Generating Self-Excited Oscillations via Two-Relay Controller

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Abstract—A tool for the design of a self-excited oscillation of a desired amplitude and frequency in linear plants by means of the variable structure control is proposed. An approximate approach based on the describing function method is given, which requires that the mechanical plant should be a low-pass filter—the hypothesis that usually holds when the oscillations are relatively fast. The proposed approach is demonstrated via the controller design and experiments on the Furuta pendulum.

Index Terms—Frequency domain methods, periodic solution, underactuated systems, variable structure systems.

I. INTRODUCTION

Overview: Traditionally, control systems are categorized into: regulators, which are supposed to maintain a certain process variable at a desired set point value, and servo systems that are supposed to track external inputs as precisely as possible. However, in practice there are some other control tasks that fall into neither of the above categories. One of those tasks is generating a functional motion: the motion having some properties important to the functionality of a certain system without involvement of set point tracking and specification of other properties of the motion.

In this technical note, we consider the control of one of the simplest types of a functional motion: generation of a periodic motion. Current representative works on periodic motions and orbital stabilization of underactuated systems involve finding and using a reference model as a generator of limit cycles (e.g., [11]), thus considering the problem of obtaining a periodic motion as a servo problem. Orbital stabilization of underactuated systems finds applications in the coordinated motion of biped robots [5], gymnastic robots, and others (see, e.g., [8], [12] and references therein).

Methodology: It is known (see for example [7], [13]) that the presence of parasitic dynamics in the sliding mode systems causes the chattering effect. Moreover, in [4], [7] it is shown that if the sliding mode (averaged) dynamics have an isolated equilibrium point, the presence of additional fast dynamics causes a periodic self-excited motion in the systems. It allows us to propose a novel method of generating a periodic motion in the system where the same behavior can be seen via second order sliding mode (SOSM) algorithms, i.e. generating self-excited oscillations using the same mechanism as the one that produces chattering.

The describing function (DF) method (see for example [2]) offers finding approximate values of the frequency and the amplitude of periodic motions in the systems with linear plants driven by the sliding mode controllers.

The proposed approach is based on the fact that all second order sliding mode (SOSM) algorithms ([3], [4]) produce chattering (periodic motions of relatively small amplitude and high frequency) in the presence of unmodelled dynamics [3], [4]. In sliding mode control, chattering is usually considered an undesirable component of the motion. In this technical note, we aim to use this property of SOSM for the purpose of generating a relatively slow motion with a significantly higher amplitude and lower frequency than respectively the amplitude and frequency of chattering.

Results of the Technical Note:

• The twisting algorithm [9] originally created as a SOSM controller—to ensure the finite-time convergence—is generalized, so that it can generate self-excited oscillations in linear closed-loop systems. The required frequencies and amplitudes of periodic motions are produced without tracking of precomputed trajectories. It allows for generating a wider (than the original twisting algorithm with additional dynamics) range of frequencies and encompassing a variety of plant dynamics.

• An approximate approach based on the describing function is proposed to find the values of the controller parameters allowing one output to obtain the desired frequencies and the output amplitudes.

• The necessary conditions for asymptotic stability of desired periodic solutions are given.

• The theoretical results are validated experimentally via the tests on the laboratory Furuta pendulum. The computed gains of the two-relay controller allow for the existence of a periodic motion of the required frequency and amplitude around the upright position (which gives the non-minimum-phase system case) in wide ranges of frequencies and amplitudes.

Organization of the Technical Note: This technical note is structured as follows: Section II introduces the problem statement. In Section III, the idea of the two-relay controller is explained via the describing function method, and formulas for computing approximate values of controller parameters are derived. In Section IV, necessary conditions for the stability of the periodic solution are provided. In Section V, the design methodology is validated via periodic motion design for the experimental Furuta pendulum. Section VI provides final conclusions.

