Technical Notes and Correspondence

Analysis of Second-Order Sliding-Mode Algorithms in the Frequency Domain

I. Boiko, L. Fridman, and M. I. Castellanos

***Abstract*—A frequency domain analysis of the second-order sliding-mode algorithms, particularly of the twisting algorithm is carried out in the frequency domain with the use of the describing function method and Tsypkin's approach. It is shown that in the presence of an actuator, the transient process may converge to a periodic motion. Parameters of this periodic motion are analyzed. A comparison of the periodic solutions in the systems with higher order sliding-mode controllers and the oscillations that occur in classical sliding-mode systems with actuators is done.**

***Index Terms*—Chattering, relay control, sliding-mode (SM) , variable structure systems.**

I. Introduction

Higher order sliding modes (SM) have received a lot of attention from the control research community over the last decade (see the bibliographies in [1]–[9]). The main advantages of the higher order sliding-mode algorithms are: A higher accuracy of resulting motions; the possibility of using continuous control laws (super twisting or twisting as a filter); the possibility of utilizing the Coulomb friction in the control algorithm [7]; and finite time convergence for the systems with arbitrary relative degree.

It is known that the first order SM in systems with actuators of relative degree *two* or higher is realized as chattering [2], [9]. For the same reason, it would be logical to expect a similar behavior from a second-order SM, as the aforementioned algorithms contain the *sign* function. The modes that occur in a relay feedback system with the plant being the order 1, 2, 3, etc. dynamics were studied in publications [10], [11]. It has been proven in those works that for the plant of order 3 and higher the point of the origin cannot be a stable equilibrium point. Similar behavior, therefore, can be expected from a system with a second-order SM algorithm. Thus, the objective of this note is to analyze the motions that occur in a system with one of the most popular second-order SM algorithm—the twisting algorithm, to show the existence of periodic motions, to estimate the parameters of those motions, and to compare the latter with the parameters of chattering in the systems with asymptotic second-order relay control [6], [10], [11] and first-order SM control [12].

Given the objective of the outlined analysis and the fact that the introduction of an actuator increases the order of the system, the analysis of corresponding Poincare maps becomes complicated. In this case, the describing function (DF) method [13] seems to be a good choice. The DF method provides a simple and efficient solution of the problem. However, the DF method provides only an approximate solution and for that reason the Tsypkin's method [10] modified to accommodate the analyzed problem is used for the analysis as well. The latter does not require involvement of the filter hypothesis and provides an exact solution of the problem of finding the parameters of self-excited oscillations in a relay feedback system. However, the twisting algorithm is not equivalent to the relay feedback control and some modifications to the Tsypkin's method need to be done to accommodate this method to the analyzed algorithm.

The note is organized as follows. At first the model of the system involving the twisting algorithm suitable for the frequency domain analysis is obtained. Then, DF model and the model suitable for deriving the Tsypkin's locus of the system are built. After that the DF analysis and the exact analysis of the system with the twisting algorithm are considered. Finally, a number of examples are considered and a comparison is done.

II. Twisting Algorithm and its DF Analysis

The twisting algorithm is one of the simplest and most popular algorithms among the second-order sliding-mode algorithms. There are two ways to use the twisting algorithm [8]: to apply it to a plant of relative degree *two*, or to apply it to a plant of relative degree *one* and introduce an integrator in series with the plant (twisting-as-a-filter). For the plants of relative degree *two* it can be formulated as follows. Let the plant (or the plant plus actuator) be given by the following differential equations:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{1}$$

where x is an n-dimenaionsl state vector, u is a scalar control, A and B are matrices of respective dimensions, and $y$is scalar and can be treated as either the sliding variable or the output of the plant. Also, let the control u of the twisting algorithm be given as follows [5], [8]:

$$u = -c_1 \operatorname{sign}(y) - c_2 \operatorname{sign}(\dot{y}) \tag{2}$$

where c_1 and c_2 are positive values, $c_1 > c_2$. Assume that a periodic motion occurs in the system with the twisting algorithm. Then the system can be analyzed with the use of the DF method. As normally accepted in the DF analysis, we assume that the plant has a magnitude characteristic of a low-pass filter. Find the DF q of the twisting algorithm as the first harmonic of the periodic control signal divided by the amplitude of $y(t)$—in accordance with the definition of the DF [13]

