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On nonlinear H^∞ sliding mode control for a class of nonlinear cascade systems

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In this work two main robust control strategies, the sliding mode control (SMC) and nonlinear H^∞ control, are integrated to function in a complementary manner for tracking control tasks. The SMC handles matched $L_\infty[0,\infty)$ type system uncertainties with known bounding functions. H^∞ control deals with unmatched disturbances of $L_2[0,\infty)$ type where the upper-bound knowledge is not available. The new control method is designed for a class of nonlinear uncertain systems with two cascade subsystems. Nonlinear H^∞ control is applied to the first subsystem in the presence of unmatched disturbances. Through solving a Hamilton-Jacoby inequality, the nonlinear H^∞ control law for the first subsystem well defines a nonlinear switching surface. By virtue of nonlinear H^∞ control, the resulting sliding manifold in the sliding phase possesses the desired L_2 gain property and to a certain extend the optimality. Associated with the new switching surface, the SMC is applied to the second subsystem to accomplish the tracking task, and ensure the L_2 gain robustness in the reaching phase. Two illustrative examples are given to show the effectiveness of the proposed robust control scheme.

Keywords: Sliding mode control; H^∞ control; Nonlinear cascade systems; L_2 gain; Matched and unmatched uncertainties

1. Introduction

Sliding mode control and H^∞ control are well recognized as two major robust control strategies. SMC can handle matched $L_\infty[0,\infty)$ type system disturbance where the upper-bound knowledge is available (Utkin 1992). On the other hand, nonlinear H^∞ control can deal with $L_2[0,\infty)$ type unmatched disturbance even if the upper-bound knowledge is not available.

In this paper, sliding mode control is incorporated with nonlinear H^∞ control for a class of cascade nonlinear systems which consists of a null space dynamics and range space dynamics. Nonlinear cascade systems exist in many real applications. One typical such field is mechatronics which consist of electro subsystem and mechanical subsystem in cascade form. In literature, various cases have been considered, for instance the stabilization of translational oscillations by a rotational actuator (TORA) system (Jankovic *et al*. 1996), the control of an underactuated surface ship with disturbances to follow a predefined path at a desired speed (Do *et al*. 2004), the control of rigid spacecraft with actuator dynamics (Dalsmo and Maas 1998), etc.

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A common practice in SMC is to design a switching surface according to the null space dynamics, which must ensure a stable sliding manifold when the system is in sliding mode (Edwards and Spurgeon 1998). For known LTI systems, such design turns to be pole-placement (Zinober 1994, Chang and Chen 2000) or LQR (Young and Ozguner 1997). For known nonlinear systems, nonlinear optimal design can be applied (Xu and Zhang 2002). However if there exist uncertaities in the null space nonlinear dynamics, switching surface design becomes extremely difficult. The challenge lies in that we need a systematic design which captures the inherent relationship between the switching surface and the sliding manifold, in the sequel yields a stable sliding manifold. Note that both switching surface and sliding manifold may be highly nonlinear in nature.

Nonlinear H^∞ control offers such a systematic design, which can allow even the presence of unmatched uncertainties of $L_2[0,\infty)$ type, and achieves a desired L_2 gain. The L_2 gain of a nonlinear system has been known to be a useful measure for stabilization and performance, e.g. the finite gain stability and H^∞ disturbance attenuation (Schaft 1991, 1992). By applying the nonlinear H^∞ control to the null space dynamics, the resulting robust control law defines a suitable switching surface. The L_2 gain property is realized when the sliding mode occurs.

In SMC, designing a suitable reaching control law is as important as designing an appropriate switching surface. Traditionally this control law is to force the system to reach and then stay on the switching surface. Nevertheless, this feature alone is no longer sufficient when the unmatched null space uncertrainties are present. The null space nonlinear dynamics may go diverging in a period shorter than the reaching time, if the L_2 gain property does not hold during the reaching phase. Obviously, L_2 gain property should be guaranteed not only for the sliding phase, but also for the reaching phase. This is achieved in the proposed control by again combining SMC and nonlinear H^∞ control.

The paper is organized as follows. In section 2, the problem is presented. In section 3, a nonlinear H^∞ switching surface is designed. Section 4 proposes a nonlinear H^∞ sliding mode control scheme. Two illustrative examples are given in section 5 to show the effectiveness of the proposed scheme. Section 6 draws the conclusions.

Notations: $\mathcal{R}^n$ denotes an n-dimension real vector space, $\|\cdot\|$ is the Euclidean norm and induced matrix norm, $L_2[0,\infty)$ is the space of square integrable functions on $[0,\infty)$, $L_\infty[0,\infty)$ is the space of uniformly bounded functions on $[0,\infty)$, $(D_{\mathbf{x}}f) = \partial f(\mathbf{x},\mathbf{y})/\partial \mathbf{x}$ and $(D_{\mathbf{y}}f) = \partial f(\mathbf{x},\mathbf{y})/\partial \mathbf{y}$ are row vectors.

2. Problem fomulation

Consider the following nonlinear cascade system

$$\begin{cases} \dot{\mathbf{x}}_1 = \mathbf{f}_1(\mathbf{x}_1,t) + B_1(\mathbf{x}_1,t)\boldsymbol{\varphi}(\mathbf{x}_2) + C_1(\mathbf{x}_1,t)\mathbf{w}_1 \\ \dot{\mathbf{x}}_2 = \mathbf{f}_2(\mathbf{x},t) + B_2(\mathbf{x},t)[I + \Delta B_2(\mathbf{x},t)][\mathbf{u} + \mathbf{w}_2(\mathbf{x},t)], \end{cases} \tag{1}$$

where $\mathbf{x} = [\mathbf{x}_1^T,\ \mathbf{x}_2^T]^T$ is a physically measurable state vector, $\mathbf{x}_1 \in \mathcal{R}^n$ is the null space dynamics and $\mathbf{x}_2 \in \mathcal{R}^m$ is the range space dynamics, $\mathbf{u} \in \mathcal{R}^m$ denotes the control input, $\mathbf{w}_1 \in \mathcal{R}^l$ is the external disturbance and $\mathbf{w}_2 \in \mathcal{R}^m$ is the matched uncertainties. The mappings $\mathbf{f}_1 \in \mathcal{R}^n$, $\boldsymbol{\varphi} \in \mathcal{R}^m$, $\mathbf{f}_2 \in \mathcal{R}^m$, $B_1 \in \mathcal{R}^{n\times m}$, $C_1 \in \mathcal{R}^{n\times l}$ and $B_2 \in \mathcal{R}^{m\times m}$ are known and smooth with respect to $\mathbf{x}$ and continuous with respect to time t. $\Delta B_2 \in \mathcal{R}^{m\times m}$ represent the uncertainties in the control input. $\partial\boldsymbol{\varphi}/\partial\mathbf{x}_2 \neq 0$ is bounded in $\mathcal{D} \subset \mathcal{R}^m \cap [0,\infty)$. The relation $m \le n$ holds for the system.

In this paper, the $\mathbf{x}_1$ subsystem in (1) is required to track the desired reference model

$$\dot{\mathbf{x}}_{1d} = \bar{\mathbf{f}}(\mathbf{x}_{1d},\mathbf{r}(t),t), \tag{2}$$

where $\mathbf{r}(t)$ is a smooth reference input. Define $\mathbf{e}_1 = \mathbf{x}_1 - \mathbf{x}_{1d}$. The error dynamics of the $\mathbf{x}_1$ subsystem can be written as

$$\begin{aligned} \dot{\mathbf{e}}_1 &= \mathbf{f}_1 + B_1\boldsymbol{\varphi} + C_1\mathbf{w}_1 - \dot{\mathbf{x}}_{1d} \\ &= \mathbf{f}_1 + B_1\boldsymbol{\varphi} + C_1\mathbf{w}_1 - \bar{\mathbf{f}}. \end{aligned} \tag{3}$$

The system (1) is assumed to satisfy the following assumptions.

