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Slow periodic motions in variable structure systems

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Singularly perturbed relay control systems (SPRCS) with stable periodic motion in reduced systems are studied here. The slow motions integral manifold of such systems consists of parts that correspond to different values of control and the solutions of SPRCS contain the jumps from one part of the slow manifold to the other. The theorems about existence and stability of the slow periodic solutions are proved. An algorithm of asymptotic representation for this periodic solutions using a boundary layer method is suggested.

1. Introduction

There is a wide class of relay control systems that work in periodic regimes. For example, such regimes arise every time in relay control systems with time delays because a time delay does not allow an ideal sliding mode to be realized and results in periodic oscillations (Drakunov and Utkin 1992, Fridman *et al.* 1993, Gouaisbaut *et al.* 1999). In controllers of exhaust gases for fuel injector automotive control systems (e.g. Choi and Hedrick 1996, Li and Yurkovitch 1999) the sensors can measure only the sign of the controlled variable with a delay. In such systems, only oscillations around zero value can occur. In the controllers for stabilization of underwater manipulators it is possible to realize only oscillations because of the manipulators properties (Bartolini *et al.* 1997). The different aspects of periodic solutions to relay control systems are discussed in (Concalves *et al.* 1998, Bernardo *et al.* 2001) using Poincaré maps.

In this paper, I will investigate the existence and stability of periodic solutions for singularly perturbed relay control systems (SPRCS). SPRCS could describe, for example, the behaviour of the fast actuators in the control systems. For example, the complete model of fuel injector systems taking into account the influence of the car motor. Knowledge about the properties of SPRCS is necessary in the controllers for stabilization of underwater manipulator fingers to take into account the influence of the elasticity of these fingers.

For *smooth* singularly perturbed systems there are two main classes for slow periodic solutions. The slow periodic solutions of the smooth singularly perturbed systems 'without jumps' are situated on slow motion integral manifolds (e.g. Wasov 1965). The other important class of periodic solutions are the relaxation solutions (Mishchenko and Rosov 1980), which contain the 'jumps' from the neighbourhood of one stable branch of the slow motions manifold to the neighbourhood of another one.

SPRCS was investigated by Fridman and Bogatyrev (1992) (existence of stable slow motion integral manifold) and Fridman (2001) (averaging and existence of stable periodic solutions). Some control algorithms for SPRCS was suggested by Sira Ramirez (1988), Heck (1991), Su (1999), Innocenti *et al.* (2000) and Castro-Linares *et al.* (2001).

The slow motions integral manifold of relay systems is discontinuous and consists of at least two parts corresponding to the different values of control (figure 1). This means that the desired periodic solution of the SPRCS should have jumps from the small neighbourhood of one sheet of an integral manifold to the neighbourhood of another one. From this viewpoint the qualitative behavior of this periodic solution will be nearer to the relaxation solution.

The main specific feature of systems with relaxation oscillations is the following: at the moment of time corresponding to the jump from the neighbourhood of one branch of the stable integral manifold to the neighbourhood of another one, the value of the righthand side is small. That is why in order to find the asymptotic repre-

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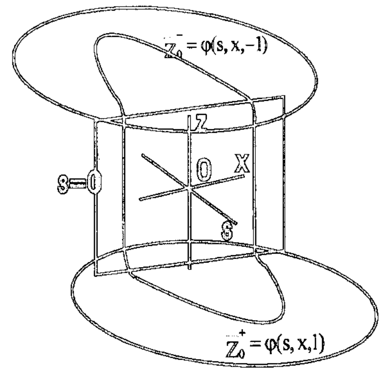

Figure 1. Two sheets of slow motions integral manifold.

sentation of the relaxation solution, it was necessary to make special asymptotic representations.

The situation with SPRCS is different. The right-hand side of an SPRCS switches immediately after the switching moment and the righthand side of the fast equations in SPRCS after this moment is very big. It allows one to use the Tikhonov theorem (Tikhonov 1952, Vasil'eva *et al.* 1995) for the proof of existence and stability of slow periodic solution to SPRCS, and the standard boundary layer function method (Vasil'eva and Butuzov 1973, Vasil'eva *et al.* 1995) for its asymptotic representation.

2. Problem statement

2.1. *Original and reduced systems' specific features*

Consider the system

$$\mu\, \mathrm{d}z/\mathrm{d}t = g(z, s, x, u), \quad \mathrm{d}s/\mathrm{d}t = h_1(z, s, x, u),$$
$$\mathrm{d}x/\mathrm{d}t = h_2(z, s, x, u), \tag{1}$$

where $z \in R^m$, $s \in R$, $x \in R^n$, $u(s) = sign(s)$, g, h_1, h_2 are sufficiently smooth functions of their arguments and μ is a small parameter.

Suppose that ignoring additional dynamics, having accepted $\mu = 0$ and expressing z_0 from the equation

$$g(z_0, s, x, u) = 0 \tag{2}$$

according the formula $z_0 = \varphi(s, x, u)$, we obtain the system

$$\begin{aligned} \mathrm{d}s/\mathrm{d}t &= h_1(\varphi(s, x, u), s, x, u) = H_1(s, x, u), \\ \mathrm{d}x/\mathrm{d}t &= h_2(\varphi(s, x, u), s, x, u) = H_2(s, x, u). \end{aligned} \tag{3}$$

We will suppose that for this system the sufficient conditions for existence of the orbitally exponentially stable isolated periodic solution hold. The parts of the periodic solution of the reduced system corresponding to the different values of control are situated in different sheets of the slow motions integral manifold of SPRCS. This means that the desired periodic solution of the original system (1) contains internal boundary layers describing the jumps from the one part of the slow motions manifold to the another one. We will find sufficient conditions for existence (see Section 3) and orbital asymptotic stability (see Section 4) of the isolated periodic solution of the original system (1), which corresponds to the periodic solution of the reduced system. Proposed results allow one to make an important conclusion. Relay control design resulting in the existence of an exponentially asymptotically stable periodic solution is robust with respect to stable unmodelled dynamics. It is not true for sliding mode systems because in those systems stable unmodelled dynamics of order 2 or more yield to unstability (e.g. Fridman and Levant 2002).

Let us denote as $\mathcal{Z}$, $\mathcal{X}$ the domains of definition for (z, s, x) and (s, x). Suppose that the following conditions are true:

1^0 $h_1, h_2, g \in C^2[\bar{\mathcal{Z}} \times [-1, 1]]$;
2^0 the function $z_0 = \varphi(s, x, u)$ for all $(s, x, u) \in \Omega = \bar{\mathcal{X}} \times [-1, 1]$ is the isolated solution of equation (2).

2.2. *Poincaré map generated by a reduced system*

For investigation of stability in (1) and (3), it is impossible to use methods based on linearization because (1) and (3) are relay systems. In the paper for investigating the periodic solutions to systems (1) and (3), the Poincaré maps of the surface $s = 0$ into itself generated by those systems are using.

Let us define first the Poincaré map of surface $s = 0$ into itself, generated by system (3) (figure 2). Consider the solution of system (3) for $u = 1$

$$\mathrm{d}\bar{s}_0^+/\mathrm{d}t = H_1(\bar{s}_0^+, \bar{x}_0^+, 1), \quad \mathrm{d}\bar{x}_0^+/\mathrm{d}t = H_2(\bar{s}_0^+, \bar{x}_0^+, 1) \tag{4}$$

with initial conditions

$$\begin{aligned} &\bar{s}_0^+(0) = 0,\ \bar{x}_0^+(0) = \xi,\ \xi \in \mathcal{V} \subset S^+ \\ &= \{\xi : H_1(0, \xi, 1) > 0\}. \end{aligned} \tag{5}$$

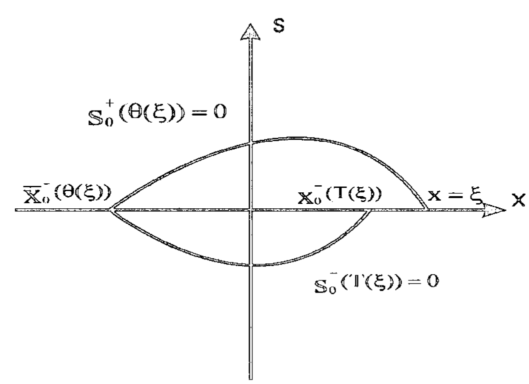

Figure 2. Poincaré map $\Psi(\xi)$.

