brevity of presentation the case $p = 1$ is considered only (the case of vector measurements can be treated similarly). We will need the following assumptions.

Assumption 2. For all $\theta \in \Theta$, there is an invertible matrix $S(\theta)$ such that the system (1) can be represented as follows:

$$x = S(\theta) z_1, \quad y = c^T z_1,$$

$$\dim(z_1) = n_1, \quad \dim(z_2) = n_2, \quad n_1 + n_2 = n,$$

$$\dot{z}_1 = A_0 z_1 + b_0 s_1(\theta)^T z_2 + a_0(\theta)^T u,$$

$$\dot{z}_2 = A_2(\theta) z_1 + A_2(\theta) z_2 + B(\theta) u,$$
where
\[ c = [10 \ldots 0]^T, \quad b_0 = [0 \ldots 01]^T, \]
\[
A_0 = \begin{bmatrix}
0 & 1 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\]
is a canonical representation, the vector functions \(a_{11}(\theta), a_{12}(\theta), b_1(\theta)\) and the matrix functions \(A_{21}(\theta), A_{22}(\theta), B_2(\theta)\) have corresponding dimensions.

Assumption 2 states that there exists a transformation coordinate, which represents the system (1) as a pair of interconnected subsystems (5) and (6). The subsystem (5) is strongly observable since it has the canonical representation \(c, A_0, b_0\) (the conditions of existence of such a transformation for linear time-invariant systems are analyzed in Bejarano and Pisano (2011)). However, the system is not necessarily detectable (the dynamics of (1) could be non-minimum phase as in Shtessel et al. (2010)) since there is no requirement on stability of the matrix function \(A_{22}(\theta)\). This relaxation may be important for application of an interval observer design method for estimation in uncertain non-minimum phase systems. It is worth stressing that for \(n_1 = 1\) this assumption is always true (at least the output coordinate can be chosen in the vector \(z_1\), i.e. the linear systems always have a strongly observable subsystem).

Remark 3. Instability of the matrix function \(A_{22}(\theta)\) does not contradict Assumption 1 (with the boundedness of the state \(x\) and the inputs \(u, v\)). Indeed, as we will show below (see Remark 7) this assumption can be introduced on a finite time interval only, then the system can be unstable (as in the third example). In addition, the matrix \(A_{22}(\theta)\) is uncertain and possibly time-varying, then due to the incertainty it may be neither stable nor unstable (as in the first example, when for the chosen initial conditions the system has bounded trajectories). Finally, the system may be non-minimum phase or unstable, but the control input \(u\) may stabilize it, guaranteeing Assumption 1, while for the observer synthesis the expression of \(u\) is not available and only the signal \(u(t)\) is measured (the control law \(u\) includes some variables which are not accessible for a locally designed observer in the distributed system, for example). Thus in all these cases the matrix \(A_{22}(\theta)\) can be unstable, but the system trajectories are bounded (at least locally in time) and Assumption 1 is satisfied.

Assumption 3. Let there exist a vector function \(f(\theta) \in \mathbb{R}^{n_2}\) such that
\[
A_{22}(\theta)z_2 + B_2(\theta)u - f(\theta)a_{12}(\theta)^Tz_2 + b_1(\theta)^Tu
= A_1z_2 + A_2z_1
\]
for some Hurwitz matrix \(A_1 \in \mathbb{R}^{n_2 \times n_2}\) and \(A_2 : \Theta \to \mathbb{R}^{n_2 \times m}\).

This assumption states that the matrix \(A_1 = A_{22}(\theta) - f(\theta)a_{12}(\theta)^T\) is Hurwitz, i.e. the matrix \(A_{22}(\theta)\) can be stabilized by an output feedback or the pair of matrices \((A_{22}(\theta), a_{12}(\theta)^T)\) is observable for all \(\theta \in \Theta\), and independent in \(\Theta\) (for an LMI verification of observability of interval matrices see the recent work Lee, Park, and Joo (2010) and references therein).