Assumption 2.1: *The control input uncertainty satisfies* $\|\Delta B_2\| = \sqrt{\lambda_{\max}(\Delta B_2^T \Delta B_2)} \le \varepsilon_{b2}(\mathbf{x},t)$, *where* $0 \le \varepsilon_{b2}(\mathbf{x},t) < 1$ *is a positive function.*

Assumption 2.2: *The matched uncertainties in* $\mathbf{x}_2$ *subsystem is norm bounded by a known function, i.e.* $\|\mathbf{w}_2\| = \sqrt{w_{2,1}^2 + \cdots + w_{2,m}^2} \le \phi_2(\mathbf{x},t)$, *where* $\phi_2(\mathbf{x},t) \ge 0$ *is a known positive function.*

Assumption 2.3: *The matrix* $(D_{\mathbf{x}_2}\boldsymbol{\varphi})B_2$ *is of full rank for* $\forall \mathbf{x}_2$.

Assumption 2.4: *There exists a smooth function* $\boldsymbol{\zeta}(\cdot)$ *such that the following matching condition holds,*

$$\mathbf{f}_1(\mathbf{x}_1,t) - \bar{\mathbf{f}}(\mathbf{x}_{1d},\mathbf{r}(t),t) = \mathbf{g}_1(\mathbf{e}_1,t) + B_1(\mathbf{x}_1,t)\boldsymbol{\zeta}(\mathbf{x}_1,\mathbf{x}_{1d},\mathbf{r}(t),t),$$

where $\dot{\boldsymbol{\xi}} = \mathbf{g}_1(\boldsymbol{\xi},t)$ *is asymptotically stable.*

$\boldsymbol{\zeta}(\cdot)$ *is a smooth function with respect to its arguments.*

According Assumption 2.4 and the error dynamics (3), system (1) with the control objective (2) can be written as

$$\dot{\mathbf{e}}_1 = \mathbf{g}_1 + B_1(\boldsymbol{\varphi} + \boldsymbol{\zeta}) + C_1\mathbf{w}_1 \quad (4)$$

$$\dot{\mathbf{x}}_2 = \mathbf{f}_2 + B_2(I + \Delta B_2)(\mathbf{u} + \mathbf{w}_2). \quad (5)$$

3. Nonlinear H^{∞} sliding mode design

The nonlinear H^{∞} control aims to obtain a prespecified performance–disturbance attenuation from the external disturbance to the state variable, i.e. the robust L_2 gain ρ_1 from $\mathbf{w}_1$ to $\mathbf{e}_1$, as defined below

$$\int_0^t \|\mathbf{e}_1\|^2\,\mathrm{d}\tau \leq \beta_1(\mathbf{e}_1(0),0) + \rho_1^2\int_0^t \|\mathbf{w}_1\|^2\,\mathrm{d}\tau, \quad \forall \mathbf{w}_1 \in L_2[0,t), \quad (6)$$

where $t \in [0,\infty)$ and $\beta_1(\mathbf{e}_1(0),0)$ is a positive real valued function depending on the initial conditions.

For $\mathbf{e}_1$ subsystem, $\boldsymbol{\varphi}(\mathbf{x}_2)$ can be regarded as a virtual control input. If $C_1\mathbf{w}_1$ is absent from the null space dynamics (4), the switching surface design can be easily accomplished by choosing $\boldsymbol{\sigma} = \boldsymbol{\varphi}(\mathbf{x}_2) + \boldsymbol{\zeta}$. Once entering the sliding mode, $\boldsymbol{\sigma} = \boldsymbol{\varphi}(\mathbf{x}_2) + \boldsymbol{\zeta} = 0$. Then the error dynamics (4) become $\dot{\mathbf{e}}_1 = \mathbf{g}_1(\mathbf{e}_1, t)$, that is, the resulting sliding manifold is asymptotically stable and $\mathbf{e}_1 \to 0$ as $t \to \infty$. If $C_1\mathbf{w}_1$ is present, $\boldsymbol{\sigma} = 0$ renders a sliding manifold

$$\dot{\mathbf{e}}_1 = \mathbf{g}_1 + C_1\mathbf{w}_1$$

which in the worst case may go divergent due to the strong effect of the disturbance. Since $C_1\mathbf{w}_1$ could be unmatched, the best we can do is to limit its effect at a prescribed level through the entire reaching and sliding phases. The $L_2[0,\infty)$ nature of $\mathbf{w}$ implies that the effect of disturbance will disappear gradually, therefore the asymptotic convergence of $\mathbf{e}_1$ still retains.

From L_2 gain property (6), we can see that nonlinear H^{∞} control is a good control candidate to reduce the disturbance effect.

Theorem 3.1: *The* L_2 *gain* (6) *is achieved for system* (4) *when the following nonlinear* H^{∞} *sliding mode holds*

$$\boldsymbol{\sigma}(\mathbf{x}_1, \mathbf{x}_{1d}, \mathbf{x}_2, t) = \boldsymbol{\sigma}_1(\mathbf{x}_1, \mathbf{x}_{1d}, t) + \boldsymbol{\varphi}(\mathbf{x}_2) = 0, \quad (7)$$

$$\boldsymbol{\sigma}_1(\mathbf{x}_1, \mathbf{x}_{1d}, t) = \frac{1}{r_1(\mathbf{x}_1, \mathbf{x}_{1d}, t)} B_1^T (D_{\mathbf{e}_1}V_1)^T + \boldsymbol{\zeta}, \quad (8)$$

where $V_1(\mathbf{e}_1, t)$, $\forall \mathbf{e}_1 \in \mathcal{R}^n$ *and* $t \geq 0$ *is a positive definite smooth solution of the following Hamilton–Jacoby inequality*

$$D_tV_1 + (D_{\mathbf{e}_1}V_1)\mathbf{g}_1 - (D_{\mathbf{e}_1}V_1)\frac{B_1B_1^T}{r_1}\left(D_{\mathbf{e}_1}V_1^T\right) + \frac{1}{4\rho_1^2}(D_{\mathbf{e}_1}V_1)C_1C_1^T(D_{\mathbf{e}_1}V_1)^T + \mathbf{e}_1^T\mathbf{e}_1 \leq 0, \quad (9)$$

with $r_1(\mathbf{x}_1, \mathbf{x}_{1d}, t) > 0$.

Proof: When in the sliding mode defined in (7), we have $\boldsymbol{\varphi} = -(1/r_1)B_1^T(D_{\mathbf{e}_1}V_1)^T - \boldsymbol{\zeta}$ with V_1 the solution of the Hamilton–Jacoby inequality (9). Hence $\boldsymbol{\varphi}$ is a nonlinear H^{∞} control law for (4) (Shen and Tamura 1995).

