Abstract—We consider the application of a min-max optimal control based on the LQ-index for a set of systems where only the output information is available. Here every system is affected by matched uncertainties, and we propose to use an output integral sliding mode to compensate the matched uncertainties right after the beginning of the process. For the case when the extended system is free of invariant zeros, a hierarchical sliding mode observer is applied. The error of realization of the proposed control algorithm is estimated in terms of the sampling step and actuator time constant. An example illustrates the suggested method of design.

Index Terms—Hierarchical sliding mode observer (HSMO), output integral sliding mode (OISM).

I. INTRODUCTION

A. Antecedents and Motivation

For the case of multi-plant there are two main approaches to control such systems. One is to decentralize the controls of each plant ([1], [2]). The other method is to design the same optimal control law for all the plants and make this control robust with respect to perturbations.

The robustification of the optimal control is one of the main problems in the modern control theory (see, e.g., [3]–[12] and references therein). In [5] and [6], a robust optimal control based in a min-max LQ-index for a multi-model system was proposed. Basically, it was considered a set of possible models for the same plant, each model is characterized by a LQ-index and the objective of the robust optimal control is to minimize the worst of the LQ-indices. However, the exact solution of this optimal control problem requires of two basic assumptions:

- the system is free from any uncertainty;
- the state vector is completely available.

Thus, for the case when we have output information only, we should ensure the compensation of the matched uncertainties. Furthermore, we need to reconstruct the original states to take advantage of the state feedback robust optimal control.

The integral sliding mode (ISM) was proposed in [13], it is usually used to compensate the matched uncertainties from the beginning of the process. The optimal control problem in the presence of matched uncertainty was considered in [7], [9]. They proposed the use of the integral sliding mode control allowing to ensure the robustness of the solution from the initial time moment. Some applications of ISM can be found in [9]–[11], [14]–[20]. The order of the motion equation in ISM is equal to the order of the original system. As a result, robustness of the trajectory for a system driven by a smooth control law can be guaranteed throughout an entire response of the system starting from the initial time instance. However, again, the main problem related to the implementation of this ISM concept consists in the requirement of the knowledge of the state vector, including the initial conditions. Thus, the ISM turns out to be useless when being applied directly and only output information is available.

B. Methodology

To realize the robust optimal output control for the multi-plant case three approaches must be modified and synthesized:

- the min-max optimal LQ control;
- the integral sliding-mode control;
- the hierarchical sliding mode observation.

In [21] and [22] were proposed two different forms for resolving the problem of matched uncertainty compensation for the case of a control based on the min-max LQ-index in the context of a multi-model system. The difference between the multi-plant and multi-model systems is the following: in the multi-model case it is considered that for the same plant different models could be realized. However, in the multi-plant systems we are considering a set of plants working simultaneously and the min-max optimal control law is applied to all plants simultaneously. On the other hand, the robust optimal control based on a min-max LQ-index requires the knowledge of a weighting vector that minimizes a functional. This vector can be found graphically for the cases of two and three plants, evidently for the other cases, a numerical method need to be used. Hence, for finding the weighting vector we will make use of the algorithm proposed in [23].

Recently it was designed an output integral sliding mode allowing to robustify the optimal LQ-control for the case when...
only the output of the system is available ([24]). Here we propose to use the advantage of the output integral sliding mode being applied to the robust optimal control in the case of a set of linear systems where only the output of the system is known and each plant is affected by matched uncertainties. Therefore, the use of the output integral sliding mode proposed in [24] seems to be an acceptable option to overcome the restriction imposed in the design of the robust optimal control based on a min-max LQ-index. In the present paper we try to modify the approach proposed in [24] for the case of multi-plant linear uncertain systems. We suggest to use an output integral sliding mode (OISM) to compensate the matched uncertainties right after the beginning of the process, namely, we try to maintain the properties of the ISM but using uniquely the output of each system instead of using all the state vector. We suggest also to design a hierarchical sliding mode observer (HSMO) (see, e.g. [25]) for the reconstruction of the state vector together with the subsequent application of the robust optimal control. The design of the observer consists in reconstructing at each step of the hierarchy a part of the vector obtained by multiplying an observability matrix by the state vector. During the realization of such observer, it is needed to use some filters. However, it is shown that the time of convergence and the observer error can be made arbitrarily small just by decreasing the sampling time and the filter constants. That is why, we take advantage of the OISM to overcome the restrictions imposed by the robust optimal control designing.

C. Basic Assumptions and Restrictions

Since for each system only a part of the state vector is available, in this work:

• we consider a finite set of plants whose trajectories are supposed to be estimated;
• each plant of the system is described by a system of linear time-invariant ODE (ordinary differential equations) with matched uncertainties that may be of a nonlinear nature;
• the performance of each plant is characterized by a LQ-index over a finite horizon;
• the optimal control action is assumed to be applied to all the plants simultaneously.

D. Main Contribution

It is shown that it is possible to apply the robust optimal control based on the min-max LQ-index for the case when only the output (not the complete state) of each system is measurable (available), and even in the presence of matched uncertainties. To apply the robust optimal control to such system, we use an output integral sliding mode that allows to:

1) compensate the matched uncertainties;
2) make the observation error arbitrarily small after (for) an arbitrary small time just by decreasing the sampling step;
3) use a numerical algorithm for the adjustment of the weights appearing in the control law of the min-max LQ problem for the multi-plant case;
4) give an estimation of the error during the closed loop control, including the error due to the presence of the actuators time constants as well as the observation error in terms of the sampling time.

E. Structure of the Paper

Section II deals with the model description and the formulation of the control law. The design of the OISM (output integral sliding mode) control that compensates the matched uncertainties is considered in the Section III. Section IV is devoted to the design of the hierarchical observer. Section V deals with constructing the min-max LQ optimal control. In Section VI it is estimated the error produced during the implementation of the closed loop control. We give an example in the Section VII which illustrates the effectiveness of the proposed approach.