$$q = \frac{\omega}{\pi A_1} \int_0^{\frac{2\pi}{\omega}} u(t) \sin \omega t dt + j \frac{\omega}{\pi A_1} \int_0^{\frac{2\pi}{\omega}} u(t) \cos \omega t dt$$

where A_1 is the amplitude of the input to the nonlinearity (of $y(t)$ in our case) and ω is the frequency of $y(t)$. However, the twisting algorithm can be analyzed as the parallel connection of two ideal relays where the input to the first relay is the sliding variable and the input to the second relay is the derivative of the sliding variable. The DF for those nonlinearities are known [13]. For the first relay the DF is: $q_1 = 4c_1/\pi A_1$, and for the second relay it is: $q_2 = 4c_2/\pi A_2$, where A_2 is the amplitude of dy/dt. Also, take into account the relationship between y and dy/dt in the Laplace domain, which gives the relationship between the amplitudes A_1 and $A_2 : A_2 = A_1 \Omega$, where Ω is the frequency of the

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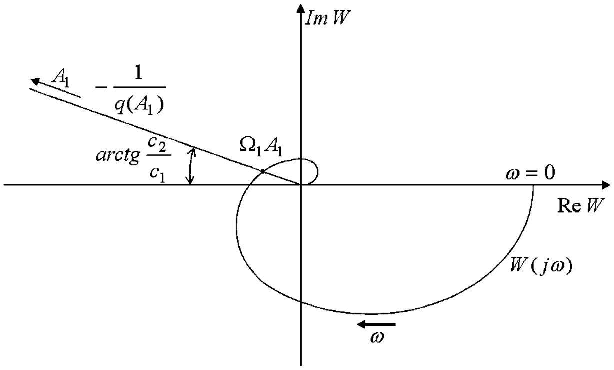

Fig. 1. Finding the periodic solution.

oscillation. As a result, taking into account the parallel connection of those relays, the DF of the twisting algorithm can be given as a sum of the DF of the first relay and the DF of the second relay multiplied by the Laplace variable

$$q = q_1 + sq_2 = \frac{4c_1}{\pi A_1} + j\Omega\frac{4c_2}{\pi A_2} = \frac{4}{\pi A_1}(c_1 + jc_2), \qquad s = j\Omega. \tag{3}$$

Let us note that the DF of the twisting algorithm depends on the amplitude value only. This suggests the technique of finding the parameters of the limit cycle—via the solution of the complex equation [13]

$$-\frac{1}{q(A_1)} = W(j\Omega) \tag{4}$$

where $W(j\omega)$ is the complex frequency response characteristic (Nyquist plot) of the plant and the function at the left-hand side is given by: $-1/q = \pi A_1(-c_1 + jc_2)/[4(c_1^2 + c_2^2)]$. Equation (4) is equivalent to the condition of the complex frequency response characteristic of the open-loop system intersecting the real axis in the point $(-1, j0)$. The graphical illustration of the technique of solving (4) is given in Fig. 1.

The function $-1/q$ is a straight line the slope of which depends on c_2/c_1 ratio. This line is located in the second quadrant of the complex plane. The point of the intersection of this function and of the Nyquist plot $W(j\omega)$ provides the solution of the periodic problem. This point gives the frequency of the oscillation Ω and the amplitude A_1. Therefore, *if the transfer function of the plant (or plant plus actuator) has relative degree higher than two a periodic motion may occur in such a system. For that reason, if an actuator of first or higher order is added to the plant with relative degree two driven by the twisting controller a periodic motion may occur in the system.*

In [3], [6], [10], and [11], the *asymptotic second-order SM relay controller* was studied. The simplest scalar example of this controller has the form: $x = -a\dot{x} - bx - k\,\mathrm{sign}(x),\ a > 0, k > 0$. It is shown in those works that this system is exponentially stable (no finite time convergence). In respect to our analysis, from Fig. 1 it also follows that *the frequency of the periodic solution for the twisting algorithm is always higher than the frequency of the asymptotic second-order sliding-mode relay controller*, because the latter is determined by the point of the intersection of the Nyquist plot and the real axis.

