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Singularly Perturbed Analysis of Chattering in Relay Control Systems

Leonid M. Fridman

***Abstract*—For sliding-mode control systems with *fast* actuators, sufficient conditions for the exponential decreasing of the amplitude of chattering and unlimited growth of frequency are found. The connection between the stability of actuators and the stability of the plant on the one hand and the stability of the sliding-mode system as the whole on the other hand is investigated. The algorithm for correction of sliding-mode equations is suggested for taking into account the presence of fast actuators.**

***Index Terms*—Singularly perturbed systems, sliding-mode control, variable structure systems.**

I. Introduction

The chattering phenomenon is one of the major problems in modern sliding-mode control (see [2], [13], and [12]). The presence of fast actuators is one basic reasons for chattering occurring in sliding-mode control systems. In [3], was shown that the behavior of sliding-mode systems with actuators is described by control systems with higher order sliding-modes and the order of sliding is the sum of a relative degrees of the plant and the actuator (the definitions and properties of the higher order sliding modes; see, for example, in [7] and [10]).

Relay control systems have the following specific features.

- In systems with the order of sliding modes more than 1, the fast switches occur and there are the limit cycles with internal fast switches in such systems [8].
- The second-order sliding modes could be asymptotically stable [1], [7].
- All sliding modes of the order three and more are unstable [1], [7].

In [2], for chattering elimination in systems with actuators with relative degree 1, the second-order suboptimal control algorithm, ensuring the finite-time convergence to the second-order sliding domain was used. In [12], for systems with uncertain actuators, the second-order sliding-mode control algorithms on dynamics sliding manifolds was implemented.

This note is devoted to chattering analysis in sliding-mode control systems with **fast** actuators. The behavior of such systems with is described by **singularly perturbed relay** systems with higher order sliding modes (SPRSHOSMs).

The chattering phenomenon for sliding-mode control systems with **fast** actuators, whose behavior is described as SPRSHOSM with the order of sliding three and more, was analyzed in [6] from the view point of averaging.

This note is devoted to analysis of the chattering phenomenon in sliding-mode control systems with **fast** actuators given by SPRSHOSM with order sliding two (SPRS2OSM). The general model of such systems is described by SPRS2OSM of the form [3]

$$\mu\, dz/dt = f(t,\, z,\, s,\, x,\, u(s)) \quad ds/dt = g_1(t,\, z,\, s,\, x)$$
$$dx/dt = g_2(t,\, z,\, s,\, x) \tag{1}$$

where $s \in R$, $x \in R^n$ are variables describing the behavior of the plant, $z \in R^m$ is vector describing the behavior of the actuator, $u(s) = \mathrm{sign}(s)$ is a relay control, f, g_1, g_2 are sufficiently smooth functions of their arguments, μ is the actuator time constant. The specific feature of (1) is the following: the equations for plant's variables s, x in (1) do not contain the relay control $u(s)$ but this control is included in equations for the fast variable z describing actuator dynamics. The derivative of the fast actuator variable z in (1) is big. That is why the real time usage of the second-order sliding-modes control algorithms with the finite time convergence is difficult due to big computational problems [2], [7], [12]. On the other hand, relay systems with second-order sliding modes could have an infinite number of switches and the time intervals between switches tend to zero, but there is no finite time convergence to the second-order sliding domain. This means that for (1), it is impossible to use the classical methods of singular perturbations theory (see [9] and [14]).

At the same time, ignoring the dynamics of the actuator, i.e., setting $\mu = 0$ and expressing $\bar{z}_0$ from the equation $f(\bar{z}_0,\, s,\, x,\, u(s)) = 0$ according to the formula $\bar{z}_0 = \varphi(s,\, x,\, u(s))$, we obtain the reduced system

$$ds/dt = g_1(t,\, \varphi(s,\, x,\, u(s)),\, s,\, x) = F_1(t,\, s,\, x,\, u(s))$$
$$dx/dt = g_2(t,\, \varphi(s,\, x,\, u(s)),\, s,\, x) = F_2(t,\, s,\, x,\, u(s)). \tag{2}$$

Here, we suppose that for (2) the sufficient conditions for existence of a stable sliding mode

$$F_1(t,\, 0,\, x,\, 1) < 0 \quad F_1(t,\, 0,\, x,\, -1) > 0 \tag{3}$$

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hold and the dynamics into this mode are described by the equations of equivalent control method (see, for example, [13])

$$dx/dt = F_2(t, 0, x, u_{eq}(t, x)) \quad F_1(t, 0, x, u_{eq}(t, x)) = 0. \quad (4)$$

In this note, two problems are considered concerning the convergence of the system with fast actuator (1) solutions to the corresponding sliding equation (4) solutions.

1) Design the Mathematical Tools for Investigating SPRS2OSM (Section II):

- Sufficient conditions for the exponential decreasing of the amplitude of chattering and the unlimited growth of frequency are found (Section II-B).
- It is shown that the exponentially stable slow-motions integral manifold of a smooth singularly perturbed system, describing the motion of original SPRS2OSM in the second-order sliding domain, is the exponentially stable slow-motions integral manifold of the original SPRS2OSM (Section II-C).
- The reduction principal theorem is proved in which the sufficient conditions of the equivalence for the stability of slow motions of plants and the stability of original systems with an actuator are found (Section II-D).