In order to show the L_2 gain property, differentiating the smooth solution $V_1(\mathbf{e}_1, t)$

$$\begin{aligned}
\dot{V}_1 &= D_tV_1 + (D_{\mathbf{e}_1}V_1)\{\mathbf{g}_1 + B_1(\boldsymbol{\varphi} + \boldsymbol{\zeta}) + C_1\mathbf{w}_1\} \\
&= D_tV_1 + (D_{\mathbf{e}_1}V_1)\mathbf{g}_1 + (D_{\mathbf{e}_1}V_1)C_1\mathbf{w}_1 \\
&\quad - (D_{\mathbf{e}_1}V_1)\frac{B_1B_1^T}{r_1}(D_{\mathbf{e}_1}V_1)^T \\
&= (D_tV_1) + (D_{\mathbf{e}_1}V_1)\mathbf{g}_1 - (D_{\mathbf{e}_1}V_1)\frac{B_1B_1^T}{r_1}(D_{\mathbf{e}_1}V_1)^T \\
&\quad + \rho_1^2\mathbf{w}_1^T\mathbf{w}_1 + \frac{1}{4\rho_1^2}(D_{\mathbf{e}_1}V_1)C_1C_1^T(D_{\mathbf{e}_1}V_1)^T \\
&\quad - \left\|\frac{1}{2\rho_1}C_1^T(D_{\mathbf{e}_1}V_1)^T - \rho_1\mathbf{w}_1\right\|^2 \\
&\leq D_tV_1 + (D_{\mathbf{e}_1}V_1)\mathbf{g}_1 - (D_{\mathbf{e}_1}V_1)\frac{B_1B_1^T}{r_1}(D_{\mathbf{e}_1}V_1)^T + \rho_1^2\mathbf{w}_1^T\mathbf{w}_1 \\
&\quad + \frac{1}{4\rho_1^2}(D_{\mathbf{e}_1}V_1)C_1C_1^T(D_{\mathbf{e}_1}V_1)^T. \quad (10)
\end{aligned}$$

If V_1 is the smooth solution of the Hamilton–Jacoby inequality (9), then (10) becomes

$$\dot{V}_1(\mathbf{e}_1, t) \leq -\mathbf{e}_1^T\mathbf{e}_1 + \rho_1^2\mathbf{w}_1^T\mathbf{w}_1. \quad (11)$$

Integrating both sides of (11) from 0 to t yields

$$V_1(\mathbf{e}_1, t) - V_1(\mathbf{e}_1(0), 0) \leq -\int_0^t \|\mathbf{e}_1\|^2\,\mathrm{d}\tau + \rho_1^2\int_0^t \|\mathbf{w}_1\|^2\,\mathrm{d}\tau. \quad (12)$$

Because $V_1(\mathbf{e}_1, t) \geq 0$, we can achieve the following H^{∞} performance from (12)

$$\int_0^t \|\mathbf{e}_1\|^2\,\mathrm{d}\tau \leq \beta_1(\mathbf{e}_1(0),0) + \rho_1^2\int_0^t \|\mathbf{w}_1\|^2\,\mathrm{d}\tau, \quad (13)$$

where $\beta_1(\mathbf{e}_1(0),0) = V_1(\mathbf{e}_1(0),0)$. □

4. Nonlinear H^∞ sliding mode control scheme

Traditionally, the SMC in range space is designed in such a way that the reaching condition is guaranteed. This requires a finite reaching time for the states $\mathbf{x}$ to reach the switching surface $\boldsymbol{\sigma} = \mathbf{0}$, and stay on it afterwards. Often a quadratic function $V_0 = \frac{1}{2}\boldsymbol{\sigma}^T\boldsymbol{\sigma}$ is selected and the reaching control law is such designed that $\dot{V}_0 \le -\|\boldsymbol{\sigma}\|$, consequently $\boldsymbol{\sigma}$ reaches zero in finite time.

It is worth to note that, in the above SMC design the behaviour of the null space dynamics is rather uncertain during the reaching phase. Since $\dim(\boldsymbol{\sigma}) = m < \dim(\mathbf{x}) = n$, V_0 is not radially unbounded in $\mathbf{x}$, i.e. V_0 is not a Lyapunov function of $\mathbf{x}$. In general $\boldsymbol{\sigma} < \infty$ does not imply the boundedness of the system states $\mathbf{x}$, unless the system enters the designated stable sliding manifold. Will the null space dynamics produce a finite escape time during reaching phase? This will not happen when the null space dynamics is global Lipschitz continuous. A linear null space dynamics is global Lipschitz continuous, thus $\mathbf{x}_1$ would not escape to infinity in any finite time. However, we have to be cautious if there exists non-global Lipschitz continuous (NGLC) components in the null space dynamics. In Xu and Zhang (2002), a control Lyapunov function approach is proposed to ensure that the states $\mathbf{x}$ are bounded during the reaching phase. This approach however requires the complete knowledge about the null space dynamics. What can we do if there are NGLC type nonlinearities as well as $L_\infty[0,\infty)$ type uncertainties in the null space subsystem?

Obviously, the SMC law has to take both NGLC and uncertain factors into consideration, so as to prevent the finite escape time phenomenon. In this section, we first show that, by incorporating the nonlinear H^∞ control into SMC, the L_2 gain property retains in the reaching phase. Next we prove that the system states are bounded for both reaching and sliding phase. Then we can derive the finite reaching time property for the system to reach the switching surface.

Theorem 4.1: *Consider the nonlinear uncertain system (4)–(5), the system achieves robust L_2 gain ρ_0 ($\rho_0 - \rho_1 > 0$) from $\mathbf{w}_1$ to $\mathbf{e}_1$ by the following sliding mode control law*

$$\mathbf{u} = \mathbf{u}_c + \mathbf{u}_s, \tag{14}$$

$$\mathbf{u}_c = -\Gamma^{-1}\Big[D_t\boldsymbol{\sigma}_1 + (D_{\mathbf{x}_{1d}}\boldsymbol{\sigma}_1)\dot{\mathbf{x}}_{1d} + S(\mathbf{f}_1 + B_1\boldsymbol{\varphi}) + D_{\mathbf{x}_2}\mathbf{f}_2 + \frac{1}{4\rho_m^2}SC_1C_1^TS^T\boldsymbol{\sigma}\Big], \tag{15}$$

$$\mathbf{u}_s = -\left(\frac{\psi}{1-\varepsilon_{b2}}\right)\frac{\Gamma^T\boldsymbol{\sigma}}{\|\Gamma^T\boldsymbol{\sigma}\|}, \tag{16}$$

where $S(\mathbf{x}_1, \mathbf{x}_{1d}, t) = D_{\mathbf{x}_1}\boldsymbol{\sigma}_1 \in \mathcal{R}^{m\times n}$, $\Gamma(\mathbf{x}, t) = (D_{\mathbf{x}_2}\boldsymbol{\varphi})B_2 \in \mathcal{R}^{m\times m}$, $\rho_m = \sqrt{\rho_0^2 - \rho_1^2}$, $\psi(\mathbf{x}, \mathbf{x}_{1d}, t) = \varepsilon_{b2}\|\mathbf{u}_c\| + (1+\varepsilon_{b2})\phi_2 + \delta$ and $\delta > 0$ is a positive constant.

Proof: We first construct a Lyapunov function

$$V(\mathbf{x}, \mathbf{x}_{1d}, t) = V_1(\mathbf{e}_1, t) + V_0(\boldsymbol{\sigma}) = V_1(\mathbf{e}_1, t) + \tfrac{1}{2}\boldsymbol{\sigma}^T\boldsymbol{\sigma} \ge 0. \tag{17}$$