Another modification of the twisting algorithm is its application to a plant with relative degree one with the introduction of the integrator [8]. This will be further referred to as the "twisting as a filter" algorithm. The above reasoning is applicable in this case as well. The introduction of the integrator in series with the plant makes the relative degree of this part of the system equal to two. As a result, any actuator introduced in the loop increases the overall relative degree to at least three. In this case, there always exists a point of intersection of the Nyquist plot of the serial connection of the actuator, the plant and the integrator and of the negative reciprocal of the DF of the twisting algorithm (Fig. 1). Thus, *if an actuator of first or higher order is added to the plant with relative degree one a periodic motion may occur in the system with the twisting as a filter algorithm*. However, if the actuator is of second or higher order there is an opportunity for reduction of the amplitude of chattering in the control signal when using twisting as a filter algorithm in comparison with the first order SM control. This reduction is achieved due to the falling character of the magnitude characteristic of the integrator introduced between the discontinuous nonlinear element and the plant. The DF analysis provides a very demonstrative proof of possible existence of a periodic solution in the system with the twisting algorithm in the case of an actuator introduction. However, the DF method is an approximate one and a rigorous analysis would be desirable. This can be provided using Tsypkin's method [10].

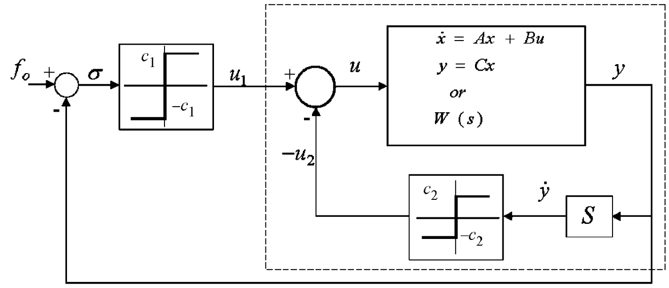

Fig. 2. Transformed system with twisting algorithm.

III. Exact Analysis of Twisting Algorithm

The Tsypkin's method [10] can provide an exact solution of the periodic problem in a relay feedback system having a plant (1) and the control given by the hysteresis relay function. The Tsypkin's locus for such a system is defined as: $\Lambda(\omega) = (1/\omega)\dot{\sigma}(\pi/\omega) + j\sigma(\pi/\omega)$, where $\sigma(t)$ is the error signal, $t = \pi/\omega$ is half a period in a periodic motion that corresponds to the switch of the relay from "+" to "−." Considering the identities $\sigma(t) = -y(t)$ and $\sigma(0) = -\sigma(\pi/\omega)$ rewrite the above formula as follows: $\Lambda(\omega) = (1/\omega)\dot{y}(0) + jy(0)$, where time $t = 0$ corresponds to the switch of the relay from "−" to "+." We shall treat the last formula as the Tsypkin's locus definition. For practical computations, the Tsypkin's locus can be represented via the plant transfer function as follows [10]:

$$\Lambda(\omega) = \frac{4c}{\pi}\sum_{k=1}^{\infty}\mathrm{Re}\ W[(2k-1)\omega] + j\frac{4c}{\pi}\sum_{k=1}^{\infty}\frac{\mathrm{Im}W[(2k-1)\omega]}{2k-1}. \tag{5}$$

With the function $\Lambda(\omega)$ computed, analysis of periodic motions in a relay feedback system becomes an easy task. The frequency of the periodic motion Ω can be found from the following equation:

$$Im\ \Lambda(\Omega) = -b \tag{6}$$

where b is the hysteresis of the relay (which is zero in our case). However, to be able to use this method for the twisting algorithm analysis, we need to transform the original problem into an equivalent one. Transposition of the second relay into the feedback around the plant allows us to build the following equivalent system (Fig. 2).

In Fig. 2, we are going to treat the part of the system denoted by the dashed line as a new plant of the relay system (the equivalent plant). However, the equivalent plant is nonlinear with the nonlinearity being the second relay. For that reason, the Tsypkin's approach needs to be modified.