Suppose that for all $\xi \in \mathcal{V}$, $t = \theta(\xi)$ is the smallest positive root of equation $\bar{s}_0^+(\theta(\xi)) = 0$, such that

$$H_1(0, \bar{x}_0^+(\theta(\xi)), 1) < 0, H_1(0, \bar{x}_0^+(\theta(\xi)), -1) < 0. \quad (*)$$

Consequently, at $t = \theta(\xi)$ the function u changes from 1 to -1, the solution of system (3) for $t > \theta(\xi)$ until the next switching moment will be the solution of the Cauchy problem

$$\mathrm{d}\bar{s}_0^-/\mathrm{d}t = H_1(\bar{s}_0^-, \bar{x}_0^-, -1), \quad \mathrm{d}\bar{x}_0^-/\mathrm{d}t = H_2(\bar{s}_0^-, \bar{x}_0^-, -1), \quad (6)$$

$$\bar{s}_0^-(\theta(\xi)) = \bar{s}_0^+(\theta(\xi)),\ \bar{x}_0^-(\theta(\xi)) = \bar{x}_0^+(\theta(\xi)).$$

Suppose now that at all $\xi \in \mathcal{V}$ for $\bar{s}_0^+(t)$, which is the first coordinate of the Cauchy problem (6) solution, there is a smallest root $T(\xi)$ of equation $\bar{s}_0^-(T(\xi)) = 0$, such that

$$\bar{x}_0^-(T(\xi)) \in \mathcal{V},\ H_1(0, \bar{x}_0^-(T(\xi)), -1) > 0,$$
$$H_1(0, \bar{x}_0^-(T(\xi)), 1) > 0. \quad (**)$$

Then $\Psi(\xi) : \xi \to \bar{x}_0^-(T(\xi))$ is the Poincaré map of the surface $s = 0$ into itself generated by system (3).

Then the fixed point of the Poincaré map ξ_0 corresponding to the periodic solution is described by equation $\Psi(\xi_0) = \xi_0$. The period and switching moment of that periodic solution we will define in form $T(\xi_0) = T_0$, $\theta(\xi_0) = \theta_0$. The conditions of periodicity take the form

$$\bar{s}_0^-(T_0) = \bar{s}_0^+(0) = 0,\ \bar{x}_0^-(T_0) = \bar{x}_0^+(0).$$

To have a continuous periodic solution at the point we should have

$$\bar{s}_0^-(\theta_0) = \bar{s}_0^+(\theta_0) = 0,\ \bar{x}_0^-(\theta_0) = \bar{x}_0^+(\theta_0).$$

Remark 1: Designing the Poincaré map $\Psi(\xi)$, one needs the knowledge of the general solutions to the reduced system (3). Generally, it is possible only when the functions H_1, H_2 are linear.

Remark 2: Assumptions (*) and (**) allow one to avoid the bifurcation due to the sliding in some part of the map (Bernardino *et al.* 2001).

2.3. *Existence and stability of the isolated periodic solutions to the reduced system*

Suppose that:

3^0 system (3) has an isolated T_0 periodic solution $(\bar{s}_0(t), \bar{x}_0(t))$, ξ_0 is an isolated fixed point of the Poincaré map $\Psi(\xi)$ which corresponds to $(\bar{s}_0(t), \bar{x}_0(t))$, moreover $\bar{s}_0(0) = 0$ and $\det \partial\Psi/\partial\xi(\xi_0) \neq 0$;

4^0 $|\mathrm{Spec}\, \partial\Psi/\partial\xi(\xi_0)| < 1$.

2.4. *Stability of the fast motions*

Suppose that

5^0 the equilibrium points

$$z_0 = \varphi(\bar{s}_0(t), \bar{x}_0(t), 1),\ t \in [0, \theta_0],$$
$$z_0 = \varphi(\bar{s}_0(t), \bar{x}_0(t), -1),\ t \in [\theta_0, T_0]$$

to systems

$$\mathrm{d}z/\mathrm{d}\tau = g(\varphi(\bar{s}_0(t), \bar{x}_0(t), 1), \bar{s}_0(t), \bar{x}_0(t), 1),$$
$$dz/\mathrm{d}\tau = g(\varphi(\bar{s}_0(t), \bar{x}_0(t), -1), \bar{s}_0(t), \bar{x}_0(t), -1)$$

are uniformly asymptotically stable on $[0, \theta_0]$ and $[\theta_0, T_0]$ correspondingly. Moreover

$$\mathrm{Re\,Spec}\, \partial g(\varphi(\bar{s}_0(t), \bar{x}_0(t), 1), \bar{s}_0(t), \bar{x}_0(t), 1)/\partial z < -\alpha < 0, t \in [0, \theta_0],$$

$$\mathrm{Re\,Spec}\, \partial g(\varphi(\bar{s}_0(t), \bar{x}_0(t), -1), \bar{s}_0(t), \bar{x}_0(t), -1)/\partial z -\alpha < 0, t \in [\theta_0, T_0].$$

2.5. *Attractivity conditions*

The slow motions integral manifold to the original system (1) consists of parts that correspond to different values of relay control. We have to ensure that right after the switching points the original system (1) solution will tend to the other sheet of stable slow-motions integral manifold corresponding to an other value of control. Suppose that:

6^0 the points $\varphi(0, \xi_0, -1)$ are $\varphi(0, \bar{x}_0(\theta_0), 1)$ are situated in the interior of the attractive domain of stable equilibrium points

$$\varphi(0, \xi_0, 1) \quad \text{and} \quad \varphi(0, \bar{x}_0(\theta_0), -1).$$

Denote as $\mathcal{L}_0(t)$ the broken line

$$\mathcal{L}_0(t) = \begin{cases} \varphi(\bar{s}_0(t), \bar{x}_0(t), 1) & \text{for} \quad t \in (0, \theta_0), \\ \varphi(\bar{s}_0(t), \bar{x}_0(t), -1) & \text{for} \quad t \in (\theta_0, T_0), \\ (1-\gamma)\varphi(0, \bar{x}_0(\theta_0), 1) + \gamma\varphi(0, \bar{x}_0(\theta_0), -1), & \\ \qquad \gamma \in [0,1] & \text{for} \quad t = \theta_0, \\ (1-\gamma)\varphi(0, \xi_0, -1) + \gamma\varphi(0, \xi_0, 1), & \\ \qquad \gamma \in [0,1] & \text{for} \quad t = 0. \end{cases}$$

In this paper sufficient conditions are found for the existence of the isolated orbitally asymptotically stable periodic solution of system (1) near to the broken line

$$(\mathcal{L}_0(t), \bar{s}_0(t), \bar{x}_0(t)).$$

An algorithm for asymptotic representation of this periodic solution is suggested.

3. Existence of a periodic solution to SPRCS

Suppose that we have chosen the norm in $\mathbf{R}^n$ in such way that the Poincaré map Ψ is a contracting map in the small neighborhood of ξ_0 and moreover $\|\partial\Psi/\partial\xi(\xi_0)\| < q < 1$.

Consider now the Poincaré map $\Phi(z, x, \mu)$ of the surface $s = 0$ into itself generating by system (1).

Lemma 1: *Under conditions* 1^0–6^0 *there is neighbourhood* Γ *of the point* $(\varphi(0, \xi_0, 1), \xi_0)$ *on the surface* $s = 0$, *such that for every* $(\eta, \xi) \in \Gamma$ *and for* $(z(t, \mu), s(t, \mu), x(t, \mu))$, *which is the solution of system* (1) *with initial conditions*

$$z(0, \mu) = \eta, \quad s(t, \mu) = 0, \quad x(0, \mu) = \xi, \tag{7}$$

for sufficiently small μ *there exist* $0 < \theta(\eta, \xi, \mu) < T(\eta, \xi, \mu)$, *such that*

$$s(\theta(\eta, \xi, \mu), \mu) = 0, \quad s(T(\eta, \xi, \mu), \mu) = 0,$$

moreover the solution $(z(t, \mu), s(t, \mu), x(t, \mu))$ *is unique on*

$$[0, T(\eta, \xi, \mu)]$$

and

$$\Phi(z, x, \mu) = (z(T(\eta, \xi, \mu), \mu),\ x(T(\eta, \xi, \mu), \mu)) \in \Gamma.$$

Proof: The functions $(z(t, \mu), s(t, \mu), x(t, \mu))$ are differentiable on η, ξ, μ at the points $t = \theta_0$, $T = T_0$ (e.g. Strygin and Sobolev 1988). Then from the implicit function theorem it follows that there exists a closed ball $\bar{U}(\alpha) \subset \mathcal{V} \in \mathbf{R}^n$ with radius α and center at the point ξ_0 such that for every $\xi \in \bar{U}(\alpha)$:

