Combined backstepping and HOSM control design for a class of nonlinear MIMO systems

Antonio Estrada¹,⁎,†, Leonid Fridman² and Rafael Iriarte²

¹CONACYT–CIDESI, Playa Pie de la Cuesta 702, Col. Desarrollo San Pablo, Queretaro, Qro., Mexico
²Department of Control and Robotics, Engineering Faculty, UNAM–CU, Mexico City, Mexico

SUMMARY

This paper presents a control design algorithm that combines backstepping and high-order sliding modes. It is known that backstepping can achieve asymptotic stability for nonlinear systems in strict-feedback form in spite of parametric uncertainties. Nevertheless, when external perturbations are also present, only practical stability can be ensured. For the same aforementioned perturbed conditions, the combined design presented in this paper can achieve finite-time exact tracking/regulation. At the same time, the semi-global or global stability obtained through backstepping is preserved, and the gains of the high-order sliding modes controller can be reduced with respect to its direct application. The design is based on recently reported combined designs that are based on the idea of virtual controls, which can contain terms based on high-order sliding modes algorithms. The proposal also extends previous results to the multiple-input–multiple-output case. Copyright © 2016 John Wiley & Sons, Ltd.

Received 2 January 2015; Revised 23 November 2015; Accepted 24 May 2016

KEY WORDS: robust control; sliding modes; backstepping; uncertain systems

1. INTRODUCTION

The high-order sliding mode (HOSM) control has been used for several different applications; robustness against perturbations is one of its main advantages. On the contrary, one of its main drawbacks is the chattering; the magnitude of this high-frequency vibrations is directly related to the control gain magnitude. Direct utilization of HOSM that compensates uncertainties and disturbances reflected in the channel of control, according to the relative degree of the controlled output, leads to the necessity of using large gains. As the chattering increases with control gains, larger gains imply more chattering, and then stronger and more expensive actuators are required [1, 2]. The proposals in [3] and [4] are model-based control design schemes that aim to reduce the HOSM control gains; they are applicable to systems with unmatched uncertainties and theoretically ensure exact tracking of a smooth reference in spite of perturbations. Nonetheless, the reduction of the HOSM gains compromises the region of attraction as the stability relies precisely on the HOSM algorithms’ properties. The present paper proposes a control scheme that can achieve global or semiglobal domain of attraction preserving the finite time exact tracking by combining two different design approaches, namely, HOSM and backstepping (BS) control.

The BS technique was first introduced in [5] and further developed in [6] by solving the problem of overparameterization. BS has become an important tool in feedback control design achieving semiglobal and global stability results for uncertain nonlinear systems. However, when uncertainties are nonvanishing at the desired equilibrium point (time varying in the case of tracking), BS can only ensure convergence to a vicinity of the origin of the error system. The BS design uses the idea of virtual control in order to obtain a controller, through a systematic step-by-step procedure, and a

⁎Correspondence to: Antonio Estrada, Playa Pie de la Cuesta 702, Col. Desarrollo San Pablo, 76125 Queretaro, Qro., Mexico.
†E-mail: xheper@yahoo.com

Copyright © 2016 John Wiley & Sons, Ltd.
Lyapunov function that ensures stability of the closed loop system. Reported on [7] is a proposal to strengthen the BS design with sliding mode control for systems in strict-feedback form. The results in [7] are further extended to the multi-input case in [8]. For an output with relative degree equal to \( r \), the BS design is applied in order to obtain either \( r - 1 \) or \( r - 2 \) transformed state equations, which are coupled with a subsystem of first or second order accordingly. In this latter subsystem, a sliding mode controller of the required first or second order is introduced. In the first stage of BS design, no sliding mode algorithm is introduced in the virtual controls. In fact, if the procedure were applied using HOSM of arbitrary order, instead of the ones of first or second order, the controller would make no use of the BS design. In such a case, an analytic expression for the \( r \)-th derivative of the system output would be required in order to compensate the derivatives of the known terms by state feedback, leaving the unknown terms for being compensated by means of an HOSM controller of order \( r \).

The main obstacle for introducing sliding mode algorithms through virtual controls is the necessity of smoothness of these virtual controls. For instance, in [9], a control design based on block control [10] and sliding modes is proposed. Sigmoid functions are used to fulfill the smoothness requirement. Sigmoid functions are also used in the virtual controls of the scheme for perturbed strict-feedback systems reported in [11]. On the aforementioned references [9] and [11], exact tracking is not ensured; it is proved convergent to a zone whose magnitude depends on the gain magnitude of the sigmoidal functions.

In [12], a scheme is proposed based on identification of perturbations. The identified perturbations are then compensated on the BS controller achieving exponential stability. By contrast, the proposals in [3] and [4] ensure finite time exact stabilization overcoming the difficulty of introducing HOSM algorithms inside the expression of virtual controls by means of a hierarchical HOSM design. In [3], local stability is obtained, and in [4], the attraction region is enhanced with the aid of the integral sliding modes reported in [13]. The main drawback of the aforementioned approaches is that the information of the nominal dynamics is used to reduce the gains on HOSM but the region of attraction still depends on the gains of the HOSM algorithms. A complete design that ensures stability by means of using the knowledge of the system was not given. The present paper enhances the results in [3] and [4] by giving a fully combined BS-HOSM design featuring advantages from both approaches. Moreover, the design is presented for an extended class of systems; the multi-input–multi-output case is considered. The proposed BS-HOSM combination enlarges the type of unknown dynamics that can be tackled by BS; specifically, it will be able to reject external perturbations and nonvanishing uncertainties. In the present proposal, practical stability is established first via BS design; in [14], similar results, stabilization with respect to a set, have been presented for more general class of nonlinear systems. The combination introduced in the present work renders a reduction of the HOSM gains compared with a direct application [1]. Additionally, the proposed scheme retains the semiglobal, or global if it is the case, stability obtained with BS but enhanced with the finite time exact regulation/tracking achieved by the HOSM approach.