Since $\mathbf{x}_{1d}$ is finite, it is easy to verify that V is radially unbounded in $\mathbf{x}$. Using the switching surface constructed in (7), (8), and under the control law (14)–(16), the derivative of V_0 is

$$\begin{aligned}
\dot{V}_0 &= \boldsymbol{\sigma}^T\dot{\boldsymbol{\sigma}} = \boldsymbol{\sigma}^T\big[D_t\boldsymbol{\sigma}_1 + (D_{\mathbf{x}_{1d}}\boldsymbol{\sigma}_1)\dot{\mathbf{x}}_{1d} + (D_{\mathbf{x}_1}\boldsymbol{\sigma}_1)\dot{\mathbf{x}}_1 \\
&\qquad + (D_{\mathbf{x}_2}\boldsymbol{\varphi})\dot{\mathbf{x}}_2\big] \\
&= \boldsymbol{\sigma}^T\big[D_t\boldsymbol{\sigma}_1 + (D_{\mathbf{x}_{1d}}\boldsymbol{\sigma}_1)\dot{\mathbf{x}}_{1d} + (D_{\mathbf{x}_1}\boldsymbol{\sigma}_1)(\mathbf{f}_1 + B_1\boldsymbol{\varphi})\big] \\
&\quad + \boldsymbol{\sigma}^T(D_{\mathbf{x}_2}\boldsymbol{\varphi})[\mathbf{f}_2 + B_2(I + \Delta B_2)(\mathbf{u} + \mathbf{w}_2)] \\
&\quad + \boldsymbol{\sigma}^T(D_{\mathbf{x}_1}\boldsymbol{\sigma}_1)C_1\mathbf{w}_1 \\
&= \boldsymbol{\sigma}^T\big[D_t\boldsymbol{\sigma}_1 + (D_{\mathbf{x}_{1d}}\boldsymbol{\sigma}_1)\dot{\mathbf{x}}_{1d} + S(\mathbf{f}_1 + B_1\boldsymbol{\varphi}) \\
&\quad + (D_{\mathbf{x}_2}\boldsymbol{\varphi})\mathbf{f}_2 + \Gamma\mathbf{u}_c\big] + \boldsymbol{\sigma}^TSC_1\mathbf{w}_1 \\
&\quad + \boldsymbol{\sigma}^T\Gamma[\Delta B_2\mathbf{u}_c + (I + \Delta B_2)(\mathbf{u}_s + \mathbf{w}_2)] \\
&\le \boldsymbol{\sigma}^T\Gamma[\Delta B_2\mathbf{u}_c + (I + \Delta B_2)\mathbf{w}_2] + \boldsymbol{\sigma}^T\Gamma(I + \Delta B_2)\mathbf{u}_s \\
&\quad - \frac{1}{4\rho_m^2}\boldsymbol{\sigma}^TSC_1C_1^TS^T\boldsymbol{\sigma} + \boldsymbol{\sigma}^TSC_1\mathbf{w}_1 \\
&\le \|\boldsymbol{\sigma}^T\Gamma\|[\varepsilon_{b2}\|\mathbf{u}_c\| + (1+\varepsilon_{b2})\phi_2] + \boldsymbol{\sigma}^T\Gamma(I + \Delta B_2)\mathbf{u}_s \\
&\quad - \frac{1}{4\rho_m^2}\boldsymbol{\sigma}^TSC_1C_1^TS^T\boldsymbol{\sigma} + \boldsymbol{\sigma}^TSC_1\mathbf{w}_1
\end{aligned} \tag{18}$$

$$\le -\delta\|\Gamma^T\boldsymbol{\sigma}\| - \frac{1}{4\rho_m^2}\boldsymbol{\sigma}^TSC_1C_1^TS^T\boldsymbol{\sigma} + \boldsymbol{\sigma}^TSC_1\mathbf{w}_1 \tag{19}$$

$$\begin{aligned}
&\le -\delta\|\Gamma^T\boldsymbol{\sigma}\| - \frac{1}{4\rho_m^2}\boldsymbol{\sigma}^TSC_1C_1^TS^T\boldsymbol{\sigma} + \boldsymbol{\sigma}^TSC_1\mathbf{w}_1 \\
&\quad - \rho_m^2\mathbf{w}_1^T\mathbf{w}_1 + \rho_m^2\mathbf{w}_1^T\mathbf{w}_1 \\
&\le -\delta\|\Gamma^T\boldsymbol{\sigma}\| - \left(\frac{1}{2\rho_m}\boldsymbol{\sigma}^TSC_1 - \rho_m\mathbf{w}_1^T\right) \\
&\quad \times \left(\frac{1}{2\rho_m}S^TC_1^T\boldsymbol{\sigma} - \rho_m\mathbf{w}_1\right) + \rho_m^2\mathbf{w}_1^T\mathbf{w}_1 \\
&\le -\delta\|\Gamma^T\boldsymbol{\sigma}\| + \rho_m^2\mathbf{w}_1^T\mathbf{w}_1.
\end{aligned} \tag{20}$$

Using the result (11) and (20), the derivative of the constructed Lyapunov function V in (17) is

$$\begin{aligned}
\dot{V} &= \dot{V}_1 + \boldsymbol{\sigma}^T\dot{\boldsymbol{\sigma}} \le -\mathbf{e}_1^T\mathbf{e}_1 + \rho_1^2\mathbf{w}_1^T\mathbf{w}_1 - \delta\|\Gamma^T\boldsymbol{\sigma}\| + \rho_m^2\mathbf{w}_1^T\mathbf{w}_1 \\
&\le -\mathbf{e}_1^T\mathbf{e}_1 + \rho_1^2\mathbf{w}_1^T\mathbf{w}_1 + (\rho_0^2 - \rho_1^2)\mathbf{w}_1^T\mathbf{w}_1 = -\mathbf{e}_1^T\mathbf{e}_1 + \rho_0^2\mathbf{w}_1^T\mathbf{w}_1.
\end{aligned} \tag{21}$$

Integrating both sides of (21), we have

$$-V(\mathbf{x}(0),\mathbf{x}_{1d}(0),0) \le V(\mathbf{x},\mathbf{x}_{1d},t) - V(\mathbf{x}(0),\mathbf{x}_{1d}(0),0) \le -\int_0^t \|\mathbf{e}_1\|^2\,\mathrm{d}\tau + \rho_0^2\int_0^t \|\mathbf{w}_1\|^2\,\mathrm{d}\tau, \quad (22)$$

$$\Rightarrow \int_0^t \|\mathbf{e}_1\|^2\,\mathrm{d}\tau \le \beta(\mathbf{x}(0),\mathbf{x}_{1d}(0),0) + \rho_0^2\int_0^t \|\mathbf{w}_1\|^2\,\mathrm{d}\tau, \quad (23)$$

where $\beta(\mathbf{x}(0),\mathbf{x}_{1d}(0),0) = V(\mathbf{x}(0),\mathbf{x}_{1d}(0),0)$. Hence (23) holds throughout the entire tracking period, i.e. the H^{∞} performance is achieved. □

Remark 1: In the nonlinear uncertain system (4), if $\mathbf{g}_1(\mathbf{e}_1,t)$ can be expressed as $G_1(\mathbf{e}_1,t)\mathbf{e}_1$, when $G_1(\mathbf{e}_1,t)$ is a matrix-valued smooth function, then the HJB inequality can be simplified into the following differential Riccati inequality

$$\tfrac{1}{2}\dot{P} + \tfrac{1}{2}\left(PG_1 + G_1^TP\right) - P\left[\frac{B_1B_1^T}{r_1} - \frac{1}{4\rho_1^2}C_1C_1^T\right]P + I_{n\times n} \le 0, \quad (24)$$

where $P(\mathbf{e}_1,t) \in \mathcal{R}^{n\times n}$ is a symmetric positive definite smooth matrix. The $\mathbf{e}_1$ subsystem warrants a robust L_2 gain ρ_1 from $\mathbf{w}_1$ to $\mathbf{e}_1$ by the nonlinear control law

$$\boldsymbol{\varphi} = -\frac{1}{r_1}B_1^TP\mathbf{e}_1 - \zeta, \quad (25)$$

which also specifies the switching surface as $\boldsymbol{\sigma}_1 = -\boldsymbol{\varphi}$.

