High-order sliding mode observers for nonlinear autonomous switched systems with unknown inputs

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Abstract

The state observation and unknown input identification problems are studied for a class of nonlinear autonomous switched systems. A bank of observers are designed using the high-order sliding mode techniques. The robustness of the high-order sliding mode observers is exploited to provide exact reconstruction of the continuous state and the discrete state even in the presence of unknown inputs. The value of the equivalent injection is used for identification of unknown inputs. Simulation results support the proposed method.

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1. Introduction

1.1. Antecedents and motivations

The switched systems, which behavior can be represented by the interaction of continuous and discrete dynamics, have been widely studied during the last decades since they can be used to describe a wide range of physical and engineering systems. Most of the attention have been...
focused on the problems of stability and stabilization with extensive and satisfactory results
(see, for example [11,5,24,25,33]). The techniques developed for switched system have been
applied to solve complex problems, for example in [9] a hybrid observer is applied to estimate
the state of mechanical oscillators.

In the context of the observation problem for switched systems, i.e. the estimation of
the continuous and discrete state, the observer design is of great interest for many control
areas (see, for example, the work by [19]). This problem has been already studied by
other authors using different kinds of approaches. The main difference is related to the
knowledge of the active discrete state or operating mode: some approaches consider only
continuous state uncertainty with known operating mode, while others assume that both
the operating mode and the continuous state are unknown. In Alessandri and Coletta [1]
and Bejarano et al. [7] a Luenberger observer approach and a high-order sliding-mode
observer for linear systems are proposed for the known operating mode case. Another
work considering the operating mode is known and that the output and input are available
is the one presented by Pina and Botto [29]. In this paper the problem of the simultaneous
state and input estimation for hybrid systems when subject to input disturbances is
addressed by an algorithm based on the moving horizon estimation method. Considering
that the continuous state is known, an algorithm for reconstructing the discrete state
in nonlinear uncertain switched systems is presented in Orani et al. [28] based on sliding-
mode control theory. On the other hand, for the unknown operating mode case, in
Bejarano and Pisano [6], based on a property of strong detectability and using a LMI
approach, are designed two state observers for some classes of switched linear systems with
unknown inputs. Considering that the output and the initial state are available, in Vu and
Liberzon [32] necessary and sufficient conditions for a switched system to be invertible
are proposed, i.e. condition for recovering the switching signal and the input uniquely.
In the same context, a nonlinear finite time observer to estimate the capacitor voltage for
multicellular converters, which have a switched behavior, is proposed by Defoort et al.
[16]. On graph theoretic approach, assuming only the knowledge of the system structure,
in Boukhobza and Hamelin [10] the authors deal with the observability of the discrete
state, the internal state and the input of switching structured linear systems with unknown
input. In Davila et al. [14], based on the nonhomogeneous high-order sliding mode
approach, a robust observer for the unknown and exogenous switching signal is proposed
to solve the problem of continuous and discrete state estimation for a class of nonlinear
switched systems.

The problem of observability definitions is also intensively studied in the literature. For
instance, Bemporad et al. [8], Chaib et al. [12] and Barbot et al. [4] analyze the observability of
hybrid systems, where the discrete state depends on the state trajectories. For the detectability
case, in De Santis et al. [15] the detectability of linear switched systems is addressed that
reduces to asymptotic stability of a suitable switched system with guards extracted from it, i.e.
switching systems whose discrete state is triggered externally.

Unknown input identification techniques are widely applied to solve control problems of
systems with disturbances, for instance in the fault detection problem (see, for example [2,17]).
Therefore for the best of our knowledge, none of the existing works present the unknown
input identification problem acting in nonlinear autonomous switched systems. However,
there are few works dealing with the observability of nonlinear switched systems with unknown
inputs (see, for example [13,28]), which aim at designing unknown input observers for some
classes of nonlinear switched systems.

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1.2. Main contributions

Motivated by these works, under assumption that both the continuous and discrete state are unknown, a robust observer is proposed that is able to estimate the continuous and discrete state in the presence of unknown inputs for nonlinear autonomous switched systems. A method for identification of unknown inputs is provided using the equivalent injection. The proposed solution is based on sliding-mode control theory.

