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Second-Order Sliding-Mode Observer for Mechanical Systems

Jorge Davila, Leonid Fridman, and Arie Levant

***Abstract*—The super-twisting second-order sliding-mode algorithm is modified in order to design a velocity observer for uncertain mechanical systems. The finite time convergence of the observer is proved. Thus, the observer can be designed independently of the controller. A discrete version of the observer is considered and the corresponding accuracy is estimated.**

***Index Terms*—Nonlinear observers, sliding modes.**

I. Introduction

The design of observers for the mechanical systems with Coulomb friction is important for the following reasons:

- linear observers do not achieve adequate performance for such systems;
- model-based observers are usually restricted to the cases when the model is exactly known;
- high-gain differentiators [2] are not exact with any fixed finite gain and feature the peaking effect with high gains: The maximal output value during the transient grows infinitely as the gains tend to infinity (see, for example, [3], [5], [12], [15], and [16]).

The sliding mode observers are widely used due to the finite-time convergence, robustness with respect to uncertainties and the possibility of uncertainty estimation (see, for example, the bibliography in the recent tutorials [3], [5], and [12]). A new generation of observers based on the second-order sliding-mode algorithms has been recently developed. In particular, asymptotic observers [13] and the asymptotic observer for systems with Coulomb friction [1], [11] were designed based on the second-order sliding-mode. These observers require the proof of a separation principle theorem due to the asymptotic convergence of the estimated values to the real ones.

A robust exact differentiator [9] featuring finite-time convergence was designed as an application of the super-twisting algorithm [8]. Its implementation does not need the separation principle to be proved. These differentiators were, for example, successfully applied in [14], [4], and [10]. A new differentiator [7] was developed, based on it. Straightforward application of such a differentiator does not benefit from the knowledge of a mathematical model of the process. If such a model is known, or the system parameters and uncertainties can be estimated (which is common for the case of mechanical systems with Coulomb friction), it is reasonable to design a system-specific observer.

An observer is proposed in this paper, which reconstructs the velocity from the position measurements, using the modification of the second-order sliding-mode super-twisting algorithm [8] with finite-time convergence. The separation principle theorem is trivial in this case, and the observer can be designed separately from the controller. Only partial knowledge of the system model is required. The discrete version of the of the proposed observer is considered, and the corresponding accuracy of the proposed observer is estimated.

II. Problem Statement

The general model of second-order mechanical systems has the form

$$M(\mathbf{q})\ddot{\mathbf{q}} + C(\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{q}} + P(\dot{\mathbf{q}}) + G(\mathbf{q}) + \Delta(t,\mathbf{q},\dot{\mathbf{q}}) = \tau \tag{1}$$

where $\mathbf{q} \in R^{\mathbf{n}}$ is a vector of generalized coordinates, $M(\mathbf{q})$ is the inertia matrix, $C(\mathbf{q},\dot{\mathbf{q}})$ is the matrix of Coriolis and centrifugal forces, $P(\dot{\mathbf{q}})$ is the Coulomb friction, which possibly contains relay terms depending on $\dot{\mathbf{q}}$, $G(\mathbf{q})$ is the term of gravitational forces, $\Delta(t,\mathbf{q},\dot{\mathbf{q}})$ is an uncertainty term and τ is the torque produced by the actuators. The control input τ is assumed to be given by some known feedback function. Note that $M(\mathbf{q})$ is invertible, since $M(\mathbf{q}) = M^T(\mathbf{q})$ is strictly positive definite. Also, other terms are supposed to be uncertain, but the corresponding nominal functions $M_n(\mathbf{q}), C_n(\mathbf{q},\dot{\mathbf{q}}), P_n(\dot{\mathbf{q}}), G_n(\mathbf{q})$ are assumed known.

Introducing the variables $x_1 = \mathbf{q}, x_2 = \dot{\mathbf{q}}, u = \tau$, the model (1) can be rewritten in the state–space form

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= f(t,x_1,x_2,u) + \xi(t,x_1,x_2,u) \quad u = U(t,x_1,x_2)\\ y &= x_1\end{aligned} \tag{2}$$

where the nominal part of the system dynamics is represented by the function

$$f(t,x_1,x_2,u) = -M_n^{-1}(x_1) \times [C_n(x_1,x_2)x_2 + P(x_2) + G_n(x_1) - u]$$

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containing the known nominal functions M_n, C_n, G_n, P, while the uncertainties are concentrated in the term $\xi(t, x_1, x_2, u)$. The solutions to the system (2) are understood in Filippov's sense [6]. It is assumed that the function $f(t, x_1, x_2, U(t, x_1, x_2))$ and the uncertainty $\xi(t, x_1, x_2, U(t, x_1, x_2))$ are Lebesgue-measurable and uniformly bounded in any compact region of the state–space x_1, x_2.