Throughout the paper, the notation \( \| G \| \) is used either for the Euclidian norm of a vector or the spectral norm of a square matrix \( G \), according to the context. The function \( \text{eigval}(G) \) returns the set of eigenvalues of the square matrix \( G \).

### 2. PROBLEM STATEMENT

Consider the following class of MIMO systems in Nonlinear block controllable (NBC)-form [15]:

\[
\begin{align*}
\dot{x}_0 &= f_0(t, x) \\
\dot{x}_1 &= f_1(t, x_1) + g_1^\Delta(t, x_1)x_2 + \omega_1(t, x_1) \\
&\vdots \\
\dot{x}_i &= f_i(t, x_i) + g_i^\Delta(t, x_i)x_{i+1} + \omega_i(t, x_i) \\
&\vdots \\
\dot{x}_r &= f_r(t, x) + g_r^\Delta(t, x)u + \omega_r(t, x)
\end{align*}
\]  

(1)
with \( g^\Delta_i(t, \xi) := g_i(t, \xi) + \Delta g_i(t, \xi) \); the term \( \Delta g_i \) stands for uncertainties on the known \( g_i \).

The vector states \( x_i \in \mathbb{R}^{m_i} \) and \( \xi_i = [x_0, \ldots, x_i] \) are such that \( m = m_1 \ldots = m_r \). That is, only \( x_0 \), which represents internal dynamics, may have a dimension different from the other \( x_i \) states. \( x = \tilde{x}_r \in \mathbb{R}^n \), with \( n = \sum_{\rho=1}^m m_i \) is the state vector, and \( u \in \mathbb{R}^m \) is the control signal. Defining the outputs \( y_i = x_i \) for \( 1 \leq i \leq n \), the vector-relative degree [16] is such that all the components of \( y_i \) have the same relative degree, denoted \( \rho(y_i) \), and fulfills that \( \rho(y_i) = r - i + 1 \). The vector functions \( f_i(t, \xi_i), \ g_i^\Delta(t, \xi_i) \in \mathbb{R}^{m \times m} \) and \( \omega_i(t, \xi_i) \) are assumed to be smooth with at least \( r - i \) bounded derivatives and \( g_i^\Delta(t, \xi_i) \) invertible \( \forall x \in D \subset \mathbb{R}^n, t \in [0, \infty) \). The terms \( \omega_i \) represent parameter uncertainties and external disturbances and will be referred in the following simply as perturbations.

The control objective is to design a controller that ensures semiglobal, and possibly global, stability for system (1) and achieves finite time exact tracking for any given smooth signal, \( \phi_0 \), with bounded \( r \)-th time derivative, by the output \( y_1 \).

### 3. DESIGN ALGORITHM AND MAIN RESULT

We propose a solution for the control problem described in Section 2. Such a solution combines the BS and HOSM techniques.

Before presenting our design proposal, some definitions must be introduced. For the sake of brevity, the dependence on \( t \) and states may be omitted.

Virtual controls, \( \phi_{i} \), and error signals, \( \sigma_{i} \), are defined as follows:

\[
\phi_{i} = \phi_{iBS} + g_i^{-1}\phi_{iQC},
\]

\[
\sigma_{i} = x_{i} - \phi_{i-1};
\]

\[
\sigma_{i} = [\sigma_{i1}, \ldots, \sigma_{im}]^{T},
\]

Then, we define \( \bar{\sigma}_{i} \) as follows:

\[
\bar{\sigma}_{i} = [\sigma_{i1}, \ldots, \sigma_{im}]^{T}.
\]

In the sequel, \( \Psi_{l-1,f}(\sigma_{i}) \) is used to denote the \( l \)-th order quasi-continuous HOSM controller [17] of each of the components of the vector \( \sigma_{i} \). To be more specific, the notation in (5) is used:

\[
\Psi_{l-1,f}(\sigma_{i}) = \begin{bmatrix}
\Psi_{l-1,f}(\bar{\sigma}_{i1}) \\
\vdots \\
\Psi_{l-1,f}(\bar{\sigma}_{im})
\end{bmatrix} = \begin{bmatrix}
\Psi_{l-1,f}(\sigma_{i1}, \sigma_{i1}^{(1)}, \ldots, \sigma_{i1}^{(l-1)}) \\
\vdots \\
\Psi_{l-1,f}(\sigma_{im}, \sigma_{im}^{(1)}, \ldots, \sigma_{im}^{(l-1)})
\end{bmatrix}.
\]  

The following iterative procedure is proposed for the construction of the controller:

\[
\phi_{1BS} = g_i^{-1}\left( \hat{\phi}_{0} - a_1\sigma_{1} - f_1 \right),
\]

\[
\phi_{1QC}^{(r-1)} = -a_1\Psi_{r-1,r}(\sigma_{1}),
\]

\[
\phi_{iBS} = g_i^{-1}\left( \hat{\phi}_{i-1} - g_i^{-1}\sigma_{i-1} - a_i\sigma_{i} - f_i \right),
\]

\[
\phi_{iQC}^{(r-1)} = -a_i\Psi_{r-i,r-i+1}(\sigma_{i}),
\]

\[
\text{for } i = 2, \ldots, r.
\]
proof of our main result; please refer to Appendix. Note that the r-th control, \( \phi_r \), obtained from the aforementioned procedure corresponds to the real input \( u \). Therefore,

\[
u = \phi_r B_S + g^{-1} \phi_r Q C. \tag{8}\]

Finally, we define the following functions:

\[
\lambda_m(Q) = \{ \min(|\lambda|) : \lambda \in \text{eigval}(Q) \}, \tag{9a}
\]

\[
\lambda_M(Q) = \{ \max(|\lambda|) : \lambda \in \text{eigval}(Q) \}, \tag{9b}
\]

where \( Q \) is a square matrix. The next theorem states the main result of the present paper.