II. PROBLEM STATEMENT

Consider a set of linear time invariant uncertain systems

\[
\begin{align*}
\dot{x}^0(t) &= A_0x^0(t) + B_0u(t) + B_0\gamma(t) + d^0(t), \\
y^0(t) &= C_0x^0(t), \\
x^0(0) &= x_0^0
\end{align*}
\]  

where \(\alpha = 1, N\) (\(N\) is a positive integer), \(x^0(t) \in \mathbb{R}^n\) is the state vector at time \(t \in [0, T]\), \(u(t) \in \mathbb{R}^m\) is the vector of control inputs and \(y^0(t) \in \mathbb{R}^p (1 \leq p < N)\) represents the output vector of each system. Every excitation vector \(d^0(t)\) is assumed to be known for all \(t \in [0, T]\). The current state \(x^0(t)\) and the initial state \(x^0_0\) are supposed to be non available. \(A_0, B_0, C_0\) are known matrices of appropriate dimensions with \(\text{rank } B_0 = m\) and \(\text{rank } C_0 = p\). Here all the plants are running in parallel.

Throughout the paper we will assume that:

A1. The vector \(\gamma(t)\) is upper bounded by a known scalar function \(q_\gamma(t)\), that is

\[
\|\gamma(t)\| \leq q_\gamma(t).
\]

A2. It is known a bound for every vector \(x_0^0\), that is

\[
\|x_0^0\| \leq \mu.
\]

A. The Control Design Challenge

Before designing an optimal control we have to make the system free from the effects of matched uncertainties. Therefore, the control design problem can be formulated as follows: design the control \(u\) in the form

\[
u = u_0 + u_1
\]

where the control \(u_1\) will compensate the uncertainty \(\gamma(t)\) just after the beginning of the process \(t = 0\), and \(u_0(\cdot)\) is the robust optimal control law minimizing the min-max LQ-index

\[
\min_{u_0 \in \mathbb{R}^m} \max_{\alpha \in \mathcal{A}} h^\alpha
\]

\[
h^\alpha := \frac{1}{2} (x^0(T), G^\alpha x^0(T))
\]

\[
+ \frac{1}{2} \int_{t=0}^{T} [(x^0(t), Q^\alpha x^0(t)) + (u_0(t), R u_0(t))] dt
\]

\[
Q^\alpha \geq 0, \quad G^\alpha \geq 0, \quad R > 0
\]

along the ‘nominal’ system trajectories

\[
x^0(t) = A_0x^0(t) + B_0u_0 + d^0.
\]
The exact solution of (5) requires the availability of all the vector states \( x^a(t) \) at any \( t \in [0,T] \) (see [6]), and the system must be free of any uncertainty. Therefore, to carry out this optimal control, we first should

1) ensure the compensation of the matched uncertainties \( \gamma(t) \); 2) design a state estimator that reconstruct every state vector \( x^a(t) \) practically from the beginning of the process.

### III. Output Integral Sliding Mode (OISM)

For each \( \alpha \in \{1,2,\ldots,N\} \), substitution of the control law (4) into (1) yields

\[
\dot{x}^a(t) = A_{\alpha} x^a(t) + B_{\alpha} (u_0 + u_1 + \gamma(t)) + d^a(t),
\]

(8)

Let us define the following extended system

\[
\dot{x}(t) = Ax(t) + B(u_0 + u_1 + \gamma) + d
\]

\[
y(t) = Cx(t)
\]

(9)

where

\[
x := \begin{bmatrix} x^1 \\ \vdots \\ x^N \end{bmatrix}, \quad A := \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & A_N \end{bmatrix}, \quad B := \begin{bmatrix} B_1 \\ \vdots \\ B_N \end{bmatrix}
\]

\[
C := \begin{bmatrix} C_1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & C_N \end{bmatrix}, \quad d := \begin{bmatrix} d^1 \\ \vdots \\ d^N \end{bmatrix}
\]

(10)

Now, to carried out the output integral sliding mode, we will assume that

\[ \text{A3. rank}((CB)+) = m \quad \text{A3. rank}((CB)+) = m \]

Thus, define the auxiliary affine sliding function \( s : \mathbb{R}^pN \rightarrow \mathbb{R}^m \) as follows:

\[
s(y(t)) := (CB)^+ y(t) - \sigma(t)
\]

(11)

where \((CB)^+ = [(CB)^T (CB)]^{-1} (CB)^T\) The term \( \sigma(t) \) includes an integral term that will be defined below. Thus, for the time derivative \( \dot{s} \) we have

\[
\dot{s} = (CB)^+ C[Ax + d] + u_0 + u_1 + \gamma - \dot{\sigma}.
\]

(12)

Define \( \sigma \) as

\[
\sigma(t) = (CB)^+ C[Ax + d] - u_0
\]

(13)

The vector \( \dot{x} \) represents the state of an observer which will be designed in the Section IV. Substituting \( \dot{s} \) into (12) gives

\[
\dot{s} = (CB)^+ CAx + u_1 + \gamma, \quad s(0) = 0.
\]

The control \( u_1 \) is suggested to be designed in the following form

\[
u_1 = -\beta(t) \frac{s(t)}{|s(t)|}
\]

(14)

with \( \beta(t) \) being a scalar gain satisfying the condition

\[
\beta(t) - q_h(t) - \|(CB)^+ CA\| |x - \dot{x}| \geq \lambda > 0
\]

where \( \lambda \) is a constant.