II. PROBLEM STATEMENT

Let firstly, the linearized plant be given by

\[
\begin{align*}
x &= Ax + Bu, & x \in \mathbb{R}^n, & y \in \mathbb{R}.
\end{align*}
\]

We assume in (1) that \( CB = 0 \), so that the relative degree of system (1) is higher than one. The following two-relay controller is proposed for the purpose of exciting a periodic motion:

\[
u = -c_1 \text{sign}(y) - c_2 \text{sign}(y)
\]

where \( c_1 \) and \( c_2 \) are parameters designed such that the scalar output of the system (the position of a selected link of the plant) has a steady periodic motion with the desired frequency and amplitude.

Let us assume that the two relay controller has two independent parameters \( c_1 \in C_1 \subseteq \mathbb{R} \) and \( c_2 \in C_2 \subseteq \mathbb{R} \), so that the changes to those parameters result in the respective changes of the frequency
\( \Omega \in \mathcal{W} \subset \mathbb{R} \) and the amplitude \( A_1 \in \mathcal{A} \subset \mathbb{R} \) of the self-excited oscillations. Then we can note that there exist two mappings \( F_1 : C_1 \times C_2 \rightarrow \mathcal{W} \) and \( F_2 : C_1 \times C_2 \rightarrow \mathcal{A} \), which can be rewritten as \( F : C_1 \times C_2 \rightarrow \mathcal{W} \times \mathcal{A} \subset \mathbb{R}^2 \). Assume that mapping \( F \) is unique. Then there exists an inverse mapping \( G : \mathcal{W} \times \mathcal{A} \rightarrow C_1 \times C_2 \). The objective is, therefore, (a) to obtain mapping \( G \) using a frequency-domain method for deriving the model of the periodic process in the system, (b) to prove the uniqueness of mappings \( F \) and \( G \) for the selected controller, and (c) to find the ranges of variation of \( \Omega \) and \( A_1 \) that can be achieved by varying parameters \( c_1 \) and \( c_2 \).

The analysis and design objectives are formulated as follows: Find parameter values \( c_1 \) and \( c_2 \) in (2) such that the system (1) has a periodic motion with the desired frequency \( \Omega \) and desired amplitude of the output signal \( A_1 \). Therefore, the main objective of this research is to find mapping \( G \) to be able to tune \( c_1 \) and \( c_2 \) values, using the models provided by the describing function method.

### III. Idea of the Method

The idea of the method is to provide the mapping from a set of desired frequencies \( \mathcal{W} \subset \mathbb{R} \) and amplitudes \( \mathcal{A} \subset \mathbb{R} \) into a set of gain values \( C \subset \mathbb{R}^2 \), that is \( G : \mathcal{W} \times \mathcal{A} \rightarrow C \). To achieve the objective, let us start with the design via the describing function method which is a useful frequency-domain tool for time-invariant linear plants to predict the existence or absence of limit cycles and estimate the frequency and amplitude when it exists. In the scenario introduced in [3], the DF method was used for analysis of chattering for the closed-loop system with the twisting algorithm where the inverse of this mapping was derived.

#### A. Describing Function of the Two-Relay Controller

Suppose that the transfer function of system (1) can be represented as follows:

\[
W(s) = C(sI - A)^{-1}B.
\]

The Describing Function (DF), \( N \), of the variable structure controller (2) is the first harmonic of the periodic control signal divided by the amplitude of \( y(t) \) [2]:

\[
N = \frac{\omega}{\pi A_1} \int_0^{2\pi/\omega} u(t) \sin \omega t dt + j \frac{\omega}{\pi A_1} \int_0^{2\pi/\omega} u(t) \cos \omega t dt
\]

(3)

where \( A_1 \) is the amplitude of the input to the nonlinearity (of \( y(t) \) in our case) and \( \omega \) is the frequency of \( y(t) \). However, the algorithm (2) can be analyzed as the parallel connection of two ideal relays where the input to the first relay is the output variable and the input to the second relay is the derivative of the output variable (see Fig. 1). For the first relay the DF is

\[
N_1 = \frac{4c_1}{\pi A_1}
\]

and for the second relay it is [2]:

\[
N_2 = \frac{4c_2}{\pi A_2}
\]

where \( A_2 \) is the amplitude of \( dy/dt \). Also, take into account the relationship between \( y \) and \( dy/dt \) in the Laplace domain, which gives the relationship between the amplitudes \( A_1 \) and \( A_2 : A_2 = A_1 \Omega \), where \( \Omega \) is the frequency of the oscillation. Using the notation of the algorithm (2) we can rewrite this equation as follows:

\[
N = N_1 + \alpha N_2 = \frac{4c_1}{\pi A_1} + j\Omega \frac{4c_2}{\pi A_2} = \frac{4}{\pi A_1} (c_1 + j c_2)
\]