**Theorem 1**

Consider system (1) and the controller (8). If the following assumptions are globally/semiglobally satisfied:

A1. Growth and gain conditions. For some positive constants \( b_i, b_{g_i}, B_r \), and \( d_i \), it holds that

\[
\| \omega_i \| \leq b_i \| \sigma_i \| + d_i, \tag{10}\]

\[
\| g_i \| \leq b_{g_i} \| \sigma_i \| \quad \forall \| \sigma_i \| \geq B_r, \tag{11}\]

\[
b_{g_i}^{-} \leq \lambda_m(g_i(t, \bar{x}_i)) - \lambda_M(\Delta g_i(t, \bar{x}_i)). \tag{12}\]

A2. The functions \( f_i, g_i, \) and \( \omega_i \) are smooth with at least \( r - i \) bounded derivatives \( \forall x \in D \subset \mathbb{R}^n, t \in [0, \infty) \).

then a proper selection of parameter matrices \( \alpha_i, a_i \) for the controller (8) ensures global/semiglobal stability of system (1) and achieves finite time exact tracking of \( \phi_0 \) by the output \( y_1 \).

**Remark 2**

The conditions in Theorem 1 imply growth conditions also for \( f_i \) and their derivatives. Specifically, the growth condition (10) for each \( \omega_i \) where uncertainties in \( f_i \) and \( g_i \) are considered implies analogous linear growth conditions for \( f_i \) and \( g_i \). Moreover, the contribution due to the internal dynamics vector state \( x_0 \) must be absorbed by the term \( d_i \). For instance, this can be ensured with the \( x_0 \) subsystem bounded-input–bounded-state.

**Remark 3**

In the proof of Theorem 1, see the Appendix, firstly, it is shown that practical stability can be ensured for system (1) by means of BS design. Once practical stability is established, finite time convergence will be concluded with the aim of HOSM control properties. It is important to note that the practical stability is obtained taking into account the terms \( \phi_i Q C \). That is, the proposed design considers the two terms acting simultaneously.

**4. EXAMPLE**

Consider the following nonlinear model of an F16 jet fighter [18] at Mach = 0.7, \( h = 10000 \) ft, \( \delta_{\text{trim}} = -0.0925 \) rad, \( \delta_{\text{trim}} = \beta_{\text{trim}} = 0.106803 \) rad, and \( \beta_{\text{trim}} = p_{\text{trim}} = q_{\text{trim}} = r_{\text{trim}} = \phi_{\text{trim}} = \delta_{\text{trim}} = \beta_{\text{trim}} = 0 \).

\[
\dot{\theta} = q \cos \phi - r \sin \phi, \tag{13}
\]

\[
\dot{\phi} = p + q \sin \phi \tan \theta + r \cos \phi \tan \theta.
\]
\[
\begin{align*}
\dot{\alpha} &= -\beta p + 0.0427 \cos \theta \cos \phi + 0.083589 + Z_a \alpha + Z_q q + \omega_a, \\
\dot{\beta} &= -0.9973r + \alpha p + 0.0427 \cos \theta \sin \phi + Y_\beta \beta + Y_p p, \\
\dot{\delta}_a &= 20(u_1 - \delta_a), \\
\dot{\delta}_e &= 20(u_2 - \delta_e), \\
\dot{\delta}_r &= 20(u_3 - \delta_r),
\end{align*}
\] (14)

\[
\begin{align*}
\dot{p} &= -0.1345pq - 0.8225qr + L_\beta \beta + L_p p + L_r r - 50.933 \delta_a + L_\delta \delta_r, \\
\dot{q} &= 0.9586pr - 0.0833(r^2 - p^2) - 1.94166 + M_\alpha \alpha + M_q q + M_\delta \delta_e + \omega_q, \\
\dot{r} &= -0.7256pq + 0.1345qr + N_\beta \beta + N_p p + N_r r + 4.125 \delta_a + N_\delta \delta_r,
\end{align*}
\] (15)

where (13) describes the dynamics of the pitch angle, (14) the dynamics of command angles (roll, attack, and sideslip), (15) is for the rotational rate, and (16) for the actuator dynamics. The parameters \( Z_1, Y_1, L_1, M_1, \) and \( N_1 \) are uncertain parameters composed of a nominal part and an additive deviation:

\[
\begin{align*}
Z_a &= -1.15 + \Delta Z_a, \quad \Delta Z_a = 0.04, \\
Z_q &= 0.9937 + \Delta Z_q, \quad \Delta Z_q = 0.0031, \\
Y_\beta &= -0.297 + \Delta Y_\beta, \quad \Delta Y_\beta = 0.0534, \\
Y_p &= 0.00085 + \Delta Y_p, \quad \Delta Y_p = -0.00005, \\
L_\beta &= -53.48 + \Delta L_\beta, \quad \Delta L_\beta = 8.024, \\
L_p &= 4.324 + \Delta L_p, \quad \Delta L_p = 0.071, \\
L_r &= -0.224 + \Delta L_r, \quad \Delta L_r = 0.055, \\
L_\delta &= 10.177 + \Delta L_\delta, \quad \Delta L_\delta = -5.089, \\
M_\alpha &= 3.724 + \Delta M_\alpha, \quad \Delta M_\alpha = 1.856, \\
M_q &= -1.26 + \Delta M_q, \quad \Delta M_q = 0.42, \\
M_\delta &= -19.5 + \Delta M_\delta, \quad \Delta M_\delta = 9.75, \\
N_\beta &= 17.67 + \Delta N_\beta, \quad \Delta N_\beta = -5.82, \\
N_p &= 0.234 + \Delta N_p, \quad \Delta N_p = 0.01, \\
N_r &= -0.649 + \Delta N_r, \quad \Delta N_r = 0.133, \\
N_\delta &= -6.155 + \Delta N_\delta, \quad \Delta N_\delta = 3.077.
\end{align*}
\]