Assumption 4. There exists a matrix \(P \in \mathbb{R}^{n_2 \times n_2}\) such that the matrix \(D = P^{-1}A_1P\) is Hurwitz and Metzler (H&M).

Under mild conditions of the main result in Raïssi et al. (2012), in the case of Assumption 3 there is a matrix \(P \in \mathbb{R}^{n_2 \times n_2}\) such that \(D\) is H&M, as it is stated in Assumption 4 (the approach proposed in Raïssi et al. (2012) to calculate the matrix \(P\) is based on solution of a Sylvester equation). In the paper Mazenc and Bernard (2011) it is shown that always there exists a time varying similarity transformation \(P(t)\) such that \(D = P(t)^{-1}A_1P(t)\) for an H&M matrix \(D\).

Remark 4. The introduced assumptions are less restrictive than usually stated to design an HOSM or an interval observer. Indeed, an application of the techniques from Barbot (2008), Pisano and Usai (2011) and Shtessel et al. (2010) for (5), (6) is blocked due to the dependence on \(\theta\) of all matrices in the system equations. The solutions from Bernard and Gouzé (2004), Moisian et al. (2009) are hard to apply for a non-detectable and non-cooperative system. However, a combination of both approaches allows us to design an interval observer for the LPV system (5), (6) with a non-minimum phase internal dynamics.

Under these assumptions it is proposed to use the differentiator (2) to estimate the state \(z_1\) and its derivative \(z_1\), then from (5) we get an improved estimate on the signal \(a_{12}(\theta)^Tz_2 + b_1(\theta)^Tu\), which can be applied for design of an interval observer for the system (6) in the new coordinates \(r = P^{-1}z_2\). Let us consider these steps consequently.

Under Assumption 2 the output \(y\) of the system (5) has \(n_1\) derivatives. Therefore according to Theorem 1 and Assumption 1, there exist parameters \(\phi_k, k = 0, n_1\) in (2) with \(s = n_1\) and \(T > 0\) such that for all \(t \geq T\):
\[
|q_k(t) - y(\theta)(t)| \leq \mu_k \sqrt{\frac{1}{m+1}}, \quad k = 0, n_1
\]
for some constant \(\mu_k, k = 0, n_1\). Thus \(z_2(t) = \hat{z}_1(t) + e_1(t)\) and \(\hat{z}_1, n_1, e_1(t) = q_1(t) + e_2(t)\) for all \(t \geq T\), where \(\hat{z}_1(t) = q_{n_1}(t) = -\mu_{n_1} \sqrt{1 + 1}\). The variables \(\hat{z}_1\) and \(q_1\) are generated by the HOSM differentiator (2), thus they are available for a designer. The errors \(e_1\) and \(e_2\) are upper bounded by some functions of \(V\). Substitution of these variables into the last equation of (5) gives \(q_{n+1} = e_1 = a_{11}(\theta)^T[\hat{z}_1 + e_1\] + \(a_{12}(\theta)^Tz_2 + b_1(\theta)^Tu\), or equivalently
\[
a_{12}(\theta)^Tz_2 + b_1(\theta)^Tu = q_{n_1} + e_2 = a_{11}(\theta)^T[\hat{z}_1 + e_1\].

Substituting this equality in the differential equation (6) we obtain
\[
\dot{z}_2 = A_1z_2 + [A_{21}(\theta) - f(\theta)a_{12}(\theta)^T][\hat{z}_1 + e_1 + e_2] + f(\theta)(q_{n_1} + e_2 + A_2z_1)u, \quad (7)
\]
which is a stable system according to Assumption 3.