The task is to design a finite-time convergent observer of the velocity $\dot{\mathbf{q}}$ for the original system (1), when only the position $\mathbf{q}$ and the nominal model are available. In other words, the state x_2 of the system (2) is to be observed, while only the state x_1 is available. Only the scalar case $x_1, x_2 \in R$ is considered for the sake of simplicity. In the vector case, the observers are constructed in parallel for each position variable x_{1j} in exactly the same way.

III. Observer Design

The proposed super-twisting observer has the form

$$\begin{aligned}\dot{\hat{x}}_1 &= \hat{x}_2 + z_1\\ \dot{\hat{x}}_2 &= f(t, x_1, \hat{x}_2, u) + z_2\end{aligned} \tag{3}$$

where $\hat{x}_1$ and $\hat{x}_2$ are the state estimations, and the correction variables z_1 and z_2 are output injections of the form

$$\begin{aligned}z_1 &= \lambda |x_1 - \hat{x}_1|^{1/2}\,\mathrm{sign}(x_1 - \hat{x}_1)\\ z_2 &= \alpha\,\mathrm{sign}(x_1 - \hat{x}_1).\end{aligned} \tag{4}$$

It is taken for the definiteness that at the initial moment $\hat{x}_1 = x_1$ and $\hat{x}_2 = 0$. Taking $\tilde{x}_1 = x_1 - \hat{x}_1$ and $\tilde{x}_2 = x_2 - \hat{x}_2$ we obtain the error equations

$$\begin{aligned}\dot{\tilde{x}}_1 &= \tilde{x}_2 - \lambda|\tilde{x}_1|^{1/2}\,\mathrm{sign}(\tilde{x}_1)\\ \dot{\tilde{x}}_2 &= F(t, x_1, x_2, \hat{x}_2) - \alpha\,\mathrm{sign}(\tilde{x}_1)\end{aligned} \tag{5}$$

where $F(t, x_1, x_2, \hat{x}_2) = f(t, x_1, x_2, U(t, x_1, x_2)) - f(t, x_1, \hat{x}_2, U(t, x_1, x_2)) + \xi(t, x_1, x_2, U(t, x_1, x_2))$. Suppose that the system states can be assumed bounded, then the existence is ensured of a constant f^+, such that the inequality

$$|F(t, x_1, x_2, \hat{x}_2)| < f^+ \tag{6}$$

holds for any possible t, x_1, x_2 and $|\hat{x}_2| \leq 2\sup|x_2|$.

Remark 1: When the accelerations in the mechanical system are bounded, the constant f^+ can be found as the double maximal possible acceleration of the system. Moreover, the estimation constant f^+ does not depend on the nominal elasticity and control terms. Such assumption of the state boundedness is true too, if, for example, system (2) is BIBS stable, and the control input $u = U(t, x_1, x_2)$ is bounded.

Let α and λ satisfy the inequalities

$$\begin{aligned}&\alpha > f^+\\ &\lambda > \sqrt{\frac{2}{\alpha - f^+}}\frac{(\alpha + f^+)(1+p)}{(1-p)}\end{aligned} \tag{7}$$

where p is some chosen constant, $0 < p < 1$.

Theorem 1: Suppose that the parameters of the observer (3), (4) are selected according to (7), and condition (6) holds for system (2). Then, the variables of the observer (3), (4) converge in finite time to the states of system (2), i.e., $(\hat{x}_1, \hat{x}_2) \to (x_1, x_2)$.

Proof: In order to prove the convergence of the state estimates to the real states, it is necessary to prove first the convergence of $\tilde{x}_1$ and $\dot{\tilde{x}}_1$ to zero. Assume at first that (6) holds all the time (it will be proved further). As follows from (5), (6), estimation errors $\tilde{x}_1$ and $\tilde{x}_2$ satisfy the differential inclusion