(4)

where \( s = j\Omega \). Let us note that the DF of the algorithm (2) depends on the amplitude value only. This suggests the technique of finding the parameters of the limit cycle—via the solution of the harmonic balance equation [2]

\[
W(j\Omega)N(a) = -1
\]

(5)

where \( a \) is the generic amplitude of the oscillation at the input to the nonlinearity, and \( W(j\omega) \) is the complex frequency response characteristic (Nyquist plot) of the plant. Using the notation of the algorithm (2) and replacing the generic amplitude with the amplitude of the oscillation of the input to the first relay this equation can be rewritten as follows:

\[
W(j\Omega) = -\frac{1}{N(A_1)}
\]

(6)

where the function at the right-hand side is given by

\[
\frac{1}{N(A_1)} = \frac{\pi A_1}{4} (c_1 + j c_2 - c_2^2 - c_1^2)
\]

Equation (5) is equivalent to the condition of the complex frequency response characteristic of the open-loop system intersecting the real axis in the point \((-1, j0)\). The graphical illustration of the technique of solving (5) is given in Fig. 2. The function 
\(-1/N\) is a straight line whose slope depends on \( c_2/c_1 \). The point of intersection of this function and of the Nyquist plot \( W(j\omega) \) provides the solution of the periodic problem.

#### B. Tuning the Parameters of the Controller

Here, we summarize the steps to tune \( c_1 \) and \( c_2 \):

\[\text{Fig. 1. Relay feedback system.}\]

\[\text{Fig. 2. Example of a Nyquist plot of the open-loop system } W(j\omega) \text{ with two relay controller.}\]
a) Identify the quadrant in the Nyquist plot where the desired frequency $\Omega$ is located, which falls into one of the following categories (sets):

$$Q_1 = \{ \omega \in \mathbb{R} : \text{Re} \{ W(j\omega) \} > 0, \text{Im} \{ W(j\omega) \} \geq 0 \}$$
$$Q_2 = \{ \omega \in \mathbb{R} : \text{Re} \{ W(j\omega) \} \leq 0, \text{Im} \{ W(j\omega) \} \geq 0 \}$$
$$Q_3 = \{ \omega \in \mathbb{R} : \text{Re} \{ W(j\omega) \} \leq 0, \text{Im} \{ W(j\omega) \} < 0 \}$$
$$Q_4 = \{ \omega \in \mathbb{R} : \text{Re} \{ W(j\omega) \} > 0, \text{Im} \{ W(j\omega) \} < 0 \}.$$

b) The frequency of the oscillations depends only on the $c_2/c_1$ ratio, and it is possible to obtain the desired frequency $\Omega$ by tuning the $\xi = c_2/c_1$ ratio

$$\xi = \frac{c_2}{c_1} = -\frac{\text{Im} \{ W(j\Omega) \}}{\text{Re} \{ W(j\Omega) \}}.$$

Since the amplitude of the oscillations is given by

$$A_1 = \frac{4}{\pi} |W(j\Omega)| \sqrt{c_1^2 + c_2^2}$$

then the $c_1$ and $c_2$ values can be computed as follows:

$$c_1 = \left\{ \begin{array}{ll}
\frac{\xi}{2} \frac{4}{|W(j\Omega)|} \cdot (\sqrt{1 + \xi^2} - 1) & \text{if } \Omega \in Q_2 \cup Q_3 \\
-\frac{\xi}{2} \frac{4}{|W(j\Omega)|} \cdot (\sqrt{1 + \xi^2} - 1) & \text{elsewhere}
\end{array} \right.$$  
$$c_2 = \xi \cdot c_1.$$  
(9)

(10)

Remark 1: It is possible to obtain the formulas for computing the exact values of $c_1$ and $c_2$ using the Locus of Perturbed Relay Systems (LPRS) method (see details in [1]). In addition, sufficient conditions for the existence and stability of desired periodic solutions could be obtained by using Poincaré sections.