### Remark 1:

Notice that, by A2, an upper bound of \( |x - \dot{x}| \) always can be estimated. Indeed, since \( |x - \dot{x}| \leq |\dot{x}| + |\dot{x}| \), using the Gronwall-Bellman inequality an upper-bound \( \Omega(t, x(0)) \) for \(|\dot{x}| \) can be calculated. Therefore, by the knowledge of \(|\dot{x}|\), \(|x - \dot{x}| \leq \Omega(t, x(0)) + |\dot{x}| \). Nevertheless, this could be a big over estimation, that is why, a better way to estimate \(|x - \dot{x}| \) is as follows. The vector \( \dot{x} \) will be given by \( \dot{x} = \dot{x} + w \) where \( \dot{x} \) represents a Luenberger observer and \( x \) is known and its norm tends to a small constant (see Sections IV-C and VI). Then \( |x - \dot{x}| < \phi(t) = \zeta \exp(-\delta t) \|\sqrt{\mu + |\dot{x}(0)|} + \rho \) for positive known constants \( \zeta \) and \( \kappa \), and \( \rho \) is any arbitrarily small positive constant. Therefore, \(|x - \dot{x}| < \phi(t) + |\dot{x}| \). Thus, even in the case when \( x \) is unstable, \(|x - \dot{x}| \) has an upper-bound which tends to \( \rho + |\dot{x}| \).

In order to ensure the sliding motion, here is selected the Lyapunov function as \( V = (1/2) \|s\|^2 \). Since \( \dot{V} = s^T \dot{s} \) and in view of (14) and (2), one gets

\[
\dot{V} = s^T \left( (CB)^+ CAx - \dot{x} - \beta \dot{s} \right)
\]

\[
\leq - \|s\| \left( \beta - \|(CB)^+ CA\| |x - \dot{x}| - q_h \right)
\]

\[
\leq - \|s\| \lambda \leq 0
\]

(15)

Therefrom, due to \( s(0) = 0 \), one obtains \( (1/2) \|s(t)\|^2 = V(s(t)) \leq V(s(0)) = (1/2) \|s(0)\|^2 = 0 \). Thus, the identities

\[
s(t) = \dot{s}(t) = 0
\]

hold for all \( t \geq 0 \), i.e., there is no reaching phase to the sliding mode.

From (12) and in view of the equality (15) the equivalent control, which is, in fact, unrealizable, is

\[
u_{1\text{eq}} = -(CB)^+ CAx - \gamma,
\]

(16)

Substitution of \( u_{1\text{eq}} \) into (9) yields

\[
\dot{x}(t) = \dot{\dot{x}}(t) + B(CB)^+ CAx(t) + Bu_0 + d(t)
\]

\[
y(t) = Cx(t)
\]

(17)

where

\[
\dot{\dot{x}} := [I - (BC)^+ C] A
\]

(18)

Thus, our first objective has been achieved, i.e., we have compensated the uncertainty \( \gamma \). The next section is devoted to the design of the sliding mode observer generating \( \dot{x} \).

### IV. Design of the Observer

Now, having the system without uncertainties, we can reconstruct the state vector. To design the observer, the pair \((\dot{A}, C)\) must be observable. The following Lemma establishes the

\[1\text{In the theory of the sliding mode the equivalent control is used for obtaining the effects over the system of the discontinuous control once the sliding motion is attained. That is, it is known that on the sliding motion the discontinuous control (14) produces the same effects over the system than the ones produced by the equivalent control (see, [26]). The equivalent control can be approximate [26] by the output of a linear first order filter with the input as in (14).]
conditions in terms of \((A, B, C)\) that specify when \((\hat{A}, C)\) is observable.

**Lemma 2:** The pair \((\hat{A}, C)\) is observable if and only if the triple \((A, B, C)\) has no invariant zeros, i.e.,

\[
\{ s \in \mathbb{C} : \text{rank} (P(s)) < Nn + m \} = \emptyset
\]  

(19)

where \(P(s)\) is the Rosenbrock’s matrix system defined as

\[
P(s) = \begin{bmatrix} sI - A & -B \\ C & 0 \end{bmatrix}.
\]  

(20)

A proof of Lemma (2) has been given in [24].

Thus, we will assume that

A4. The triple \((A, B, C)\) has no zeros.

It is well-known that the conditions \(\text{rank}(CB) = m\) (assumption A3) and \(pN = m\) imply that the triple \((A, B, C)\) has invariant zeros; therefore, A3 and A4 imply \(pN > m\).

The observer will be based on the recovering of the vectors \(Cx(t), \hat{CA}x(t)\), and so on, until we get \(\hat{C}A^{l-1}x(t)\). Afterwards, the aim is to recover the vector \(\hat{H}x(t)\) where

\[
\hat{H}^T = \begin{bmatrix} C^T & (\hat{C}A)^T & \cdots & (\hat{C}A^{l-1})^T \end{bmatrix}.
\]  

(21)

Here \(I\) is defined as the observability index, that is, the least positive integer such that \(\text{rank}(H) = n\) (see, e.g., [27]). Thus, after pre-multiplying \(Hx(t)\) by \(H^+\), the state vector \(x(t)\) can be reconstructed by \(x(t) = H^+ \underbrace{Hx(t)}_{\text{gotten one-line}}\), where \(H^+ = (H^T H)^{-1} H^T\) is the pseudo-inverse of \(H\).

Before designing the observer we need to ensure a bound required by the sliding mode algorithm. Design the following dynamic system:

\[
\dot{x}(t) = \hat{A}x(t) + Bu_0(t) + B(CB)^+CAx(t) + LB(y(t) - CX(t)) + d(t)
\]  

(22)

where \(L\) must be designed such that the eigenvalues of \(\hat{A} := (A - LC)\) have negative real part. Let \(r(t) = x(t) - \hat{x}(t)\), from (17) and (22), the dynamic equations governing \(r(t)\) are

\[
\dot{r}(t) = [\hat{A} - LC]r(t) = \hat{A}r(t).
\]  

(23)

Since the eigenvalues of \(\hat{A}\) have negative real part, (23) is exponentially stable, i.e., there exist some constants \(\gamma, \eta > 0\) such that

\[
||r(t)|| \leq \gamma \exp(-\eta t) \left( \sqrt{N} \mu + ||\hat{x}(0)|| \right).
\]  

(24)

The Luenberger observer used here ensures the stability of the new vector state \(r = x - \hat{x}\). This allows to calculate an upper-bound for \(||r||\) which is needed for achieving the sliding motions in the observation process.

