Non-linear sliding surface: towards high performance robust control

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Abstract: The study proposes a method to design a non-linear sliding surface to achieve better transient response for a class of single-input and single-output (SISO) non-linear uncertain system represented in a Brunowsky canonical form. The proposed surface can also be used for linear uncertain systems with matched perturbations. The proposed surface increases the damping ratio of the closed-loop system from its initial low value; as the output approaches the setpoint from its initial value. Initially, the system is lightly damped resulting in a quick response and as the output approaches the setpoint, the system is overdamped to avoid overshoot. The existence of sliding mode is proved and a new control law is proposed to enforce sliding motion. The scheme is able to achieve low overshoot and short settling time simultaneously which is not possible with a linear sliding surface. To ease the synthesis of the non-linear surface, linear matrix inequalities-based algorithm is proposed. Effectiveness of the proposed scheme is illustrated by the simulation results.

1 Introduction

Many practical systems call for an improvement in transient performance along with the steady-state accuracy. For example, many electro-mechanical, robotics and power converter systems require a quick response without any overshoot. It is a well-understood fact that a low overshoot can be achieved at the cost of a high settling time. However, a short settling time is also necessary for a quick response. Thus, most of the design schemes make a trade-off between these two transient performance indices and the damping ratio is chosen as a fixed number. Notable exceptions exist, of course, in [1], the authors proposed a seminal idea of composite non-linear feedback (CNF) for a class of second-order systems subject to actuator saturation. Composite non-linear control uses a variable-damping ratio to achieve high performance. Initially, it keeps damping ratio to a low value to ensure quick response and as the output approaches the setpoint the damping ratio is increased and thus overshoot is avoided. Subsequently, CNF controller was extended for general higher-order single-input and single-output (SISO) and multiple-input and multiple-output (MIMO) systems in [2–4] for the state-feedback and output-feedback cases.

The second important requirement is robustness towards uncertainty, because most of the physical systems are represented by approximate mathematical models to ease analysis. However, model mismatches and discrepancies are inevitable between the actual plant and the mathematical model used for the controller design. These mismatches along with external unknown disturbances affect the performance of the system. To tackle the effect of parameter variations and external disturbances, many robust control techniques have been developed. One of the robust control techniques is variable structure control (VSC); recently many successful practical applications of VSC have established the importance of sliding mode theory which has mainly been developed in the last three decades. This fact is also witnessed by many special issues of learned journals focusing on SMC [5, 6]. To relax the need for measuring the entire state vector, an output feedback-based sliding mode control system is proposed in [7–9], which widens the scope of SMC. In SMC, the sliding surface decides closed-loop dynamics, therefore it should be designed such that it addresses all requirements. As mentioned earlier, high performance in an uncertain environment is one of the key requirements. To enhance the performance of the system with SMC algorithms, a time-varying switching surface is proposed by many researchers in [10–14]. In [15], a strategy based on fuzzy logic is devised to change the parameters of switching surfaces of higher-order systems. Some researchers proposed non-linear surfaces to improve the performance [16].

The variable-damping ratio improves the system performance significantly. This paper proposes a method to
design a non-linear sliding surface for a class of non-linear uncertain system. A non-linear sliding surface is designed by using the principle of a variable-damping concept. Using a non-linear sliding surface, the damping ratio of a system can be changed from its initial low value to a final high value. The initial low value of damping ratio results in a quick response and the later high damping avoids overshoot. Thus, the proposed surface ascertains the reduction in settling time without any overshoot. Furthermore, the system’s damping ratio changes continuously as per the chosen function. The proposed approach inherits the robustness of SMC and delivers high performance owing to change of damping ratio by means of the non-linear sliding surface. During sliding mode, because of the order reduction, the system response is unaffected by m poles. For the systems of order higher than two, the damping ratio is specified by considering the contribution of dominant poles. However, non-dominant poles always affect the system’s response to some extent depending on their relative locations with respect to the dominant poles. Because of the order-reduction property of SMC, m non-dominant poles will not contribute to the system response and thus, the performance specifications can be met more closely. The proposed non-linear sliding surface achieves high performance and robustness unlike a sliding surface designed by assigning eigenvalues or by minimising a quadratic index that normally lead to a linear-sliding surface. To ease the synthesis of the proposed non-linear surface a linear matrix inequality (LMI)-based algorithm is also proposed. Some of our results based on non-linear surfaces are documented in [17–20].