1.3. Structure of the paper

The paper has the following structure. Section 2 deals with the problem statement. Section 3 presents the main assumptions underlying the feasibility of the proposed procedure, and the continuous and discrete state observer. In Section 4 the unknown input identification and the high frequency classical method to recover the equivalent injection are presented. Section 5 presents briefly the multiple output case. Two simulation examples are given in Section 6. Finally, some concluding remarks are given in Section 6.

2. Problem statement

Consider the following class of nonlinear autonomous switched system

\[
\dot{x} = f_i(x) + Fw(t,x),
\]

\[
y = h(x),
\]

\[
i(x) = \begin{cases}
1, & \forall x|C(x) \in C_1, \\
2, & \forall x|C(x) \in C_2, \\
\vdots & \\
q, & \forall x|C(x) \in C_q,
\end{cases}
\]

where \(x \in \mathcal{X} \subseteq \mathbb{R}^n\) is the continuous state vector, the unknown discrete state \(i(x) \in \mathcal{Q} = \{1, \ldots, q\}\) is determined by a smooth continuous scalar function \(C(x) : \mathcal{X} \to \mathcal{C} \subseteq \mathbb{R}\). This function determines which of the \(q\) dynamics is acting on the system. On the other hand, \(y \in \mathcal{Y} \subseteq \mathbb{R}\) is the output, \(w \in \mathcal{W} \subseteq \mathbb{R}\) is the unknown input term. It is assumed that the unknown input term is a piecewise continuous function in \(t\) and bounded Lipschitz function in \(x\), i.e. \(|w(t,x_0) - w(t,x_1)| < M_0(x_0 - x_1), \forall t \geq 0\). The vector fields \(f_i(x) : \mathcal{X}_i(x) \to \mathbb{R}^n\) are considered smooth, \(F \in \mathbb{R}^{n}\) is a constant column matrix, and \(h(x) : \mathcal{X} \to \mathcal{Y}\) is a smooth function defined on an open set of \(\mathbb{R}^n\). The properties of the system (1)–(2) are described in Section 3.

The aims of this paper are to design a finite time converging observer capable of estimating the continuous and discrete state in spite of the unknown inputs and to establish a method for identifying the unknown input to the system (1)–(2) using the equivalent injection.

3. Preliminaries

**Notation:** The following notation is used along the paper. The pseudo-inverse matrix of \(F \in \mathbb{R}^{k \times m}\) is defined as \(F^+ \in \mathbb{R}^{m \times n}\). Then, if \(\text{rank}(F) = n \to FF^+ = I\), and if \(\text{rank}(F) = m \to F^+F = I\). With reference to a scalar function \(h(x)\) with vector argument \(x\) defined in an
open set $\Omega \in \mathbb{R}^n$ such that $h(x) : \mathbb{R}^n \to \mathbb{R}$, denote $dh(x) = \partial h(x)/\partial x = [\partial h(x)/\partial x_1 \cdots \partial h(x)/\partial x_n]$. Let $\mathcal{U}$ denote a subset of a topological space, $P(\mathcal{U})$ denotes the set of all subsets of $\mathcal{U}$, and $TU$ denotes the tangent bundle of $\mathcal{U}$.

Now, some basic definitions for hybrid automaton, hybrid trajectory, execution, and dwell time are recalled in this section.

**Definition 1** ([Lygeros et al. [26]]): A hybrid automaton $\mathcal{H}$ is a collection $\mathcal{H} = (Q, X, f, Init, D, E, G, R)$, where $Q$ is the finite set of discrete variables; $X$ is the finite set of continuous variables; $f : Q \times X \to T X$ is a vector field; $Init \subseteq Q \times X$ is the set of initial states; $D : Q \to P(X)$ is a domain; $E \subseteq Q \times Q$ is the set of edges; $G : E \to P(X)$ is the guard condition, and $R : E \times X \to P(X)$ is the reset map.