Applying the transformation of coordinates \(r = P^{-1}z_2\), the system (7) can be rewritten as follows
\[
\dot{r} = Dr + G_1(\theta)(\hat{z}_1 + e_1) + G_2(\theta)(q_{n_1} + e_2) + G_3(\theta)u, \quad (8)
\]
where \(G_1(\theta) = P^{-1}[A_{22}(\theta) - f(\theta)a_{12}(\theta)^T], G_2(\theta) = P^{-1}f(\theta)\) and \(G_3(\theta) = P^{-1}A_1(\theta)\). The dynamics of (8) is cooperative and stable, and all uncertain functions or variables in the right hand side of (8) belong to an interval for \(\theta \in \Theta\):
\[
G_1(\theta) \in G_1, \quad j = 1, 3; \quad |u(t)| \leq U;
\]
\[
|e_{1,i}(t)| \leq E_{1,i}, \quad i = 1, n_1;
\]
\[
|e_{2,i}(t)| \leq E_{2,i}, \quad i = 1, \mu_n \sqrt{1 + 1}
\]
for all \(t \geq T\), where the matrices \(G_j, E_j, j = 1, 3\) are known. Therefore the following interval observer can be synthesized for (8):
\[
\dot{\hat{r}} = D\hat{r} + \left(G_{1i} - G_{1i}\right)\hat{z}_1 + \left(G_{3i} - G_{3i}\right)\hat{z}_i
+ (\hat{G}_{1i} + G_{1i})\hat{e}_1 + (\hat{G}_{3i} + G_{3i})\hat{q}_{n_1}
+ (G_{2i} - G_{2i})e_1 + (G_{3i} - G_{3i})\hat{e}_2 + (G_{3i} + G_{3i})U, \quad (9)
\]
\[
\dot{\hat{r}} = D\hat{r} + \left(G_{1i} - G_{1i}\right)\hat{z}_1 + \left(G_{3i} - G_{3i}\right)\hat{z}_i
- (G_{1i} + G_{1i})\hat{e}_1 + (G_{3i} + G_{3i})\hat{q}_{n_1}
+ (G_{2i} - G_{2i})\hat{e}_1 + (G_{3i} + G_{3i})\hat{e}_2 - (G_{3i} + G_{3i})U, \quad (10)
\]
the properties (3), (4) have been used to calculate (9), (10). Introducing the interval estimation errors $\tau = \hat{\tau} - r$, $\xi = r - \xi$, we obtain
\[
\dot{\xi} = D\xi + \xi, \quad \dot{\hat{\tau}} = D\hat{\tau} + \hat{\tau},
\]
where $\tau = (G^+ - G^-) \hat{x}^2 + (G^+ - G^-) \hat{x}^1 + (G^+ + G^-) \hat{x}^1 + (G^+ - G^-) q_n^1 + (G^+ - G^-) q_n^2 + (G^+ + G^-) \hat{q}_2 + (G^+ - G^-) U - G^0 \left( \hat{x}_1 + e_1 \right) - G^0 \left( q_n^1 + e_2 \right) - G^0 \left( q_n^2 + e_2 \right) \right) \hat{x}_1 + (G^+ - G^-) q_n^1 + (G^+ - G^-) q_n^2 + (G^+ + G^-) \hat{q}_2 + (G^+ - G^-) U - G^0 \left( \hat{x}_1 + e_1 \right) - G^0 \left( q_n^1 + e_2 \right) - G^0 \left( q_n^2 + e_2 \right).$

Remark 6. It is an arithmetic exercise to verify that under Assumptions 1 and 2 (and the result of Theorem 1) the residual terms $\tau$ and $\xi$ are element-wise positive and bounded. Then using the results of monotone system theory (Smith, 1995) we prove that for all $t \geq T$
\[
\xi(t) \leq r(t) \leq \hat{\tau}(t)
\]
and the estimates $\hat{\tau}(t), \hat{\tau}(t)$ are bounded, provided that
\[
\hat{\tau}(t) \leq r(t) \leq T(t).
\]
(11)

The former relation for the initial conditions can be easily satisfied since $|\|x\| | \leq X$ under Assumption 1. Using property (3) we get for all $t \geq T$:
\[
\xi_2(t) = \xi_2(x) \leq \xi_2(t), \quad \xi_2(t) = P^\tau(t) - P^{-\hat{\tau}}(t), \quad \xi_1(t) \leq \xi_1(t), \quad \xi_1(t) = \xi_1(x) - \xi(t), \quad \xi(t) = \hat{x}(t) + \xi(t).
\]