1.1 Main contribution

In this work, we propose the following.

- A non-linear sliding surface for a class of non-linear uncertain SISO system to achieve low overshoot and short settling time simultaneously.
- Existence of sliding mode around the proposed non-linear sliding surface and a new control law.
- An LMI-based technique to ease the synthesis of the proposed non-linear surface.

The brief outline of the paper is as follows. Section 2 contains the structure of non-linear sliding surface and its stability. Section 3 proposes a new SMC algorithm and its existence. Example and simulation results are presented in Section 4 followed by the conclusion in Section 5.

2 Non-linear sliding surface

This section discusses the design of a non-linear sliding surface for a class of non-linear uncertain SISO system. Consider the system of order \( n \) described by the following equation

\[
\dot{x}(t) = f(\bar{x}, t) + g(\bar{x}, t)u(t)
\]

\[
y(t) = h(\bar{x})
\]

Here \( f(\bar{x}, t), h(\bar{x}) \) and \( g(\bar{x}, t) \) are uncertain smooth functions satisfying the classical condition for the existence and uniqueness. With the relative degree \( n \), the above non-linear system, using appropriate transformation, can be transformed into the following form [21]

\[
\dot{x}_i = x_{i+1} \quad i = 1 \cdots (n-1)
\]

\[
\dot{x}_n = p_{d2}(x, t) + B_2 u
\]

\[
y(t) = Cx(t)
\]

where \( x(t) = [x_1, x_2, \cdots, x_n]^T \) represents the state vector, and \( p_{d2}(x, t) \) is an uncertain smooth function that satisfies the classical condition for existence and uniqueness. Furthermore, the function satisfies

\[
p_{d2}(x, t) \leq Q
\]

\[
Q := Q_1 + Q_2 ||x(t)||
\]

where \( Q > 0 \) and \( Q_1 \) and \( Q_2 \) are positive constants. The above definition relaxes the need of precise modeling of the plant. The above non-linear system is in a Brunowsky canonical form. \( B_2 \) is a non-zero scalar.

The aforementioned system can be represented in the so-called regular form as

\[
\dot{z}_1(t) = A_{11}z_1(t) + A_{12}z_2(t)
\]

\[
\dot{z}_2(t) = A_{21}z_1(t) + A_{22}z_2(t) + B_2 u(t) + p_{d2}(x, t)
\]

\[
y(t) = Cz(t)
\]

where

\[
z_1 := [x_1, x_2, \ldots, x_{n-1}]^T
\]

\[
z_2 := x_n
\]

\[
A_{11} = \begin{bmatrix} 0 & I_{n-2} \\ 0 & 0 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad A_{21} = [0, \ldots, 0]
\]

\[
A_{22} = 0
\]

where \( z_1(t) \in \mathbb{R}^{n-1}, z_2(t) \in \mathbb{R} \). For notational simplicity, let us define the following

\[
A_{reg} := \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}
\]

\[
c_1 := F - \Psi(y)A_{12}^TP
\]

\[
c^T := [c_1, 1]
\]

Let the sliding surface for the system be defined as follows

\[
s(z, t) := c^Tz(t) = [c_1, 1][z_1(t)\ z_2(t)]
\]

\[
= [F - \Psi(y)A_{12}^TP, 1][z_1(t)\ z_2(t)]
\]

\( F \) is chosen such that \( (A_{11} - A_{12}F) \) has stable eigenvalues and the dominant poles have a low-damping ratio. \( \Psi(y) \) is a non-positive differentiable function in \( y \), used to change the damping ratio. \( P \) is a positive definite matrix that is chosen based on the desired final damping ratio \( \xi_2 \), and also
satisfies the Lyapunov equation
\[(A_{11} - A_{12}F)^T P + P(A_{11} - A_{12}F) = -W\]  
(9)
for some positive definite matrix \(W\). Using this non-linear sliding surface, the damping ratio of the system is increased from its initial value \(\xi_1\) to the final value \(\xi_2\). The formulation to calculate the matrix \(P\), based on the required \(\xi_2\), is discussed later in this section.