Note that the r_1 function and the parameter ρ_1 may affect the solvability of the inequality (24) in practical design. In general, the selection of r_1 and ρ_1 depends on the designed system specifications. For example, given a value of ρ_1, if P is selected as a constant matrix and G_1 is such a stable matrix that the term $\frac{1}{2}(PG_1 + G_1^TP)$ is stable enough to deal with the positive terms $I_{n\times n}$ and $1/(4\rho_1^2)C_1C_1^T$ in (24), then the selection of $r_1(\cdot)$ is not very critical since the eigenvalues of term BB^T satisfy $\lambda_i(BB^T) \ge 0,\ i = 1,\ldots,n$. Though depending on the specific system dynamics, in general r_1 and ρ_1 should be selected such that their contributions to the inequality (24) makes the left part to be negative definite.

As the second step, let us prove the boundedness of the system states, which is directly related to the uncertainty $\mathbf{w}_1$. The following Corollary considers two types of $\mathbf{w}_1$ commonly encountered in control problems.

Corollary 4.2: *With the proposed controller* (14)–(16):

(a) *if* $\mathbf{w}_1 \in L_2[0,\infty)$, *all the system states are bounded*;
(b) *if* $\mathbf{w}_1 \in L_2[0,\infty) \cap L_\infty[0,\infty)$, $\lim_{t\to\infty}\mathbf{e}_1(t) = 0$ *and* $\mathbf{x}_2$ *is bounded.*

Proof: (a) If $\mathbf{w}_1 \in L_2[0,\infty)$, then $\int_0^t\|\mathbf{w}_1\|^2\,\mathrm{d}\tau \le M_d$, where M_d is a finite constant. From (22),

$$V(\mathbf{x},\mathbf{x}_{1d},t) - V(\mathbf{x}(0),\mathbf{x}_{1d}(0),0) \le -\int_0^t \|\mathbf{e}_1\|^2\,\mathrm{d}\tau + \rho_0^2\int_0^t \|\mathbf{w}_1\|^2\,\mathrm{d}\tau, \quad (26)$$

$$\Rightarrow V(\mathbf{x},\mathbf{x}_{1d},t) \le V(\mathbf{x}(0),\mathbf{x}_{1d}(0),0) + \rho_0^2M_d. \quad (27)$$

Because $V(\mathbf{x},\mathbf{x}_{1d},t)$ is radially unbounded in $\mathbf{x}$, (27) means that $\mathbf{x}$ is bounded.

(b) If $\mathbf{w}_1 \in L_2[0,\infty) \cap L_\infty[0,\infty)$, then we have $\int_0^t\|\mathbf{w}_1\|^2\,\mathrm{d}\tau \le M_d$ and $\|\mathbf{w}_1\| \le \epsilon_d$, where ϵ_d is a constant. The inequality (21) becomes

$$\frac{\mathrm{d}V(\mathbf{x},\mathbf{x}_{1d},t)}{\mathrm{d}t} \le -\|\mathbf{e}_1\|^2 + \rho_0^2\epsilon_d^2,$$

which shows that $\|\mathbf{e}_1\| \le \rho_0\epsilon_d$ is bounded. Thus from the system dynamics (4), $\dot{\mathbf{e}}_1$ is bounded and as a result $\mathbf{e}_1$ is uniformly continuous. Note that in (23), $\int_0^t\|\mathbf{e}_1\|^2\,\mathrm{d}\tau$ is bounded because $\mathbf{w}_1 \in L_2[0,\infty)$. Using Barbalat's Lemma (Narendra and Annaswamy 1989), it is straightforward to reach the conclusion that $\lim_{t\to\infty}\mathbf{e}_1(t) = 0$. □

Now we are in a position to derive the finite reaching time property, which is indispensable in any SMC.

Theorem 4.3: *Under the sliding mode control law* (14)–(16), *system* (4) *can reach the switching surface* $\boldsymbol{\sigma} = \mathbf{0}$ *in finite time when* $\mathbf{w}_1 \in L_2[0,\infty)$.

Proof: From (19), the derivative of V_0 is

$$\begin{aligned}\dot{V}_0 = \boldsymbol{\sigma}^T\dot{\boldsymbol{\sigma}} &\le -\delta\|\Gamma^T\boldsymbol{\sigma}\| + \boldsymbol{\sigma}^TSC_1\mathbf{w}_1 - \frac{1}{4\rho_m^2}\boldsymbol{\sigma}^TSC_1C_1^TS^T\boldsymbol{\sigma}\\ &\le -\delta\|\Gamma^T\boldsymbol{\sigma}\| + \|\mathbf{w}_1\|\|SC_1\|\|\boldsymbol{\sigma}\|\\ &\le -\delta\underline{\mu}(\Gamma)\|\boldsymbol{\sigma}\| + \bar{\mu}(SC_1)\|\mathbf{w}_1\|\|\boldsymbol{\sigma}\|\end{aligned}$$

$$\Rightarrow \frac{\boldsymbol{\sigma}^T\dot{\boldsymbol{\sigma}}}{\sqrt{\boldsymbol{\sigma}^T\boldsymbol{\sigma}}} \le -\delta\underline{\mu}(\Gamma) + \bar{\mu}(SC_1)\|\mathbf{w}_1\|, \quad (28)$$

where $\bar{\mu}(A)$ and $\underline{\mu}(A)$ are the maximum and the minimum singular values of a matrix A. Note that $\mathbf{w}_1 \in L_2[0,\infty) \Rightarrow \int_0^\infty\|\mathbf{w}_1\|^2\,\mathrm{d}\tau < \infty$. Therefore $\forall\varepsilon_0 > 0$, $\exists t_0 > 0$, and $t_2 > t_1 > t_0$, such that $\int_{t_1}^{t_2}\|\mathbf{w}_1\|^2\,\mathrm{d}\tau < \varepsilon_0$. Using Hölder inequality (Lusternik and Sobolev 1961),

$$\int_{t_1}^{t_2}\|\mathbf{w}_1\|\cdot 1\,\mathrm{d}\tau \le \left(\int_{t_1}^{t_2}\|\mathbf{w}_1\|^2\,\mathrm{d}\tau\right)^{1/2}\left(\int_{t_1}^{t_2}1\,\mathrm{d}\tau\right)^{1/2} < \sqrt{\varepsilon_0(t_2 - t_1)}. \quad (29)$$

From Assumption 2.3 Γ is of full rank. From Corollary 4.2, $\mathbf{x}$ is bounded. Hence there exist positive constants $\underline{\mu}_c \le \underline{\mu}(\Gamma)$ and $\bar{\mu}_c \ge \bar{\mu}(SC_1)$. (28) becomes

$$\frac{\boldsymbol{\sigma}^T\dot{\boldsymbol{\sigma}}}{\sqrt{\boldsymbol{\sigma}^T\boldsymbol{\sigma}}} \le -\delta\underline{\mu}_c + \bar{\mu}_c\|\mathbf{w}_1^T\|. \tag{30}$$

By integrating both sides of the inequality (30) and using (29),

$$\begin{aligned}\int_{t_1}^{t_2}\frac{\boldsymbol{\sigma}^T\dot{\boldsymbol{\sigma}}}{\sqrt{\boldsymbol{\sigma}^T\boldsymbol{\sigma}}}\mathrm{d}t &= \int_{t_1}^{t_2}\frac{\dot{V}_0}{\sqrt{2V_0}}\mathrm{d}t \le -\int_{t_1}^{t_2}\delta\underline{\mu}_c\mathrm{d}t + \int_{t_1}^{t_2}\bar{\mu}_c\|\mathbf{w}_1\|\mathrm{d}t\\ \sqrt{2V_0}\,\Big|_{V_0(t_1)}^{V_0(t_2)} = \sqrt{2V_0(t_2)} - \sqrt{2V_0(t_1)} &< -\delta\underline{\mu}_c(t_2-t_1) + \sqrt{\varepsilon_0}\bar{\mu}_c\sqrt{t_2-t_1}.\end{aligned} \tag{31}$$

In the following part, it is shown that t_2 is finite such that sliding mode can be reached at time $t=t_2>t_1$ which means that $V_0(t_2)=0$. Denote $z=\sqrt{t_2-t_1}>0$, $q_0=\delta\underline{\mu}_c>0$, $q_1=\sqrt{\varepsilon_0}\bar{\mu}_c>0$ and $q_2=\sqrt{2V_0(t_1)}>0$. Let $V_0(t_2)=0$, (31) can be rewritten as a linear quadratic inequality as

$$q_0z^2 - q_1z - q_2 < 0. \tag{32}$$

Solving the inequality (32), we have

$$\begin{aligned}0 < z = \sqrt{t_2-t_1} &< \frac{q_1+\sqrt{q_1^2+4q_0q_2}}{2q_0}\\ \Rightarrow t_2 &< t_1 + \left(\frac{q_1+\sqrt{q_1^2+4q_0q_2}}{2q_0}\right)^2,\end{aligned}$$

which shows that t_2 is finite. □

Remark 2: Due to the presence of unmatched disturbance of $L_2[0,\infty)$ type, it is not possible to specify the reaching time t_2.