Defining $\xi = [\xi_2 \xi_1]^T, \xi = [\xi_2 \xi_1]^T$ and using (4) we can formulate the interval estimates for the state $x$:
\[
\begin{array}{l}
\hat{x}^T - \xi^T - \xi^T - \xi^T + \xi^T \leq x = S(\theta) z \\
\end{array}
\]
\[
\leq \hat{x}^T - \xi^T - \xi^T - \xi^T + \xi^T,
\]
(12)

which is satisfied for all $t \geq T$. Thus we have proven the following theorem.

Theorem 5. Let Assumptions 1–4 hold for the system (1). Then there exist the set of parameters $\lambda_k, k = 0, n_1$ in (2) and a constant $T > 0$ such that for all $t \geq T$ the interval estimate (12) is true, provided that the condition (11) is satisfied for (9), (10).

Remark 6. The Assumptions 3 and 4 can be skipped if we assume the existence of a vector function $f(\theta) \in \mathbb{R}^n$ such that
\[
[a_2(\theta) x_2 + b_2(\theta) u] - f(\theta) [a_1(\theta) x_2 + b_1(\theta) u]
\]
\[
= \lambda_1(\theta) x_2 + \lambda_2(\theta) u
\]
for some Hurwitz and Metzler matrix function $\lambda_1 : \Theta \rightarrow \mathbb{R}^{n_2 \times n_2}$ and some $\lambda_2 : \Theta \rightarrow \mathbb{R}^{n_2 \times n_3}$. Next, the result of Theorem 5 can be obtained using the same technique and an interval observer from the paper Raissi et al. (2010).

Remark 7. Since the interval estimates are obtained in a finite time, then Assumption 1 can be relaxed introducing the requirement on boundedness of the state $x$ during a finite time interval only for $t \in [0, T_2)$ with $T_2 \leq T$, where $T$ is defined in Theorem 5. Implicitly the conditions of Theorem 5 mean that the interval observer (9), (10) has to be activated for $t \geq T$ only. The time $T$ can be detected on-line using the property
\[
\sup_{t \in [0, T_2)} |\psi(t) - \psi(t)| \leq \theta,
\]
where $\theta > 0$ is a constant dependent on the discretization step used for computation of (2) ($\theta = 0$ under assumption that the differential equation (2) is solved without a computational error) (Levant, 2003).

4. Examples

To illustrate the improvement achieved in interval estimation by application of HOSM techniques consider three examples. Two examples deal with non-minimum phase systems, the conventional techniques for the interval observer design (Bernard & Gouzé, 2004; Moisan et al., 2009; Raissi et al., 2012) cannot be applied in this case. And one minimum phase example is presented in order to compare the estimation accuracy for different interval observers.

4.1. Non-minimum phase second order system

In order to explain the peculiarities of the proposed solution, let us start with analysis of a second order academic example:
\[
\begin{align*}
\dot{x}_1 &= -a_{11}(\theta) x_1 + a_{12}(\theta) x_2 + b_1(\theta) u; \\
\dot{x}_2 &= a_{21}(\theta) x_1 + a_{22}(\theta) x_2 + b_2(\theta) u; \\
y &= x_1,
\end{align*}
\]
where $x_1 \in \mathbb{R}, x_2 \in \mathbb{R}$ are the state variables, and for all $\theta \in \Theta$
\[
0.5 \leq a_{11}(\theta) \leq 1.5, \quad 1 \leq a_{12}(\theta) \leq 3, \\
-1 \leq a_{21}(\theta) \leq 1, \quad -0.5 \leq a_{22}(\theta) = 0.5a_{12}(\theta) - 1 \leq 0.5, \\
0.75 \leq b_1(\theta) \leq 1, \quad 0.5 \leq b_2(\theta) \leq 1, \quad U = 0.25,
\]
$V = 0.03$.