According to the definition of the Tsypkin's locus, the imaginary part of $\Lambda(\omega)$ can be found as the output of the plant at the switch time if a periodic square-wave pulse signal u_1 is applied to the plant. Note also that the signal u_2 is also applied to the plant and the output of the plant y can be considered as a sum of two outputs y_1 and y_2, each of them is a response to the control u_1 and u_2, respectively. Moreover, y_2 can be obtained by time shifting and scaling of y_1.

Introduce the following function that can be helpful for $\Lambda(\omega)$ computing. Let $L(c, \omega, \theta)$ be the function that denotes a linear plant output at the instant $t = \theta T$ (with T being the period) if a periodic square-wave pulse signal of amplitude c is applied to the plant: $L(c, \omega, \theta) = y(t)|_{t=2\pi\theta/\omega}$, where $\theta \in [-0.5; 0.5]$, $\omega \in [0, \infty]$. Positive values of θ correspond to the time following the switching instant, negative values—to the time preceding the switching instant. If we compare this formula with the Tsypkin's locus definition we would find that: $Im\Lambda(\omega) = L(c, \omega, 0)$. Analysis of the Fourier series of a linear plant response leads to the following expression for $L(\omega, \theta)$:

$$L(c,\omega,\theta) = \frac{4c}{\pi}\sum_{k=1}^{\infty}\frac{1}{2k-1} \times \{\sin[(2k-1)2\pi\theta]\cdot \mathrm{Re}\, W[(2k-1)\omega] + \cos[(2k-1)2\pi\theta]\cdot \mathrm{Im}\, W[(2k-1)\omega]\}. \quad (7)$$

With the formula of $L(c, \omega, \theta)$ available, we can obtain an expression for $Im\Lambda(\omega)$ of the equivalent plant as a sum of the plant responses to two square wave signals at the time of the first relay switch from "$-$" to "+")

$$\mathrm{Im}\Lambda(\omega) = L(c_1, \omega, 0) + L(c_2, \omega, \theta) \quad (8)$$

In formula (8), the value of time shift θ between the switches of the first and second relays is unknown. It can be found from the following equation: $\dot{y}(\theta) = 0$, which is the condition of the second relay switch. This condition can be expressed via the function $L(c, \omega, \theta)$ as follows (now we consider time $t = 0$ being the time of the second relay switch from "$-$" to "+"):'

$$L_1(c_1, \omega, -\theta) + L_1(c_2, \omega, 0) = 0 \quad (9)$$

In (9), L_1 is the function $L(c, \omega, \theta)$ for which the transfer function in formula (7) is $W_1(s) = sW(s)$ (transfer function from the control to dy/dt). Therefore, the methodology of analysis of the periodic motions in the system with the twisting algorithm is as follows. At each frequency from the analyzed range, (9) is solved for the time shift θ (in parts of the period) between the switches of the two relays, where function $L(c, \omega, \theta)$ is computed as per (7). After that the imaginary part of $\Lambda(\omega)$ is computed as per (8). With the imaginary part available, the frequency of the oscillations is found from (6). Local stability of the periodic solution can be analyzed as per [14]. Since we are not using the real part of Tsypkin's locus in the analysis, a simplified approach to its computing can be applied. It can be approximately computed as the first term in the series (5), which represents the first harmonic approximation and results in the following formula: $\mathrm{Re}\Lambda(\omega) \approx \mathrm{Re}[W(j\omega)/(1 + j\omega \cdot q_2(A_2) \cdot W(j\omega))]$, where q_2 is the DF of the second relay: $q_2 = 4c_2/(\pi A_2)$, $A_2 = 4c_1\omega|W(j\omega)|/\pi$.