2) Chattering Analysis in Sliding-Mode Systems With Fast Actuators Based on Decomposition Tools Designing in the Section II: It is shown (Section III-A) that the definition of the motions in sliding mode, according to the equivalent control method, corresponds to the presence of fast actuators in the control system. In Section III-B, the connection between the stability of the actuators and the stability of the plant on the one hand and the stability of the sliding-mode system as a whole on the other hand is investigated. The algorithm for correcting the sliding-mode equations is suggested in Section III-C for taking into account the presence of fast actuators. In Section III-D, it is shown that whenever the sliding motions of the plant are stable, but not asymptotically stable, *it is obligatory to make a correction* to the sliding-mode equations taking into account the presence of fast actuators in the system.

II. Mathematical Tools

A. System Transformation Into a Convenient Form for the Analysis

We shall develop the mathematical tools for the case, when solutions of the relay control system (1) are determined uniquely. It is true (see [4]) for a wide class of such systems in which f is linearly depending on *relay* function $U(t, x, u(s))$ satisfying the inequality

$$U_1|s| \le sU(t, x, u(s)) \le U_2|s|, \qquad U_2 > U_1 > 0$$

for all (t, s, x).

Let us make three substitutions of variables in (1).

1) Here, we consider the case when, in (1), there exists a stable second-order sliding mode. In such a case, $g'_{1_z}(t, z, s, x) \neq 0$. We will consider the behavior of (1) in the small neighborhood of the second-order sliding domain $s = ds/dt = 0$. That is why it is reasonable to consider ds/dt as the system (1) state variable. Suppose that z_m is the last coordinate for the vector z and $g'_{1_{z_m}}(t, z, s, x) \neq 0$. Then we can introduce the variable $\sigma = ds/dt = g_1(t, z, s, x)$ instead of z_m in (1).
2) It is reasonable to consider sliding-mode control systems with a stable fast actuator. Then according to the boundary layer method [14], and conditions (3), ensuring the existence of a stable first-order sliding-mode in the reduced system (2) (when the actuator is ideal), one can conclude that the solution of (1) starting far from switching surface $s = 0$ will reach the neighborhood of the switching surface with radius $O(\mu)$ after a finite time. It allows us to examine only solutions of (1), starting in the neighborhood of the switching surface with radius $O(\mu)$, and to prove that such solutions of (1) will not leave this neighborhood. With this aim we have to introduce the new variable $\xi = s/\mu$ instead of variable s in (1), characterizing the behavior of (1) in the $O(\mu)$ neighborhood of the switching surface. Then, (1) takes the form

$$\begin{aligned} \mu\, d\overline{z}/dt &= f_1(t, \overline{z}, \sigma, \mu\xi, x) + d(t, x)U(t, x, u(\xi)) \\ \mu\, d\sigma/dt &= f_2(t, \overline{z}, \sigma, \mu\xi, x) + b(t, x)U(t, x, u(\xi)) \\ \mu\, d\xi/dt &= \sigma \quad dx/dt = g_2(t, \overline{z}, \sigma, \mu\xi, x). \end{aligned} \quad (5)$$

3) Let us eliminate relay control from the first equation of (5). Then, after the variables substitution $\overline{\eta} = \overline{z} - d(t, x)\sigma/b(t, x)$, system (1) takes the canonical form

$$\begin{aligned} \mu\, d\overline{\eta}/dt &= v_1(t, \overline{\eta}, \sigma, \mu\xi, x, \mu) \\ \mu\, d\sigma/dt &= v_2(t, \overline{\eta}, \sigma, \mu\xi, x, \mu) + b(t, x)U(t, x, u(\xi)) \\ \mu\, d\xi/dt &= \sigma \quad dx/dt = v_3(t, \overline{\eta}, \sigma, \mu\xi, x). \end{aligned} \quad (6)$$

The specific feature of (6) is that only the second-order sliding mode can occur in it, and the motion in this mode is determined by

$$\mu\, d\overline{\eta}/dt = v_1(t, \overline{\eta}, 0, 0, x, \mu) \quad dx/dt = v_3(t, \overline{\eta}, 0, 0, x, \mu). \quad (7)$$

System (7) has an asymptotically stable slow-motion integral manifold $\eta = h(t, x, \mu)$ if the following conditions are held [9].

I) The equation $v_1(t, \overline{\eta}, 0, 0, x, 0) = 0$ has an isolated solution $\overline{\eta} = h_0(t, x)$ at all $(t, x) \in \mathbf{R} \times \mathbf{R}^n$.

II) Functions v_i $(i = 1, 3)$, h_0 have second-order continuous derivatives in the domain $\overline{\Omega} = \{(t, \overline{\eta}, x, \mu) \in \mathbf{R} \times \mathbf{R}^{m-1} \times \mathbf{R}^n \times [0, \mu_0]\colon |\overline{\eta} - h_0(t, x)| < \delta\}$, where $\delta > 0$, $|.|$ is the Euclidean norm.