The unit vector control law $\mathbf{u}_s$ in (16) may incur chattering when the system reaches the sliding mode in finite time. In order to eliminate the chattering phenomenon, $\mathbf{u}_s$ is modified as below

$$\mathbf{u}_s = -\left(\frac{\psi}{1-\varepsilon_{b2}}\right)\frac{\Gamma^T\boldsymbol{\sigma}}{\|\Gamma^T\boldsymbol{\sigma}\| + \epsilon e^{-\nu t}}, \tag{33}$$

where ϵ and ν are positive constants. In the following Corollary, we show that the L_2 gain property is retained by the smoothing control law (33).

Corollary 4.4: *Consider the uncertain nonlinear system (4)–(5), with $\mathbf{w}_1 \in L_2[0,\infty)$, the controller in (14), (15) and (33) guarantees that: a finite L_2 gain performance is achieved, all the variables are bounded. Moreover, if $\mathbf{w}_1 \in L_2[0,\infty)\cap L_\infty[0,\infty)$, the tracking error $\mathbf{e}_1$ converges to zero asymptotically.*

Proof: According to the proof in Theorem 4.1, from (18), using the smoothing control (33), the derivative of V_0 becomes

$$\begin{aligned}\dot{V}_0 &\le (\psi-\delta)\|\boldsymbol{\sigma}^T\Gamma\| - \boldsymbol{\sigma}^T\Gamma(I+\Delta B_2)\left(\frac{\psi}{1-\varepsilon_{b2}}\right)\\ &\quad\times\frac{\Gamma^T\boldsymbol{\sigma}}{\|\Gamma^T\boldsymbol{\sigma}\|+\epsilon e^{-\nu t}}\\ &\quad-\frac{1}{4\rho_m^2}\boldsymbol{\sigma}^TSC_1C_1^TS^T\boldsymbol{\sigma} + \boldsymbol{\sigma}^TSC_1\mathbf{w}_1\\ &\le \psi\|\boldsymbol{\sigma}^T\Gamma\| - \frac{\psi\|\Gamma^T\boldsymbol{\sigma}\|^2}{\|\Gamma^T\boldsymbol{\sigma}\|+\epsilon e^{-\nu t}} - \left(\frac{\boldsymbol{\sigma}^TSC_1}{2\rho_m} - \rho_m\mathbf{w}_1^T\right)\\ &\quad\times\left(\frac{S^TC_1^T\boldsymbol{\sigma}}{2\rho_m} - \rho_m\mathbf{w}_1\right) + \rho_m^2\mathbf{w}_1^T\mathbf{w}_1\\ &\le \frac{\psi\|\Gamma^T\boldsymbol{\sigma}\|\epsilon e^{-\nu t}}{\psi\|\Gamma^T\boldsymbol{\sigma}\|+\epsilon e^{-\nu t}} + \rho_m^2\mathbf{w}_1^T\mathbf{w}_1 \le \epsilon e^{-\nu t} + \rho_m^2\mathbf{w}_1^T\mathbf{w}_1.\end{aligned} \tag{34}$$

Using (11) and (34), it is straightforward to get

$$\begin{aligned}\dot{V} &\le -\mathbf{e}_1^T\mathbf{e}_1 + \rho_1^2\mathbf{w}_1^T\mathbf{w}_1 + \epsilon e^{-\nu t} + \rho_m^2\mathbf{w}_1^T\mathbf{w}\\ &= -\mathbf{e}_1^T\mathbf{e}_1 + \rho_0^2\mathbf{w}_1^T\mathbf{w}_1 + \epsilon e^{-\nu t}.\end{aligned} \tag{35}$$

By integrating both sides of (35), we have

$$\begin{aligned}V(\mathbf{x},\mathbf{x}_{1d},t) &\le V(\mathbf{x}(0),\mathbf{x}_{1d}(0),0)\\ &\quad-\int_0^t\|\mathbf{e}_1\|^2\,\mathrm{d}\tau + \rho_0^2\int_0^t\|\mathbf{w}_1\|^2\,\mathrm{d}\tau + \frac{\epsilon}{\nu}(1-e^{-\nu t}),\end{aligned} \tag{36}$$

$$\begin{aligned}\Rightarrow \int_0^t\|\mathbf{e}_1\|^2\,\mathrm{d}\tau &\le \beta(\mathbf{x}(0),\mathbf{x}_{1d}(0),0) + \rho_0^2\int_0^t\|\mathbf{w}_1\|^2\,\mathrm{d}\tau\\ &\quad+\frac{\epsilon}{\nu}(1-e^{-\nu t}).\end{aligned} \tag{37}$$

From the inequality (37), a finite L_2 gain performance is achieved. Moreover, as shown in Corollary 4.2, the inequality (36) implies that all the states of the system are bounded. If $\mathbf{w}_1 \in L_2[0,\infty)\cap L_\infty[0,\infty)$, we have $\lim_{t\to\infty}\mathbf{e}_1(t)=0$. □

5. Illustrative examples

In this section, we present two examples. In the first example, it is shown that the proposed nonlinear H^∞ sliding mode control can be successfully applied to a nonlinear cascade system with unmatched uncertainties. In the second example, the proposed method is compared with the suboptimal VSC method. It shows that the existence of NGLC terms in the null space dynamics may lead to divergence during the reaching phase if the control system does not possess the robust L_2 gain property for the entire tracking period.

A. Example 1

Consider a nonlinear uncertain cascade system

$$\begin{cases} \dot{\mathbf{x}}_1 = \mathbf{f}_1(\mathbf{x}_1, t) + \mathbf{b}_1\varphi(x_2) + C_1(\mathbf{x}_1, t)\mathbf{w}_1 \\ \dot{x}_2 = f_2(\mathbf{x}, t) + b_2(\mathbf{x}, t)[1 + \Delta b_2(\mathbf{x}, t)][u + w_2(\mathbf{x}, t)], \end{cases} \tag{38}$$

where

$$\mathbf{f}_1 = G_1 \cdot \mathbf{x}_1 = \begin{bmatrix} 0 & 1 \\ -2 & -4 \end{bmatrix}\begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix}, \quad \mathbf{b}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$
$$C_1 = \begin{bmatrix} \cos(x_{12}) & \sin(x_{12}) \\ \sin(x_{11}) & \cos(x_{11}) \end{bmatrix},$$
$$\mathbf{w}_1 = \left[e^{-0.1t}, -e^{-0.5t}\right]^T, \quad f_2 = x_{11}\sin(x_2),$$
$$\Delta b_2 = 0.2\cos(x_{12}), \quad w_2 = \sin(\pi t),$$
$$b_2 = 1,\ \varphi(x_2) = x_2,\ x_{11}(0) = x_{12}(0) = 0.5,$$
$$x_2(0) = 0.2,\ \varepsilon_{b,2} = 0.2 \text{ and } \phi_2 = 1.$$

The target trajectory is $x_{11d} = 0.2\sin(\pi t)$ and $x_{12d} = \dot{x}_{11d} = 0.2\pi\cos(\pi t)$. The error dynamics of $\mathbf{x}_1$ subsystem in (4) can be expressed as

$$\dot{\mathbf{e}}_1 = G_1\mathbf{e}_1 + \mathbf{b}_1[\varphi + \zeta(t)] + C_1\mathbf{w}_1,$$

where $\zeta(t) = -\dot{x}_{12d} - 2x_{11d} - 4x_{12d}$.