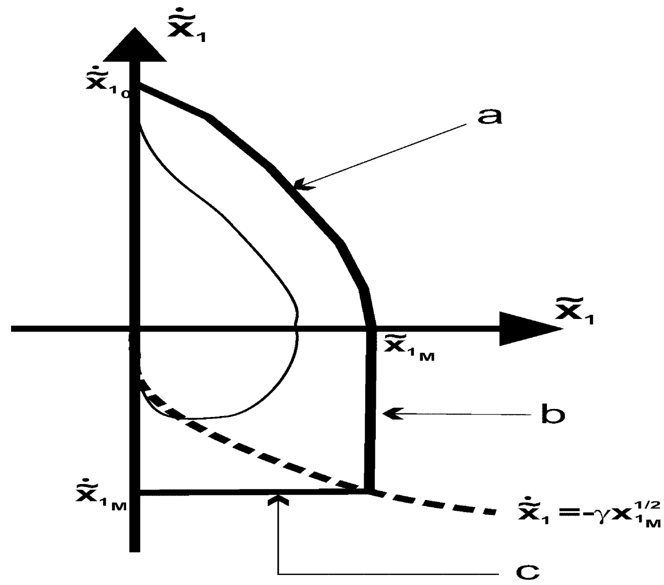

Fig. 1. Majorant curve for the finite-time convergent observer.

$$\begin{aligned}\dot{\tilde{x}}_1 &= \tilde{x}_2 - \lambda|\tilde{x}_1|^{1/2}\,\mathrm{sign}(\tilde{x}_1)\\ \dot{\tilde{x}}_2 &\in [-f^+, +f^+] - \alpha\,\mathrm{sign}(\tilde{x}_1).\end{aligned} \tag{8}$$

Here and further, all differential inclusions are understood in the Filippov sense, which means that the right hand side is enlarged in some points in order to satisfy the upper semicontinuity property [6], in particular the second formula of (8) turns into $\dot{\tilde{x}}_2 \in [-\alpha - f^+, \alpha + f^+]$ with $\tilde{x}_1 = 0$. Note that the solutions of (8) exist for any initial condition and are infinitely extendible in time [6]. Computing the derivative of $\dot{\tilde{x}}_1$ with $\tilde{x}_1 \neq 0$ obtain

$$\ddot{\tilde{x}}_1 \in [-f^+, f^+] - \left(\frac{1}{2}\lambda\frac{\dot{\tilde{x}}_1}{|\tilde{x}_1|^{1/2}} + \alpha\,\mathrm{sign}\tilde{x}_1\right). \tag{9}$$

The trivial identity $d|x|/dt = \dot{x}\,\mathrm{sign}\,x$ is used here. Inclusion (9) does not "remember" anything on the real system, but can be used to describe the majorant curve drawn in Fig. 1. Note that at the initial moment $\tilde{x}_1 = 0$ and $\tilde{x}_2 = x_2 - 0 = x_2$. The trajectory enters the half-plane $\tilde{x}_1 > 0$ with a positive initial value of x_2 and the half-plane $\tilde{x}_1 < 0$, otherwise.

Let $\tilde{x}_1 > 0$ then with $\dot{\tilde{x}}_1 > 0$ the trajectory is confined between the axis $\tilde{x}_1 = 0, \dot{\tilde{x}}_1 = 0$ and the trajectory of the equation $\ddot{\tilde{x}}_1 = -(\alpha - f^+)$; see Fig. 1 line (a). Let $\tilde{x}_{1_M}$ be the intersection of this curve with the axis $\dot{\tilde{x}}_1 = 0$. Obviously, $2(\alpha - f^+)\tilde{x}_{1_M} = \dot{\tilde{x}}_{1_0}^2$, where $\dot{\tilde{x}}_{1_0} > 0$ is the value of $\dot{\tilde{x}}_1$ with $\tilde{x}_1 = 0$. It is easy to see that for $\tilde{x}_1 > 0, \dot{\tilde{x}}_1 > 0$

$$\ddot{\tilde{x}}_1 \leq f^+ - \alpha\,\mathrm{sign}\,\tilde{x}_1 - \frac{1}{2}\lambda\frac{\dot{\tilde{x}}_1}{|\tilde{x}_1|^{1/2}} < 0.$$

Thus, the trajectory approaches the axis $\dot{\tilde{x}}_1 = 0$. The majorant curve for $\tilde{x}_1 > 0, \dot{\tilde{x}}_1 \geq 0$ is described by the equation (see Fig. 1)

$$\dot{\tilde{x}}_1^2 = 2(\alpha - f^+)(\tilde{x}_{1_M} - \tilde{x}_1) \quad \text{with} \quad \dot{\tilde{x}}_1 > 0.$$

The majorant curve for $\tilde{x}_1 > 0, \dot{\tilde{x}}_1 \leq 0$ consists of two parts. In the first part the point instantly drops down from $(\tilde{x}_{1_M}, 0)$ to the point $(\tilde{x}_{1_M}, -(2/\lambda)(f^+ + \alpha)\tilde{x}_{1_M}^{1/2})$, where, in the "worst case," the right-hand side of inclusion (9) is equal to zero [see Fig. 1, line (b)]. The second part of the majorant curve is the horizontal segment between the points $(\tilde{x}_{1M}, -(2)/(\lambda)(f^+ + \alpha)\tilde{x}_{1M}^{1/2}) = (\tilde{x}_{1M}, \dot{\tilde{x}}_{1M})$ and $(0, \dot{\tilde{x}}_{1M})$ [see Fig. 1, line (c)].