III) $\mathrm{Re\,Spec}\, \partial v(t, h_0(t, x), 0, 0, x, \mu)/\partial\overline{\eta} < -\kappa \le 0$ for all $(t, x, \mu) \in \mathbf{R} \in \mathbf{R}^n \times [0, \mu_0]$.

After the substitution of variables $\eta = \overline{\eta} - h(t, x, \mu)$ and expansion in the series toward to η, σ, ξ degrees at the point $(0, 0, 0)$, (6) takes the form

$$\begin{aligned} \mu\, d\eta/dt = {} & B_{11}(t, x, \mu)\eta + B_{12}(t, x, \mu)\sigma \\ & + \mu B_{13}(t, x, \mu)\xi + \varphi_1(t, \eta, \sigma, \mu\xi, x, \mu) \\ \mu\, d\sigma/dt = {} & B_{21}(t, x, \mu)\eta + B_{22}(t, x, \mu)\sigma \\ & + \mu B_{23}(t, x, \mu)\xi + \varphi_2(t, \eta, \sigma, \mu\xi, x, \mu) \\ & + b(t, x)U(t, x, u(\xi)) \quad \mu\, d\xi/dt = \sigma \end{aligned} \quad (8)$$

$$dx/dt = \varphi_3(t, \eta, \sigma, \mu\xi, x) \quad (9)$$

where $\varphi_1(t, \eta, 0, 0, x, \mu) = 0$ and everywhere in $\Omega = \{(t, \eta, x, \mu) \in \mathbf{R} \times \mathbf{R}^{m-1} \times \mathbf{R}^n \times [0, \mu_0]\colon |\eta| < \delta\}$ for nonlinear terms the following conditions hold: $\varphi_2(t, \eta, \sigma, \mu\xi, x, \mu) = \varphi_2(t, 0, 0, 0, x, \mu) + o(|y|)$, $\varphi_1(t, \eta, \sigma, \mu\xi, x, \mu) = o(|y|)$ by $y = (\eta, \sigma, \xi) \to 0$.

B. Exponential Stability of Fast Motions

Suppose that for all $(t, x) \in \mathbf{R} \times \mathbf{R}^n$, conditions I)–III) are satisfied and, moreover

IV) $B_{22}(t, x, 0) < -\alpha < 0$, $b(t, x) < -\alpha < 0$, $|\varphi_2(t, 0, 0, 0, x, \mu)| < \alpha U_1$.

Let us denote as $y = (\eta^T, \sigma, \xi)^T$, and as $(y(t, \mu), x(t, \mu))$. the corresponding coordinates of (8) and (9) solution with initial conditions

$$y(0, \mu) = y_0 \quad x(0, \mu) = x_0.$$

Lemma 1: If conditions I)–IV) are true, there exist constants $K_1 > 0$, $K_2 > 0$, $\gamma > 0$ and W some neighborhood of the origin in the state space of variables y such that for all $(y_0, x_0) \in \Omega' = \mathbf{R}^+ \times W \times \mathbf{R}^n$ and $\mu \in (0, \mu_0]$ the following inequality holds:

$$|y(t, \mu)|_* \leq K_1 |y_0|_* e^{\gamma t/\mu} \leq K_2 e^{-\gamma t/\mu}$$
$$|y|_* = \sqrt{|\eta|^2 + |\sigma|^2 + |\xi|}. \tag{10}$$

Lemma 1 was proved in [1] and [5] with the help of Lyapunov function

$$E = \eta^T S \eta + \sigma^2 - \xi[2b(t, x)U(t, x, u(\xi)) + 2\varphi_2(t, \eta, \sigma, \mu\xi, x, \mu) + B_{22}\sigma + 2\eta^T S B_{12} + 2B_{21}\eta]$$

where $S(t, x, \mu)$ is positive definite solution of equation $SB_{11} + B_{11}^T S = -I_{m-1}$.

C. Decomposition Theorem

Consider a solution of (8) and (9), only starting in Ω'. Then, the $x(t, \mu)$ coordinate of the solution of (8) and (9) will be a solution of the initial problem

$$dx/dt = \Phi(t, y(t, \mu), x, \mu), \qquad x(0) = x_0$$
$$\Phi(t, y(t, \mu), x, \mu) = \varphi_3(t, \eta(t, \mu), \sigma(t, \mu), \mu\xi(t, \mu), x(t, \mu)).$$

Let us represent $x(t, \mu)$ as $x(t, \mu) = \overline{x}(t, \mu) + \Pi x(t, \mu)$, such that

$$d\overline{x}/dt = \Phi(t, 0, \overline{x}, \mu), \qquad \overline{x}(0) = \overline{x}_0 \tag{11}$$
$$d\Pi x/dt = \Phi(t, y(t, \mu), \overline{x} + \Pi x, \mu) - \Phi(t, 0, x, \mu), \tag{12}$$
$$\Pi x(0) = \Pi_0 x, \qquad \overline{x}_0 + \Pi_0 x = x_0. \tag{13}$$

To define the solutions of (11)–(13), it is necessary to choose $(x_0, \Pi_0 x)$.