IV. STABILITY OF PERIODIC SOLUTIONS

We shall consider that the harmonic balance condition still holds for small perturbations of the amplitude and the frequency with respect of the periodic motion. In this case the oscillation can be described as a damped one. If the damping parameter will be negative at a positive increment of the amplitude and positive at a negative increment of the amplitude then the perturbation will vanish, and the limit cycle will be asymptotically stable.

Theorem 1: Suppose that for the values of the $c_1$ and $c_2$ given by (9) and (10) there exists a corresponding periodic solution to the system (1), (2). If

$$\frac{d \text{arg} W}{d \ln \omega} \bigg|_{\omega=\Omega} < -\frac{c_1 c_2}{c_1^2 + c_2^2}$$  
(11)

then the above-mentioned periodic solutions to the system (1), (2) is orbitally asymptotically stable.

Proof: The approach for the stability analysis of the periodic motions is similar to the one proposed in [10]. Writing the harmonic balance equation of the perturbed motion:

$$\begin{align*}
\{ N_1(A_1 + \Delta A_1) + j\Omega (\Delta \sigma + j\Delta \Omega) \} N_2(A_2 + \Delta A_2) \times W(\Delta \sigma + j(\Omega + \Delta \Omega)) &= -1 \\
&= \frac{\partial N_1}{\partial \Delta A_1} (\Delta A_1 = 0) \cdot W(j\Omega) + \frac{\partial N_2}{\partial \Delta A_1} (\Delta A_1 = 0) \cdot j\Omega W(j\Omega) \\
&+ \frac{d W}{d \Omega} \bigg|_{\omega=\Omega} \left( \frac{d \Delta \sigma}{d \Delta A_1} + j \frac{d \Delta \Omega}{d \Delta A_1} \right) \cdot N_1(A_1) \\
&+ N_2(A_2) \left( \frac{d \Delta \sigma}{d \Delta A_1} + j \frac{d \Delta \Omega}{d \Delta A_1} \right) W(j\Omega) \\
&+ N_2(A_2) \left( \Omega \frac{d W}{d \Omega} \bigg|_{\omega=\Omega} \left( \frac{d \Delta \sigma}{d \Delta A_1} + j \frac{d \Delta \Omega}{d \Delta A_1} \right) \right) = 0
\end{align*}$$

where

$$\frac{\partial N_1}{\partial \Delta A_1} = -\frac{4c_1}{\pi A_1^2} \quad \text{and} \quad \frac{\partial N_2}{\partial \Delta A_1} = -\frac{4c_2}{\pi A_2^2} \Omega = -\frac{4c_2}{\pi A_1^2} \Omega.$$

Thus, the following equation is obtained:

$$\begin{align*}
-\frac{4c_1}{\pi A_1^2} W(j\Omega) &= j\frac{4c_2}{\pi A_1^2} W(j\Omega) \\
&= \left( \frac{d \Delta \sigma}{d \Delta A_1} + j \frac{d \Delta \Omega}{d \Delta A_1} \right) \\
&\times \left\{ -N_1(A_1) \frac{d W}{d \Omega} \bigg|_{\omega=\Omega} - N_2(A_2) \left( \Omega \frac{d W}{d \Omega} \bigg|_{\omega=\Omega} \right) \right\} \\
&= \left( \frac{d \Delta \sigma}{d \Delta A_1} + j \frac{d \Delta \Omega}{d \Delta A_1} \right) \\
&\times \left\{ c_1 \frac{d W}{d \Omega} \bigg|_{\omega=\Omega} + c_2 \frac{d W}{d \Omega} \bigg|_{\omega=\Omega} \right\} \\
&= \frac{1}{A_1} \left( \frac{d \ln W}{d s} \bigg|_{\omega=\Omega} + \frac{c_1 c_2}{\Omega} \frac{d W}{d \Omega} \bigg|_{\omega=\Omega} \right)
\end{align*}$$