2.1 Choice of non-linear function \(\Psi(y)\)

The non-linear function \(\Psi(y)\) is used to change the system’s closed-loop damping ratio from its initial low value to a final high value as the output varies from its initial value and approaches the setpoint (the origin for the regulator case). The non-linear function should have the following two properties:

- It should change from 0 (or a very small value) to \(-\beta\) as the output approaches the setpoint from its initial value, where \(\beta > 0\).
- It should be differentiable with respect to \(y\) to ensure the existence of the sliding mode.

One possible choice of \(\Psi(y)\), which is modified from [3], which has the above properties is

\[\Psi(y) = -\frac{\beta}{1 - e^{-1}}(e^{-(1-\gamma_0)(1-\gamma_0)^y}) - e^{-1}\]  
(10)
where \(\gamma_0 = y(0)\) and \(\beta\) is used as a tuning parameter. This parameter contributes to decide the final damping ratio along with the matrix \(P\). It should be noted that the choice of \(\Psi(y)\) is not unique and any function with the above-mentioned properties can be used. Another possible choice is as follows

\[\Psi(y) = -\beta e^{-\hat{k}y^2}\]  
(11)
where \(\hat{k}\) is a positive constant that should have a large value to ensure a small initial value of \(\Psi\).

2.1.1 Stability of sliding surface: During the sliding mode (i.e. \(s(z, t) = 0\)), from (8)

\[z_2(t) = -Fz_2(t) + \Psi(y)A_{12}^TPz_1(t)\]  
(12)
From (4) and (12), the system equation during sliding mode becomes

\[\dot{z}_1 = (A_{11} - A_{12}F + \Psi(y)A_{12}A_{12}^TP)z_1(t)\]  
(13)
To prove the stability in sliding mode, it should be shown that the subsystem (13) is stable. The stability of (13) is proved in the following theorem.

Theorem 2.1: If \((A_{11} - A_{12}F)\) is stable and \(\Psi(y)\) is defined by (10) or (11), then the subsystem in (13) is stable.

Proof: Let a Lyapunov function for the system in (13) be defined as

\[V(z) = z_1^T(t)Pz_1(t)\]
Then, it follows that

\[\dot{V}(z) = \dot{z}_1^T(t)Pz_1(t) + z_1^T(t)P\dot{z}_1(t)\]
\[= z_1^T(t)(A_{11} - A_{12}F)^TPz_1(t) + z_1^T(t)P(A_{11} - A_{12}F)z_1(t)\]
\[+ 2\Psi(y)z_1^T(t)PA_{12}A_{12}^TPz_1(t)\]
\[= z_1^T(t)((A_{11} - A_{12}F)^TP + P(A_{11} - A_{12}F))z_1(t)\]
\[+ 2\Psi(y)z_1^T(t)PA_{12}A_{12}^TPz_1(t)\]
\[= z_1^T(t)((A_{11} - A_{12}F)^TP + P(A_{11} - A_{12}F)\]
\[+ 2\Psi(y)PA_{12}A_{12}^TPz_1(t)\]
\[= z_1^T(t)(-W + 2\Psi(y)PA_{12}A_{12}^TP)z_1(t)\]
Therefore we have

\[\dot{V}(z) = z_1^T(t)(-W + 2\Psi(y)MM^T)z_1(t)\]

where \(M := PA_{12} \in \mathbb{R}^{(n-1)}\). As \(PA_{12}A_{12}^TP = MM^T \geq 0\) and the function \(\Psi(y)\) is negative by definition, therefore matrix \(2\Psi(y)MM^T\) is negative semi-definite. The matrix \(-W\) is negative definite and the addition of a negative definite and a negative semi-definite matrix always results in a negative definite matrix. Therefore we can write \(\dot{V}(z) < 0\), which completes the proof.