Theorem 2: Suppose that for all $(t, y, x), (t, \overline{y}, \overline{x}) \in \Omega'$ conditions I–IV are true, inequality $|\Phi(t, y, x) - \Phi(t, \overline{y}, \overline{x})| \leq M(|y - \overline{y}| + |x - \overline{x}|)$ is satisfied and

$$\mu M/\gamma < 1, \qquad KM/(\gamma - \mu M) < C. \tag{14}$$

Then, for any initial points $(y_0, x_0) \in \Omega'$ the solutions of (8) and (9) can be represented as slow and fast parts in the form: $(y(t, \mu), x(t, \mu)) = (0, \overline{x}(t, \mu)) + (\Pi y(t, \mu), \Pi x(t, \mu))$. So $\overline{x}(t, \mu)$ is the solution of (11) with initial conditions $\overline{x}(0) = \overline{x}_0$ while $x_0 = \overline{x}_0 + O(\mu)$. The fast part of this solution $\{\Pi y(t, \mu), \Pi x(t, \mu)\}$ satisfies the inequality

$$\mu|\Pi y(t, \mu)| + |\Pi x(t, \mu)| < \mu(C + K)e^{-\gamma t}. \tag{15}$$

The proof of this theorem is given in the Appendix.

D. Reduction Principle Theorem

Theorem 2 and (15) yield the following reduction principle theorem.

Theorem 3: If, under the conditions of Theorem 2, the function $\overline{x}(t, \mu)$ is the solution of (11), then $(0, 0, 0, \overline{x}(t, \mu))$ is the solution of (8) and (9), and this solution will be stable (unstable, asymptotically stable) if and only if $\overline{x}(t, \mu)$ is stable (unstable, asymptotically stable).

III. Analysis of Chattering in Sliding Mode Systems With Fast Actuators

A. Systems Containing Relay Control Nonlinearly

Consider the control system

$$ds/dt = -u \quad dx/dt = (u^2 - 1)x \quad x, s \in \mathbf{R} \quad \mathbf{u}(\mathbf{s}) = \mathrm{sign}(\mathbf{s}) \tag{16}$$

containing the relay control $u(s)$ nonlinearly. There is a stable sliding mode in (16). Defining solutions in the sliding domain (16) are not unique. For example, on one hand, extension definition of (16) solutions into the sliding mode according to [4] takes the form $dx/dt = x$ with an unstable zero solution. On the other hand, extension of the definition of (16) into the sliding mode according to the equivalent control method takes the form $dx/dt = -x$ with an asymptotically stable zero solution.

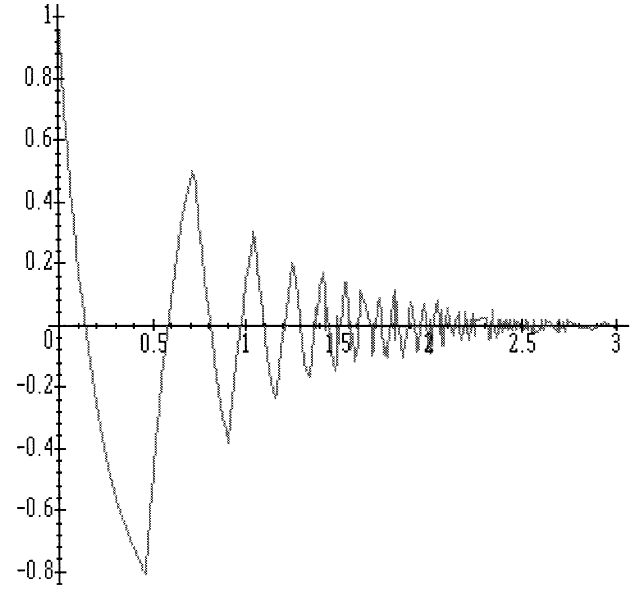

Fig. 1. Exponential decreasing of z.

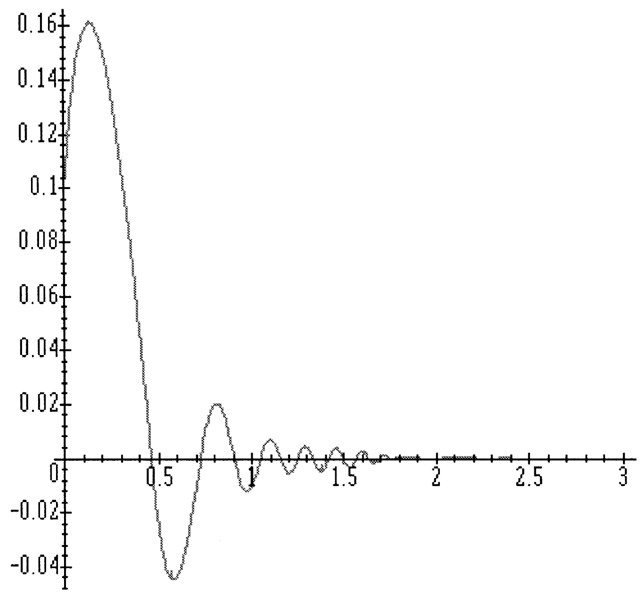

Fig. 2. Exponential decreasing of s.