As we can see, the system (13) is already in the form (5), (6) with $x_1 = z_1$ and $x_2 = z_2$ (the matrix $S(\theta)$ equals to the identity, and Assumption 2 is satisfied). For simulation we use
\[
\begin{align*}
a_{11}(\theta) &= 1 + 0.5 \sin(3\pi t), \quad a_{12}(\theta) = 2 + \sin(2\pi t), \\
a_{21}(\theta) &= 0.5, \quad b_1(\theta) = 0.5, \\
b_2(\theta) &= 0.875 + 0.125 \cos(0.5t),
\end{align*}
\]
\[
u(t) = U \sin(2\pi t), \quad u(t) = V \sin(10t), \quad \theta = [x_1 x_2]^T.
\]

For the system (13) with the chosen parameters and the given input $u$ the state is bounded (Assumption 1 holds). The initial condition uncertainty is $-4 \leq x_2(0) \leq 4$. It is easy to verify that for $f = 0.5$ we have
\[
[a_{22}(\theta) x_2 + b_2(\theta) u] - f[a_{12}(\theta) x_2 + b_1(\theta) u] = \Delta_1 x_2 + \Delta_2(\theta) u
\]
for $\Delta_1 = -1$ and $0.25 \leq \Delta_2(\theta) \leq 0.625$ (Assumption 3 is satisfied). Since $\Delta_1 < 0$ Assumption 4 is true with the matrix $P$ equal to the identity.

Therefore, according to Theorem 5 we may use the differentiator (2) to estimate $x_1$ and $\dot{x}_1$, for which $s = 2$ can be reduced to the conventional super-twisting differentiator (Levant, 1998):
\[
\dot{q}_0 = -\lambda_0 \left[ |\psi(t) - \psi(t)| \text{sgn}[q_0 - \psi(t)] + q_1 \right], \quad \dot{q}_1 = -\lambda_1 \text{sgn}[q_0 - \psi(t)],
\]
where in our example $\lambda_0 = 20, \lambda_1 = 50, \dot{z}_1 = q_0 = 0, \dot{\xi}_1 = 1.1V$. In this case the finite time $T = 0.1$. Next, the interval observer (9), (10) generates the required set-membership estimates for the variable $x_2 = r (t) \leq 1.75, G_2 = 0.5$ and $G_3(\theta) = \Delta_2(\theta)$:
\[
\tilde{r} = -T + 1.75 \tilde{x}_1^+ - 0.75 \tilde{x}_1^- + 1.75 \tilde{e}_1 + 0.5 \tilde{q}_1 + 0.5 \tilde{e}_2 + 0.625U, \\
\tilde{r} = -T - 0.75 \tilde{x}_1^+ + 1.75 \tilde{x}_1^- - 1.75 \tilde{e}_1 + 0.5 \tilde{q}_1 - 0.5 \tilde{e}_2 - 0.625U.
\]

The results of this interval estimation are shown in Fig. 1, where the red solid line represents the variable $x_2$, while the dash and
the dash–dot blue lines correspond to interval estimates $x_2$ and $x_3$ respectively. The interval estimates $x_2$, $x_3$ envelop the state trajectory of the plant $x_2$ and at some time instants the trajectory $x_2$ touches the bounds $x_2$, $x_3$ justifying that the obtained estimates are not conservative. The width of the estimated interval $[x_2, x_3]$ is proportional to the current system incertitude.

It is worth noting that to the best knowledge of the authors, other existing approaches cannot solve the problem of interval estimation for (13). In particular, application of a conventional interval observer design method (Moisan et al., 2009; Raïssi et al., 2010) blocked by the non-minimum phase condition ($-0.5 \leq a_{22}(θ) \leq 0$).