**A. Auxiliary Dynamic Systems and Output Injections**

The essential purpose in the design of the observer is to recover the vectors

\[
x^{A^k}(t), \quad k = 1, \ldots, l - 1.
\]

First, to recover \(CAx(t)\), let us introduce an auxiliary state vector \(x^{A^1}_m(t)\) governed by the following dynamics equations:

\[
x^{A^1}_m(t) = \dot{A}x^{A^1}_m(t) + Bu_0(t) + (CB)^+CAx(t) + L(CL)^{-1}v^1(t) + d(t)
\]  

(25)

where \(x^{A^1}_m(0)\) satisfies \(Cx^{A^1}_m(0) = y(0)\) and \(L\) is any matrix such that \(\text{det}(CL) \neq 0\). The vector \(\hat{x}(t)\) represents the observer we will design below. For the variable \(s^1 \in \mathbb{R}^N\) defined by

\[
s^1(y(t), \hat{x}_m(t)) = Cx(t) - Cx^{A^1}_m(t)
\]  

(26)

we have

\[
s^1(y(t), \hat{x}_m(t)) = CA(x(t) - \hat{x}(t)) - v^1(t)
\]  

(27)

with \(v^1(t)\) defined as \(v^1 = M_1(s^1/||s^1||)\). Here the scalar gain \(M_1\) must satisfy the condition \(M_1 > ||CAx||/||x||\) to obtain the sliding mode regime. A bound of \(||x||\) can be estimated using (24). Hence, with such a scalar gain \(M_1\), we get the identities \(s^1(t) = 0, s^1(t) = 0, \forall t \geq 0\). Thus, from (26) we obtain that

\[
Cx(t) = C\hat{x}(t), \quad \forall t \geq 0
\]  

(28)

and from (27), the equivalent output injection is

\[
v^{A^1}_m(t) = CAx(t) - C\hat{x}(t), \quad \forall t > 0.
\]  

(29)

Thus, in principle, \(CAx(t)\) can be recovered from (29).

Now, the next step is to recover the vector \(CA^2x(t)\). Let us design the second auxiliary state vector \(x^{A^2}_m(t)\) generated by

\[
x^{A^2}_m(t) = \dot{A}^2x(t) + ABu_0(t) + L(CL)^{-1}v^2(t) + \tilde{A}B(CB)^+CAx(t) + d(t)
\]  

where \(x^{A^2}_m(0)\) satisfies \(CA^{A^1}(0) + v^{A^1}_m(0) - Cx^{A^2}_m(0) = 0\). Again, for \(s^2 \in \mathbb{R}^N\) defined by \(s^2(v^{A^1}_m(t) - CA^{A^1}(t) + v^{A^1}_m(t) - Cx^{A^2}_m(t)\), and in view of (29), we have that \(s^2\) takes the form

\[
s^2(v^{A^1}_m(t) - CA^{A^1}(t) + v^{A^1}_m(t) - Cx^{A^2}_m(t)\)
\]  

(30)

Hence, the time derivative of \(s^2\) is

\[
s^2(v^{A^1}_m(t) - CA^{A^1}(t) + v^{A^1}_m(t) - Cx^{A^2}_m(t) - v^2(t).
\]  

(31)

Now, take the output injection \(v^2(t)\) as

\[
v^2 = M_2 \frac{s^2}{||s^2||}, \quad M_2 > ||CA^2x||/||x||
\]  

(32)

which implies that

\[
s^2(t) = s^2(t) = 0,
\]  

(33)

In view of (33) and (31), \(v^{A^2}_m(t)\) is

\[
v^{A^2}_m(t) = CA^2x(t) - C\tilde{x}(t), \quad t > 0
\]  

(34)

and the vector \(CA^2x(t)\) can be recovered from (34).
Thus, iterating the same procedure, all the vectors $\mathbf{C} \tilde{A}^k \mathbf{x}$ can be recovered. In a summarizing form, the procedure above goes as follows:

a) the dynamics of the auxiliary state $\mathbf{x}_k(t)$ at the $k$-th level is governed by

$$\dot{\mathbf{x}}_k(t) = \tilde{A}^k \dot{x}(t) + \tilde{A}^{k-1} \mathbf{B} u(t) + \tilde{L}(\mathbf{C} \mathbf{L})^{-1} v^k + \tilde{A}^{k-1} \mathbf{B} (\mathbf{C} \tilde{A}^k \mathbf{x}(t) + d(t))$$

with $\tilde{L}$ being any constant matrix so that $\det(\mathbf{C} \mathbf{L}) \neq 0$, and the output injection $v^k$ at the $k$-th level is

$$v^k = M_k \frac{s^k}{\|s^k\|}, \quad M_k > \|\mathbf{C} \tilde{A}^k\| \|\mathbf{x}\|$$

where $M_k$ is a scalar gain. A bound of $\|\mathbf{p}\|$ can be found using (24).

b) Define $s^k$ at the $k$-level of the hierarchy as:

$$s^k(t) = \begin{cases} \frac{\mathbf{y} - \mathbf{C} \mathbf{x}^k_1}{\|\mathbf{C} \mathbf{x}^k_1\|}, & k = 1 \\ \frac{\mathbf{C} \mathbf{x}^k_{k-1} - \mathbf{C} \mathbf{x}^k_1}{\|\mathbf{C} \mathbf{x}^k_{k-1} - \mathbf{C} \mathbf{x}^k_1\|}, & k > 1 \end{cases}$$

where $v_{\text{eq}}^{k-1}$ is the equivalent output injection whose general expression will be obtained in the following Lemma, but $\mathbf{x}^k_0(0)$ should be chosen such that $s^k(0)$ satisfies

$$s^k(0) = 0, \quad k = 1, \ldots, l - 1.$$  

**Lemma 3:** If the auxiliary state vector $\mathbf{x}_k$ and the variable $s^k$ are designed as in (35) and (37), respectively, then

$$v_{\text{eq}}^k(t) = \mathbf{C} \tilde{A}^k [\mathbf{x}(t) - \dot{x}(t)]$$

for all $t \geq 0$ at each $k = \frac{l - 1}{l - 1}$.