Suppose that a relay control is transmitted to the plant via a fast actuator and a complete model of a system, taking into account the presence of a fast actuator, has the form

$$\mu\, dz/dt = -z - u \quad ds/dt = z \quad dx/dt = (2z^2 - 1)x \tag{17}$$

where $z \in \mathbf{R}$ is the actuator variable and μ is the actuator time constant. For (17), Theorems 2 and 3 are true. This means that the fast variables z, s are exponentially decreasing (in Figs. 1 and 2 $\mu = 0.2$, $z(0) = 1$, $s(0) = 0.1$), equation $dx/dt = -x$ of the equivalent control method is approximately described by the slow motions in (17), and the zero solution of (17) is asymptotically stable.

B. Stability of Actuators and Absence of Chattering

In this section, we investigate the correlation between the natural conditions of stability of fast actuators in sliding-mode control systems and the existence of the stable first-order sliding mode for a reduced system, describing the behavior of the plant without actuator on the one hand, and sufficient conditions for exponential decreasing of fast oscillations (absence of chattering) in the original system on the other hand.

Consider the simplest case, when the behavior of the plant is described by

$$dx/dt = Ax + Bu, \qquad x \in \mathbf{R}^n \quad u = \mathrm{sign}(s), \qquad s = Cx \in \mathbf{R}. \tag{18}$$

Suppose that the relay control u ensures the stable first-order sliding mode on the switching surface $s = 0$, and consequently $CB < 0$. Consider the case when relay control is transmitted to the plant via a fast actuator, with behavior described by

$$\mu\, dz/dt = Dz + Fx + bu(s), \qquad z \in \mathbf{R}^{\mathbf{m}} \tag{19}$$

where μ is the actuator time constant. This means that the system model taking into account the presence of a fast actuator has the form

$$\mu\, dz/dt = Dz + Fx + bu(s) \quad dx/dt = Ax + BKz. \tag{20}$$

It is natural to suppose the following.

- The actuator is stable, which means that

$$\mathrm{Re\,Spec}\, D < 0. \tag{21}$$

- System (20), for $\mu = 0$, turn to (18) and consequently

$$KD^{-1}b = -1, \qquad -CBKD^{-1}b = CB < 0. \tag{22}$$

Transform (20) to the canonical form (see Section II-A)

$$\begin{aligned}\mu\, dz_1/dt &= D_{11}z_1 + D_{12}\sigma + F_{11}s + F_{12}x_1\\ \mu\, d\sigma/dt &= D_{21}z_1 + D_{22}\sigma + F_{11}s + F_{12}x_1 + du(s)\\ ds/dt &= \sigma \quad dx_1/dt = B_{11}z_1 + B_{12}\sigma + A_{33}s + A_{34}x_1\end{aligned} \tag{23}$$

$z_1 \in \mathbf{R}^{m-1}$, $x_1 \in \mathbf{R}^{n-1}$, $\sigma \in \mathbf{R}$. For (23), the conditions of stability of the second-order sliding mode are

$$d < 0, \qquad D_{22} < 0. \tag{24}$$

Inequality $\mathrm{Re\,Spec}\, D_{11} < 0$ ensures exponential decreasing of actuator variables in the second-order sliding domain. The following proposition is obvious.

Proposition 4: When the actuator is the single-input—single-output system ($m = 1$, $z \in \mathbf{R}$) and the condition of stability of fast actuator (21) and conditions of existence of stable first-order sliding mode for the reduced system (18) are held, the amplitude of chattering in (20) is exponentially decreasing.

However, it is not true just for $m = 2$. Condition (22) for (23) means that

$$(0\ 1)D^{-1}\begin{pmatrix}0\\ d\end{pmatrix} = d\,\frac{\det(D_{11})}{\det D} > 0.$$

Now, from condition (21), follows that $\det D > 0$. Conditions (21), (22) mean that d and $\det(D_{11})$ have the same sign. This means that, for $m = 2$, (21) and (22) do not ensure exponential decreasing of chattering.

Proposition 5: Let $m = 2$. If (21), (22), and $D_{11} < 0$ or $d < 0$ are held, then the amplitude of chattering in (20) is exponentially decreasing.

The stability of the fast actuator (21) and of the second-order sliding mode (24) still does not guarantee the absence of chattering if $\dim z > 1$. Consider the system

$$\begin{aligned}&\mu\, dz_1/dt = z_1 + z_2 + \eta + D_1x \quad \mu\, dz_2/dt = 2z_2 + \eta + D_2x\\ &\mu\, d\eta/dt = 24z_1 - 60z_2 - 9\eta + D_3x + k\,\mathrm{sign}\, s\\ &ds/dt = \eta \quad dx/dt = F(z_1,\, z_2,\, \eta,\, s,\, x)\end{aligned}$$

where $z_1,\, z_2,\, \eta,\, s$ are scalars, $k < 0$. It is easy to check that the spectrum of the matrix is $\{-1,\, -2,\, -3\}$, and condition (24) hold for this system. On the other hand, motions in the second-order sliding mode are described by

$$\begin{aligned}&\mu\, dz_1/dt = z_1 + z_2 + D_1x \quad \mu\, dz_2/dt = 2z_2 + D_2x\\ &dx/dt = F(z_1,\, z_2,\, 0,\, 0,\, x).\end{aligned}$$

The fast dynamics in this system are unstable and the absence of chattering in the original system cannot be guaranteed.