Case 1: $\rho_1 = 1.8$

In $\mathbf{x}_1$ subsystem, according to Remark 1, we first choose $V_1(\mathbf{e}_1, t) = \frac{1}{2}\mathbf{e}_1^T P\mathbf{e}_1$, where P is determined by the differential Riccati inequality (24). When $\dot{P} = 0$, $\rho_1 = 1.8$ and $r_1 = 0.1$, from the linear algebraic matrix inequality

$$\tfrac{1}{2}(PG_1 + G_1^T P) - P\left[\frac{\mathbf{b}_1\mathbf{b}_1^T}{r_1} - \frac{1}{4\rho_1^2}C_1C_1^T\right]P + I_{2\times 2} \le 0, \tag{39}$$

and using the singular value of the matrix C_1, which is 2, we can get a symmetric positive definite smooth matrix

$$P = \begin{bmatrix} 5.4398 & 0.65719 \\ 0.65719 & 0.262 \end{bmatrix}.$$

Thus from (25), we have $\varphi(\mathbf{x}_1, \mathbf{x}_{1d}, t) = -(1/r_1)\times \mathbf{b}_1^T P\mathbf{e}_1 - \zeta(t)$. The switching surface is $\sigma = x_2 + \sigma_1 = x_2 - \varphi(\mathbf{x}_1, t) = x_2 + [0\ \ 10]P\mathbf{e}_1 - \zeta(t)$. From Theorem 4.1, choose $\Gamma = 1$, $\delta = 0.5$, $\rho_0 = 3.5$, $\rho_m = \sqrt{\rho_0^2 - \rho_1^2} = 3$ and $S = [0\ \ 10]P$, the system possesses the robust L_2 gain ρ_0 from $\mathbf{w}_1$ to $\mathbf{e}_1$ by the controller $u = u_c + u_s$, where

$$u_c = -\left\{D_t\sigma_1 + [0\ \ 10]P(\mathbf{f}_1 + \mathbf{b}_1x_2) + f_2 + \frac{1}{4\rho_m^2}SC_1C_1^TS^T\sigma\right\}, \tag{40}$$

$$u_s = -\frac{\psi}{0.8}\cdot\frac{\sigma}{|\sigma| + e^{-0.1t}}, \quad \psi = 0.2|u_c| + 1.2. \tag{41}$$

Case 2: $\rho_1 = 3$

When $\rho_1 = 3$, according to the design procedure in Case 1, $\rho_m = \sqrt{\rho_0^2 - \rho_1^2} = 1.8$ and

$$P = \begin{bmatrix} 2.7095 & 0.2889 \\ 0.2889 & 0.21183 \end{bmatrix}.$$

The resulted control law has the same form as in (40) and (41).

In figure 1, it is shown that the tracking error approaches zero asymptotically, i.e., $\lim_{t\to\infty}\mathbf{e}_1(t) = 0$. It is noted that the smaller the value ρ_1 (e.g. $\rho_1 = 1.8$), the better the transient performance in attenuating the effect of external disturbance in $\mathbf{x}_1$ subsystem. Figure 2 shows the integral of the error signal $\int_0^t \|\mathbf{e}_1\|^2\,d\tau$.

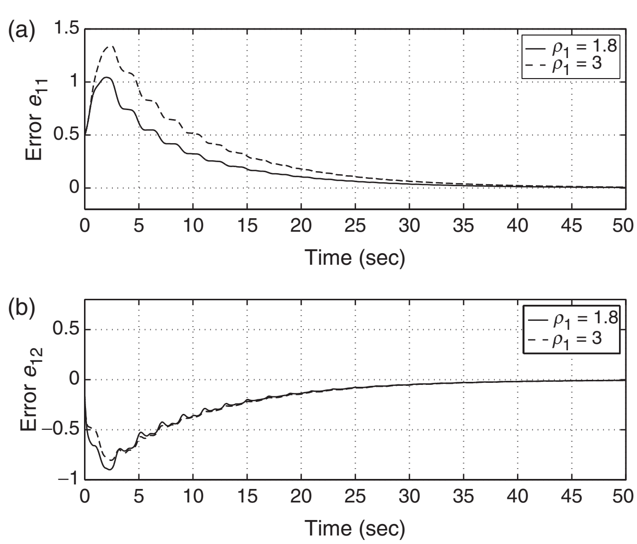

Figure 1. The evolution of the tracking error $\mathbf{e}_1$: (a) e_{11}; (b) e_{12} (Solid line – $\rho_1 = 1.8$; Dashed line – $\rho_1 = 3$).

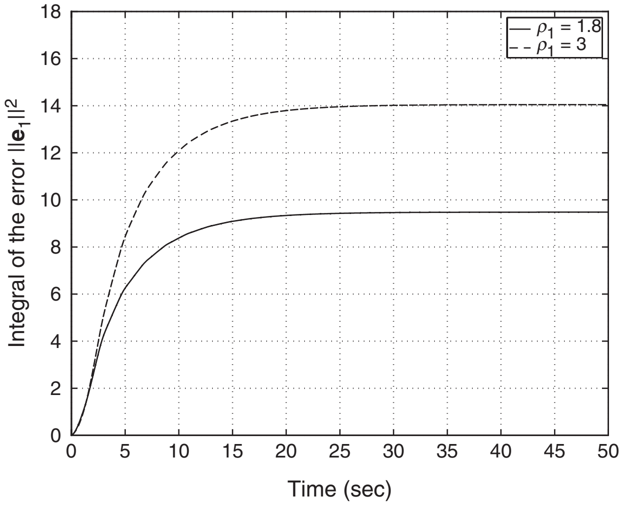

Figure 2. The integral $\int_0^t \|\mathbf{e}_1\|^2 \,\mathrm{d}\tau$ of the tracking error $\mathbf{e}_1$ (Solid line – $\rho_1 = 1.8$; Dashed line – $\rho_1 = 3$).

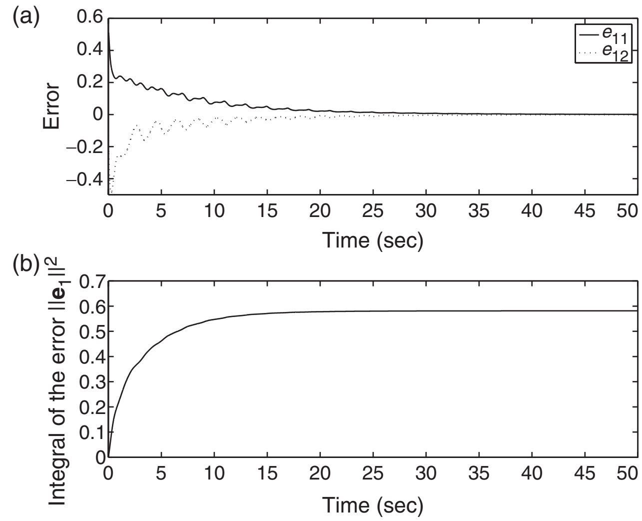

Figure 3. (a) The evolution of the tracking error $\mathbf{e}_1$ under the proposed controller; (b) The integral of the tracking error.