Condition (7) implies that

$$\frac{\left|\dot{\tilde{x}}_{1M}\right|}{\left|\dot{\tilde{x}}_{10}\right|} < \frac{1-p}{1+p} < 1.$$

Let us denote as $\dot{\tilde{x}}_{1_0}, \dot{\tilde{x}}_{1M} = \dot{\tilde{x}}_{1_1}, \dot{\tilde{x}}_{1_2}, \ldots, \dot{\tilde{x}}_{1_i}, \ldots$ the consequent crossing points of the system (5) trajectory starting at $(0, \dot{\tilde{x}}_{1_0})$ with the $\tilde{x}_1 = 0$ axis. Last inequality ensures the convergence of the state $(0, \dot{\tilde{x}}_{1_i})$ to $\tilde{x}_1 = \dot{\tilde{x}}_1 = 0$ and, moreover, the convergence of $\Sigma_0^\infty |\dot{\tilde{x}}_{1_i}|$.

Consider the dynamics of $\tilde{x}_2$ to prove the finite-time convergence. Obviously, $\tilde{x}_2 = \dot{\tilde{x}}_1$ at the moments when $\tilde{x}_1 = 0$ and, taking into account that

$$\dot{\tilde{x}}_2 = F(x_1, x_2, \hat{x}_2, u) - \alpha \text{ sign } \tilde{x}_1$$

obtain that

$$0 < \alpha - f^+ \leq |\dot{\tilde{x}}_2| \leq \alpha + f^+$$

holds in a small vicinity of the origin. Thus

$$\left|\dot{\tilde{x}}_{1_i}\right| \geq (\alpha - f^+)t_i$$

where t_i are the time intervals between the successive intersection of the trajectory with the axis $\tilde{x}_1 = 0$. Hence

$$t_i \leq \frac{\left|\dot{\tilde{x}}_{1_i}\right|}{(\alpha - f^+)}$$

and the total convergence time is estimated by

$$T \leq \sum \frac{\left|\dot{\tilde{x}}_{1_i}\right|}{(\alpha - f^+)}.$$

Therefore, T is finite and the estimated states converge to the real states in finite time.

The previous proof was based on inequality (6). As follows from the aforementioned consideration, sufficiently large f^+ provides for $|\tilde{x}_2| \leq |\tilde{x}_{2,0}| = |x_2(t_0)|$, where t_0 is the initial time. It implies that $|\hat{x}_2| \leq |\tilde{x}_{2,0}| + |x_2| \leq 2 \sup |x_2|$. Hence, the suggested choice of f^+ is valid. ∎

Remark 2: Finite-time convergence of the observer allows to design the observer and the control law separately, i.e., the separation principle is satisfied. The only requirement for its implementation is the boundedness of the function $F(t, x_1, x_2, \hat{x}_2, u)$ in the operational domain. If the applied controller is known to stabilize the process, one of the admissible ways is to choose the observer dynamics fast enough to provide for the exact evaluation of the velocity before leaving some preliminarily chosen area, where the stabilization is assured. It is easily performed by simulation (see the following example).

Remark 3: The standard 2-sliding-mode-based differentiator [9] can be also implemented here to estimate the velocity. At the same time, the proposed observer requires smaller gains and is more accurate, since the elasticity term $M^{-1}(\mathbf{q})G(\mathbf{q})$ does not influence the gain choice.

Remark 4: Another way to choose α and λ is to take $\alpha = a_1 f^+, \lambda = a_2(f^+)^{1/2}$ with some predetermined proper a_1, a_2. In particular, $a_1 = 1.1, a_2 = 1.5$ is a valid choice [9].

The previous analysis is valid for the ideal version of the observer. Let f, x, z_1, z_2 be measured at discrete times with the time interval δ, and let t_i, t_{i+1} be successive measurement times. Consider a discrete modification of the observer (the Euler scheme)

$$\begin{aligned}\hat{x}_1(t_{i+1}) &= \hat{x}_1(t_i) + (\hat{x}_2(t_i) + \lambda|x_1(t_i) - \hat{x}_1(t_i)|^{1/2} \\ &\quad \times \text{sign } (x_1(t_i) - \hat{x}_1(t_i)))\delta \\ \hat{x}_2(t_{i+1}) &= \hat{x}_2(t_i) + (f(t_i, x_1(t_i), \hat{x}_2(t_i), u(t_i)) \\ &\quad + \alpha \text{ sign } (x_1(t_i) - \hat{x}_1(t_i)))\delta \end{aligned} \tag{10}$$

where $\hat{x}_1(t_i), \hat{x}_2(t_i)$ are the estimated variables.