- $\|\partial\Psi/\partial\xi(\xi)\| < q' < 1$;
- for $(\bar{s}_0^+(t), \bar{x}_0^+(t))$ the solution of system (4) with initial condition $\bar{s}_0^+(0) = 0, \quad \bar{x}_0^+(0) = \xi$ there exist $\theta(\xi)$, such that $\bar{s}_0^+(\theta(\xi)) = 0$ and

$$\mathrm{d}\bar{s}_0^+/\mathrm{d}t(\theta(\xi)) = H_1(0, \bar{x}_0^+(\theta(\xi)), 1) < 0,$$

$$H_1(0, \bar{x}_0^+(\theta(\xi)), -1) < 0;$$

- the point $\varphi(0, \bar{x}_0^+(\theta(\xi)), 1)$ is an internal point of the attractive domain of the stable equilibrium point $\varphi(0, \bar{x}_0^+(\theta(\xi)), -1)$;
- for $(\bar{s}_0^-(t), \bar{x}_0^-(t))$, the solution of system (6) with initial conditions

$$\bar{s}_0^-(\theta(\xi)) = \bar{s}_0^+(\theta(\xi)); \quad \bar{x}_0^-(\theta(\xi)) = \bar{x}_0^+(\theta(\xi)),$$

  there exists $T(\xi)$ such that

$$\bar{s}_0^-(T(\xi)) = 0,$$
$$\mathrm{d}\bar{s}_0^-/\mathrm{d}t(T(\xi)) = H_1(0, \bar{x}_0^-(T(\xi)), 1) > 0;$$

- $\bar{x}_0^-(T(\xi)) \in U(q'\alpha)$;
- $\bar{W} = co\, \varphi(0, \bar{U}(\alpha), -1)$ is situated in the interior of the attractive domain for the stable equilibrium point $\varphi(0, \bar{x}_0^-(T(\xi)), 1)$.

Then from Tikhonov's theorem (e.g. Vasil'eva *et al.* 1995) and the implicit function theorem it follows for every $(\eta, \xi) \in \bar{W} \times \bar{U}(\alpha)$ there exists $\mu_0(\eta, \xi)$, such that for all $\mu \in [0, \mu_0(\eta, \xi)]$:

- for solution the $(z^+(t, \mu), s^+(t, \mu), x^+(t, \mu))$ of system

$$\mu\, \mathrm{d}z^+/\mathrm{d}t = g(z^+, s^+, x^+, 1),$$

$$\mathrm{d}s^+/\mathrm{d}t = h_1(z^+, s^+, x^+, 1),\ \mathrm{d}x^+/\mathrm{d}t = h_2(z^+, s^+, x^+, 1)$$

  with initial condition (7), there exists $\theta(\eta, \xi, \mu)$ the smallest positive root of equation

$$s^+(\theta(\eta, \xi, \mu), \mu) = 0,$$

$$\begin{aligned}\mathrm{d}s^+(\theta(\eta, \xi, \mu), \mu)/\mathrm{d}t \\ = h_1(z^+(\theta(\eta, \xi, \mu), \mu), 0, x^+(\theta, \mu), 1) < 0,\end{aligned}$$
$$h_1(z^+(\theta(\eta, \xi, \mu), \mu), 0, x^+(\theta, \mu), -1) < 0;$$

  the point $z^+(\theta(\eta, \xi, \mu), \mu)$ is situated in the attractive domain

$$\varphi(0, \bar{x}_0^+(\theta(\eta, \xi, \mu), \mu)), -1);$$

  for $(z^-(t, \mu), s^-(t, \mu), x^-(t, \mu))$, which is the solution of system

$$\mu\, \mathrm{d}z^-/\mathrm{d}t = g(z^-, s^-, x^-, -1),$$

$$\mathrm{d}s^-/\mathrm{d}t = h_1(z^-, s^-, x^-, -1),$$
$$\mathrm{d}x^-/\mathrm{d}t = h_2(z^-, s^-, x^-, -1)$$

  with initial conditions

$$z^-(\theta(\eta, \xi, \mu), \mu) = z^+(\theta(\eta, \xi, \mu), \mu),$$
$$s^-(\theta(\eta, \xi, \mu), \mu) = s^+(\theta(\eta, \xi, \mu), \mu) = 0,$$
$$x^-(\theta(\eta, \xi, \mu), \mu) = x^+(\theta(\eta, \xi, \mu), \mu),$$

  there exists the smallest positive root $T(\eta, \xi, \mu) > \theta(\eta, \xi, \mu)$ of equation $s^-(T(\eta, \xi, \mu), \mu) = 0$, for which

$$\mathrm{d}s^-/\mathrm{d}t(T, \mu) = h_1(z^-(T, \mu), 0, x^+(T, \mu), -1) > 0,$$

  and $(z^-(T(\eta, \xi, \mu), \mu), x^-(T(\eta, \xi, \mu), \mu))$ is situated in

$$(\varphi(0, U((1 + q')\alpha/2), -1), U((1 + q')\alpha/2)).$$

Moreover,

$$\begin{aligned}\Phi(\eta, \xi, 0) &= \lim_{\mu\to 0} \Phi(\eta, \xi, \mu) \\ &= (\varphi(0, x^-(T(\xi)), 1), x^-(T(\xi))),\end{aligned}$$

and for $\xi = \xi_0$ we will have

$$\Phi(\varphi(0,\xi_0,1),\xi_0,0) = (\varphi(0,\xi_0,1),\xi_0).$$

Then from the compactness of $\bar{W} \times \bar{U}(\alpha)$ it follows that there exists such μ_0, such that for every $\mu \in [0,\mu_0]$

$$\Phi(\eta,\xi,\mu) = (\Phi_1(\eta,\xi,\mu),\Phi_2(\eta,\xi,\mu)) = (z^-(T(\eta,\xi,\mu),\mu), x^-(T(\eta,\xi,\mu),\mu))$$

the Poincaré map $\Phi(\eta,\xi,\mu)$ of the surface $s=0$ into itself, generated by system (1), is correctly defined on the set $\Gamma = \bar{W} \times \bar{U}(\alpha)$ and transforms it into itself. This means that $\Phi(\eta,\xi,\mu)$ for all $\mu \in [0,\mu_0]$ has on Γ a fixed point, corresponding to the periodic solution of (1) in the small neighbourhood of the broken line $(\mathcal{L}_0(t),\bar{s}_0(t),\bar{x}_0(t))$. ∎

4. Uniqueness and stability of periodic solution to SPRCS

Theorem 2: *Under conditions* 1^0–6^0 *for sufficiently small* μ *there exists an orbitally asymptotically stable periodic solution in the small neighbourhood of the broken line* $(\mathcal{L}_0(t),\bar{s}_0(t),\bar{x}_0(t))$ *with period* $T(\mu)\to 0$, *for* $\mu \to 0$ *and boundary layers for* $t=0$ *and in the small vicinity of* $t=\theta_0$.

Proof: The derivatives of the Poincaré map with respect to initial conditions η,ξ are smoothly dependent on derivatives of the functions

$$z^+(\theta(\eta,\xi,\mu),\mu), x^+(\theta(\eta,\xi,\mu),\mu),$$
$$z^-(T(\eta,\xi,\mu),\mu), x^-(T(\eta,\xi,\mu),\mu)$$

and $\theta(\eta,\xi,\mu)$, $T(\eta,\xi,\mu)$ with respect to initial conditions, which are smoothly dependent on η,ξ (Strygin and Sobolev 1988).

Let us consider the new variable $\chi = \eta - \varphi(0, x^-(T(\xi)), -1)$. Then we will consider the auxiliary operator

$$\begin{aligned}\Lambda(\chi,\xi,\mu) &= (\Lambda_1(\chi,\xi,\mu),\Lambda_2(\chi,\xi,\mu)) \\ &= (\Phi_1(\chi+\varphi(0,x^-(T(\xi)),-1),\xi,\mu) \\ &\quad - \varphi(0,x^-(T(\xi)),-1), \\ &\quad \Phi_2(\chi+\varphi(0,x^-(T(\xi)),-1),\xi,\mu)).\end{aligned}$$

It is necessary to remark that for $\mu=0$ the point $(0,\xi_0)$ is a fixed point of the operator Λ , and Λ itself for sufficiently small $\beta,\bar{\mu}$ transforms into itself the set

$$M(\beta,\alpha,\bar{\mu}) = \{(\chi,\xi,\mu) : \|\chi\| < \beta, x \in \bar{U}(\alpha), \mu \in [0,\bar{\mu}]\}.$$

Let us find the derivative of Λ with respect to χ and ξ. For $\mu=0$ the value $\Lambda(\chi,\xi,0)$ does not depend on χ, and $\Lambda_1(\chi,\xi,0)$ does not depend on ξ. This means that

$$\frac{\partial \Lambda}{\partial(\chi,\xi)} = \begin{pmatrix} O(\mu) & O(\mu) \\ O(\mu) & \partial\Psi/\partial\xi(\xi_0)+O(\mu)\end{pmatrix}.$$

Now it is possible to choose $\beta,\bar{\mu}>0$ such that for some $q_1 (q_1<1)$

$$\sup_{M(\beta,\alpha,\bar{\mu})} \left\| \frac{\partial\Lambda}{\partial(\chi,\xi)} \right\| < q_1 < 1.$$

This means that $\Lambda(\chi,\xi,\mu)$ is a contractive operator on $M(\beta,\alpha,\bar{\mu})$ and has an unique fixed point which corresponds to the orbitally asymptotically stable periodic solution of system (1). ∎

5. Algorithm of asymptotic representation for the periodic solution to SPRCS

Suppose that $h_1,h_2,g \in C^{k+3}[\bar{\mathcal{Z}}\times[-1,1]]$, and conditions 1^0–6^0 hold.