### 4.2. Minimum phase second order system

Now consider a second order system for which we can apply a conventional approach for design of interval observers. In this example we would like to show the estimation accuracy improvement. Consider system (13) with

$$
\begin{align*}
a_{11}(θ) &= 0.5, & a_{12}(θ) &= 0, \\
-0.5 \leq a_{21}(θ) &\leq 0.5, & a_{22}(θ) &= -1, \\
b_1(θ) &= 2, & 0.5 \leq b_2(θ) &\leq 1, & U &= 1, & V &= 0.1
\end{align*}
$$

for $θ \in Θ$. Again Assumption 2 is satisfied. For simulation we use $a_{21}(θ) = 0.5 \sin(θ)$, $b_2(θ) = 0.75 + 0.25 \cos(0.5t)$, $u(t) = U \sin(2t)$, $v(t) = V \sin(10t)$.

For the system (13) with the chosen parameters and the given input $u$ the state is bounded (Assumption 1 holds). We assume that $-2 \leq x_2(0) \leq 2$. For $f(θ) = 0.5b_2(θ)$ the equality $f(θ)b_1(θ) = b_2(θ)$ holds (from the subsystem (5) we evaluate the value of $u$ and we have

$$
[a_{22}(θ)z_2 + b_2(θ)u] - f(θ)[a_{12}(θ)z_2 + b_1(θ)u] = \Delta_1z_2,
$$

where $Δ_1 = -1, Δ_2(θ) = 0$ and Assumption 3 is satisfied. Since $Δ_1 < 0$, as previously, Assumption 4 is true with the identity matrix $P$. For differentiation we again use (14) with the same values of parameters. In accordance with the result of Theorem 5 the interval observer takes the form for $x_2 = r (-0.25 \leq G_1(θ) = P^{-1}[a_{22}(θ) + f(θ)a_{11}(θ)] \leq 1, 0.25 \leq G_2 = f(θ) \leq 0.5$ and $G_3(θ) = 0)$:

$$
\begin{align*}
\tilde{r} &= -\tilde{r} + \tilde{x}_1^+ - 0.25\tilde{x}_1^- + \tilde{e}_1 + 0.5\tilde{q}_1^+ + 0.25\tilde{q}_1^- + 0.5\tilde{e}_2, \\
\tilde{\dot{r}} &= -\tilde{x}_1^+ - 0.25\tilde{x}_1^- - \tilde{e}_1 - 0.25\tilde{q}_1^+ + 0.5\tilde{q}_1^- - 0.5\tilde{e}_2.
\end{align*}
$$

Note that in this case for the $x_2$ subsystem an interval observer can be designed directly using (Raïssi et al., 2010) ($x_2 = r$ and $x_3$ is fixed).

![Fig. 1. The results of simulation for the non-minimum phase example.](image1)

![Fig. 2. The results of simulation for the minimum phase example.](image2)

### 4.3. Pendulum system

Consider the model of an inverted pendulum linearized around the upper unstable equilibrium:

$$
\begin{align*}
\dot{x}_1 &= x_2, & \dot{x}_2 &= -gm\sin x_1 - k_1(θ)x_2 + u(t); & \psi = x_1 + v; \\
\dot{x}_3 &= x_4, & \dot{x}_4 &= \ell^{-1}g(M + m)x_3 - k_2x_4 + u(t),
\end{align*}
$$

where $x_1 \in \mathbb{R}, x_3 \in \mathbb{R}$ are deviations of the cart position [m] and velocity [m/s] respectively, $x_3 \in [-\pi, \pi)$, $x_4 \in \mathbb{R}$ are the angular position and velocity in [rad/s]; $M$ and $m$ are known masses of the cart and the pendulum point respectively [kg], $\ell$ is a fixed known length of the pendulum link [m], $g = 9.8$ is the gravity acceleration [m/s²], $k_2 > 0$ and $k < k_1(θ) \leq k$ are friction coefficients ($k_1$ is unknown and time-varying); $u = u_0 + d$ [m/s²], $u_0 \in \mathbb{R}$ is an exciting input, $-d \leq d \leq \bar{d}$ is an external bounded disturbance and $\psi \in \mathbb{R}$ is the measurement noise [m]. The cart subsystem has one negative eigenvalue and zero eigenvalue, the pendulum subsystem has one positive and one negative eigenvalue (the system is unstable at the upper equilibrium). Only the cart position $x_1$ is available for measurements. The positive parameters $\bar{d}, k$ and $\bar{k}$ are given.