Theorem 2: Suppose that the function f is uniformly bounded and condition (6) holds. Then, the observation algorithm (10) with parameters (7) ensures the convergence of the estimation errors to the domain $|\tilde{x}_1| \leq \gamma_1\delta^2, |\tilde{x}_2| \leq \gamma_2\delta$ where γ_1, γ_2 are some constants, depending on the observer parameters.

Proof: Let $t \in [t_i, t_{i+1})$, where t_i, t_{i+1} are successive measurement times, $t_{i+1} - t_i = \delta$, and t is the current time. The observer (10) may be rewritten in the continuous time as follows:

$$\begin{aligned}\dot{\hat{x}}_1 &= \hat{x}_2(t_i) + \lambda|x_1(t_i) - \hat{x}_1(t_i)|^{1/2} \text{ sign } (x_1(t_i) - \hat{x}_1(t_i)) \\ \dot{\hat{x}}_2 &= f(t_i, x_1(t_i), \hat{x}_2(t_i), u(t_i)) + \alpha \text{ sign } (x_1(t_i) - \hat{x}_1(t_i)).\end{aligned} \tag{11}$$

Hence, the errors satisfy the differential inclusion

$$\begin{aligned}\dot{\tilde{x}}_1 &= \tilde{x}_2(t_i) + x_2 - x_2(t_i) - \lambda|\tilde{x}_1(t_i)|^{1/2} \text{ sign } (\tilde{x}_1(t_i)) \\ \dot{\tilde{x}}_2 &\in [-f^+, f^+] - \alpha \text{ sign } (\tilde{x}_1(t_i)).\end{aligned}$$

Let $|f + \xi| \leq f_1^+$, then

$$\begin{aligned}\dot{\tilde{x}}_1 &\in \tilde{x}_2(t_i) + [-f_1^+, f_1^+]\delta - \lambda|\tilde{x}_1(t_i)|^{1/2} \text{ sign } (\tilde{x}_1(t_i)) \\ \dot{\tilde{x}}_2 &\in [-f^+, f^+] - \alpha \text{ sign } (\tilde{x}_1(t_i)).\end{aligned} \tag{12}$$

It may be considered as (8) with measurement errors. Indeed, let D be some compact region around the origin O of the space $\tilde{x}_1, \tilde{x}_2$. As follows from the proof of Theorem 1, all trajectories of (8) starting in D converge in some finite time T to the origin O. During this time they do not leave some larger homogeneous disk $B_{R_0} = \{(\tilde{x}_1, \tilde{x}_2) : |\tilde{x}_1|^{1/2} + |\tilde{x}_2| \leq R_0\}$. Let $M(R) = \max\{\tilde{x}_2 - \lambda|\tilde{x}_1|^{1/2} \text{ sign } (\tilde{x}_1) : (\tilde{x}_1, \tilde{x}_2) \in B_R\}$. Due to the homogeneity property $M(R) = mR$ holds, where the constant $m > 0$ can be easily calculated. Thus, obviously

$$|\tilde{x}_1(t) - \tilde{x}_1(t_i)| \leq mR_0\delta \quad |\tilde{x}_2(t) - \tilde{x}_2(t_i)| \leq (f^+ + \alpha)\delta$$

in B_{R_0} and, denoting $f_2^+ = f^+ + f_1^+ + \alpha$, obtain that the trajectories of (12) satisfy the inclusion

$$\begin{aligned}\dot{\tilde{x}}_1 &\in \tilde{x}_2 + [-f_2^+, f_2^+]\delta - \lambda|\tilde{x}_1 + [-2m, 2m]R_0\delta|^{1/2} \\ &\quad \times \text{sign } (\tilde{x}_1 + [-2m, 2m]R_0\delta) \\ \dot{\tilde{x}}_2 &\in [-f^+, f^+] - \alpha \text{ sign } (\tilde{x}_1 + [-2m, 2m]R_0\delta)\end{aligned} \tag{13}$$

while $(\tilde{x}_1, \tilde{x}_2) \in B_{2R_0}$. With δ being zero, the dynamics (13) coincides with (8), whose trajectories converge in finite time to the origin. Due to the continuous dependence of the Filippov solutions on the graph of the differential inclusion, with sufficiently small δ the trajectories of (13) starting in D terminate in the time T in some small compact vicinity $\tilde{D} \subset D$ of the origin without leaving B_{2R_0} on the way. Let Ω be the compact set [6] of all points belonging to the trajectory segments starting in $\tilde{D}$ and corresponding to the closed time interval $T, \tilde{D} \subset \Omega$. With δ small enough $\tilde{D} \subset \Omega \subset D$, since the origin O is invariant for (13).