Denote by $y=(z^{\mathrm{T}},s^{\mathrm{T}},x^{\mathrm{T}})^{\mathrm{T}}$. Then we will find the asymptotic representation of the periodic solution of system (1) on the time segment $[0,T(\mu)]$, period $T(\mu)$ and switching point $\theta(\mu)$

$$Y_k(t,\mu) = \sum_{i=0}^{k}[\bar{y}_i(t) + \Pi_i^+ y(\tau) + \Pi_i^- y(\tau_{k+1})]\mu^i; \quad (8)$$

$$S_k(t,\mu) = \sum_{i=0}^{k}[\bar{s}_i(t) + \Pi_i^+ s(\tau) + \Pi_i^- s(\tau_k)]\mu^i;$$

$$X_k(t,\mu) = \sum_{i=0}^{k}[\bar{x}_i(t) + \Pi_i^+ x(\tau) + \Pi_i^- x(\tau_k)]\mu^i;$$

$$\theta(\mu) = \theta_0 + \mu\theta_1 + \cdots + \mu^k\theta_k + \cdots;$$

$$T(\mu) = T_0 + \mu T_1 + \cdots + \mu^k T_k + \cdots;$$

$$\Theta(\mu) = T(\mu) - \theta(\mu),$$

where

$$\tau = t/\mu,\ \tau_k = (t - \tilde{\Theta}_{k+1}(\mu)))/\mu,$$

$$\tilde{\theta}_{k+1}(\mu) = \theta_0 + \mu\theta_1 + \cdots + \mu^{k+1}\theta_{k+1};$$

$$\tilde{\Theta}_{k+1}(\mu) = \Theta_0 + \mu\Theta_1 + \cdots + \mu^{k+1}\Theta_{k+1};$$

$$\tilde{T}_k(\mu) = T_0 + \mu T_1 + \cdots + \mu^k T_k;$$

$$\|\Pi_i^- y(\tau)\| < Ce^{-\gamma\tau}, C,\gamma > 0,$$

$$\Pi_i^- y(\tau) \equiv 0 \quad for \quad \tau < 0;$$

$$\|\Pi_i^+ y(\tau_{k+1})\| < Ce^{-\gamma\tau_{k+1}},$$

$$\Pi_i^+ y(\tau_{k+1}) \equiv 0 \quad for \quad \tau_{k+1} < 0.$$

Let us denote

$$\bar{y}_0(t) = \begin{cases} \bar{y}_0^+(t) = (\varphi(\bar{s}_0^+(t),\bar{x}_0^+(t),1),\bar{s}_0^+(t),\bar{x}_0^+(t)) \\ \qquad\qquad \text{for} \quad t \in [0,\theta_0]; \\ \bar{y}_0^-(t) = (\varphi(\bar{s}_0^-(t),\bar{x}_0^-(t),1),\bar{s}_0^-(t),\bar{x}_0^-(t)) \\ \qquad\qquad \text{for} \quad t \in [\theta_0,T_0]. \end{cases}$$

Function $\Pi_0^+ z(\tau)$ is defined by equation

$$d\Pi_0^+ z/d\tau = g(\Pi_0^+ z + \varphi(0, \bar{x}_0^+(0), 1), 0, \bar{x}_0^+(0), 1),$$
$$\Pi_0^+ z(0) = \varphi(0, \bar{x}_0^+(0), -1) - \varphi(0, \bar{x}_0^+(0), 1),$$

and $\Pi_0^- z(\tau)$ by

$$d\Pi_0^- z/d\tau = g(\Pi_0^- z + \varphi(0, \bar{x}_0^-(\theta_0), 1), \bar{s}_0^-(\theta_0), \bar{x}_0^-(\theta_0), \theta_0),$$

$$\Pi_0^- z(0) = \varphi(0, \bar{x}_0^-(\theta_0), 1) - \varphi(0, \bar{x}_0^+(\theta_0), -1).$$

To find $\bar{s}_1^{\pm}(t), \bar{x}_1^{\pm}(t), \bar{z}_1^{\pm}(t)$ we have the system of linear equations

$$\begin{aligned}
\bar{z}_1^+(t) &= -[g_z'^+]^{-1}(g_s'^+ \bar{s}_1^+ + g_x'^+ \bar{x}_1^+ + g_1^+(t)),\\
d\bar{s}_1^+/dt &= h_{1z}'^+(t)\bar{z}_1^+(t) + h_{1s}'^+ \bar{s}_1^+(t) + h_{1x}'^+ \bar{x}_1^+(t),\\
d\bar{x}_1^+/dt &= h_{2z}'^+(t)\bar{z}_1^+(t) + h_{2s}'^+ \bar{s}_1^+(t) + h_{2x}'^+ \bar{x}_1^+(t),\\
\bar{z}_1^-(t) &= -[g_z'^-]^{-1}(g_s'^- \bar{s}_1^- + g_x'^- \bar{x}_1^- + g_1^-(t));\\
d\bar{s}_1^-/dt &= h_{1z}'^-(t)\bar{z}_1^-(t) + h_{1s}'^- \bar{s}_1^-(t) + h_{1x}'^- \bar{x}_1^-(t),\\
d\bar{x}_1^-/dt &= h_{2z}'^-(t)\bar{z}_1^-(t) + h_{2s}'^- \bar{s}_1^-(t) + h_{2x}'^- \bar{x}_1^-(t).
\end{aligned} \tag{9}$$

Here the upper index $\pm$ means that we have found the value of corresponding functions at the points $(\varphi(\bar{s}_0^{\pm}(t), \bar{x}_0^{\pm}(t), \pm 1), \bar{s}_0^{\pm}(t), \bar{x}_0^{\pm}(t), \pm 1)$. To find $\Pi_1^{\pm} z$, $\Pi_1^{\pm} s$, $\Pi_1^{\pm} x$ we will have the linear system

$$\begin{aligned}
d\Pi_1^+ z/d\tau &= g_z'\Pi_1^+ z + g_s'\Pi_1^+ s + g_x'\Pi_1^+ x + \Pi_1^+ g(\tau),\\
d\Pi_1^+ s/d\tau &= \Pi_0^+ h_1 = h_1(\bar{z}_0^+(0) + \Pi_0^+ z, 0, \bar{x}_0^+(0), 1)\\
&\quad - h_1(\bar{z}_0^+(0), 0, \bar{x}_0^+(0), 1),\\
d\Pi_1^+ x/d\tau &= \Pi_0^+ h_2 = h_2(\bar{z}_0^+(0) + \Pi_0 z, 0, \bar{x}_0^+(0), 1)\\
&\quad - h_2(\bar{z}_0^+(0), 0, \bar{x}_0^+(0), 1),\\
d\Pi_1^- z/d\tau &= g_z'\Pi_1^- z + g_s'\Pi_1^- s + g_x'\Pi_1^- x + \Pi_1^- g(\tau),\\
d\Pi_1^- s/d\tau &= \Pi_0^- h_1 = h_1(\bar{z}_0^-(\theta_0) + \Pi_0^- z, 0, \bar{x}_0^-(\theta_0), -1)\\
&\quad - h_1(\bar{z}_0^-(\theta_0), 0, \bar{x}_0^-(\theta_0), -1),\\
d\Pi_1^- x/d\tau &= \Pi_0^- h_2 = h_2(\bar{z}_0^-(\theta_0) + \Pi_0^- z, 0, \bar{x}_0^-(\theta_0), -1)\\
&\quad - h_2(\bar{z}_0^-(\theta_0), 0, \bar{x}_0^-(\theta_0), -1),
\end{aligned}$$

where $\bar{z}_0^+(0) = \varphi(0, x_0, 1)$, $\bar{z}_0^-(\theta_0) = \varphi(0, \bar{x}_0^+(\theta_0), -1)$. The upper index $+$ means that the derivatives of function g are computed at the point

$$(\bar{z}_0^+(0) + \Pi_0^+ z, 0, \bar{x}_0, 1),$$

and the upper index $-$ means that the derivatives of function g are computed at the point

$$(\bar{z}_0^-(\theta_0) + \Pi_0^- z, 0, \bar{x}_0^-(\theta_0), -1).$$

The initial conditions for the boundary layer functions for the slow coordinates can be found from the expressions

$$\Pi_1^+ s(0) = \int_\infty^0 \Pi_0^+ h_1(\Theta)d\Theta, \quad \Pi_1^+ x(0) = \int_\infty^0 \Pi_0^+ h_2(\Theta)d\Theta,$$

$$\Pi_1^- s(0) = \int_\infty^0 \Pi_0^- h_1(\Theta)d\Theta, \quad \Pi_1^- x(0) = \int_\infty^0 \Pi_0^- h_2(\Theta)d\Theta.$$

Then $\bar{s}_1^+(0) = -\Pi_1^+ s(0)$; $\bar{s}_1^-(\theta_0) = -\Pi_1^- s(0)$.