C. Algorithm for Correction of Equivalent Control Method

According to Theorem 2, the slow motions in system with fast actuator (1) are described by equations for the motions on the slow-motions integral manifold of a smooth singularly perturbed system describing (1) dynamics into the second-order sliding domain. It allows for the formulation of the following algorithm.

Step 1. Design of Algebraic–Differential Equations for Description of Motions in (1) in the Second-Order Sliding Mode: Suppose that the stable second-order sliding mode exists in (1). Then, motions in this mode are determined by equations of the equivalent control method

$$\begin{aligned}&\mu\, dz/dt = f(t,\, z,\, 0,\, x,\, \overline{u}_{eq}(t,\, z,\, x,\, \mu)) = \hat{f}(t,\, z,\, x,\, \mu)\\ &dx/dt = g_2(t,\, z,\, 0,\, x)\end{aligned} \tag{25}$$

$$ds/dt = g_1(t,\, z,\, 0,\, x) = 0 \tag{26}$$

where the equivalent control $\overline{u}_{eq}(t,\, z,\, x,\, \mu)$ for second-order sliding one can find from

$$\begin{aligned}&d^2s/dt^2(t,\, z,\, 0,\, x,\, \overline{u}_{eq},\, \mu)\\ &\qquad = g'_{1_z}f/\mu + g'_{1_\xi}g_1 + g'_{1_x}g_2|_{(t,\, z,\, 0,\, x,\, \overline{u}_{eq},\, \mu)} = 0.\end{aligned} \tag{27}$$

Step 2. Design of Differential Equations for the Description of Motions in (1) in the Second-Order Sliding Mode: Let us express one of the vector z coordinates from (26) to reduce the algebraic–differential system (25)–(27) to the system of differential equations. Let it be, for example, its last coordinate z_m, and the corresponding expression has the form $z_m = p(t,\, \overline{z},\, x)$, where $\overline{z} \in R^{m-1}$ is the vector consisting of the first $(m-1)$ coordinates of the vector z. Then, (25) may be represented in the form

$$\mu\, d\overline{z}/dt = \overline{f}(t,\, \overline{z},\, x,\, \mu) \quad dx/dt = \overline{g}_2(t,\, \overline{z},\, x) \tag{28}$$

where $\overline{f}$ consists of the first $(m-1)$ coordinates of function $\hat{f}$ at the point $(t,\, \overline{z},\, p(t,\, \overline{z},\, x),\, x,\, \mu)$.

Step 3. Design of Corrected Equations of the Equivalent Control Method: System (28) is a smooth singularly perturbed system. If in such systems the fast variables are uniformly exponentially stable, then there exists the slow-motions integral manifold in the following form: $\overline{z} = h(t,\, x,\, \mu)$. Motion on that manifold is described by

$$d\overline{x}/dt = \overline{g}_2(t,\, h(t,\, \overline{x},\, \mu),\, \overline{x}) \quad \overline{z} = h(t,\, \overline{x},\, \mu). \tag{29}$$

In Theorem 2 (Section II), the sufficient conditions are found ensuring that the x coordinate of the solutions of (1) will differ from the solutions of (29) up to the fast decreasing exponent. In this sense, slow motion in (28) is precisely described by (29), and we will call (29) precise equations of the equivalent control method. Function $h(t,\, x,\, \mu)$ could be expressed as an asymptotic series $h(t, x, \mu) = \sum_0^\infty \mu^k h_k(t, x)$ from

$$\mu[h_t' + h_x' g_2(t, h(t, x, \mu), x)] = \overline{f}(t, h(t, x, \mu), x, \mu). \quad (30)$$

Function $h_0(t, x)$ is determined by $f(t, h_0, x, 0) = 0$. This means that for $\mu = 0$, (29) coincides with the equivalent control method equation (4). With $\mu = 0$, (29) differs from (4) only in the terms, which correspond to the presence of fast actuators in the original system (1). From Theorem 3, it follows that the problems of investigating the stability for the zero solution of (1) and (29) are equivalent.

D. When Is Correction of the Equivalent Control Method Obligatory?

Consider the sliding-mode control system

$$\begin{aligned} &ds/dt = -u(s) \quad dx_1/dt = x_2 \quad dx_2/dt = u(s) - x_1 \\ &u(s) = -\text{sign}(s). \end{aligned} \quad (31)$$

There exists a stable first-order sliding mode for (31). Then, the sliding-mode dynamics are described by

$$dx_1/dt = x_2 \quad dx_2/dt = -x_1. \quad (32)$$

It is obvious that the solutions of this system are stable but not asymptotically stable. Suppose that the relay control $u(s)$ is transmitted to the plant with the help of a fast actuator, whose behavior is described by variables z_1, z_2. The complete mathematical model of control system has the form

$$\begin{aligned} &\mu\, dz_1/dt = -z_1 - x_1 \quad \mu dz_2/dt = -z_2 - \text{sign}(s) \\ &ds/dt = z_2 \quad dx_1/dt = x_2 \\ &dx_2/dt = (a+1)z_1 - z_2 + ax_1 \end{aligned} \quad (33)$$

where a is the scalar parameter determining actuator/plant connection. It is easy to see that, for (33), Theorems 2 and 3 are true and slow dynamics for (33) with precision level $o(\mu)$ are described by (32). On the other hand, for dynamics in the second-order sliding mode for (33), one has

$$\begin{aligned} &\mu\, dz_1/dt = -z_1 - x_1 \quad dx_1/dt = x_2 \\ &dx_2/dt = (a+1)z_1 + ax_1. \end{aligned} \quad (34)$$