For an exciting input $u_0$ the pendulum is unstable, but its solutions stay bounded during some small finite time interval. We are going to show that the proposed observer is able to generate the interval estimates in less than 0.1 [s] that is acceptable for further stabilizing control application (Efimov, Raïssi, & Zolghadri, 2011; Fridman, Levant, & Davila, 2007) (see also Remark 7). Another solution is to use an HOSM algorithm for bounded logarithmic derivatives (this is true for linear systems).

Obviously in this example the cart subsystem is strongly observable ($z_1 = [x_1, x_2]$, $z_2 = [x_1, x_3]$) and using the HOSM differentiator of the third order ($s = 2$) we are able to estimate $x_2$ and $x_3$:

$$
\begin{align*}
x_2(t) &= q_1(t) + e_1(t), & \dot{x}_2(t) &= q_2(t) + e_2(t).
\end{align*}
$$

Substitution of these estimates in the second equation gives:

$$
\begin{align*}
q_2(t) + e_2(t) + k_1(θ)[q_1(t) + e_1(t)] - u_0 &= d - gm\sin x_1,
\end{align*}
$$

where $x_1(t) \in [\psi(t) - V, \psi(t) + V]$: 

$$
\begin{align*}
\dot{r} &= -\tilde{r} + 0.5|\psi| + 0.5V + U, \\
\dot{\tilde{r}} &= -\tilde{r} - 0.5|\psi| - 0.5V - U.
\end{align*}
$$

As we see the observer (16) depends directly on the worst case estimate $U$ for the input $u$, while the observer (15) is based on an estimated value of the input $u$ calculated in the subsystem (5) using (14). Such a substitution leads to accuracy of estimation improvement as confirmed by the results of simulation for both observers presented in Fig. 2 (the solid line represents $x_2(t)$, the dash lines correspond to the interval estimates of the observer (15) and the dash–dot lines show the estimates calculated by (16)).
Using this relation the equations of the link subsystem can be rewritten as follows:

\[
\begin{align*}
\dot{x}_3 &= x_4 + f_1(d - gm x_3) - f_1[q_2 + e_2 + k_1(\theta)(q_1 + e_1)] - u_0, \\
\dot{x}_4 &= -1g[M + m]x_3 - k_2 x_4 + u_0 + d,
\end{align*}
\]

where \( f_1 > 0 \) is chosen high enough to ensure that the Metzer matrix

\[
\begin{bmatrix}
-f_1 gm & 1 \\
-1g[M + m] & -1k_2
\end{bmatrix}
\]

is Hurwitz (this matrix describes the dynamics of the link subsystem). In this case all conditions of the Theorem 5 are satisfied for \( f(\theta) = [f, 0]^T \) and the interval observer takes the form:

\[
\begin{align*}
\dot{\hat{x}}_3 &= x_4 + f_1[-d - gm x_3] \\
&- f_1[q_2 + e_2 + k_1(\theta)] - k_2 \hat{x}_4 + u_0 - d, \\
\dot{\hat{x}}_4 &= \hat{x}_3 + f_1[-d - gm x_3] \\
&- f_1[q_2 - e_2 + k_1(\theta)] - \hat{k}_2 \hat{x}_4 + u_0 - d,
\end{align*}
\]

The results of this observer application for (the units are described above)

\[
M = 5, \quad m = 1, \quad \ell = 2, \quad \sigma = 0.02, \quad k_1 = 0.2, \quad k = 0.1, \quad \hat{k} = 0.4.
\]

are shown in Fig. 3, the results of interval estimation for the variable \( x_3 \) are given in Fig. 3, a and for the variable \( x_4 \) in Fig. 3, b (solid lines represent the variables \( x_3 \) and \( x_4 \), the dash lines correspond to upper and lower estimates). As we can conclude from this figure, the pendulum is unstable, but the sliding-mode interval observer is able to evaluate the set of admissible values for both variables very quickly in a finite time (see the zoomed inclusion), this information can be used for control (Efimov et al., 2011).