Obviously, Ω is an invariant set attracting the trajectories of (12) starting in D. Check now that it is a globally attracting set. Define a homogeneous parameter-time-coordinate transformation

$$t \longmapsto \eta t \quad (\tilde{x}_1, \tilde{x}_2) \longmapsto (\eta^2 \tilde{x}_1, \eta \tilde{x}_2)$$
$$(R_0, \delta) \longmapsto (\eta R_0, \eta\delta) \tag{14}$$

and let $G_\eta(\tilde{x}_1, \tilde{x}_2) = (\eta^2 \tilde{x}_1, \eta \tilde{x}_2)$. It is easily seen that (14) preserves (13), i.e., the trajectories are preserved. Choose such $\eta > 1$ that $G_\eta \Omega \subset D$, then the trajectories of the inclusion

$$\begin{aligned}\dot{\tilde{x}}_1 &\in \tilde{x}_2 + [-f_2^+, f_2^+]\eta\delta - \lambda|\tilde{x}_1 + [-2m, 2m]R_0\eta^2\delta|^{1/2} \\ &\quad \times \text{sign}\,(\tilde{x}_1 + [-2m, 2m]R_0\eta^2\delta) \\ \dot{\tilde{x}}_2 &\in [-f^+, f^+] - \alpha\,\text{sign}\,(\tilde{x}_1 + [-2m, 2m]R_0\eta^2\delta)\end{aligned} \tag{15}$$

starting in $G_\eta D$ terminate following time ηT in $G_\eta \Omega \subset D$ without leaving $G_\eta B_{2R_0} = B_{2\eta R_0}$ on the way. Comparing (13) and (15) obtain that (15) describes the solutions of (12) in $B_{2\eta R_0}$, but with redundantly enlarged "noise level" due to the replacement of δ by $\eta\delta > \delta$. Hence, the solutions of (12) satisfy (15) in $B_{2\eta R_0}$. Therefore, the trajectories of (12) starting in $G_\eta D$ terminate following time ηT in $G_\eta \Omega \subset D$ and proceed into Ω in time T. Representing the whole plane $\tilde{x}_1, \tilde{x}_2$ as $\mathbf{R}^2 = \cup G_\eta^k D$ obtain the global finite-time convergence to the set Ω.

It is easy to see that (12) is invariant with respect to the transformation $t \longmapsto \eta t, (\tilde{x}_1, \tilde{x}_2) \longmapsto (\eta^2 \tilde{x}_1, \eta \tilde{x}_2), \delta \longmapsto \eta\delta$. Let Ω satisfy the inequalities $|\tilde{x}_1| \le a_1, |\tilde{x}_2| \le a_2$ with some discretization interval δ_0. Applying the transformation with $\eta = \delta/\delta_0$ obtain that with arbitrary $\delta > 0$ the invariant set satisfies the inequalities $|\tilde{x}_1| \le \gamma_1\delta^2, |\tilde{x}_2| \le \gamma_2\delta$ with $\gamma_1 = a_1/\delta_0^2, \gamma_2 = a_2/\delta_0$. ■

IV. Example

Consider a pendulum system with Coulomb friction and external perturbation given by the equation

$$\ddot{\theta} = \frac{1}{J}\tau - \frac{g}{L}\sin\theta - \frac{V_s}{J}\dot{\theta} - \frac{P_s}{J}\,\text{sign}(\dot{\theta}) + v \tag{16}$$

where the values $M = 1.1, g = 9.815, L = 0.9, J = ML^2 = 0.891, V_S = 0.18, P_s = 0.45$ were taken and v is an uncertain external perturbation, $|v| \le 1$. It was taken $v = 0.5\sin 2t + 0.5\cos 5t$ in simulation. Let it be driven by the twisting controller

$$\tau = -30\,\text{sign}(\theta - \theta_d) - 15\,\text{sign}(\dot{\theta} - \dot{\theta}_d) \tag{17}$$

where $\theta_d = \sin t$ and $\dot{\theta}_d = \cos t$ are the reference signals. The system can be rewritten as