Then, the slow motion integral manifold of (33) and (34) takes the form $z_1 = p_1(\mu)x_1 + p_2(\mu)x_2$, where $p_{ij}(\mu) = p_{i0} + p_{i1}\mu + \cdots + p_{ik}\mu^k + \cdots$, $i = 1, 2$. The functions $p_{ij}(\mu)$ can be found from

$$\mu(p_1\ p_2)\left(\begin{pmatrix} 0 \\ a+1 \end{pmatrix}(p_1\ p_2) + \begin{pmatrix} 0 & 1 \\ a & 0 \end{pmatrix}\right) = -(p_1\ p_2) - (1\ 0) \quad (35)$$

and, consequently, $(p_{10}\ \ p_{20}) = (-1\ 0)$, $(p_{11}\ \ p_{21}) = (0\ 1)$. This means that the slow motion in (33) is described by

$$dx_1/dt = x_2 \quad dx_2/dt = -x_1 + \mu(a+1)x_2 + O(\mu^2). \quad (36)$$

From Theorems 2 and 3, it follows that variables s and $ds/dt = z_2$ are asymptotically decreasing, but for $a > -1$, the zero solution of (33) is unstable and for the $a < -1$ this solution is asymptotically stable.

This means that in the case when the spectrum of sliding-mode equations is critical, the presence of fast actuators can change the behavior of a system from stability to instability or asymptotic stability. One can conclude that for the investigation of stability in the critical case, the correction of sliding-mode equations is obligatory.

IV. Conclusion

1) The sufficient conditions under which the oscillations in the sliding-mode control systems, with fast actuators, whose behavior are described by SPRS2OSM, have the following structure.
   - The oscillations in the second-order sliding mode, which are described by a smooth singularly perturbed system of differential equations, and the slow-motion integral manifold of this system is the stable slow-motion integral manifold of the original system.
   - The oscillations in the second-order sliding mode, which are described by a smooth singularly perturbed system of differential equations, and the slow-motion integral manifold of this system is the stable slow-motion integral manifold of the original system.

Due to this fact, it was shown that it is possible to design chattering-free sliding-mode control systems with fast actuators in the case when the order of sliding in a complete model is 2.

2) It is proved that in the general case, when the plant contains the relay control nonlinearly, the equations of the equivalent control method for the sliding motions of the plant are approximately describing the slow motion in the original SPRS2OSM and correspond to the presence of fast actuators in a sliding-mode control system.
3) The connection between the stability of the actuators and the stability of the plant on one hand and the stability of the sliding-mode system as a whole on the other hand is investigated.
4) The algorithm for the correction of the sliding-mode equation is proposed. In the case when the linear part of the sliding-mode equations has a critical spectrum, it is obligatory to correct the equations of the sliding motion in order to take into account the presence of fast actuators in the system, because the presence of such devices may cause change to the system behavior from stability to asymptotic stability or instability.

Appendix
Proof of Decomposition Theorem

Consider the (11) and (12). Let us design an integral manifold of (11) and (12) in the form $\mathcal{S} = \{(t, x, \Pi x) \in \mathbf{R}^+ \times \mathbf{R}^n \times \mathbf{R}^n : \Pi x = \mathcal{H}(t, x, \mu)\}$, where the function $\mathcal{H}(t, x, \mu)$ is continuous on $\mathbf{R}^+ \times \mathbf{R}^n \times [0, \mu_0]$ and the following inequality is true:

$$\sup|\exp(\gamma t/\mu)\mathcal{H}(t, x, \mu)| < \mu d \quad (t, x) \in \mathbf{R} \times \mathbf{R}^n. \quad (37)$$

The constant $d > 0$ in (37) will be defined later. Denote as $\mathcal{U}$ the metric space of continuous functions $\mathbf{R}^+ \times \mathbf{R}^n \times [0, \mu_0] \to \mathbf{R}^n$, satisfying (37) with the metric $\rho(\mathcal{H}, \overline{\mathcal{H}}) = \sup|\exp(\gamma t/\mu)(\mathcal{H}(t, x, \mu) - \overline{\mathcal{H}}(t, x, \mu))|$, for $(t, x, \mu) \in \mathbf{R}^+ \times \mathbf{R}^n \times [0, \mu_0]$. The space $\mathcal{U}$ is a complete metric space. The function $\Pi x = \mathcal{H}(t, x, \mu) \in \mathcal{U}$ is the solution of