5. Conclusion

The objective of this technical note is to present an approach for improvement of estimation accuracy for interval observers designed for LPV systems. Applying the HOSM differentiation, the information from a strongly observable subsystem is used to decrease the level of uncertainty in the rest of the underlying system. That allows us to improve the estimation accuracy of an interval observer designed for LPV systems, and enlarge the class of LPV systems having an interval observer. The efficiency of the proposed technique has been demonstrated through non-minimum phase examples.

References


Denis Efimov received the Ph.D. degree in Automatic Control from the Saint-Petersburg State Electrical Engineering University (Russia) in 2001, and the Dr.Sc. degree in Automatic Control in 2006 from Institute for Problems of Mechanical Engineering RAS (Saint-Petersburg, Russia). From 2001 to 2008 he was a postdoctoral researcher at the Institute for Problems of Mechanical Engineering RAS, Control of Complex Systems Laboratory. From 2006 to 2011 he was working in the LSS (Supélec, France), the Montoore Institute (University of Liege, Belgium) and the Automatic Control group at IMS lab (University of Bordeaux 1, France). In 2011 he joined the Non-A team at INRIA Lille Center. He is a member of the IFAC TC on Adaptive and Learning Systems and a senior member of IEEE. His main research interests include nonlinear oscillation analysis, observation and control, switched and hybrid system stability.

Leonid Fridman received the M.S. degree in mathematics from Kuibyshev (Samara) State University, Samara, Russia, in 1976, the Ph.D. degree in applied mathematics from the Institute of Control Science, Moscow, Russia, in 1988, and the Dr.Sc. degree in control science from Moscow State University of Mathematics and Electronics, Moscow, Russia, in 1998. From 1976 to 1999, he was with the Department of Mathematics, Samara State Architecture and Civil Engineering University. From 2000 to 2002, he was with the Department of Postgraduate Study and Investigations at the Chihuahua Institute of Technology, Chihuahua, Mexico. In 2002, he joined the Department of Control Engineering and Robotics, Division of Electrical Engineering of Engineering Faculty at National Autonomous University of Mexico (UNAM), Mexico. His research interests are variable structure systems. Prof. Fridman is the Associate Editor of the International Journal of System Science, the Journal of the Franklin Institute, the Conference Editorial Board of the IEEE Control Systems Society, a Member of TC on Variable Structure Systems and Sliding Mode Control of the IEEE Control Systems Society and TC on Discrete Events and Hybrid Systems of IFAC. He is an author and editor of five books and nine special issues and more than 300 technical papers on sliding mode control. He is a winner of the Scopus prize for the best cited Mexican Scientists in Mathematics and Engineering 2010. He has worked as an invited professor in 17 universities and research centers in France, Germany, Italy, Israel and Spain.

Tarek Raïssi received the Engineer degree from École Nationale d'Ingénieurs de Tunis in 2000, the DEA in automatic control from École Centrale de Lille in 2001 and the Ph.D. degree from University Paris XII in 2004. From 2005 to 2011 he was an Associate Professor at the University of Bordeaux. He is currently at the Conservatoire National des Arts et Métiers, Paris, France. He is a member of the IFAC TC on Modelling, Identification and Signal Processing. His research interests include fault detection and isolation, nonlinear systems estimation, interval analysis.

Ali Zolghadri is a Full Professor of control engineering with the University of Bordeaux, France. Dr. Zolghadri's main research interest is narrowing the gap between real world control engineering requirements and theoretical analysis and design techniques. His areas of expertise include model-based fault diagnosis, fault-tolerant control and guidance, health management and operational autonomy for complex safety-critical systems. He has published around 150 publications including journal articles, book chapters and communications. He is a co-holder of five patents.

Ramatou Seydou obtained her master's degree in aeronautical maintenance institute, Bordeaux University, France, in 2009. She is currently a Ph.D. student at IMS lab, Bordeaux University, France. The topic of her Ph.D. is design and development of set membership techniques and their application to fault detection and diagnosis.