**Proof:** It was shown that the following identity holds

$$v_{\text{eq}}^k(t) = \mathbf{C} \tilde{A}^k [\mathbf{x}(t) - \dot{x}(t)] \quad \forall t > 0.$$

Now, suppose that the equivalent output injection $v_{\text{eq}}^{k-1}$ is as (39), then the substitution of $v_{\text{eq}}^{k-1}$ into (37) gives

$$s^k(v_{\text{eq}}^{k-1}(t), \mathbf{x}_k(t)) = \mathbf{C} \tilde{A}^{k-1} \mathbf{x}(t) - \mathbf{C} \mathbf{x}^k_0(t).$$

The derivative of (40) yields

$$s^k(t) = \mathbf{C} \tilde{A}^k [\mathbf{x}(t) - \dot{x}(t)] - v^k(t)$$

Thus, selecting $v^k(t)$ as in (36) one gets

$$s^k(t) \equiv 0, \quad s^k(t) \equiv 0 \quad \text{for all } t \geq 0.$$

Therefore, (42) and (41) implies (39).

**B. Observer in Its Algebraic Form**

Now, we can design an observer with the properties required in the problem statement. From (28) and (39), we obtain the following algebraic equations arrangement:

$$\mathbf{C} \mathbf{x}(t) = \mathbf{C} \dot{x}(t) + [\mathbf{y}(t) - \mathbf{C} \mathbf{x}(t)]$$

$$\mathbf{C} \tilde{A} \mathbf{x}(t) = \mathbf{C} \tilde{A} \dot{x}(t) + v_{\text{eq}}^1(t)$$

$$\vdots$$

$$\mathbf{C} \tilde{A}^{l-1} \mathbf{x}(t) = \mathbf{C} \tilde{A}^{l-1} \dot{x}(t) + v_{\text{eq}}^{l-1}(t).$$

Thus, (43) yields the matrix equation

$$\mathbf{H} \mathbf{x}(t) = \mathbf{H} \dot{x}(t) + v_{\text{eq}}(t), \quad \forall t > 0$$

where $\mathbf{H}$ was defined in (21) and

$$v_{\text{eq}}^T = \begin{bmatrix} (\mathbf{y}(t) - \mathbf{C} \mathbf{x}(t))^T & (v_{\text{eq}}^1)^T & \cdots & (v_{\text{eq}}^{l-1})^T \end{bmatrix}.$$  

Since the pair $(\tilde{A}, \mathbf{C})$ is observable, the matrix $\mathbf{H}$ has rank $N_n$. Thus, after pre-multiplying $\mathbf{H}^+$ by (44), we obtain

$$\mathbf{x}(t) \equiv \dot{\mathbf{x}}(t) + \mathbf{H}^+ v_{\text{eq}}(t), \quad \forall t > 0.$$  

Thus, the observer can be designed as

$$\dot{\mathbf{x}}(t) := \dot{\mathbf{x}}(t) + \mathbf{H}^+ v_{\text{eq}}(t).$$

Now, we can formulate the following theorem.

**Theorem 4:** Under the assumptions A1–A4

$$\dot{\mathbf{x}}(t) \equiv \mathbf{x}(t) \quad \forall t > 0.$$  

**Proof:** It follows directly from (46) and (47).

**Remark 5:** It must be noticed that the solution of (44) has the form (46) which, in fact, is the unique solution of (46). Indeed, let us define $\hat{x} := \mathbf{x} + \mathbf{H}^+ v_{\text{eq}}$. Then, by substituting $\mathbf{x}$ and $\mathbf{H} \mathbf{x}$ into the norm $\|\mathbf{H} \mathbf{x} - \mathbf{H} \hat{x}\|$, we have

$$\|\mathbf{H} \mathbf{x} - \mathbf{H} \hat{x}\| = \|\mathbf{H} (\mathbf{x} + \mathbf{H}^+ v_{\text{eq}}) - (\mathbf{H} \hat{x} + v_{\text{eq}})\| = \|\mathbf{H} \mathbf{H}^+ - I\| \|\mathbf{x} - \hat{x}\|.$$

Since $\mathbf{H} (\mathbf{x} - \hat{x}) = v_{\text{eq}}$, the substitution of $v_{\text{eq}}$ into the previous equation, and taking into account that $\mathbf{H}^+ \mathbf{H} = \mathbf{H}$, yields

$$\|\mathbf{H} \mathbf{x} - \mathbf{H} \hat{x}\| = \|\mathbf{H} \mathbf{H}^+ - I\| \|\mathbf{x} - \hat{x}\| = 0.$$

Hence, we have that $\hat{x}$ given by (47) is a solution of (44). Furthermore, since $\mathbf{H}$ has full column rank, the solution is unique.