A qualitative analysis of the periodic solution shows that in (8) the second addend represents the increment of $\Lambda(\omega)$ (at frequency ω) caused by the introduction of the second relay. Since the switching signal of the second relay is the derivative of the output, the phase shift between the two relay switches is close to $\pi/2$. The sign of θ is positive because the derivative signal leads with respect to the output, and both the output and its derivative are inverted. From (7), we can see that at a frequency close to the frequency of the self-excited oscillations the following holds: $L(c_2, \omega, \theta) < 0$. This means that

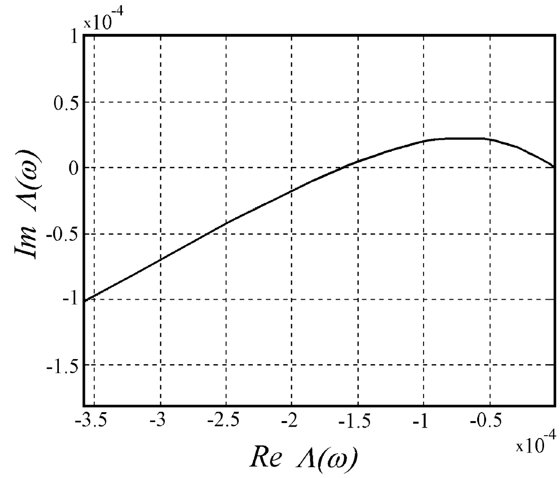

Fig. 3. Tsypkin's locus of the equivalent actuator-plant $\omega \in [50\ \mathrm{s}^{-1}; \infty)$.

the Tsypkin's locus of the system with the twisting algorithm is always located below the corresponding locus of the system with the asymptotic second-order SM relay controller, which totally confirms the conclusion of the DF analysis: *the frequency of the periodic solution for the twisting algorithm is always higher than the frequency of the asymptotic second-order SM relay controller*.

IV. Examples of Analysis and Comparison of Results

At first, consider an example of analysis of the system with a relay feedback control. This will serve as a benchmark for the comparison with other types of control. Let the plant be given by: $\dot{x}_1 = x_2$, $\dot{x}_2 = -x_1 - x_2 + u_a$ and the actuator by: $\dot{u}_a + u_a = u$. Carry out analysis of periodic motions in the systems with the asymptotic second-order SM relay control and with the twisting control algorithm. Compute the Tsypkin's locus of the actuator-plant and in accordance with (6) obtain the frequency of the oscillations: $\Omega = 9.357\ \mathrm{s}^{-1}$. Now, carry out analysis of periodic motions in the system with the twisting algorithm. Suppose the relay amplitudes are $c_1 = 0.8$ and $c_2 = 0.6$ (that provides the same amplitude of the fundamental frequency of the control signal as the unity amplitude). Compute and plot $\Lambda(\omega)$ of the equivalent actuator-plant, which for the range $[50\ \mathrm{s}^{-1}; \infty)$ is depicted in Fig. 3. The frequency of the periodic process found as a solution of (8) with the use of Tsypkin's approach is $\Omega = 77.70\ \mathrm{s}^{-1}$. The simulations of the system with the given actuator-plant and the relay algorithm as well as with the twisting algorithm provide a very good match to the exact analysis.

An analysis of a number of combinations of first/second-order actuators, first/second relative degree plants, and twisting/twisting-as-a-filter algorithms was done, and the results are presented in Table I. The review of the results shows that a good match between the DF analysis, the Tsypkin's analysis and the simulations takes place. A periodic motion occurs if the relative degree of the actuator-plant is higher than two. The frequency of the periodic solution for the twisting algorithm is always higher than the frequency for the second-order SM asymptotic relay control (for the same actuator-plant)—the fact that was predicted by both: the DF analysis and by the Tsypkin's locus analysis. Also, a comparison between the twisting-as-a-filter algorithm and the classical first order SM control is done (in both cases the plant is of first order). The frequency of chattering of the twisting as a filter algorithm is always lower than the frequency of the first order SM control—the fact, which can also be explained by the above analysis, if the plant is viewed as the original plant plus an integrator, with the twisting algorithm applied to that combined plant. The amplitudes of the oscillations