Next are the expressions for \( \omega_a \) and \( \omega_q \) that represent external disturbances and will be added together with the uncertainties

\[
\omega_a = 0.4 \sin(0.5t),
\] (17)

\[
\omega_q = 0.3 \sin(0.5t) - 1.2 \sin(2t).
\] (18)

In order to rewrite the aforementioned system in the NBC-form (1), we define \( y_1 = x_1 = \begin{bmatrix} \phi & \alpha & \beta \end{bmatrix}^T, \quad x_2 = \begin{bmatrix} p & q & r \end{bmatrix}^T, \quad \text{and} \quad x_3 = \begin{bmatrix} \delta_a & \delta_e & \delta_r \end{bmatrix}^T. \) Then, the functions \( f_j \) and \( g_i \) are given by the following:

\[
\begin{align*}
f_1(x_1, \theta) &= \begin{bmatrix} \dot{\alpha} \\
0.0427 \cos \theta \cos \phi + 0.083589 + Z_a \alpha \\
0.0427 \cos \theta \sin \phi + Y_\beta \beta \\
0.1345pq - 0.8225qr + L_\beta \beta + L_p p + L_r r \end{bmatrix}, \\
f_2(x_1, x_2, \theta) &= \begin{bmatrix} \dot{p} \\
0.9586pr - 0.0833(r^2 - p^2) - 1.94166 + M_\alpha \alpha + M_q q \\
-0.7256pq + 0.1345qr + N_\beta \beta + N_p p + N_r r \end{bmatrix}.
\end{align*}
\]
\[ g_1(x_1, \theta) = \begin{bmatrix} 1 & \sin \phi \tan \theta & \cos \phi \tan \theta \\ -\beta & Z_q & 0 \\ \alpha + Y_p & 0 & -0.9973 \end{bmatrix}, \quad g_2 = \begin{bmatrix} -50.933 & 0 & L_{\delta_r} \\ 0 & M_{\delta_e} & 0 \\ 4.125 & 0 & N_{\delta_r} \end{bmatrix}. \]

The references for the command angles are the output to the next second-order filters:

\[ H_\alpha(s) = \frac{4}{s^2 + 3s + 4}, \quad H_\beta(s) = \frac{16.98}{s^2 + 6.18s + 16.98}, \quad H_\phi(s) = \frac{2}{s^2 + 2.2s + 2}. \]

The desired signals are produced by a step input, \( \alpha^* \), to the \( H_\alpha \) filter, changing amplitude from 0 to 0.7 at second 2 and returning to 0 at second 6. The input to the other filters is equal to 0. The vector of reference signals \( \phi_0 = diag (H_\phi, H_\alpha, H_\beta) [0, \alpha^* 0]^T \), and the error vector \( \sigma_i = x_i - \phi_{i-1} = [\sigma_{i1}, \sigma_{i2}, \sigma_{i3}]^T \). Then, applying the proposed design, we have the following:

- **Step 1.**
  \[ \phi_1 = \phi_{1BS} + g_1^{-1} \phi_{1QC}, \quad \phi_{1BS} = g_1^{-1} \{-f_1 - a_1 \sigma_1 + \hat{\phi}_{0}\}, \quad \dot{\phi}_{1QC} = -\alpha_1 \Psi_{2,3}(\sigma_1). \]

- **Step 2.**
  \[ \phi_2 = \phi_{2BS} + g_2^{-1} \phi_{2QC}, \quad \phi_{2BS} = g_2^{-1} \{-f_2 - a_2 \sigma_2 + \hat{\phi}_{1}\}, \quad \dot{\phi}_{2QC} = -\alpha_2 \Psi_{1,2}(\sigma_2). \]

![Figure 1. Tracking performance of BS control for unperturbed system.](image1)

![Figure 2. Tracking performance of BS and BS-HOSM control for perturbed system.](image2)
Figure 3. Command angles, from top to bottom: $\phi$, $\alpha$, and $\beta$.

Figure 4. Rotational rates, from top to bottom: $p$, $q$, and $r$.

Figure 5. Deflections, from top to bottom: $\delta_{u}$, $\delta_{v}$, and $\delta_{r}$. 
The corresponding derivatives of the virtual controls $\dot{\phi}_0$, $\dot{\phi}_1$, $\dot{\phi}_2$ and the ones corresponding to the elements of $\sigma_1$, $\sigma_2$, and $\sigma_3$ were obtained through the robust exact differentiators of the proper order reported in [19]. The terms $g_i^{T}\sigma_{i-1}$ were omitted on purpose to test the robustness of the scheme. The expressions for the quasi-continuous HOSM algorithms [17] are written next for the reader convenience:

$$
\Psi_{0,1}(\sigma) = \text{sign}(\sigma), \quad \Psi_{1,2}(\sigma) = \frac{\tilde{\sigma} + |\sigma|^{1/2}\text{sign}(\sigma)}{|\sigma| + |\sigma|^{1/2} \text{sign}(\sigma)}, \quad \Psi_{2,3}(\sigma) = \frac{\tilde{\sigma} + 2(\sigma + |\sigma|^{2/3} - \frac{1}{2}(\sigma + |\sigma|^{2/3}\text{sign}(\sigma))}{|\sigma| + 2(|\sigma| + |\sigma|^{2/3})^{1/2}}.
$$

The simulation results are shown in Figures 1–2. The gains $a_i = \text{diag} \left( \begin{bmatrix} a_{i1} & a_{i2} & a_{i3} \end{bmatrix} \right)$ and $\alpha_i = \text{diag} \left( \begin{bmatrix} \alpha_{i1} & \alpha_{i2} & \alpha_{i3} \end{bmatrix} \right)$, with $a_{i1} = 8$, $a_{i2} = 30$, $a_{i3} = 80$, for the BS terms, whereas for the quasi-continuous HOSM, $\alpha_{i1} = 3$, $\alpha_{i2} = 10$ and $\alpha_{i3} = 8$ were chosen ($i = 1, 2, 3$).