**C. Observer Realization**

The realization of the observer described in (47) requires the availability of the equivalent output injection $v_{\text{eq}}^k$. However, the non-idealities in the implementation of $v_{\text{eq}}^k$ cause the, so-called, chattering movements. Nevertheless, $v_{\text{eq}}^k$ can be indirectly measured, namely, the first order-low pass-filter

$$\tau \tau_{\text{eq}}^k(t) + v_{\text{eq}}^k(t) = v^k(t); \quad v_{\text{eq}}^k(0) = 0$$

gives an approach of $v_{\text{eq}}^k$ (see [26]). That is,

$$\lim_{\tau \tau_{\text{eq}}^k(t) \rightarrow \tau_{\text{eq}}^k(t)} v_{\text{eq}}^k(t), \quad t > 0,$$  

where $\delta$ is the sampling time used in the computations during the realization of the observer. So, we can select

$$\tau = \delta \gamma (0 < \gamma < 1).$$

Hence, to realize the HSM observer we should:

1) use a sampling interval $\delta$ very small;

2) substitute $v_{\text{eq}}^k(t)$ into (46) and (45) by $v_{\text{eq}}^k(t)$;

3) chose $\mathbf{x}_0(0)$ in such a way that

$$\mathbf{y}(0) - \mathbf{C} \mathbf{x}_0(0) = 0, \quad \text{for } k = 1$$

$$\mathbf{C} \tilde{A}^{k-1} \hat{x}(0) - \mathbf{C} \mathbf{x}_0(0) = 0, \quad \text{for } k > 1$$

so we ensure the identity $s^k(0) = 0, k = 1, \ldots, l - 1$.

Thus, with the extended vector formed by the filter outputs, i.e.

$$v_{\text{eq}}^T := \begin{bmatrix} (\mathbf{y}(t) - \mathbf{C} \mathbf{x}(t))^T & (v_{\text{eq}}^1)^T & \cdots & (v_{\text{eq}}^{l-1})^T \end{bmatrix}$$

the observer $\dot{\mathbf{x}}(t)$ must be redefined as

$$\dot{\mathbf{x}}(t) := \dot{\mathbf{x}}(t) + \mathbf{H}^+ v_{\text{eq}}(t).$$
Remark 6: Since the use of filters causes the error \( \varepsilon = v_{\text{eq}} - v_{\text{eq}} \), (44) becomes \( Hx(t) = H\varepsilon(t) + v_{\text{eq}}(t) + \varepsilon \) (an estimate of \( \varepsilon \) is given in Section VI). Hence, \( \varepsilon \) defined in (50) satisfies the identity \( \varepsilon = \arg \min_{x \in \mathbb{R}^n} [Hx - H\hat{x} - v_{\text{eq}}] \).

Remark 7: It must be noticed that a step by step observer could be used instead of the hierarchical observer.

V. MIN-MAX OPTIMAL CONTROL DESIGN

In this section we return back to the problem of the optimal control \( u_0 \) which resolves the problem (5). Substitution of (48) into (17) yields the sliding motion equations for the state \( x \) that takes the form

\[
\dot{x}(t) = Ax(t) + Bu_0(x) + d.
\]

Now the solution for the min-max optimal problem (5) can be given. Thus, according to [5], [6], the control solving (5) for (7) is of the form

\[
u_0^*(x, t) = -R^{-1}B^T(P_\lambda x + p_\lambda)
\]

where the matrix \( P_\lambda \in \mathbb{R}^{n \times n} \) is the solution of the parameterized differential matrix Riccati equation

\[
\dot{P}_\lambda + P_\lambda A + A^T P_\lambda - P_\lambda BR^{-1} B^T P_\lambda + \Delta Q = 0 \quad \text{(52)}
\]

and the shifting vector \( p_\lambda \) satisfies

\[
\dot{p}_\lambda + A^T p_\lambda - P_\lambda BR^{-1} B^T p_\lambda + P_\lambda d = 0; \quad p_\lambda(T) = 0
\]

where the weighting vector \( \lambda \) belongs to the simplex \( S^N \)

\[
S^N := \left\{ \lambda \in \mathbb{R}^n : \lambda_\alpha \geq 0, \sum_{\alpha=1}^N \lambda_\alpha = 1 \right\}
\]

and the matrices \( Q, G, \) and \( A \) denote

\[
Q := \begin{bmatrix} Q_1 & 0 & \cdots & 0 \\ 0 & \ddots & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & Q_N \end{bmatrix}, \quad G := \begin{bmatrix} G_1 & 0 & \cdots & 0 \\ 0 & \ddots & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & G_N \end{bmatrix} \quad \text{(53)}
\]

\[
A := \begin{bmatrix} A_1 \lambda_1 I_{m \times n} & 0 \cdots 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots 0 & A_N \lambda_N I_{m \times n} \end{bmatrix}.
\]

The matrix \( A = A(\lambda^*) \) is defined by (54) with the weighting vector \( \lambda = \lambda^* \) solving the following finite dimensional optimization problem:

\[
\lambda^* = \arg \min_{\lambda \in S^N} J(\lambda) \quad \text{(55)}
\]

The weighting vector \( \lambda^* \) can be generated by means of the sequence \( \{\lambda^k\} \) defined by

\[
\lambda^{k+1} = \pi \left\{ \lambda^k + \frac{\gamma^k}{J(\lambda^k) + \epsilon} F(\lambda^k) \right\}, \quad \lambda^0 \in S^N, k = 0, 1, 2, \ldots, \quad F(\lambda^k) = [h_{1\lambda}^k \cdots h_{N\lambda}^k]^T, \quad J(\lambda^k) := \max_{\alpha \in S^N} h_{\lambda}^\alpha \quad \text{(56)}
\]

where \( \epsilon \) is an arbitrary strictly positive (small enough) constant and \( \pi \{ \} \) is the projector to the simplex \( S^N \), i.e. for each \( z \in \mathbb{R}^N \)

\[
||\pi(z) - z|| < ||\lambda - z||, \quad \forall \lambda \in S^N, \lambda \neq \pi(z).
\]

The following theorem gives the conditions under which the sequence \( \{\lambda^k\} \) generated by (56) converges to the optimal weighting vector \( \lambda^* \).