Remark 2.1: In (8), the matrix gain \(F\) is designed for a low damping ratio using pole placement technique. For higher-order systems, formulae of damping ratio and settling time are not defined explicitly in terms of system parameters. However, two complex-conjugate dominant poles with the desired damping ratio and settling time can be chosen and the remaining poles can be placed away from the dominant poles to reduce their effect on the system response. The dominant pole concept is also widely used in the literature to specify the damping ratio of higher-order SISO and MIMO systems.

2.2 Insight into change in damping ratio

In the previous section, the structure of the non-linear sliding surface and its stability have been discussed. The objective of this section is to build a comprehension of how the poles and damping ratio change with the corresponding change in the function \(\Psi\) during the sliding mode. The approach presented here is similar to the approach used in gain scheduling controller [21]. For simplicity of presentation, a third-order single-input system is used to describe how the damping ratio changes. However, the proposed non-linear sliding surface can also be used with higher-order plants.

Let us assume, without loss of generality, a third-order uncertain system described by (5). A suitable \(F\) can be chosen so that the matrix \(A_{11} - A_{12}F\) becomes Hurwitz and has a small damping ratio. Consider this system to be in the canonical form as described by (5) and thus

\[A_{12} = \begin{bmatrix} 0 & 1 \end{bmatrix}\]  
(14)
Also, define

$$A_{11} - A_{12}F := \begin{bmatrix} 0 & 1 \\ -a_1 & -a_2 \end{bmatrix}$$

(15)

where $a_1$ and $a_2$ are positive constants. Let us define the positive-definite matrices $P$ and $W$ which satisfy (9) as

$$P := \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}$$

and

$$W := \begin{bmatrix} w_1 & w_2 \\ w_2 & w_3 \end{bmatrix}$$

(16)

(17)

By using the above values of matrices, (13) can be rewritten as

$$\dot{z}_1 = \begin{bmatrix} 0 & 1 \\ -a_1 & -a_2 \end{bmatrix} z_1(t) + \begin{bmatrix} 0 & 0 \\ \Psi p_2 & \Psi p_3 \end{bmatrix} z_1(t)$$

which can be simplified as

$$\dot{z}_1 = \begin{bmatrix} 0 & 1 \\ -a_1 + \Psi p_2 & -a_2 + \Psi p_3 \end{bmatrix} z_1(t)$$

(18)

The eigenvalues of the above closed-loop system can be computed as

$$\lambda_1 = -\frac{a_2}{2} + \Psi p_3 + \frac{1}{2} \sqrt{(a_2 - \Psi p_3)^2 - 4a_1 + 4\Psi p_2}$$

(19a)

$$\lambda_2 = -\frac{a_2}{2} + \Psi p_3 - \frac{1}{2} \sqrt{(a_2 - \Psi p_3)^2 - 4a_1 + 4\Psi p_2}$$

(19b)

To obtain the above equations in terms of elements of matrix $W$, the following formulae can be derived easily by using (9) as

$$p_2 = \frac{w_1}{2a_1}$$

(20)

$$p_3 = \frac{w_1}{2a_2} + \frac{w_2}{2a_1 a_2}$$

(21)

As discussed earlier, the function $\Psi$ changes from its initial value zero to some negative value. As the function $\Psi$ changes, this results in the change in location of closed-loop poles. Now, it can be easily seen that, as $\Psi \rightarrow -\beta$ and $\beta \rightarrow \infty$, the closed-loop poles in (19) can be written as

$$\lim_{\Psi \rightarrow -\infty} \lambda_1 = -\frac{a_2 w_1}{w_1 + a_1 w_3}$$

$$\lim_{\Psi \rightarrow -\infty} \lambda_2 = -\infty$$

It can be concluded, for this general third-order system, that during sliding mode, as the function $\Psi$ changes from its initial value to $-\infty$, the closed-loop system’s damping ratio increases to $\infty$. To analyse a higher-order system a reduced-order model, which is formed by the use of only dominant poles, can be used. Similarly, when the function $\Psi$ changes from 0 to $-\beta$, where $\beta$ is some finite value, the corresponding increase in the damping ratio can also be proved.