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{1}{J}\tau - \frac{g}{L}\sin x_1 - \frac{V_s}{J}x_2 - \frac{P_s}{J}\,\text{sign}(x_2) + v.\end{aligned}$$

Thus, the proposed velocity observer (see Remark 3) has the form

$$\begin{aligned}\dot{\hat{x}}_1 &= \hat{x}_2 + 1.5(f^+)^{1/2}|\tilde{x}_1|^{1/2}\,\text{sign}(x_1 - \hat{x}_1) \\ \dot{\hat{x}}_2 &= \frac{1}{J_n}\tau - \frac{g}{L_n}\sin x_1 - \frac{V_{sn}}{J_n}\hat{x}_2 + 1.1f^+\,\text{sign}(x_1 - \hat{x}_1)\end{aligned}$$

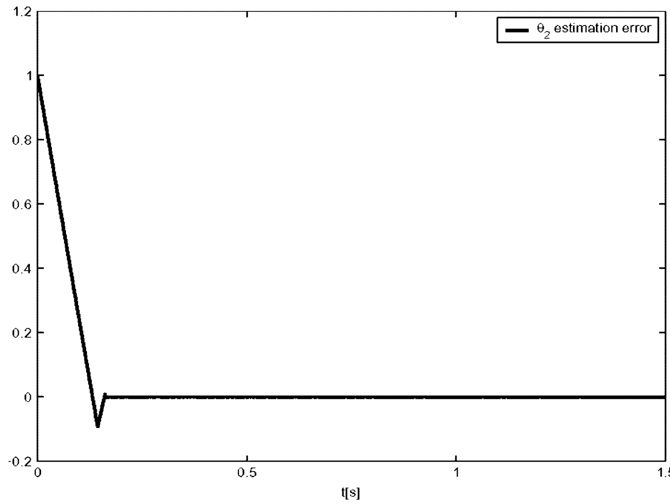

Fig. 2. Estimation error for x_2.

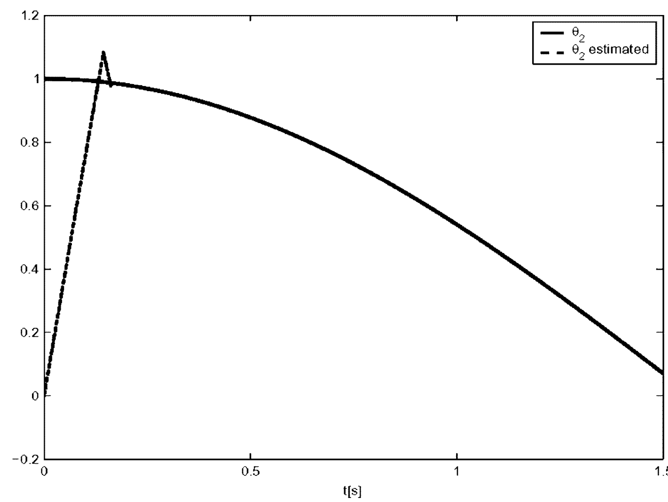

Fig. 3. Real and estimated velocity.

where $M_n = 1, L_n = 1, J_n = M_n L_n^2 = 1, V_{sn} = 0.2, P_{sn} = 0.5$ are the "known" nominal values of the parameters, and f^+ is to be assigned. Assume also that it is known that the real parameters differ from the known values by not more than 10%. The initial values $\theta = x_1 = \hat{x}_1 = 0$ and $\dot{\theta} = x_2 = 1, \hat{x}_2 = 0$ were taken at $t = 0$. Identifying 0 and 2π obtain that θ belongs to a compact set (a ring). Thus, obviously, dynamic system (16) is BIBS stable. Easy calculation shows that the given controller provides for $|\tau| \le 45$, and the inequality $|\dot{\theta}| \le 70$ is ensured, when the nominal values of parameters and their maximal possible deviations are taken into account. Taking $|x_2| \le 70, |\hat{x}_2| \le 140$ obtain that $|F| = |(1/J)\tau - (g/L)\sin x_1 - (V_s/J)x_2 - (P_s/J)\,\text{sign}\,(x_2) + v - (1/J_n)\tau + (g/L_n)\sin x_1 + (V_{sn}/J_n)\hat{x}_2| < 60 = f^+$. Therefore, the observer parameters $\alpha = 66$ and $\lambda = 11.7$ were chosen. Simulation adjustment (see Remark 1) shows that $f^+ = 6$ and the respective values $\alpha = 6.6$ and $\lambda = 4$ are sufficient. Note that the terms $(MgL/J)\sin x_1$ and $(1/J)\tau$ would be fully taken into account for the choice of the differentiator parameters [9] causing much larger coefficients to be used. The performance of the observer with the sampling interval $\delta = 0.000\,01$ is shown in Fig. 2. The finite-time convergence of the estimated velocity to the real one is demonstrated in Fig. 3, and