Equating the first-order terms in the asymptotic representation of equations $s(T(\mu), \mu) = 0$ and $s(\theta(\mu), \mu) = 0$ correspondingly, we will have

$$\begin{aligned}
\Theta_1 H_1(0, \xi_0, -1) + \bar{s}_1^-(T_0) &= 0,\\
\theta_1 H_1(0, \bar{x}_0^+(\theta_0), 1) + \bar{s}_1^+(\theta_0) &= 0.
\end{aligned} \tag{10}$$

θ_1 and Θ_1 can be uniquely expressed through $\bar{s}_1^+(0)$ and $\bar{s}_1^-(\theta_0)$ by formulas

$$\Theta_1 = -[H_1(0, \xi_0, -1)]^{-1}\bar{s}_1^-(T_0),$$
$$\theta_1 = -[H_1(0, \bar{x}_0^+(\theta_0), 1)]^{-1}\bar{s}_1^+(\theta_0).$$

Substituting these expressions in the continuity and periodicity conditions we will have

$$\bar{x}_1^-(\theta_0) + \Pi_1^- x(0) = \bar{x}_1^+(\theta_0) + \theta_1 H_2(0, \bar{x}_0^+(\theta_0), 1), \tag{11}$$

$$\bar{x}_1^+(0) + \Pi_1^+ x(0) = \bar{x}_1^-(T_0) + \Theta_1 H_2(0, \xi_0, -1). \tag{12}$$

It is necessary to remark that the values $\bar{x}_1^+(\theta_0), \bar{x}_1^-(T_0)$ are linearly depending on $\bar{x}_1^+(0), \bar{x}_1^-(\theta_0)$. Then, expressing $\bar{x}_1^-(\theta_0)$ via $\bar{x}_1^+(0)$ from equation (12) and substituting this expression in (11), we will have a linear equation for $\bar{x}_1^+(0)$. The determinant of this system coincides with $\det(\partial\Psi/\partial\xi)(\xi_0)$. This means that the initial conditions $\bar{s}_1^+(0), \bar{x}_1^+(0), \bar{s}_1^-(\theta_0), \bar{x}_1^-(\theta_0)$ can be found uniquely. To find the first approximation of s, x it is necessary to define the functions $\bar{s}_i^+(t), \bar{x}_i^+(t), i = 0, 1$ on segment $[0, \tilde{T}_1(\mu)]$ as

$$\bar{y}_i(t) = \begin{cases} \bar{y}_i^+(t) = (\bar{z}_i^+(t), \bar{s}_i^+(t), \bar{x}_i^+(t)) \\ \qquad\qquad \text{for} \quad t \in [0, \tilde{\theta}_1(\mu)]; \\ \bar{y}_i^-(t) = (\bar{z}_i^-(t), \bar{s}_i^-(t), \bar{x}_i^-(t)) \\ \qquad\qquad \text{for} \quad t \in [\tilde{\theta}_1(\mu), \tilde{T}_1(\mu)], \ i = 0, 1. \end{cases}$$

Initial conditions for $\Pi_1^{\pm} z$ are uniquely defined by equations

$$\bar{z}_1^+(0) + \Pi_1^+ z(0) = \bar{z}_1^-(T_0) + \Theta_1 d\bar{z}_0^-/dt(T_0),$$
$$\bar{z}_1^-(\theta_0) + \Pi_1^- z(0) = \bar{z}_1^+(\theta_0) + \theta_1 d\bar{z}_0^+/dt(\theta_0).$$

To find the first approximation of variable z it is necessary to find θ_2 and substitute it in the function $\Pi_1^- z(\tau_2)$.

Suppose now, that we have found the functions

$$z_j^{\pm}(t), s_j^{\pm}(t), x_j^{\pm}(t), \Pi_j^{\pm} z(\tau), \Pi_j^{\pm} s(\tau), \Pi_j^{\pm} x(\tau)$$

and constants $\theta_j, \Theta_j, j = 1, ..., k-1$.

Then to find $\bar{s}_k^{\pm}(t), \bar{x}_k^{\pm}(t), \bar{z}_k^{\pm}(t)$ we will have the system of linear differential equations

$$\begin{aligned}
\bar{z}_k^{+}(t) &= -[g_z'^{+}]^{-1}(g_s'^{+}\bar{s}_k^{+} + g_x'^{+}\bar{x}_k^{+} + g_k^{+}(t)),\\
\mathrm{d}\bar{s}_k^{+}/\mathrm{d}t &= h_{1z}'^{+}(t)\bar{z}_k^{+}(t) + h_{1s}'^{+}\bar{s}_k^{+}(t) + h_{1x}'^{+}\bar{x}_k^{+}(t) + h_{1k}^{+}(t),\\
\mathrm{d}\bar{x}_k^{+}/\mathrm{d}t &= h_{2z}'^{+}(t)\bar{z}_k^{+}(t) + h_{2s}'^{+}\bar{s}_k^{+}(t) + h_{2x}'^{+}\bar{x}_k^{+}(t) + h_{2k}^{+}(t),\\
\bar{z}_k^{-}(t) &= -[g_z']^{-1}(g_s'^{-}\bar{s}_k^{-} + g_x'^{-}\bar{x}_k^{-} + g_k^{-}(t));\\
\mathrm{d}\bar{s}_k^{-}/\mathrm{d}t &= h_{1z}'^{-}(t)\bar{z}_k^{-}(t) + h_{1s}'^{-}\bar{s}_k^{-}(t) + h_{1x}'^{-}\bar{x}_k^{-}(t) + h_{1k}^{-}(t),\\
\mathrm{d}\bar{x}_k^{-}/\mathrm{d}t &= h_{2z}'^{-}(t)\bar{z}_k^{-}(t) + h_{2s}'^{-}\bar{s}_k^{-}(t) + h_{2x}'^{-}\bar{x}_k^{-}(t) + h_{2k}^{-}(t),
\end{aligned} \tag{13}$$

here the upper index $\pm$ means that we have found the value of corresponding functions at the points

$$(\varphi(\bar{s}_0^{\pm}(t), \bar{x}_0^{\pm}(t), \pm 1), \bar{s}_0^{\pm}(t), \bar{x}_0^{\pm}(t), \pm 1).$$

The functions $g_k^{\pm}(t), h_{1k}^{\pm}(t), h_{2k}^{\pm}(t)$ are uniquely determined functions depending on

$$\bar{z}_j^{\pm}(t), \bar{s}_j^{\pm}(t), \bar{x}_j^{\pm}(t), \theta_j,\ \Theta_j, j = 1, ..., k-1.$$

For $\Pi_k^{\pm}z$, $\Pi_k^{\pm}s$, $\Pi_k^{\pm}x$ we will have the linear system

$$\begin{aligned}
&\mathrm{d}\Pi_k^{+}z/\mathrm{d}\tau = g_z'\Pi_k^{+}z + g_s'\Pi_k^{+}s + g_x'\Pi_k^{+}x + \Pi_k^{+}g(\tau),\\
&\mathrm{d}\Pi_k^{+}s/\mathrm{d}\tau = \Pi_{k-1}^{+}h_1; \quad \mathrm{d}\Pi_k^{+}x/d\tau = \Pi_{k-1}^{+}h_2;\\
&\mathrm{d}\Pi_k^{-}z/\mathrm{d}\tau = g_z'\Pi_k^{-}z + g_s'\Pi_k^{-}s + g_x'\Pi_k^{-}x + \Pi_k^{-}g(\tau),\\
&\mathrm{d}\Pi_k^{-}s/\mathrm{d}\tau = \Pi_{k-1}^{-}h_1; \quad \mathrm{d}\Pi_k^{-}x/\mathrm{d}\tau = \Pi_{k-1}^{-}h_2;
\end{aligned}$$

where the upper index $+$ means that the derivatives of function g are computed at the point

$$(\bar{z}_0^{+}(0) + \Pi_0^{+}z, 0, \bar{x}_0, 1),$$

the upper index $-$ means that the derivatives of function g are computed at the point

$$(\bar{z}_0^{-}(\theta_0) + \Pi_0^{-}z, 0, \bar{x}_0^{-}(\theta_0), -1),$$

$\Pi_{k-1}^{\pm}h_1$, $\Pi_{k-1}^{\pm}h_2$ are the functions dependent on the $\Pi_j^{\pm}z(\tau), \Pi_j^{\pm}s(\tau), \Pi_j^{\pm}x(\tau), j = 1, ..., k-1$ only.