### 2.3 LMI-based synthesis of the non-linear surface

During the sliding mode, the dynamics of the system are decided by the subsystem (16). The subsystem is stable for any non-positive value of $\Psi(y)$. The function $\Psi(y)$ changes from 0 to $-\beta$ as the output changes from its initial value to zero. Even for any intermediate value of $\Psi(y)$, the closed-loop system (13) is stable. As explained in the above section, the introduction of this function changes the damping ratio of the system from its initial value $\xi_1$ to the final value $\xi_2$, where $\xi_2 > \xi_1$. At $t = 0$, the function $\Psi(y) = 0$, and the damping ratio is contributed by $F$ which is designed for a low damping ratio $\xi_1$. When the output is closer to the origin, $\Psi(y)$ contributes significantly to increase the damping ratio of the system. When the output reaches the origin, the steady-state value of $\Psi(y)$ becomes $\Psi(y) = -\beta$; therefore the subsystem (13) can be written as

$$\dot{z}_1(t) = (A_{11} - A_{12}F - \beta A_{12}A_{12}^T P)z_1(t)$$

(22)

which decides the final damping ratio. Thus, the parameter $\beta$ and the matrix $P$ should be so designed that the dominant poles of (22) have the desired damping ratio $\xi_2$. Equation (22) can be rewritten as

$$\dot{z}_1(t) = (A_{11} - A_{12}(F + \beta A_{12}^T P))z_1(t)$$

(23)

Let the required gain be $K_2$ for the desired final damping ratio $\xi_2$, which may be computed by the pole-placement technique. Then, the resulting constraint equation is

$$K_2 = F + \beta A_{12} P$$

in order to realise the desired damping ratio in (23), the above equation should be satisfied; and which can be equivalently expressed as

$$A_{12}^T P - \frac{K_2 - F}{\beta} = 0$$

(24)

To achieve the desired damping ratio, a matrix $P$ is needed which satisfies (24). To ensure stability of the closed-loop system, the matrix $P$ also needs to satisfy (9). However, it may not always be possible for the matrix $P$ to satisfy constraints (9) and (24) simultaneously. Therefore one may relax constraint (24) as follows

$$||H|| \leq \varepsilon$$

(25)

for a sufficiently small $\varepsilon > 0$, where

$$H := A_{12}^T P - \frac{K_2 - F}{\beta}$$

(26)

It may be noted that the above non-linear inequality can be converted to a linear inequality by using Schur complement.
as follows
\[
\begin{bmatrix}
    eI & H \\
    H^T & eI
\end{bmatrix} > 0
\] (27)

Therefore we can cast the optimisation problem as follows
Choose P to minimise \(\varepsilon\) such that
\[
P > 0
\] (28a)
\[
(A_{11} - A_{12}F)^TP + P(A_{11} - A_{12}F) < 0
\] (28b)
\[
\begin{bmatrix}
    eI & H \\
    H^T & eI
\end{bmatrix} > 0
\] (28c)

The above LMIs can be easily solved to minimise \(\varepsilon\) by using many commercial computational tools. It should be noted that the freedom to choose the parameter \(\beta\) can also be used to minimise the objective function. This approach automates the tuning procedure and reduces the number of manual iterations required for tuning.

3 Existence of sliding mode

The sliding surface discussed in the previous section is non-linear and surface parameters are changing continuously with time. A control law should be chosen in such a way that from any initial condition, the system trajectory is attracted towards the sliding surface and then slides along the surface. The existence of such a control law is discussed in the following theorem.

Theorem 3.1: The control law
\[
u = -B_2^T[c^TA_{reg}z + KS + Q \, \text{sign}(s) - \frac{d\Psi(y)}{dt} A_{12}Pz_1(t)]
\] (29)

enforces the trajectory of (4) to move from any initial condition to the sliding surface in finite time and remain on it thereafter. In the above control law, the scalar \(Q\) is chosen from the maximum bound of the uncertainty that satisfies (3) and this implies
\[
Q > (\rho_{d2}(z,t))_{max}
\] (30)

\(K\) is a positive constant.