TABLE I
RESULTS OF COMPUTING AND SIMULATIONS

	Twisting controller	Twisting controller	Asymptotic second order SM relay controller	Asymptotic second order SM relay controller
Plant $W_p(s)$	$W_p(s)=\frac{1}{s^2+s+1}$	$W_p(s)=\frac{1}{s^2+s+1}$	$W_p(s)=\frac{1}{s^2+s+1}$	$W_p(s)=\frac{1}{s^2+s+1}$
Actuator $W_a(s)$	$W_a(s)=\frac{1}{0.01s+1}$	$W_a(s)=\frac{1}{0.0001s^2+0.01s+1}$	$W_a(s)=\frac{1}{0.01s+1}$	$W_a(s)=\frac{1}{0.0001s^2+0.01s+1}$
W(s)	$W=W_aW_p$	$W=W_aW_p$	$W=W_aW_p$	$W=W_aW_p$
Ω (DF analysis)	77.05	54.64	10.05	10.00
Ω (modified Tsypkin's analysis)	77.70	54.53	9.36	9.12
Ω (simulations)	77.68	54.53	9.36	9.13
Amplitudes of chattering of plant output	1.67e-4	4.83e-4	0.0146	0.0155

	Twisting as a filter	Twisting as a filter	First order SM	First order SM
Plant $W_p(s)$	$W_p(s)=\frac{s+1}{s^2+s+1}$	$W_p(s)=\frac{s+1}{s^2+s+1}$	$W_p(s)=\frac{s+1}{s^2+s+1}$	$W_p(s)=\frac{s+1}{s^2+s+1}$
Actuator $W_a(s)$	$W_a(s)=\frac{1}{0.01s+1}$	$W_a(s)=\frac{1}{0.0001s^2+0.01s+1}$	$W_a(s)=\frac{1}{0.01s+1}$	$W_a(s)=\frac{1}{0.0001s^2+0.01s+1}$
W(s)	$W={W_aW_p}/{s}$	$W={W_aW_p}/{s}$	$W=W_aW_p$	$W=W_aW_p$
Ω (DF analysis)	75.00	53.52	infinite	100.00
Ω (modified Tsypkin's analysis)	75.62	53.44	infinite	99.27
Ω (simulations)	75.51	53.41	converging to infinity	99.26
Amplitudes of chattering	2.53 e-6	9.48e-6	0	1.30e-4

in Table I were obtained only analytically. In all the examples, the obtained amplitude values reflect the relationship between the chattering frequency and the magnitude of the transfer function at this frequency.

V. CONCLUSION

A second-order SM algorithm known as twisting is analyzed with the use of the DF and Tsypkin's methods. It is proved that if the combined relative degree of the actuator and the plant is higher than two a periodic motion may occur in the system with the twisting algorithm. An approach to the analysis of the periodic motion is proposed that involved a modification of Tsypkin's approach to accommodate the second relay. The parameters of the periodic motion are obtained approximately—with the use of the DF method, and exactly—with the use of the modified Tsypkin's approach. The performed analysis as well as the theoretical proof shows that the frequency of the oscillations grows and the amplitude of chattering decreases due to the use of the twisting algorithm in comparison with the asymptotic second-order sliding-mode control algorithm. Also, the frequency of the oscillations of the twisting-as-a-filter algorithm is always lower than the frequency of the first order SM control.

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Stability Analysis for Linear Systems Under State Constraints

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***Abstract*—This note revisits the problem of stability analysis for linear systems under state constraints. New and less conservative sufficient conditions are identified under which such systems are globally asymptotically stable. Based on these sufficient conditions, iterative linear matrix inequality (LMI) algorithms are proposed for testing global asymptotic stability of the system. In addition, these iterative LMI algorithms can be adapted for the design of globally stabilizing state feedback gains.**

***Index Terms*—Nonlinear systems, stability, state constraints, state saturation.**

I. Introduction and Problem Statement

In this note, we will investigate stability analysis of two classes of linear systems under state constraints, which were recently studied in [4], [6]–[8], and [10]. The first class of systems are defined as follows:

$$\dot{x} = h(Ax) \tag{1}$$

where $x \in \mathbf{D}^n = \{x = (x_1, x_2, \ldots, x_n)^{\mathrm{T}} \in \mathbf{R}^n : -1 \le x_i \le 1, i \in [1, n]\}$, $A = [a_{ij}] \in \mathrm{R}^{n\times n}$, and

$$h(Ax) = \begin{bmatrix} h_1\left(\sum_{j=1}^n a_{1j}x_j\right) \\ h_2\left(\sum_{j=1}^n a_{2j}x_j\right) \\ \vdots \\ h_n\left(\sum_{j=1}^n a_{nj}x_j\right) \end{bmatrix}$$

with, for each $i \in [1, n]$

$$h_i\left(\sum_{j=1}^n a_{ij}x_j\right) = \begin{cases} 0, & \text{if } |x_i| = 1 \text{ and } \left(\sum_{j=1}^n a_{ij}x_j\right)x_i > 0 \\ \sum_{j=1}^n a_{ij}x_j, & \text{otherwise.} \end{cases}$$