The initial conditions for the boundary layer functions are dependent on the inequalities

$$\Pi_k^{+}s(0) = \int_{\infty}^{0} \Pi_{k-1}^{+}h_1(\Theta)\mathrm{d}\Theta,$$

$$\Pi_k^{+}x(0) = \int_{\infty}^{0} \Pi_{k-1}^{+}h_2(\Theta)\mathrm{d}\Theta,$$

$$\Pi_k^{-}s(0) = \int_{\infty}^{0} \Pi_{k-1}^{-}h_1(\Theta)\mathrm{d}\Theta,$$

$$\Pi_k^{-}x(0) = \int_{\infty}^{0} \Pi_{k-1}^{-}h_2(\Theta)\mathrm{d}\Theta.$$

Then $\bar{s}_k^{+}(0) = -\Pi_k^{+}s(0); \quad \bar{s}_k^{-}(\theta_0) = -\Pi_k^{-}s(0)$.

To find the initial condition functions $\bar{x}_k^{+}(0), \bar{x}_k^{-}(\theta_0)$ we have to equate the kth asymptotic representation in equalities $s(T(\mu), \mu) = 0$ and $s(\theta(\mu), \mu) = 0$. Then we will have

$$\begin{aligned}
\Theta_k H_1(0, \xi_0, -1) + \bar{s}_k^{-}(T_0) + \mathcal{S}_k^{-} &= 0,\\
\theta_k H_1(0, \bar{x}_0^{+}(\theta_0), 1) + \bar{s}_k^{+}(\theta_0) + \mathcal{S}_k^{+} &= 0.
\end{aligned} \tag{14}$$

Expressing θ_k, Θ_k and substituting the corresponding formulas in the conditions of continuity and periodicity we obtain

$$\bar{x}_k^{-}(\theta_0) + \Pi_k^{-}x(0) = \bar{x}_k^{+}(\theta_0) + \theta_k H_2(0, \bar{x}_0^{+}(\theta_0), 1) + \mathcal{X}_k^{+} \tag{15}$$

$$\bar{x}_k^{+}(0) + \Pi_k^{+}x(0) = \bar{x}_k^{-}(T_0) + \Theta_k H_2(0, \xi_0, -1) + \mathcal{X}_k^{-}. \tag{16}$$

Here $\mathcal{S}_k^{\pm}$, $\mathcal{X}_k^{\pm}$ is the function depending on $\bar{s}_j^{+}(\theta_0)$, $\bar{x}_j^{+}(\theta_0)$, $\bar{s}_j^{-}(T_0)$, $\bar{x}_j^{-}(T_0)$, $j = 1, \ldots, k-1$.

θ_k and Θ_k could be uniquely expressed trough $\bar{x}_k^{+}(0)$ and $\bar{x}_k^{-}(\theta_0)$ in the form

$$\Theta_k = -[H_k(0, \xi_0, -1)]^{-1}[\bar{s}_k^{-}(T_0) + \mathcal{S}_k^{-}],$$

$$\theta_k = -[H_k(0, \bar{x}_0^{+}(\theta_0), 1)]^{-1}[\bar{s}_k^{+}(\theta_0) + \mathcal{S}_k^{+}].$$

Here $\bar{x}_k^{+}(\theta_0), \bar{x}_k^{-}(T_0)$ are linearly dependent on $\bar{x}_k^{+}(0), \bar{x}_k^{-}(\theta_0)$. Then substituting the formulas for θ_k, Θ_k into the conditions of continuity (15) and periodicity (16) we will have the system of algebraic equations which depends linearly on $\bar{x}_k^{+}(0), \bar{x}_k^{-}(\theta_0)$. Now it is possible to express $\bar{x}_k^{-}(\theta_0)$ via $\bar{x}_k^{+}(0)$ from the periodicity conditions (16). Then, substituting this expression in (15), we will have the system of algebraic equations linear in $\bar{x}_k^{+}(0)$. The determinant of this system coincides with $\det(\partial\Psi/\partial\xi)(\xi_0)$.

This means that $\bar{s}_k^{+}(0), \bar{x}_k^{+}(0), \bar{s}_k^{-}(\theta_0), \bar{x}_k^{-}(\theta_0)$ are defined uniquely. Now to find $S_k(t, \mu), X_k(t, \mu)$ its necessary to define functions $\bar{s}_i^{+}(t), \bar{x}_i^{+}(t)$ on the segment $[0, \tilde{\theta}_k(\mu)]$ as

$$\bar{y}_i(t) = \begin{cases} \bar{y}_i^{+}(t) = (\bar{z}_i^{+}(t), \bar{s}_i^{+}(t), \bar{x}_i^{+}(t)) \\ \qquad \text{for} \quad t \in [0, \tilde{\theta}_k(\mu)]; \\ \bar{y}_i^{-}(t) = (\bar{z}_i^{-}(t), \bar{s}_i^{-}(t), \bar{x}_i^{-}(t)) \\ \qquad \text{for} \quad t \in [\tilde{\theta}_k(\mu), \tilde{T}_k(\mu)],\ i = 0, \ldots, k. \end{cases}$$

The initial conditions for $\Pi_k^{\pm}z$ are uniquely defined from equations

$$\bar{z}_k^{+}(0) + \Pi_k^{+}z(0) = \bar{z}_k^{-}(T_0) + \Theta_k \mathrm{d}\bar{z}_0^{-}/\mathrm{d}t(T_0) + Z_k^{-},$$

$$\bar{z}_k^{-}(\theta_0) + \Pi_k^{-}z(0) = \bar{z}_k^{+}(\theta_0) + \theta_k \mathrm{d}\bar{z}_0^{+}/\mathrm{d}t(\theta_0) + Z_k^{+},$$

where $Z_k^{\pm}$ are the functions depending on $\bar{z}_j^{+}(\theta_0), \bar{z}_j^{-}(T_0)$, $j = 1, \ldots, k-1$.

To finish with a kth order asymptotic representation it is necessary to find the value θ_{k+1} and introduce this value into the function $\Pi_k^{-}z(\tau_{k+1})$.

Therem 3: *Under conditions* 1^0–6^0

$$|\tilde{T}_k(\mu) - T(\mu)| < C\mu^{k+1}$$

uniformly on $[0, \hat{T}(\mu)]$, *where*

$$\hat{T}(\mu) = max\{T(\mu); \tilde{T}_{k+1}(\mu)\},$$

the following inequalities are true

$$\|y(t,\mu) - Y_k(t,\mu)\| < C\mu^{k+1};$$

$$\|(s(t,\mu), x(t,\mu)) - (S_k(t,\mu), X_k(t,\mu))\| < C\mu^{k+1}. \quad (17)$$

Theorem 3 follows from Theorem 2 and Lemma 4, which will be proved in Appendix 1.

6. Example

Let us show the existence of orbitally asymptotically stable periodic solutions for a singularly perturbed relay system in form

$$\mu\, dz/dt = -z - u; \quad ds/dt = x + u/2;$$
$$dx/dt = -x + z, \quad u = \mathrm{sign}[s(t)], \quad (18)$$

where $z, s, x \in R$, μ is small parameter. For $\mu = 0$ system (18) has the form

$$\bar{z}_0 = -u,\ d\bar{s}_0/dt = \bar{x}_0 + u/2;\ d\bar{x}_0/dt = -\bar{x}_0 - u. \quad (19)$$

Then the solution of system (19) with initial conditions

$$\bar{x}_0^+(0) = \xi > 0, \quad \bar{s}_0^+(0) = 0$$

takes the form

$$\bar{x}_0{}^+(t,\xi) = e^{-t}(\xi + 1) - 1;$$
$$\bar{s}_0^+(t,\xi) = (1 - e^{-t})(\xi + 1) - t/2.$$

System (19) is symmetrical with respect to the origin. Consequently, the semiperiod θ_0 of its periodic solution and fixed point ξ_0 of the Poincar*é* map $\Psi(\xi)$ (figure 2) are defined by equations

$$\bar{s}_0^+(\theta_0, \xi_0) = 0, \quad \bar{x}_0^+(\theta_0, \xi_0) = -\xi_0,$$

and consequently

$$\xi_0 = \frac{1 - e^{-\theta_0}}{1 + e^{-\theta_0}}; \quad \xi_0 = \theta_0/4.$$

Then $\theta_0 \approx 3.83, \xi_0 \approx 0.96$.