Proof: For a quadratic function, \(V_1 = 0.5s^TPs\)
\[
\Rightarrow \dot{V}_1 = s^T \dot{s}
\]
\[
\dot{V}_1 = s^T \left[F \dot{z}_1 - \frac{d\Psi(y)}{dt} A_{12}Pz_1(t) - \Psi(y)A_{12}Pz_1(t) + \dot{z}_2 \right]
\]
\[
= s^T \left[ (F - \Psi(y)A_{12})P \dot{z}_1 - \frac{d\Psi(y)}{dt} A_{12}Pz_1(t) + A_{21}z_1(t) \right]
\]
\[
+ A_{22}z_2(t) + B_2u + \rho_{d2}(z,t)
\]
By using (4) and (5), it leads to
\[
\dot{V}_1 = s^T \left[c^TA_{reg}z + B_2u + \rho_{d2}(z,t) - \frac{d\Psi(y)}{dt} A_{12}Pz_1(t) \right]
\]

Using the control law (29), it can be seen that
\[
\dot{V}_1 = s^T \left[ c^TA_{reg}z + \rho_{d2}(z,t) - \frac{d\Psi(y)}{dt} A_{12}Pz_1(t) - c^TA_{reg}z - KS \right]
\]
\[
- Q \, \text{sign}(s) + \frac{d\Psi(y)}{dt} A_{12}Pz_1(t) \right]
\]
\[
= s^T \left[ - KS - Q \, \text{sign}(s) + \rho_{d2}(z,0) \right]
\]

Considering (30), it is straightforward to verify \(\dot{V}_1 < -\eta |s|\) for some \(\eta > 0\). This ensures finite time convergence of \(s\) and thus it completes the proof.

It should be noted that using the control law (29) the state trajectory will hit the surface in finite time. After hitting the sliding surface, the trajectory will slide along the surface and the system is invariant to parameter variations and external disturbances satisfying the matching condition.

Remark 3.1: The control law (29) requires the total derivative of the output. For many systems, the derivative of the output can be calculated in the form of a state (i.e. if input and output subspaces are orthogonal to each other, then because of this, for SISO case \(CB = 0\), here \(C\) is the output matrix and \(B := [0 \ 0 \cdots B_1]^T\)). When it does not exist explicitly, it can be easily computed using robust differentiator proposed by Levant [22] or Utkin [23]. It is worth noting that recently a large number of practical applications have witnessed the efficient use of these techniques to obtain the derivative of a signal.

4 Example and simulation results

In this section, ship roll stabilisation is presented to illustrate the proposed method. Following control objectives need to be achieved.

- System should have low overshoot.
- System should have short settling time so it settles quickly.
- Controlled system should be robust against disturbances.

Simulation results are presented for different fixed damping ratio sliding surfaces. These results are compared with the proposed non-linear sliding surface. Following parameters are considered [24]

- Fin time constant \(T_f = 1.0\) s.
- Ship roll natural frequency \(\omega_n = 1.414\) rad/s.
- Ship roll damping ratio \(\zeta_r = 0.248\).
- Ship steady-state gain \(K_s = 0.5\).

With the above parameters the system model has the below form
\[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} =
\begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
-2 & -2.7 & -1.7
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} +
\begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix} u(t)
\]
\[
+ \begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix} \rho_{d2}(t)
\]
\[
y = x_1
\] (31)

By following the procedure outlined in Section 2, we can
obtain the following

\[ z_1 = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T, \quad z_2 = x_3, \quad A_{11} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \]

\[ A_{12} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad A_{21} = \begin{bmatrix} -2 & -2.7 \end{bmatrix}, \quad A_{22} = -1.7, \quad B_2 = 1 \]

\[ \rho_{22}(x,t) = \sin(10t). \] output equation is \( y = x_1. \)

Step 1: Design the non-linear sliding surface. Non-linear sliding surface is composed of a constant and a non-linear term. Initially the non-linear term is zero, therefore the constant term decides the initial damping ratio \( (\xi_1) \) and the settling time. Sliding surface can be given by the following equation

\[ s = c_1 z_1 + c_2 z_2 \tag{32} \]

As per the discussion in Section 2, let \( c_2 = 1, \) and \( c_1 \) is made of two components linear and non-linear.

\[ c_1 = F - \Psi(y)A_{12}^T P \]