$$\begin{aligned} \mathcal{H} &= \mathcal{P}(\mathcal{H}) \\ \mathcal{P}(\mathcal{H})(t, \hat{x}, \mu) &= -\int_t^\infty [\Phi(\theta, y(\theta, \mu), \phi(\theta, \mu) \\ &\quad + \mathcal{H}(\theta, \phi(\theta, \mu), \mu), \mu) \\ &\quad - \Phi(\theta, 0, \phi(\theta, \mu), \mu)]\, d\theta \end{aligned} \quad (38)$$

where $\phi(\theta, \mu)$ is the solution of Cauchy problem $d\phi/d\theta = \Phi(\theta, 0, \phi, \mu)$, $\phi(t) = \hat{x}$. Let us show that operator $\mathcal{P}$ from (38) transforms $\mathcal{U}$ into itself. Taking into account (37) and (38), one can conclude that

$$\begin{aligned} &|\exp(\gamma t/\mu)\mathcal{P}(\mathcal{H})(t, \hat{x}, \mu)| \\ &\quad \le M \exp(\gamma t/\mu) \int_t^\infty [|\mathcal{H}(\theta, \phi(\theta, \mu), \mu)| + |y(\theta, \mu)|]\, d\theta \\ &\quad < \frac{M}{\gamma}[\mu d + C|y_0|_*]. \end{aligned}$$

Now it is possible to choose such d that for any $y_0 \in W$ the inequality $(M/\gamma)[\mu d + C|y_0|_*] \le d$ is true. This means that operator $\mathcal{P}$ transforms the space $\mathcal{U}$ into itself. Similarly

$$\begin{aligned}&\exp(\gamma t/\mu)|\mathcal{P}(\mathcal{H})(t, \hat{x}, \mu) - \mathcal{P}(\overline{\mathcal{H}})(t, \hat{x}, \mu)| \\ &\quad \le \exp(\gamma t/\mu) \int_t^\infty |\Phi(\theta, \phi(\theta, \mu) + \mathcal{H}, y(\theta, \mu), \mu) \\ &\qquad - \Phi(\theta, \phi(\theta, \mu) + \overline{\mathcal{H}}, y(\theta, \mu), \mu)| \, d\theta \\ &\quad \le \exp(\gamma t/\mu) \int_t^\infty M|\mathcal{H} - \overline{\mathcal{H}}| \, d\theta \le \mu \frac{M}{\gamma} \rho(\mathcal{H}, \overline{\mathcal{H}})\end{aligned}$$

which means that operator $\mathcal{P}$ is a contraction operator on $\mathcal{U}$. Then, the operator $\mathcal{P}$ has the unique fixed point corresponding to the function $\Pi x = \mathcal{H}(t, \hat{x}, \mu)$. Moreover, from (37), one can conclude that the inequality $|\mathcal{H}(t, \hat{x}, \mu))| < \mu \, d \exp(-\gamma t/\mu)$ holds for all $(t, \hat{x}, \mu) \in \mathbf{R}^+ \times \mathbf{R}^n \times (0, \mu_0]$.

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Two-Channel Decentralized Integral-Action Controller Design

A. N. Gündeş and A. B. Özgüler

***Abstract*—We propose a systematic controller design method that provides integral-action in linear time-invariant two-channel decentralized control systems. Each channel of the plant is single-input–single-output, with any number of poles at the origin but no other poles in the instability region. An explicit parametrization of all decentralized stabilizing controllers incorporating the integral-action requirement is provided for this special case of plants. The main result is a design methodology that constructs simple low-order controllers in the cascaded form of proportional-integral and first-order blocks.**

***Index Terms*—Decentralized control, integral-action, stability.**

I. INTRODUCTION

We consider decentralized controller design with integral-action for linear time-invariant (LTI) plants, whose unstable poles can only occur at the origin. These plant models occur in many applications and are common in process control [7]. The decentralized controller structure is preferred for simplicity of implementation and the integral-action in the controllers achieves asymptotic tracking of step-input references applied at each input. We apply and explicitly define the parametrization of all decentralized controllers and incorporate integral-action into the controllers for this important class of plants, where the 2×2 plant transfer-function matrix may have simple or multiple poles at the origin in any or all of its entries.

The theory of decentralized control has produced relatively few systematic and explicit design methods despite the wide practical demand. The main difficulty is that the decentralized structure imposed on the free parameter of the set of all stabilizing controllers renders the optimization problem nonconvex [10]. Alternatively, when viewed as a problem of making the plant stabilizable and detectable from one of its channels, the decentralized stabilizing controllers are constructed relying on genericity arguments [2], [9], [12]. The decentralized controller parametrizations obtained previously (see, for example [5] and [8]) all characterize controllers at the conceptual level and do not provide explicit descriptions. The usual computational methods that would be used to convert such conceptual designs to explicit descriptions would typically produce unnecessarily high-order controllers since the standard (robust) control designs are not tailored to special type of plants as considered here.

The integral-action problem for the case of stable plants has been considered in the decentralized setting with single-input–single-output channels in [7], and [1], and design procedures were proposed for achieving reliable stability under the possible failure of controllers in [6]. For the case of unstable plants, controller designs were presented in [3] based on choosing the free design parameter to achieve a desired sensitivity function for a suitable diagonal or triangular model of the plant. However, explicit decentralized integral-action controller designs for plants with integrators are not available.

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