Moreover

$$(\partial\Psi/\partial\xi)^{1/2}(\xi_0) = \frac{1 - e^{-\theta_0} - \theta_0}{e^{\theta_0} - \theta_0 - 1} \approx -0.07.$$

This means that for system (18) the conditions of Theorems 2 and 3 hold.

The slow part of the zero approximation of the desired periodic solution for coordinate z has the form

$$\bar{z}_0(t) = \begin{cases} \bar{z}_0^+(t) = 1 & \text{for} \quad 0 \le t \le \theta_0, \\ \bar{z}_0^-(t) = -1, & \text{for} \quad \theta_0 \le t \le T_0. \end{cases}$$

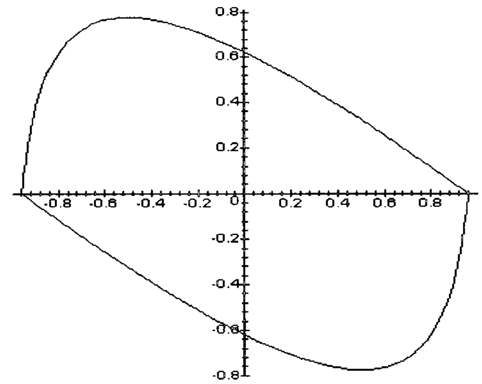

Figure 3. Periodic solution of a reduced system.

The zero-order boundary layer functions satisfy the equations

$$d\Pi_0^+ z/d\tau = -\Pi_0^+ z; \quad \Pi_0^+ z(0) = 2; \quad \Pi_0^+ z(\tau^+) = 2e^{-\tau^+};$$

$$d\Pi_0^- z/d\tau = -\Pi_0^- z; \quad \Pi_0^- z(0) = -2; \ \Pi_0^- z(\tau^-) = -2e^{-\tau^-}.$$

The equations for the first-order terms of the regular part take the form

$$\bar{z}_1^\pm = 0; \quad d\bar{s}_1^\pm/dt = x_1^\pm; \quad d\bar{x}_1^\pm/dt = -\bar{x}_1^\pm,$$

and consequently

$$\bar{x}_1^+(t, \bar{x}_1^+(0)) = \bar{x}_1^+(0)e^{-t};$$
$$\bar{s}_1^+(t) = (1 - e^{-t})\bar{x}_1^+(0) + \bar{s}_1^+(0);$$
$$\bar{x}_1^-(t, \bar{x}_1^-(\theta_0)) = \bar{x}_1^-(\theta_0)e^{-(t-\theta_0)};$$
$$\bar{s}_1^-(t) = (1 - e^{-(t-\theta_0)})\bar{x}_1^-(\theta_0) + \bar{s}_1^-(\theta_0).$$

Now the first order boundary layer terms are described by equations

$$\Pi_1^+ s(\tau) \equiv 0; \quad \Pi_1^+ s(0) = 0;$$
$$\Pi_1^+ x(\tau) = \int_\infty^\tau \Pi_0^+ z(\Theta)d\Theta; \quad \Pi_1^+ x(0) = -2;$$
$$\Pi_1^- s(\tau) \equiv 0; \quad \Pi_1^- s(0) = 0;$$
$$\Pi_1^- x(\tau) = \int_\infty^\tau \Pi_0^+ z(\Theta)d\Theta; \quad \Pi_1^- x(0) = 2.$$

Then $\bar{s}_1^+(0) = \bar{s}_1^-(\theta_0) = 0$.

Equations for θ_1 and Θ_1 in that case have the form

$$\Theta_1(\xi_0 - 1/2) + \bar{s}_1^-(T_0) = 0;$$
$$\theta_1(\bar{x}_0^+(\theta_0) + 1/2) + \bar{s}_1^+(\theta_0) = 0,$$

and consequently θ_1 and Θ_1 can be expressed via $\bar{x}_1^+(0)$, $\bar{x}_1^-(\theta_0)$ according to the formulas

$$\theta_1 = -\frac{(1 - e^{-\theta_0})\bar{x}_1^+(0)}{\bar{x}_0^+(\theta_0) + 1/2}; \quad \Theta_1 = -\frac{(1 - e^{-\theta_0})\bar{x}_1^-(\theta_0)}{\xi_0 - 1/2}.$$

Taking into account the symmetry of (19) we will have

$$\theta_1 = 2\bar{x}_1^+(0)\frac{1-e^{-2\theta_0}}{1-3e^{-\theta_0}}.$$

Then the condition of continuity has the form

$$-\bar{x}_1^+(0) + \Pi_1^-(0) = \bar{x}_1^+(0)\left(-3e^{-\theta_0} + \frac{e^{-2\theta_0}}{1-3e^{-\theta_0}}\right).$$

Now

$$\bar{x}_1^+(0) = 2\frac{1-3e^{-\theta_0}}{e^{-2\theta_0}-6e^{-\theta_0}+1} \approx 2.15;$$

$$\theta_1 = 4\frac{1-e^{-2\theta_0}}{e^{-2\theta_0}-6e^{-\theta_0}+1} \approx 4.60.$$

The asymptotic representation of the semiperiod for the desired periodic solution has the form

$$T(\mu) \approx 3.83 + 4.60\mu + O(\mu^2).$$

The results of asymptotic representation of the x coordinate for system (18) solution are shown in the figures 4 and 5.

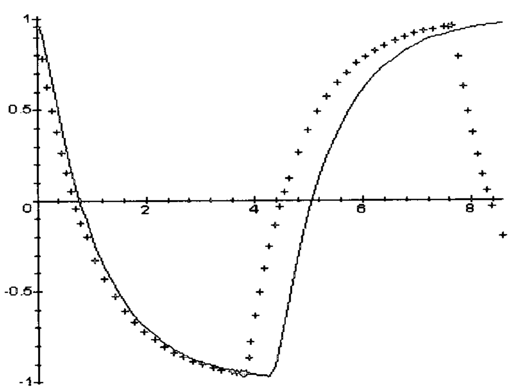

Figure 4. x coordinate for the periodic solution of a reduced (points) and an original (line) system for $\mu = 0.1$.

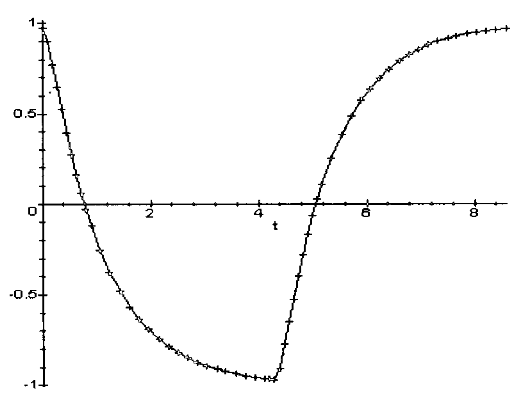

Figure 5. x coordinate for a periodic solution of original (line) system and its asymptotic (points) for $\mu = 0.1$.

7. Conclusions

The singularly perturbed relay control systems (SPRCS) having exponentially orbitally stable periodic motions for the reduced systems have been studied. It is shown that the slow motions integral manifold of SPRCS consists of parts corresponding to different values of control. Sufficient conditions are found for existence of the isolated exponentially orbitally stable periodic solutions. It is proved that such periodic solutions contain the jumps from one part of the slow manifold to the other. An algorithm for the asymptotic representation of this periodic solutions basing on the boundary functions method is suggested.

It allows one to conclude that relay control design based on the existence of exponentially stable periodic solutions is robust with respect to stable unmodeled dynamics. It is not true for sliding mode systems (e.g. Fridman and Levant 2002).