Theorem 8 (23)): Let \( \lambda^* \) be the minimum point of \( J(\lambda) \). If the following requirements are fulfilled

1) the sequence \( \{\lambda^k\} \) is generated by (56);
2) for any \( \lambda \neq \lambda^* \) in \( S^N \) the following inequality holds:

\[
(\lambda - \lambda^*, F(\lambda^*) - F(\lambda^*)) < 0
\]

and the identity in (57) is possible if and only if \( \lambda = \lambda^* \);
3) there exists a constant \( L \) such that, for all \( \alpha \in I, \) and for any \( \mu, \lambda \in S^N \)

\[
|l_\alpha^\mu - l_\alpha^\lambda| \leq J(\lambda)L|\mu - \lambda|
\]

4) the gain sequence \( \{g^k\} \) satisfies

\[
g^k > 0, \quad \sum_{k=0}^\infty g^k = \infty, \quad \sum_{k=0}^\infty (\gamma^k)^2 < \infty
\]

then

\[
\lim_{k \to \infty} \lambda^k = \lambda^*.
\]

Since the observation error can be made arbitrarily small after any arbitrarily small time, the estimated state \( \hat{x} \) can be used instead of \( x \). Therefore, the control \( u_0 \), should be designed as

\[
u_0(\hat{x}, t) = u_0^*(\hat{x}, t) = -R^{-1}B^T[P_\lambda \hat{x} + p_\lambda]
\]

with \( \hat{x} \) being designed according to (47).

A. Control Algorithm

The proposed control algorithm can be summarized as follows:

1) Design the control \( u_0 \) according to (14).
2) Design the matrix \( L \) such that the eigenvalues of \( \hat{A} := (\hat{A} - LC) \) have negative real part.
3) Design the auxiliary systems \( \hat{x}^\lambda \) as in (35) with the sliding surfaces \( s^k \) as in (37) and compute the constants \( M_k, k = 1, \ldots, l - 1 \).
4) Design the state estimator \( \hat{x} \) according to (50).
5) Calculate the matrix \( P_{\lambda^*} \) and the vector \( p_{\lambda^*} \) according to (52) and (53) respectively.
6) Use the sequence (56) for finding \( \lambda^* \), using \( \hat{x} \) instead of \( x \).
7) Design \( u_0 \) according to (59).

VI. ERROR ESTIMATION DURING IMPLEMENTATION

of the Closed Loop Control

We have seen that the filters cause some errors in the state estimation. Evidently those errors affect directly the controller since we have used the estimated states instead of the original ones. Hence, here we will calculate the estimation of the error appearing during the realization of the closed loop control, that
is, the errors due to the actuators plus the error due to the observation process. The control error due to the devices uses in the implementation of the control, including the output integral sliding mode control, is of the order $O(\mu)$, where $\mu$ is a constant characterizing the control execution depending generally from actuators time constants. Now, let us estimate the order of the error due to the observation process. As we saw, the observer design is based on the recursively use of the filters of the form (49). First, let us recall the following lemma regarding the error induced for such sort of filters.

**Lemma 9 ([26]):** If in the differential equation

$$\tau \ddot{z} + z = h(t) + H(t)\dot{s}$$

(60)

where $\tau$ is a constant and $z$, $h$, and $s$ are $m$-dimensional vector functions such that:

1) the functions $h(t)$ and $H(t)$, and their first order derivatives are bounded in magnitude by a certain number $M$ and

2) $\|s(t)\| \leq \xi (\xi$ is a constant positive value) then for any pair of positive numbers $\Delta t$ and $v$ there exists a number $d(v, \Delta t, z(0))$ such that

$$\|z(t) - h(t)\| \leq v$$

with $0 < \tau \leq d$, $\xi/\tau \leq d$ and $t \geq \Delta t$.

Indeed, $\|z(t) - h(t)\|$ satisfies the following inequality:

$$\|z(t) - h(t)\| \leq \|z(0) - h(0)\| \exp(-t/\tau) + M(\tau + \xi) + 3M\left(\frac{\xi}{\tau}\right).$$

In our case, the expression (60) can be related with the expression

$$\tau_1 v_{aw}^1 + v_{aw}^1 = \frac{v_{aw}^1}{v_{eq}} - s^1$$

obtained from the (49) and (27). Thus, in our case $h(t)$ refers to the equivalent output injection. Furthermore, the error due to sliding mode control affects directly the performance of the first sliding mode in the observation process, and it is known that the sampling step $\delta$ induces an error of the order $O(\delta)$ in the variable $s^1$ during the sliding motion. Hence, it is reasonable to accept that the error in the sliding variable $s^1$ is of the order $O(\mu) + O(\delta)$, that is, defining $\Delta := \mu + \delta$, we have that the constant $\xi$ in the lemma (9) is $\xi = O(\Delta) = O(\mu) + O(\delta)$. Therefore, choosing $\tau = O(\Delta^{1/2})$, we have that the error in the first step of the observation scheme is of order $O(\Delta^{1/2})$, that is $v_{aw}^1 - v_{eq}^1 = O(\Delta^{1/2})$. As it was mentioned in the Section IV-C, we must substitute $v_{aw}^1$ by $v_{aw}^2$ into the variable $s^2$ in (8). Thus, we can consider that during the sliding motion, $s^2$ will be bounded for a constant of order $O(\Delta^{1/2})$, and consequently, by using a constant of the filter $\tau_2 = O(\Delta^{1/4})$, the error induced for the second filter will be $v_{aw}^2 - v_{eq}^2 = O(\Delta^{1/4})$. Following a likewise analysis, we obtain an error of the order $O(\Delta^{1/2})$ in the $k$-th step for the reconstruction of the observer. Thus, it turns out to be that the observation error is of the order $O(\Delta^{1/2})$, recalling that $\delta$ is the least integer such that the matrix $H$ in (21) has rank $n_N$. Thus, we can say that, during the realization of the control process, the total error $e_c$ of the closed loop control is

$$e_c = O(\mu) + O(\Delta{1/2}).$$

VII. EXAMPLE

Consider the case $N = 3$ where the parameters are given by

$$A_1 = \begin{bmatrix} -2 & 0.5 & 1 \\ 0.5 & 1.2 & -2 \\ 1 & 2 & -1.5 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -0.3 & 1.5 & -0.15 \\ -1 & 0.12 & 2 \\ 1 & -2 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 0.4 & -1 & 0.3 \\ 0.5 & -0.4 & 0.3 \\ 0.5 & 0.6 & -1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1.5 \\ 2 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 0.5 \\ 0.2 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$  