Then for the inequality

$$\frac{d \Delta \sigma}{d \Delta A_1} < 0$$  
(12)

to be true, the following should hold:

$$\text{Re} \frac{1}{A_1 \Omega} < 0 \quad \text{or} \quad \text{Re} \Lambda < 0.$$  
(13)

Then for the real part of $\Lambda$, we can write

$$\text{Re} \frac{d \ln W}{d s} \bigg|_{\omega=\Omega} + \frac{c_1 c_2}{\Omega \left( c_1^2 + c_2^2 \right)} < 0.$$  
(14)

Representing the transfer function in the exponential format and taking the derivative with respect to $s$ leads to the following inequality:

$$\frac{d \text{arg} W}{d \omega} < -\frac{c_1 c_2}{\Omega \left( c_1^2 + c_2^2 \right)}$$  
(15)

or finally to formula (11). Therefore, the stability of the periodic motion is determined just by the slope of the phase characteristic of the plant.
which must be steeper than a certain value for the oscillation to be asymptotically stable.

V. EXPERIMENTAL STUDY: THE FURUTA PENDULUM

A. Experimental Setup

In this section, we present experimental results using the laboratory Furuta pendulum, produced by Quanser Consulting Inc., depicted in Fig. 3. It contains a 24-Volt DC motor that is coupled with an encoder and is mounted vertically in the metal chamber. The L-shaped arm, or hub, is connected to the motor shaft and pivots between ±180 degrees. At the end, a suspended pendulum is attached. The pendulum angle is measured by the encoder. As described in Fig. 3, the arm rotates about its pivot and its angle is denoted as q1, while the pendulum attached to the arm rotates about its pivot and its angle is denoted as q2. The experimental setup includes a PC equipped with an NI-M series data acquisition card connected to the Educational Laboratory Virtual Instrumentation Suite (NI-ELVIS) workstation from National Instrument. The controller was implemented using Labview programming language allowing debugging, virtual oscilloscope, automation functions, and data storage during the experiments. The sampling frequency for control implementation has been set to 400 Hz.

B. Experimental Results

Generally the equations of motion of the Furuta Pendulum are as follows:

\[ M(q)\ddot{q} + P(q, \dot{q}) = B_1\tau \]  

where \( q = (q_1, q_2)^T \) is a vector that includes the arm rotation angle \( (q_1) \) and the pendulum angle \( (q_2) \), \( \tau \in \mathbb{R} \) is the applied torque; \( B_1 = [0, 1]^T \) is the input that maps the torque input to the joint coordinates space; \( M(q) \in \mathbb{R}^{2x2} \) is the symmetric positive-definite inertia matrix; and \( P(q, \dot{q}) \in \mathbb{R}^{2} \) is the vector that contains the Coriolis, centrifugal, gravity, and friction torques (the formulae for the \( M(q) \) and \( P(q, \dot{q}) \) for the Furuta pendulum are given in Appendix).

Experiments were carried out to achieve the orbital stabilization of the unactuated link (the pendulum) \( y = q_2 \) around the equilibrium point \( q^* = (\pi, 0) \). The equation of motion of the Furuta pendulum (16) is linearized around \( q^* \in \mathbb{R}^2 \) and by virtue of the instability of the linearized open-loop system, a state-feedback controller \( \tau = -Kx + u \) and \( x = (q - q^*, \dot{q})^T \in \mathbb{R}^4 \), is designed such that the compensated system has an overshoot of 8 and gain crossover frequency at 10 [rad/s]

![Fig. 3. Experimental Furuta pendulum system.](Image)

![Fig. 4. Bode plot of the open-loop system.](Image)

TABLE I

<table>
<thead>
<tr>
<th>Desired ( \Omega )</th>
<th>Desired ( A_1 )</th>
<th>( c_1 )</th>
<th>( c_2 )</th>
<th>Exp. ( \Omega )</th>
<th>Exp. ( A_1 )</th>
</tr>
</thead>
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<td>0.23</td>
<td>6.28</td>
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<td>0.14</td>
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<td>9.00</td>
<td>0.35</td>
</tr>
</tbody>
</table>