Appendix 1: Asymptotic representations of SPRCS solutions with the finite number of switchings

Consider the solution of the Cauchy problem for system (1) with initial conditions (1). Suppose that for system (1) conditions 1^0–3^0 are true and moreover:

4* there exists a smallest positive root of equation $\bar{s}_0^+(\theta_0) = 0$, where $(\bar{s}_0^+(t), \bar{x}_0^+(t))$ is the solution of system (4) with initial conditions $\bar{s}_0^+(0) = 0$, $\bar{x}_0^+(0) = \xi$ and moreover

$$\mathrm{d}\bar{s}_0^+/\mathrm{d}t(\theta_0) = H_1(0, \bar{x}_0^+(\theta_0), 1) < 0$$
$$H_1(0, \bar{x}_0^+(\theta_0), -1) < 0.$$

5* the point $\varphi(0, \bar{x}_0^+(\theta_0), 1)$ is the internal point of the attractive domain for the equilibrium point $\varphi(0, \bar{x}_0^+(\theta_0), -1)$;

6* for every $t \in [\theta_0, T]$ there exists an unique solution of system (6) $(\bar{s}_0^-(t), \bar{x}_0^-(t))$ with initial conditions $\bar{s}_0^-(\theta_0) = 0$, $\bar{x}_0^-(\theta_0) = \bar{x}_0^+(\theta_0)$, moreover for every $t \in [\theta_0, T]$ sign $\bar{s}_0^-(t) < 0$, $(\varphi(\bar{s}_0^-(t), \bar{x}_0^-(t), -1), \bar{s}_0^-(t), \bar{x}_0^-(t)) \in Z$.

Then from Tikhonov's theorem (e.g. Vasil'eva *et al.* 1995), the implicit function theorem and conditions 4* it follows that for sufficiently small μ there exists a unique solution of the Cauchy problem (1), (7) for $u = 1$ until the switching moment $\theta(\mu)$ $(\theta(\mu) \to \theta_0$ for $\mu \to 0)$ in which this solution crosses the surface $s = 0$. Moreover, from condition 6* it follows that the solution of the Cauchy problem (1), (7) is situated in the domain $s < 0$ for every $t \in [\theta(\mu), T]$. This means that the solution of the Cauchy problem (1), (7) on $[0, T]$ is reduced to the sequential solution of two Cauchy problems:

(i) solution $(z^+(t,\mu), s^+(t,\mu), x^+(t,\mu))$ problem (1),(7) for $u = 1$;

(ii) solution $(z^-(t,\mu), s^-(t,\mu), x^-(t,\mu))$ of system (1) for $u = -1$ with initial conditions

$$z^-(\theta(\mu),\mu) = z^+(\theta(\mu),\mu);$$
$$s^-(\theta(\mu),\mu) = s^+(\theta(\mu),\mu) = 0;$$
$$x^-(\theta(\mu),\mu) = x^+(\theta(\mu),\mu). \tag{20}$$

Let us find the asymptotic representation of the switching moment $\theta(\mu)$ in (5), and the asymptotic representation of the solution in (5).

The coefficients of representation (5) can be found from equation

$$s^+(\theta_0 + \mu\theta_1 + \cdots + \mu^i\theta_i + \cdots, \mu) = 0.$$

Suppose that $\bar{s}_0^+(t), \bar{s}_1^+(t), \ldots, \bar{s}_i^+(t), \ldots$ the coefficients of the asymptotic representation for the Cauchy problem (i) using the boundary layer method are found. Then it is possible to rewrite equation (7) in the form

$$\bar{s}_0^+(\theta_0 \cdots + \mu^i\theta_i + \cdots) + \mu\bar{s}_1^+(\theta_0 + \cdots + \mu^i\theta_i + \cdots)$$
$$+ \cdots + \mu^i\bar{s}_i^+(\theta_0 + \cdots + \mu^i\theta_i + \cdots) + \cdots = 0.$$

The boundary layer functions $\Pi^+ s(\theta(\mu)/\mu)$ are exponentially small, which is why it is possible not to take those functions into account. Then to find θ_i we have the linear equations in the form

$$\theta_i H_1(0, \bar{x}_0^+(\theta_0), 1) + p_i(\theta_0, \theta_1, \ldots, \theta_{i-1}) = 0,$$

where p_i are the functions, depending on $\theta_0, \theta_1, \ldots, \theta_{i-1}$ only. $H_1(0, \bar{x}_0^+(\theta_0), 1) < 0$. Consequently constants $\theta_0, \theta_1, \ldots, \theta_i, \ldots$ can be found uniquely.

Suppose that we have found $\theta_0, \theta_1, \ldots, \theta_{k+1}$ and the coefficients of the regular part in the asymptotic representation for the Cauchy problem (i) $\bar{y}_i^+(t)$, $(i = 1, \ldots, k)$.

Let us find some special asymptotic approximation for

$$\bar{Y}_k^+(\tilde{\theta}_k(\mu),\mu) = \sum_{i=0}^{k} \bar{y}_i^+(\tilde{\theta}_k(\mu))\mu^i$$

which is the segment of the regular part in the asymptotic representation $y^+(\tilde{\theta}_k(\mu),\mu)$. Taking into account the asymptotic representation for functions $\bar{y}_i^+(\tilde{\theta}_k(\mu))$ of degree μ, consider $\hat{Y}_k^+(\tilde{\theta}_k(\mu),\mu)$ instead of $\bar{Y}_k^+(\tilde{\theta}_k(\mu),\mu)$. Then

$$\bar{Y}_k^+(\tilde{\theta}_k(\mu),\mu) = \bar{y}_0^-(\theta_0) + \mu(\bar{y}_1^+(\theta_0) + \theta_1 d\bar{y}_0^+/dt(\theta_0)) + \cdots$$

Let us denote by $\hat{Y}_k^-(\tilde{\theta}_k(\mu),\mu)$ the segment of this series up to the μ^k degree. Now to find the asymptotic solution of the Cauchy problem (1), (20) it is necessary to use the asymptotic representation of system (1) for $u = -1$ with initial conditions

$$y^-(\tilde{\theta}_{k+1}(\mu),\mu) = \hat{Y}_k^+(\tilde{\theta}_k(\mu),\mu). \tag{21}$$

Lemma 4: *Under conditions 1^0–3^0 and 4*–6* there exists a μ_0, such that for every $\mu \in [0,\mu_0]$ there exists a solution of Cauchy problem (1), (7) on $t \in [0,T]$ and uniformly on $[0,T]$ inequality (17) is true.*

Proof: Suppose that for sufficiently small μ $\theta(\mu) < \tilde{\theta}_{k+1}(\mu)$. For some $K_1 > 0$

$$|\theta(\mu) - \tilde{\theta}_{k+1}(\mu)| < K_1\mu^{k+2},$$
$$\|y^+(\theta(\mu),\mu) - \hat{Y}_k^+(\theta(\mu),\mu)\| < K_1\mu^{k+1}.$$

Then the solution of the Cauchy problem (1), (7) on the segment $[\theta(\mu), \tilde{\theta}_{k+1}(\mu)]$ the solution (1), (7) coincides with the solution of (1), (20) for $u = -1$ and for all $t \in [\theta(\mu), \tilde{\theta}_{k+1}(\mu)]$

$$\|z^-(t,\mu) - z^+(\theta(\mu),\mu)\|$$
$$= \int_{\theta(\mu)}^{t} \|g(z^-(\tau,\mu), s^-(\tau,\mu), x^-(\tau,\mu), 1)d\tau/\mu\|$$
$$< M_1|\tilde{\theta}_{k+1}(\mu) - \theta(\mu)|/\mu;$$
$$\|(s^-(t,\mu), x^-(t,\mu)) - (s^+(\theta(\mu),\mu), x^+(\theta(\mu),\mu))\|$$
$$< M_2|\tilde{\theta}_{k+1}(\mu) - \theta(\mu)|;$$
$$\|\bar{Y}_k^+(t,\mu) - \hat{Y}_k^+(\tilde{\theta}_k(\mu),\mu)\| < K_2\mu^{k+1},\ K_2 > 0,$$
$$M_1 = \sup_{\bar{Z}\times[-1,1]} \|g(z,s,x,u)\|,$$
$$M_2 = \sup_{\bar{Z}\times[-1,1]} \|(h_1((z,s,x,u), h_2(z,s,x,u))\|.$$

This means that there exists $K_3 > 0$, such that

$$\|\hat{Y}_k^-(\tilde{\theta}_k(\mu),\mu) - y^+(\tilde{\theta}_{k+1}(\mu),\mu)\| < K_3\mu^{k+1}.$$

This inequality ensures inequalities (17) on $[\theta(\mu),\ \tilde{\theta}_{k+1}(\mu)]$. Now from the boundary layer method (Vasil'eva *et al.* 1995) it follows that the asymptotic representations of the Cauchy problems (1), (21) and (1), (21) on $t \in [\tilde{\theta}_{k+1}(\mu), T]$ coincide up to the terms μ^k on the segments $[0,\theta(\mu)]$, $[\tilde{\theta}_{k+1}(\mu), T]$ and consequently the inequalities in (17) are true on the segment $[0,T]$.

The proof for the case $\tilde{\theta}_{k+1}(\mu) < \theta(\mu)$ can be make analogously.

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