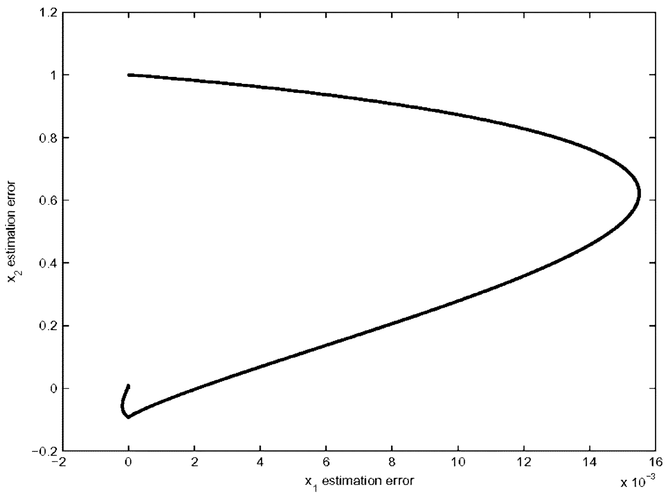

Fig. 4. Graph of $\tilde{x}_1$ versus $\tilde{x}_2$.

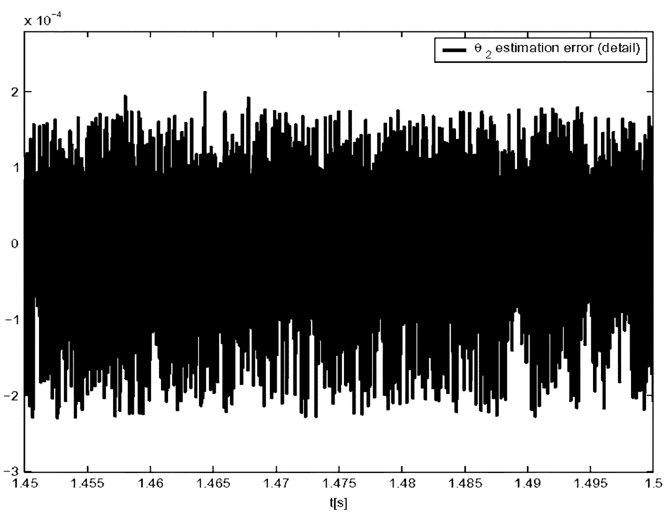

Fig. 5. Error of x_2 estimation (detail) with the sampling interval $\delta = 0.000\,01$.

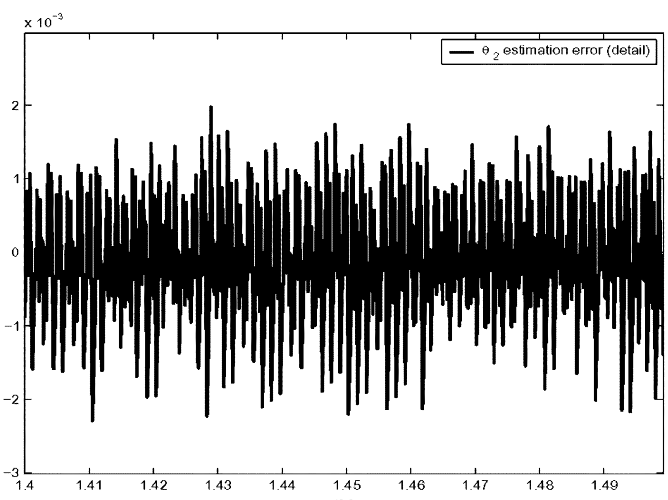

Fig. 6. Error of x_2 estimation (detail) with the sampling interval $\delta = 0.0001$.

Fig. 4 shows the convergence in the plane $\tilde{x}_1$ vs $\tilde{x}_2$. A detail of the estimation error graph is shown in Fig. 5. Fig. 6 demonstrates that the tenfold increase of the sampling time interval up to $\delta = 0.0001$ causes the proportional increase of the estimation error. This corresponds to Theorem 2.

V. CONCLUSION

The super-twisting second-order sliding-mode algorithm was modified in order to design a velocity observer for mechanical systems. The finite-time convergence of the observer is proved. Consequently, the separation principle is automatically satisfied, i.e., a controller and the observer can be separately designed. The gains of the proposed observer can be chosen ignoring the elasticity terms.

For the discrete realization of the observer the corresponding accuracy is estimated.

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