(see Bode diagram in Fig. 4 for the open-loop system). Thus, the matrices \( A, B, \) and \( C \) of the linear system (1) are

\[
A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -6.591 & 125.685 & -6.262 & 25.525 \\ 3.031 & -112.408 & 2.879 & -11.737 \end{bmatrix}, \quad 
B = \begin{bmatrix} 0 \\ 0 \\ 56.389 \\ -25.930 \end{bmatrix}, \quad 
C = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T
\]

For the experiments, we set initial conditions sufficiently close to the equilibrium point \( q^* \in \mathbb{R}^2 \). The output \( y = q_2 \) is driven to a periodic motion for several desired frequencies and amplitudes. The frequencies \( (\Omega) \) and amplitudes \( (A_1) \) obtained from experiments by using the values of \( c_1 \) and \( c_2 \) computed by means of the DF are given in Table I. Inequality (11) holds for the chosen frequencies and amplitudes, thus asymptotical stability of the periodic orbit was established by Theorem 1.

In Fig. 5, experimental oscillations for the output \( y \), for fast \( (\Omega_1 = 25$ [rad/s]) \) and slow motion \( (\Omega_2 = 10$ [rad/s]) \) are displayed. Note that certain imperfections appear in the slow motion graphics in Fig. 5, which are attributed to the Coulomb friction forces, and the dead zone. Also, in some modes natural frequencies of the pendulum mechanical structure are excited, and manifested as higher-frequency vibrations.

VI. CONCLUSION

For generation of self-excited oscillations with desired output amplitude and frequencies, a two-relay controller is proposed. Values of the controller parameters are computed through application of the DF-based model of periodic motions. Necessary conditions for
specified by applying the Euler-Lagrange formulation \([6]\), where

\[
M_1(q) = I_v r^2 \cos^2(q_1) \\
M_2(q) = -\frac{1}{2} M_p r I_v \cos(q_1) \cos(q_2) \\
M_{22}(q) = J_p + M_p l^2 \\
P_1(q, \dot{q}) = -2 M_p r^2 \cos(q_1) \sin(q_1) \dot{q}_1^2 + \frac{1}{4} M_p r I_v \cos(q_1) \sin(q_2) \dot{q}_2^2 \\
P_2(q, \dot{q}) = \frac{1}{2} M_p r I_v \sin(q_1) \cos(q_2) \dot{q}_1^2 + M_g g l \sin(q_2)
\]

where \(\dot{q}_{\mathrm{ave}} = 0.027 \text{ [Kg]}\) is mass of the pendulum, \(l_v = 0.133 \text{ [m]}\) is the length of pendulum center of mass from pivot, \(L_p = 0.191 \text{ [m]}\) is the total length of pendulum, \(r = 0.0826 \text{ [m]}\) is the length of arm pivot to pendulum pivot, \(g = 9.810 \text{ [m/s}^2\text{]}\) is the gravitational acceleration constant, \(J_p = 1.23 \times 10^{-4} \text{ [Kg - m}^2\text{]}\) is the pendulum moment of inertia about its pivot axis, and \(J_{\dot{q}_\mathrm{ave}} = 1.10 \times 10^{-4} \text{ [Kg - m}^2\text{]}\) is the equivalent moment of inertia about motor shaft pivot axis.

**Appendix**

**A. Dynamic Model of Furuta Pendulum**

The equation motion of Furuta pendulum, described by (16), was specified by applying the Euler-Lagrange formulation \([6]\), where

\[
M(q) = \begin{bmatrix} M_1(q) & M_2(q) \\ M_2(q) & M_{22}(q) \end{bmatrix} \quad P(q, \dot{q}) = \begin{bmatrix} P_1(q, \dot{q}) \\ P_2(q, \dot{q}) \end{bmatrix}
\]

with

the local orbital asymptotic stability of the desired periodic motion are also obtained from the DF-based model of periodic motions. The effectiveness of the proposed design procedures is supported by experiments carried out on the Furuta pendulum from Quanser.

**References**