Figure 1 shows the angle $\alpha$ controlled by BS without perturbations. After, perturbations were added, and it can be seen in Figure 2 that the classical BS control does not achieve the desired tracking. The proposed combined scheme is applied obtaining the tracking, dotted line in Figure 2, in spite of the perturbations (parametric uncertainty and external disturbances).

Figures 3–5 show the evolution of the states $x_1$, $x_2$ and deflections control $x_3$ for the proposed BS-HOSM scheme.

Three additional simulation tests were performed in order to make a comparison between the BS-HOSM scheme and other two previously reported results. One of these schemes is referred as hierarchical HOSM, in the spirit of [3], that constructs a nonlinear sliding surface introducing HOSM algorithms in the virtual controllers but, in contrast with [3] and the present work, makes no use of the nominal information of the system. Thus, the controls task relies completely on the HOSM control gains. Results from the application of another control scheme, referred as sliding

![Figure 6. Command angle $\delta_c$ for hierarchical high-order sliding mode control.](image-url)
Figure 7. Command angle $\delta_e$ for sliding mode block control.

Figure 8. Command angle $\delta_e$ for combined backstepping high-order sliding mode control.

Figure 9. Error $\sigma_1$ for hierarchical high-order sliding mode control.
mode block control (SMBC), based on the results reported in [1] are also presented. The obtained
expression for the latter controller is the following:

\[ u = -\hat{B}_r^{-1}U_0 \text{sign}(Z_3) \]

\[ Z_3 = g_1 g_2 x_3 + g_1 (f_2 + g_2) + f^{(1)} + k_1 \phi_1^{(1)} + k_2 Z_2 - \phi_0^{(2)} + g_1^{(1)} x_2 \]  

(19)

with \( \hat{B}_r = 20g_1 g_2 \) and the high gains matrices defined as follows \( k_1 = a_1, k_2 = a_2 \) and \( U_0 = a_3 \).

The initial conditions for the comparison simulations were set to 0 for all the states because of
the local stability of the hierarchical HOSM controller and the gain dependency of the region of
attraction for the SMBC [1]. Under the same conditions, a second simulation with the combined
BS-HOSM control was performed. It is important to note that all the gains of the HOSM algorithm
were adjusted for the hierarchical HOSM control scheme in order to achieve the tracking goal with
similar accuracy to the one obtained by the combined BS-HOSM scheme. Note that while the tuning
was made through simulation, the resultant gains \( k_1 \) and \( k_2 \) of the SMBC are equivalent to those
of the BS and BS-HOSM controllers. Indeed, those three schemes use high-gain error feedback in
virtual controls with error variables very similar to the design that involves compensation of known

![Figure 10. Error \( \sigma_1 \) for sliding mode block control.](image)

![Figure 11. Error \( \sigma_1 \) for combined backstepping high-order sliding mode control.](image)
COMBINED BACKSTEPPING AND HOSM CONTROL DESIGN

terms plus the aforementioned stabilizing high-gain terms. On the contrary, the use of nominal information on the real control reduces the discontinuous high-gain $\alpha_3$ of the BS-HOSM scheme as compared with the corresponding one of the SMBC gain. The gains $\alpha_i$ used are shown in Table I. The deflection control actions, $\delta_e$, are depicted in Figures 6–8, whereas the error signals $\sigma_1$ are shown in Figures 9–11.

Note that the error in Figure 10, which corresponds to $\sigma_1$ for SMBC, is not vanishing. This is expected according to [1] Theorem 1, where only ultimate boundedness of solutions is concluded. By contrast, hierarchical HOSM and BS-HOSM schemes theoretically achieve exact tracking. Nevertheless, the reduced discontinuous gain $\alpha_3$ for the BS-HOSM induces less high-frequency vibration as compared with hierarchical HOSM. As a consequence, the BS-HOSM scheme provides better accuracy as can be seen from Figures 9 and 11.

5. CONCLUSION

This paper presents a control scheme for a class of nonlinear MIMO systems. The control proposal is based on combining BS and HOSM control design. The gains of the HOSM part of the controller are reduced as compared with a direct hierarchical HOSM application, that is, without compensation of known nominal terms. Additionally, semi-global results can be made global with proper assumptions on the growing rate of the system functions and their corresponding derivatives in contrast with constant global bounds previously reported. The obtained controller achieves finite time exact tracking of smooth signals in spite of matched and unmatched perturbations. Simulation results of the application of the so-called SMBC are also included with the aim of showing the difference between schemes. The presented results address stability problems of previously reported works, and the approach can be applied to a more general class of systems.

APPENDIX

The proof of Theorem 1 is divided into two parts.

a.1. Backstepping design and practical stability

Consider a system in the form (1) that fulfills the assumptions, a relaxed version of standard assumptions from BS design, made in Theorem 1. The following inequalities for $\phi_{iBS}$ and $\phi_{iQC}$ are not additional assumptions; they hold because of the previous ones and by design. There exist positive constants $b_{\phi_i}$ and $d_{\phi_j}$ such that

$$\max \left\{ \left\| \frac{d^j}{dt^j} (g_i^{-1} \phi_{iQC}) \right\|, \left\| \frac{d^j}{dt^j} (\phi_{iQC}) \right\| \right\} \leq d_{\phi_j}; \quad \text{with } j = 0, \ldots, r - 1$$  \hspace{1cm} (A1)

$$\max \left\{ \left\| \frac{\partial \phi_{iBS}}{\partial \sigma_j} \right\|, \left\| \frac{\partial \phi_{iBS}}{\partial x_j} \right\| \right\} \leq b_{\phi_i}; \quad \text{with } j = 1, \ldots, i; \quad i = 1, \ldots, r - 1$$  \hspace{1cm} (A2)

hold over $D \subset \mathbb{R}^n$ (the region of validity of the assumptions in Theorem 1). The following composite Lyapunov function candidate is used through the design

$$V_{ic} = \sum_{j=1}^{i} V_j,$$  \hspace{1cm} (A3)