In the simulations, we used $\gamma(t) = \sin(t)$ and a sampling time $\delta = 10^{-4}$. The Table I shows the components of the
vector $\lambda^k$, $k = 1, \ldots, 35$, calculated using the sequence (56); it also shows the performance indexes of each plant $h^{\rho^k}$, and the index $J(\lambda^k)$. The trajectories for the three plants are shown in Figs. 1–3. They represent a comparison between the trajectories of the original state vector and the trajectories of the estimated states. The estimation error $(e^3 = x^3 - \tilde{x}^3, \alpha = 1,2,3)$ for two different sampling times is graphed in Figs. 4 and 5. Since the first two components of the state vector are available, then there is presented only the third component of the error vector. Fig. 6 shows a comparison between the control law $u_0$ when the state vector is completely available and when only the output information is available. This was done for three different sampling times and we can see how by reducing the sampling time decreases the error between the control designed for the nominal system (the state vector is known and there is no uncertainties) and the control using OISM and HSMO. Clearly, this is a consequence of the fact that by reducing the sampling time we reduce the state estimation error.

It is known that (52), (53) must be pre-computed backward using a numerical method. However, the difference here with the classical optimal control is that these equations are parameterized by a weighting vector $\lambda$. The calculation of the optimal weighting vector $\lambda^*$ is not standard. That is why in the example we present a table with the values obtained using the method described in Section V for the calculation of $\lambda^*$ (see also [23]).
Fig. 6. For three different sampling times {$\delta$}, a comparison between $u_0(x)$ and $u_{\delta}(x)$.

VIII. CONCLUSION

Here we considered the problem of the realization of a robust output optimal control based on the min-max LQ-index. For the case when only the output (not the complete state) of each system is measurable (available), it was shown that by making use of the output integral sliding mode (OISM) and the hierarchical sliding mode observer (HSMO) it is possible to apply the robust optimal control even in the presence of matched uncertainties. This was done under the supposition that there is no invariant zeros in the system. Both techniques (OISM and HSMO) allow: first, to compensate the matched uncertainties right after the initial time (independently of the observation process), and second, to reconstruct the state vector. Using a low-pass filter for the HSMO realization, we have shown that the state estimation error can be made arbitrarily small for any arbitrarily small time just by decreasing the sampling time and the filter constant. This means that the state estimation error depends on the sources available for a designer to make the sampling time and the constant of the filter small enough. Consequently, under the suitable assumptions both OISM and HSMO might be promising techniques for making feasible the use of an optimal control as we have presented in this manuscript with the realization of an optimal control based on the min-max LQ-index. The simulations exemplify how the trajectories of the control realization are nearer the trajectories of the control for the nominal system (all states are known and there are no uncertainties) as the sampling time is smaller.

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Leonid M. Fridman (M’98) received the M.S. degree in mathematics from Kuibyshev State University, Samara, Russia, in 1976, the Ph.D. degree in applied mathematics from the Institute of Control Science, Moscow, Russia, in 1988, and the Dr.Sci. degree in control science from Moscow State University of Mathematics and Electronics, Moscow, Russia, in 1998.

From 1976 to 1999, he was with the Department of Mathematics, Samara State Architecture and Civil Engineering Academy. From 2000 to 2002, he was with the Department of Postgraduate Study and Investigations at the Chihuahua Institute of Technology, Chihuahua, Mexico. In 2002, he joined the Department of Control, Division of Electrical Engineering of Engineering Faculty at National Autonomous University of Mexico (UNAM), México. He is an Editor of three books and five special issues on sliding mode control. He has published over 200 technical papers. His research interests include variable structure systems and singular perturbations.

Dr. Fridman is Associate Editor of the Conference Editorial Board of IEEE Control Systems Society, Member of TC on Variable Structure Systems and Sliding mode control of IEEE Control Systems Society.

Alexander S. Poznyak (M’97) received the M.S. degree from Moscow Physical Technical Institute (MPhTI), Moscow, Russia, in 1970 and the Ph.D. and Dr.Sci. degrees from the Institute of Control Sciences, Russian Academy of Sciences, Moscow, in 1978 and 1989, respectively.

From 1973 to 1993, he served this institute as Researcher and Leading Researcher, and in 1993 he accepted a post of Full Professor (3-E) at CINVESTAV, IPN, Mexico, where he is the head of the Automatic Control Department. He is the Director of 30 Ph.D thesis’s (23 in Mexico). He has published more than 140 papers in different international journals and nine books including *Adaptive Choice of Variants* (Moscow, Russia: Nauka, 1986), *Learning Automata: Theory and Applications* (Elsevier-Pergamon, 1994), *Learning Automata and Stochastic Programming* (Berlin, Germany: Springer-Verlag, 1997), *Self-learning Control of Finite Markov Chains* (Marcel Dekker, 2000) and *Differential Neural Networks: Identification, State Estimation and Trajectory Tracking* (Singapore: World Scientific, 2001).

Dr. Poznyak is a Member of Mexican Academy of Sciences and System of National Investigators (SNI-3). He is Associate Editor of the Iberoamerican International Journal on Computations and Systems. He was also an Associate Editor of CDC, ACC and a Member of Editorial Board of IEEE CSS. He was a member of the Evaluation Committee of SNI (Ministry of Science and Technology) responsible for Engineering Science and Technology Foundation in Mexico.