The desired H^∞ performance (6) has been achieved with the prescribed attenuation level ρ_1.

B. Example 2

To make comparisons with the suboptimal VSC method in (15), the following cascade system with matched uncertainties is considered

$$\begin{cases} \dot{\mathbf{x}}_1 = \mathbf{f}_1(\mathbf{x}_1, t) + B_1(\mathbf{x}_1, t)\boldsymbol{\varphi}(\mathbf{x}_2) + C_1(\mathbf{x}_1, t)\mathbf{w}_1 \\ \dot{\mathbf{x}}_2 = \mathbf{f}_2(\mathbf{x}, t) + B_2(\mathbf{x}, t)[I + \Delta B_2(\mathbf{x}, t)][\mathbf{u} + \mathbf{w}_2(\mathbf{x}, t)] \end{cases} \tag{42}$$

where $\mathbf{f}_1$, $\mathbf{w}_1$, $\mathbf{x}_1(0)$ and $\varepsilon_{b,2}$ are the same as in example 1, $C_1 = \mathrm{diag}(1 + x_{11}^2, 0.5)$, $\boldsymbol{\varphi}(\mathbf{x}_2) = \mathbf{x}_2$, $B_1 = B_2 = I_{2\times 2}$, $\mathbf{f}_2 = [x_{11}\sin(x_{21}), x_{12}\sin(x_{22})]^T$, $\Delta B_2 = \mathrm{diag}[0.2\cos(x_{12}), 0.2\sin(x_{21})]$, $\mathbf{w}_2 = [\sin(\pi t), \cos(\pi t)]^T$, $\mathbf{x}_2(0) = [0.2, 0.2]^T$ and $\phi_2 = 1.5$.

The designed parameters of the proposed controller are

$$\rho_1 = 0.3, \quad \rho_0 = 3.5, \quad r_1 = \frac{4\rho_1^2}{(1 + x_{11}^2)^2}, \quad P = \begin{bmatrix} p_{11} & p_0 \\ p_0 & p_{22} \end{bmatrix},$$

$$p_0 = \frac{(-2 + \sqrt{4 + 2\gamma})}{\gamma}, \quad p_{22} = \frac{\left[-4 + \sqrt{16 + 2\gamma(1 + p_0)}\right]}{\gamma},$$

$$p_{11} = \gamma p_0 p_{22} + 4p_0 + 2p_{22} \quad \text{and} \quad \gamma = \frac{0.75 + 2x_{11}^2 + x_{11}^4}{2\rho_1^2}.$$

In the construction of the suboptimal VSC, the uncertainty $\mathbf{w}_1$ is not taken into consideration because its upbound is not available. According to Xu and Zhang (2001), the switching surface is $\boldsymbol{\sigma} = P_s\mathbf{e}_1 + \mathbf{x}_2 + \boldsymbol{\zeta}$, where

$$P_s = \begin{bmatrix} 0.774 & 0.098 \\ 0.098 & 0.146 \end{bmatrix}$$

with $Q = R = I$ for the optimal control task $\inf_{\upsilon(\cdot)} \int_0^\infty [\mathbf{e}_1^T Q \mathbf{e}_1 + \upsilon^T R \upsilon]\mathrm{d}t$, and $\boldsymbol{\zeta} = [0, \ -\dot{x}_{12d} - 2x_{11d} - 4x_{12d}]^T$.

Figure 3 shows the tracking error and its integral of the proposed nonlinear H^∞ sliding mode controller. It is straightforward to see that the proposed control scheme achieves asymptotic convergence. On the contrary, when applying the suboptimal VSC controller (Xu and Zhang 2001), the system diverges as shown in figure 4. This is due to the existence of the NGLC term $1 + x_{11}^2$ in C_1, which results in the finite escape time phenomenon.

6. Conclusions

By synthesizing sliding mode control and nonlinear H^∞ techniques, a novel nonlinear H^∞ sliding mode control scheme is developed for tracking control problems. Several fundamental issues of SMC have been explored in this work. First, by means of the nonlinear H^∞ method, a nonlinear switching surface is constructed, which ensures a stable sliding manifold even in the presence of unmatched uncertainties. Second, a new reaching control law in conjunction with a Lyapunov function is proposed to obtain the L_2 gain property for the entire tracking period, as a result guarantee the system behaviour in the reaching phase. Third, the

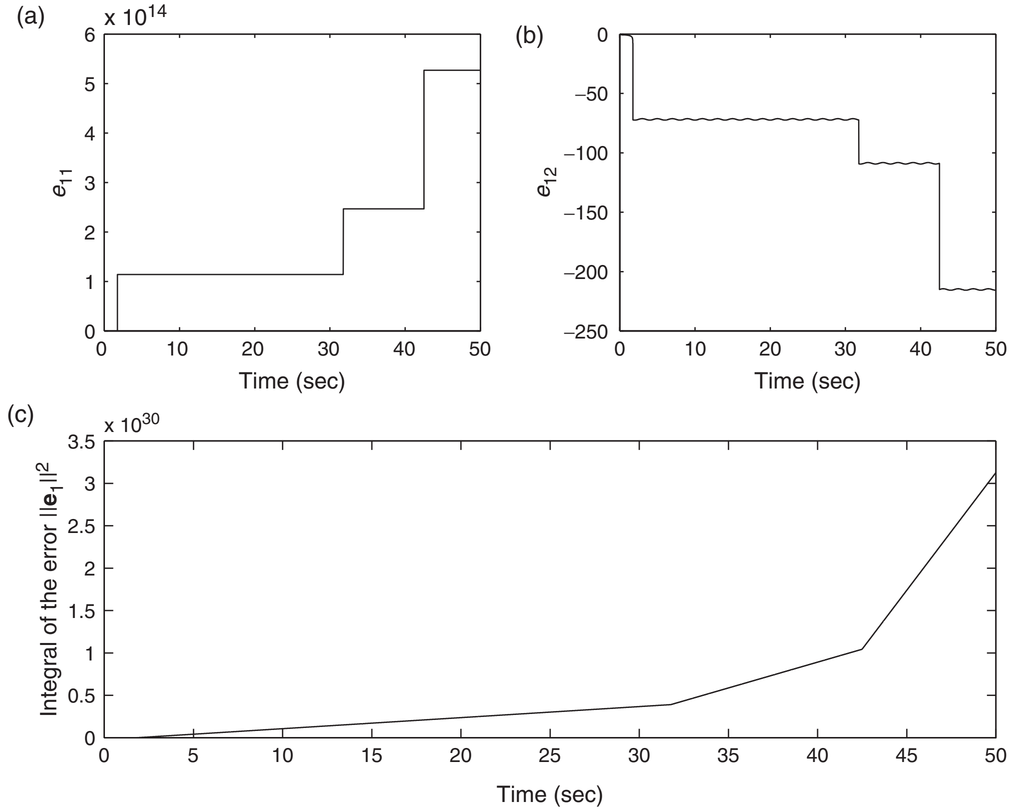

Figure 4. The performance under the conventional suboptimal VSC controller: (a) The evolution of the tracking error e_{11}; (b) The evolution of the tracking error e_{12}; (c) The integral $\int_0^t \|\mathbf{e}_1\|^2 \, d\tau$ of the tracking error $\mathbf{e}_1$.

nature and effect of $L_2[0,\infty)$ and $L_\infty[0,\infty)$ type system uncertainties have been made clear. In the sequel appropriate control mechanisms can be devised to effectively attenuate or eliminate these influences.

Focusing on theoretical analysis, in this work only numerical examples were provided to demonstrate the effectiveness of the proposed method. The extension of the proposed approach to practical nonlinear cascade systems will be our next step.

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