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Such systems are defined on a closed hypercube as all state variables are constrained to the unit hypercube $\mathbf{D}^n$. For this reason, system (1) is sometimes referred to as a linear system subject to state saturation. Clearly, saturation occurs in the state x_i if $|x_i| = 1$ and $(\sum_{j=1}^n a_{ij}x_j)x_i > 0$.

The other class of systems are systems with partial state constraints and are described as

$$\begin{cases} \dot{x} = Ax + By \\ \dot{y} = h(Cx + Ey) \end{cases} \tag{2}$$

where $x \in \mathbf{R}^{n-m}$ with $n \ge m$, $y \in \{(y_1, y_2, \ldots, y_m)^{\mathrm{T}} : -1 \le y_i \le 1, i \in [1, m]\}$, A, B, C, and E are real matrices of appropriate dimensions, and

$$h(Cx + Ey) = \begin{bmatrix} h_1\left(\sum_{j=1}^{n-m} c_{1j}x_j + \sum_{k=1}^m e_{1k}y_k\right) \\ h_2\left(\sum_{j=1}^{n-m} c_{2j}x_j + \sum_{k=1}^m e_{2k}y_k\right) \\ \vdots \\ h_m\left(\sum_{j=1}^{n-m} c_{mj}x_j + \sum_{k=1}^m e_{mk}y_k\right) \end{bmatrix} \tag{3}$$

with, for each $i \in [1, m]$, (4), as shown at the bottom of the next age, holds.

We note that the class of (2) reduces to the class of (1) if $m = n$. These two classes of systems are encountered in a variety of applications, including signal processing, recurrent neural networks and control systems, and have been studied extensively (see, e.g., [3]–[6], [8], [12], and the references therein). In this note, we revisit the problem of stability analysis for these two classes of systems. In particular, we are interested in conditions under which such systems are globally asymptotically stable at the origin. Here, by global asymptotic stability of the origin we mean that the origin is locally asymptotically stable within $\mathbf{D}^n$ (or $\mathbf{R}^{n-m} \times \mathbf{D}^m$), rather than the usual $\mathbf{R}^n$, being the domain of attraction.

Global asymptotic stability of these systems has been studied in [4], [8], and [10]. For second order systems in the form of (1), necessary and sufficient conditions for global asymptotic stability were established in [4] and [10]. For higher order systems in the form of either (1) or (2), various sufficient conditions for the global asymptotic stability were identified. Under the sufficient condition of [8], any system trajectory starting from inside $\mathbf{D}^n$ will never reach the boundary of $\mathbf{D}^n$, i.e., the state never saturates. This saturation avoidance sufficient condition leads to a degree of conservatism. Using a Lyapunov function $V: \mathbf{D}^n \to \mathbf{R}$ that satisfies

$$\left[\frac{\partial V}{\partial x}(x)\right] h(Ax) \le \left[\frac{\partial V}{\partial x}(x)\right] Ax \tag{5}$$

[4] arrives at a sufficient condition that is less conservative than that of [8].

Motivated by the observation that the hypothesis (5) might be a source of conservatism, we will in this note re-examine global asymptotic stability of such systems by exploring the special property of the function h. The sufficient conditions we thus arrive at are given in terms of matrix inequalities, which are shown to be less conservative than those of [8] and [4]. Based on these new sufficient conditions, iterative LMI algorithms are proposed for testing global asymptotic stability. In addition to the stability analysis, the proposed sufficient conditions and the iterative LMI algorithms can be readily adapted for designing globally stabilizing feedback gains for the following systems:

$$\dot{x} = h(Ax + Bu) \quad u = Fx \tag{6}$$

where $x \in \mathbf{R}^n$ and $u \in \mathbf{R}^m$.