\( F \) is designed for the low damping ratio. For initial settling time \( t_s = 2 \) s and initial damping ratio \( \xi_1 = 0.4, \) matrix \( F \) can be found from pole-placement approach as \( F = \begin{bmatrix} 25 & 4 \end{bmatrix} \). Let \( \beta = 50 \) and solving Lyapunov equation with

\[ W = 0.34 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \]

for \( P \) gives

\[ P = \begin{bmatrix} 0.0071 & -0.0170 \\ -0.0170 & 0.1105 \end{bmatrix} \]

Function \( \Psi(y) \) is designed with \( \beta = 50 \) as follows

\[ \Psi(y) = -50e^{-100y^2} \]

Therefore the sliding surface becomes

\[ s = [F - \Psi(y)A_{12}^T P \ 1][z_1 \ z_2]^T \tag{33} \]

Step 2: Design of control law. Control law can be given as

\[ u = -B_2^{-1} \left( c_1^T A z + ks + Q \text{sign}(s) - \frac{d\Psi(y)}{dt} A_{12}^T P z_1 \right) \tag{34} \]

Following values are chosen \( k = 20, \ Q = 1.01, \ x_0 = [0.1 \ 0 \ 0]^T. \) The value of \( Q \) is chosen by considering maximum bound of uncertainty which is \( (\rho_{22})_{\text{max}} = 1. \)

4.1 Design of linear sliding surface

Performance of the proposed non-linear sliding surface is compared with the controller designed with different linear-sliding surfaces. System output response is plotted for different linear surfaces and the proposed non-linear surface. When non-linear sliding surface is used, poles of the closed-loop system changes as output approaches to the origin. Following three different linear-sliding surfaces are designed

1. Case 1: Linear surface 1 with \( \xi = 0.7, \ t_s = 2 \) s.
2. Case 2: Linear surface 2 with \( \xi = 0.6, \ t_s = 2 \) s.
3. Case 3: Linear surface 3 with \( \xi = 0.8, \ t_s = 2 \) s.

Fixed sliding surface with given settling time and damping ratio is designed by pole-placement approach.

\[ s_i = c_1^T z = [F_i \ 1][z_1 \ z_2]^T \tag{35} \]

In the above equation \( F_i \) is computed by pole-placement approach for each of the three cases mentioned above with corresponding damping ratio and settling time. Control law for each of the three cases mentioned above is given as follows

\[ u_i = -B_2^{-1}(c_1^T A z + ks_i + Q \text{sign}(s_i)) \tag{36} \]

In the above control law, parameters \( Q \) and \( k \) are the same as (34). The system is simulated with the same initial condition and disturbance for different linear-sliding surfaces.

Fig. 1 Response of output \( x_1 \) with different sliding surfaces
4.2 Comparison with different linear-sliding surfaces

In Fig. 1, response of the output $y = x_1$ is plotted when non-linear sliding surface is used and different linear-sliding surfaces. The plot clearly shows that with non-linear sliding surface performance improves significantly. Furthermore, it shows very clearly that peak overshoot and settling time both can be minimised simultaneously with non-linear sliding surface. With the proposed non-linear surface, output settles in 0.8 s without any overshoot. Responses obtained with different linear surfaces suffer from either high overshoot or long settling time. With the proposed non-linear surface both can be minimised simultaneously. Fig. 2 shows evolution of non-linear switching function with time. Fig. 3 shows the plot of input, it can be seen that because of non-linear sliding surface gain increases as state reaches nearer to the origin and thus the settling time improves.

5 Conclusion

The design of non-linear sliding surface is proposed to improve the performance. The existence of sliding mode on such a surface is proved. Ship roll stabilisation example clearly shows that the performance significantly improves. The proposed algorithm is able to achieve low overshoot and short settling time simultaneously for a class of uncertain non-linear system. Robustness and high performance both can be achieved with the proposed scheme. Disturbance rejection property of the proposed method is also shown. Also, we note that it is straightforward to extend our results for MIMO case.

6 References

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Q1 Please confirm the affiliation details of First author.
Q2 Please provide the full affiliation details of Third author.
Q3 Please confirm the expansion of SISO, MIMO.
Q4 Please check and confirm the change of ‘$t_{s1} = 2ec$ to $t_{s1} = 2 s$’ in the sentence ‘For initial setting . . .’.