with $V_j = \frac{1}{2} \sigma_j^T \sigma_j$;  \hspace{1cm} (A4)

thus, $\frac{\partial V}{\partial \sigma} = \sigma_j^T$. Recall that $a_i = a_{im}^{k \times m}$, then

$$a_{im} = \lambda_m(a_i).$$  \hspace{1cm} (A5)
Consider the dynamical equation for $\sigma_1 = x_1 - \phi_0$:
\[
\dot{\sigma}_1 = f_1(t, x_1) + g_1^A(t, x_1)x_2 - \dot{\phi}_0 + \omega_1(t, x_1),
\]
and according to (A3), the next Lyapunov function candidate is proposed:
\[
V_{1c} = V_1 = \frac{1}{2}\sigma_1^T\sigma_1.
\]
Then, computing $\dot{V}_{1c}$ under the assumption that $x_2 = \phi_1$ leads to
\[
\dot{V}_{1c} = \sigma_1^T \left[ f_1 + g_1^A \phi_1 - \dot{\phi}_0 + \omega_1 \right].
\]
(A6)

$\phi_{1BS} = g_1^{-1}(-f_1 - a_1\sigma_1 + \dot{\phi}_0)$, according to (6) with $\dot{\phi}_0 = 0$ (Remark 4), which substituted in (A6), yields
\[
\dot{V}_{1c} = -a_{1m}\|\sigma_1\|^2 + \sigma_1^T(\phi_1QC + \phi_0 + \omega_1 + \Delta g_{1}\phi_1).
\]
(A7)

Given that $f_1$ is smooth and continuous, then there always exists some constant $a_{f1}$ such that $f_1 \leq a_{f1}\|\sigma_1\|$, $\forall\|\sigma\| > B_r$ hold for any bounded region $\Omega$. Define $\delta_{g1} = \|\Delta g_1 g_1^{-1}\| \leq \frac{\delta_{M}(\Delta g_1)}{\lambda_m(g_1)}$; due to the condition (12), it holds that $\delta_{g1} < 1$. Considering previously and applying (A1) and (A2), we obtain
\[
\dot{V}_{1c} \leq -a_{1m}\|\sigma_1\|^2 + \sigma_1^T(\phi_1QC + \phi_0 + \omega_1 + \Delta g_{1}\phi_1)
\]
\[
\leq -a_{1m}\|\sigma_1\|^2 + \|\sigma_1\|(d_1 - a_1m\|\sigma_1\| + d_1)
\]
\[
+ \|\sigma_1\|\delta_{g1}(a_{f1}\|\sigma_1\| + a_{1m}\|\sigma_1\| + d_1)
\]
\[
\leq -(a_{1m} - B_1 - \delta_{g1}a_{f1} - \delta_{g1}a_{1m})\|\sigma_1\|^2 + (d_1 - a_{1m}\|\sigma_1\| + \delta_{g1}d_1)\|\sigma_1\|.
\]
That is,
\[
\dot{V}_{1c} \leq -a_{1m}\|\sigma_1\|^2 + a_{1d}\|\sigma_1\|.
\]
(A8)

Therefore, $\dot{V}_{1c}$ is negative definite for $a_{1d} > 0$ and $\sigma_1$ sufficiently large. It is necessary that $\delta_{g1} < 1$ (ensured by (12)) in order to obtain $a_{1d} > 0$. It is desirable that uncertainties in $g_1$ are small enough to have $\delta_{g1} \ll 1$. In the rest of the proof, the uncertainties $\Delta g_{1}$ are not considered in order to avoid unnecessary complexity.

Remark 4
Note that the terms within $a_{1d}$ hamper the possibility of ensuring asymptotic stability to $\sigma_1 = 0$ from (A8). On the other hand, practical stability can be ensured even without including any estimation for $\dot{\phi}_0$ in $\phi_{1BS}$, provided that the former is bounded.

In the sequel of the practical stability proof, we will consider an estimation $\hat{\phi}_1$ based on the knowledge of the system. Nevertheless, it is worth mentioning that such estimations can be substituted for different ones or not to include any, as long as the high-gain parameters, $a_i$, are selected large enough for ensuring practical stability.

Now, consider the dynamical equation for $\sigma_2 = x_2 - \phi_1$:
\[
\dot{\sigma}_2 = f_2(t, \bar{x}_2) + g_2(t, \bar{x}_2)x_3 - \dot{\phi}_1 + \omega_2(t, \bar{x}_2)
\]
with \( x_3 = \phi_2 = \phi_{2BS} + g_2^{-1}\phi_{2QC} \) and the next composite Lyapunov function candidate

\[
V_{2c} = V_1 + \frac{1}{2}\sigma_2^T\sigma_2.
\]

For the derivative of \( V_{2c} \) along the trajectories of the system, the error originated from \( x_2 \neq \phi_1 \) will be considered. In order to do that, the term \( g_1\phi_1 \) is added and subtracted from the right hand side of the \( \dot{\sigma}_1 \) equation. Thus, we have the following:

\[
\dot{V}_{2c} = \sigma_1^T \left[ f_1 + g_1\phi_1 - \dot{\phi}_0 + \omega_1 \right] + \sigma_2^T g_1(x_2 - \phi_1) + \sigma_2^T \left( f_2 + g_2\phi_2 - \dot{\phi}_1 + \omega_2 \right), \quad (A9)
\]

and using the next expression for the derivative of \( \phi_1 \):

\[
\dot{\phi}_1 = \frac{\partial \phi_{1BS}}{\partial \sigma_1} \dot{\sigma}_1 + \frac{\partial \phi_{1BS}}{\partial x_1} \dot{x}_1 + \frac{\partial \phi_{1BS}}{\partial t} + \frac{d}{dt}g_1^{-1}\phi_{1QC},
\]

we obtain the following:

\[
\dot{V}_{2c} = \dot{V}_{1c} + \sigma_1^T g_1\sigma_2 + \sigma_2^T \left( f_2 + g_2\phi_{2BS} - \frac{\partial \phi_{1BS}}{\partial t} - \frac{\partial \phi_{1BS}}{\partial \sigma_1} (f_1 + g_1 x_2 - \dot{\phi}_0) \right.
\]
\[
- \frac{\partial \phi_{1BS}}{\partial x_1} (f_1 + g_1 x_2)
\]
\[
\left. + \frac{\partial \phi_{1BS}}{\partial \sigma_1} \right) g_1(x_2 - \phi_1)
\]
\[
+ \sigma_2^T \left( \phi_{2QC} = \frac{\partial \phi_{1BS}}{\partial \sigma_1}\omega_1 - \frac{\partial \phi_{1BS}}{\partial x_1}\omega_1 - \frac{d}{dt}g_1^{-1}\phi_{1QC} + \omega_2 \right).
\]

Then, we choose \( \phi_{2BS} \) according to (7) with \( \dot{\phi}_1 \) as the sum of those terms that can be analytically obtained, that is,

\[
\phi_{2BS} = g_2^{-1} \left( \dot{\phi}_1 - a_2\sigma_2 - f_2 - g_1^T \sigma_1 \right), \quad (A11)
\]

By substituting (A11) in (A10) and applying the inequalities (10) and (A1)–(A2), the next expression for \( \dot{V}_{2c} \) is obtained:

\[
\dot{V}_{2c} \leq -a_1^*\|\sigma_1\|^2 + a_1d\|\sigma_1\|^2 - a_{2m}\|\sigma_2\|^2
\]
\[
+ \|\sigma_2\| (d_{\phi_20} + 2b_{\beta\phi_1}(b_1\|\sigma_1\| + d_1) + d_{\phi_11} + b_2\|\tilde{\sigma}_2\| + d_2).
\]

Using \( \|\sigma_2\|/\|\tilde{\sigma}_2\| < \|\tilde{\sigma}_2\|^2 \), in the aforementioned inequality, yields

\[
\dot{V}_{2c} \leq -a_1^*\|\sigma_1\|^2 + 2b_{\beta\phi_1}b_1\|\sigma_1\|^2 + a_{2m}\|\sigma_2\|^2
\]
\[
+ a_{1d}\|\sigma_1\| + b_2\|\tilde{\sigma}_2\|^2 + (d_{\phi_20} + 2b_{\beta\phi_1}d_1 + d_{\phi_11} + d_2)\|\sigma_2\|. \quad (A12)
\]

Applying quadratic forms, (A12) can be expressed as follows:

\[
\dot{V}_{2c} \leq - \left[ \begin{array}{c} \|\sigma_1\| \\ \|\sigma_2\| \end{array} \right]^T \left[ \begin{array}{cc} a_1^* & -b_{\beta\phi_1}b_1 \\ -b_{\beta\phi_1}b_1 & a_{2m} \end{array} \right] \left[ \begin{array}{c} \|\sigma_1\| \\ \|\sigma_2\| \end{array} \right]
\]
\[
+ a_{1d}\|\sigma_1\| + b_2\|\tilde{\sigma}_2\|^2 + (d_{\phi_20} + 2b_{\beta\phi_1}d_1 + d_{\phi_11} + d_2)\|\sigma_2\|. \quad (A13)
\]

Then, choosing \( a_2 \) such that \( a_1^*a_{2m} > (b_{\beta\phi_1}b_1)^2 \) ensures the positive definiteness of the square matrix in (A13). Let us define the adjustable parameter \( a_{2m}^* \) as follows:

\[
a_{2m}^* = \lambda_m \left[ \begin{array}{cc} a_1^* & -b_{\beta\phi_1}b_1 \\ -b_{\beta\phi_1}b_1 & a_{2m} \end{array} \right].
\]

Then, the quadratic form in (A13) can be upper bounded by \(-a_{2m}^*\|\sigma_1\|^2 + \|\sigma_2\|^2 \geq -a_{2m}^*\|\tilde{\sigma}_2\|^2 \). Thus, the following inequality is obtained:
By applying the aforementioned procedure for each remaining state, \( \sigma_i; i \in [3, r] \), practical stability of the whole system can be concluded; at each, the corresponding inequality with the following form can be obtained:

\[
\dot{V}_{ic} \leq -a_i^* \| \tilde{\sigma}_i \|^2 + a_{i,dd} \| \tilde{\sigma}_i \|,
\]

(A15)

where \( a_i^* = a_{i,m}^* - b_2 \) and \( a_{i,dd} = \max\{a_{i,d}, d_{\phi_0} + 2b_2 \phi_1 d_1 + \phi_{11} + d_2 \} \). That is, \( \dot{V}_{ic} \) is negative definite outside a ball of radius inversely proportional to \( a_i^* \).

- By applying the aforementioned procedure for each remaining state, \( \sigma_i; i \in [3, r] \), practical stability of the whole system can be concluded; at each, the corresponding inequality with the following form can be obtained:

\[
\dot{V}_{ic} \leq -a_i^* \| \tilde{\sigma}_i \|^2 + a_{i,dd} \| \tilde{\sigma}_i \|,
\]

(A15)

where \( a_i^* \) are chosen by design, whereas \( a_{i,d} \) depends on the perturbation bounds. Therefore, we state the following.

**Proposition 5**

There exists a bounded ball \( B_\sigma \), such that the trajectories of system (1), in closed loop with controller (8), and according to the selection of parameters \( a_i \), fulfill that \( \| \sigma_1, \ldots, \sigma_r \| \leq B_\sigma \), for all \( t > T_{BS} \), for a bounded time \( T_{BS} \).

The magnitude of \( B_\sigma \) and \( T_{BS} \) depends on the selected parameters \( a_i \). That concludes the first part, practical stability, of the proof.

**a.2. Exact tracking**

According to the proposal in [3], the virtual controller terms \( \phi_{i, QC} \) are constructed as follows (we use the notation introduced in Section 3):

\[
\phi_{i, QC}^{(r-i)} = -a_i \Psi_{r-i, r-i+1}(\sigma_i);
\]

moreover, if Assumption A2 is fulfilled not only for \( r - i \) but also for \( r - i + k \) derivatives and, correspondingly, \( \phi_{i, 0} \) has \( r + k \) bounded derivatives, then the order of each \( \phi_{i, QC} \) term may be increased in order to obtain smoother control signals. We introduce the next expression where this smoothing possibility is considered:

\[
\phi_{i, QC}^{(r+k-i)} = -a_i \Psi_{r+k-i, r+k-i+1}(\sigma_i),
\]

(A16)

\[
\phi_{r, QC}^{(k)} = -a_r \Psi_{r+k, r+k+1}(\sigma_r).
\]

(A17)

To show that finite time exact tracking is achieved, consider the closed loop system:

\[
\dot{\sigma}_i = f_i + g_i \phi_i + \omega_i - \hat{\phi}_{i-1} - g_i \sigma_i,
\]

\[
\dot{\sigma}_r = f_r + g_r u + \omega_r - \hat{\phi}_{r-1}.
\]

(A18)

By applying Proposition 5, the trajectories of (A18) are bounded, and all \( t > T_{BS} \) are inside a ball \( B_\sigma \subset R^n \). The derivatives of the functions of the system are also bounded. Let us recall that \( \sigma_j = [\sigma_j, \ldots, \sigma_{jm}]^T, \tilde{\sigma}_j = [\sigma_j, \ldots, \sigma_j], \) and \( \Psi_{j-1,j}(\tilde{\sigma}) = \Psi_{j-1,j}(\sigma), \sigma^{(j-1)}_j, \ldots, \sigma^{(j-1)}_j \) according to the notation introduced in Section 3. Thus, for each scalar element \( \sigma_{ij}, i \in [1, r], \) and \( j \in [1, m] \), we have the following:

\[
\sigma_{ij}^{(r+k-i+1)} = h_{ij}^*(t, \tilde{\sigma}_i) + g_{ij}^*(t, \tilde{\sigma}_i) u_{ij},
\]

(A19)

with \( u_{ij} = -a_i \Psi_{r-k-i, r-k-i+1}(\tilde{\sigma}_{ij}), \)

\[
h_{ij}^*(t, \tilde{\sigma}_i) = \sigma_{ij}^{(r+k-i+1)}|_{u_{ij}=0},
\]

(A20)

\[
g_{ij}^*(t, \tilde{\sigma}_i) = \frac{\partial}{\partial u_{ij}} \sigma_{ij}^{(r+k-i+1)}.
\]
and there exists positive constants $K_{mij}$, $K_{Mij}$, and $C_{ij} > 0$ such that the inequalities

$$0 < K_{mij} \leq \frac{\partial}{\partial t} \sigma_{ij}^{(r+k-i+1)} \leq K_{Mij}$$

$$|\sigma_{ij}^{(r+k-i+1)}|_{u_{ij}=0} \leq C_{ij}$$

(A21)

hold. Therefore, (A19) and (A21) imply the following differential inclusion:

$$\sigma_{ij}^{(r+k-i+1)} \in [-C_{ij}, C_{ij}] + [K_{mij}, K_{Mij}]u_{ij}.$$  

(A22)

In this paper, the quasi-continuous controllers as reported in [17] are used for the scalars $u_{ij}$. Define $R = r + k - i$ and let $h = 0, \ldots, R - 1$. Then, for each scalar $\sigma_{ij}$, $i = 1, \ldots, r$, and $j = 1, \ldots, m$, the corresponding HOSM controller (A20) is constructed as follows [17]:

$$\phi_{0,R} = \sigma_{ij} \quad N_{0,R} = |\sigma_{ij}| \quad \Psi_{0,R} = \varphi_{0,R}/N_{0,R} = \text{sign}\sigma_{ij},$$  

(A23)

$$\varphi_{h,R} = \sigma_{ij}^{(h)} + \beta_{h}N_{h-1,R}^{(R-h)/(R-h+1)}\Psi_{h-1,R}.$$  

(A24)

$$N_{h,R} = |\sigma_{ij}^{(h)}| + \beta_{h}N_{h-1,R}^{(R-i)/(R-i+1)}\Psi_{i,R} = \varphi_{i,R}/N_{i,R}.$$  

(A25)

The magnitude of the positive constants $K_{mij}$, $K_{Mij}$, and $C_{ij} > 0$ depends on the radius of the ball $B_d$ of Proposition 5. Recall that by assumption, the vector functions $f_i(t, \overline{x}_i), g_i^{\alpha}(t, \overline{x}_i) \in \mathbb{R}^{m \times m}$ and $\omega_i(t, \overline{x}_i)$ are smooth with at least $r - i + k$ bounded derivatives $\forall x \in D \subset \mathbb{R}^n, t \in [0, \infty)$, and $\phi_{0,R}$ is smooth with at least $r + k$ bounded derivatives. Thus, the inequalities (A22) can be established, with the corresponding constants, due to the fact that trajectories have been constrained to evolve in a bounded ball. The ball $B_d$ is defined by the combined effect of the gains $a_i$ of the BS design and the magnitude of perturbations $\omega_i$. In the latter, it is considered also the effect of the quasi-continuous terms during their finite time transient. As a matter of fact, an accurate knowledge of the system improves the possibility of reducing $B_d$ by means of the proper use of such information on the BS design. Finally, according to [17], (A20), the finite time stability of (A22) and (A20), is ensured. The stable $r + k - i + 1$-th order sliding mode is established for every $\sigma_{ij}$. That concludes the proof of Theorem 1.

ACKNOWLEDGEMENTS

The first author gratefully acknowledges the support of the UC MEXUS-CONACYT postdoctoral fellowship program. The present contribution was partially financed by PAPIIT IN113613, PAPIIT IN112915, and CONACYT 132125 grants.

REFERENCES