According to **Definition 1**, the system (1)–(2) is a particular case of a hybrid automaton with the following properties:

- $Q \in Q = \{1, \ldots, q\}$,
- $X \in \mathcal{X} = \bigcup_{i(x) \in Q} X_{i(x)}$ with $X_{i(x)} = \{x | C(x) \in \mathcal{C}_{i(x)}\}$,
- $f_{i(x)} : X_{i(x)} \to T X_{i(x)} \subseteq \mathbb{R}^n$,
- $Init = \{Q \times X\}$,
- $D = \bigcup_{i(x) \in Q} D_{i(x)}$, with $D_{i(x)} = \{x | x \in X_{i(x)}\}$,
- $E = \{(k, i) | k, i = 1, 2 \ldots, q, \forall k \neq i\}$,
- $G = C(x) : X \to C \subseteq \mathbb{R}$, where

$$\mathcal{C} = \bigoplus_{i(x) \in Q} \mathcal{C}_{i(x)}, \tag{3}$$
$$\mathcal{C}_k \cap \mathcal{C}_i = \emptyset, \forall k, i = 1, \ldots, q, k \neq i, \tag{4}$$

- $R(k, i, x) = x$, i.e. the identity map for $x$.

**Definition 2** ([Lygeros et al. [26]]): A hybrid time trajectory is a finite or infinite sequence of intervals of the real line $\tau = \{I_i\}_{i=0}^N$ such that:

- $I_i = [\tau_i, \tau'_i]$, for all $i < N$;
- if $N < \infty$, then either $I_N = [\tau_N, \tau'_N]$ or $I_N = [\tau_N, \tau'_N)$;
- $\tau_i \leq \tau'_i = \tau_{i+1}$ for all $i$.

In other words, a hybrid time trajectory is a sequence of intervals of the real line, whose end points overlap. For a hybrid time trajectory $\tau = \{I_i\}_{i=0}^N$, define $\langle \tau \rangle$ as the set \{1, ..., $N$\} if $N$ is finite, and \{1, 2, ...\} if $N = \infty$ and $|\tau| = \sum_{i \in \langle \tau \rangle} (\tau'_i - \tau_i)$.

**Definition 3** ([Liberzon [24]]): The minimal dwell time is a number $T_\delta > 0$ such that the class of admissible switching signals satisfy the property that the switching times $t_1, t_2, \ldots$ fulfill the inequality $t_{j+1} - t_j \geq T_\delta$ for all $j$.

---

2The function $C(x)$ determines which of the $q$ dynamics is acting on the system according to the known convex disjoint subsets $C_{i(x)}$. According to the properties (3)–(4) the discrete state is distinguishable, i.e. for every value of the continuous state $x(t)$ there is only one single value of the discrete state $i(x)$ according to Eqs. (1) and (2).
In this paper we study the systems whose hybrid time trajectories satisfies the minimal dwell time definition. Moreover, it is assumed that the dwell time is sufficiently large or it is possible to estimate it (see simulation example 1 and [30] for the estimation of the switching times for linear switched systems).

**Definition 4** (Lygeros et al. [26]). An execution of a hybrid automaton \( \mathcal{H} \) is a collection \( \xi = (\tau, q, x) \), where \( \tau \) is a hybrid time trajectory, \( q : \langle \tau \rangle \rightarrow Q \) is a map, and \( x = \{x^i : i \in \langle \tau \rangle\} \) is a collection of differentiable maps \( x^i : I \rightarrow \mathcal{X} \) such that

- \((q(0), x^0(0)) \in \text{Init} \);  
- for all \( t \in [\tau_i, \tau'_i) \), \( x^i(t) = f(q(t))(x^i(t)) \) and \( x^i(t) \in \mathcal{X} \);  
- for all \( i \in \langle \tau \rangle \), \( e = (q(i), q(i+1)) \in E, x^i(\tau'_i) \in G(e) \), and \( x^{i+1}(\tau_{i+1}) \in R(e, x^i(\tau'_i)) \).

The execution of a hybrid automaton is a similar concept to the solution of a continuous dynamic systems. Notice that for any infinite execution of the system (1)–(2) it is necessary that \( |\tau| = \infty \). Zeno executions are not allowed. The zeno phenomena can be described by an infinite execution with \( |\tau| < \infty \).

The following definition of observability for a Hybrid Automaton is adapted from Kang et al. [21].

**Definition 5.** Consider the system (1)–(2) and the variable \( x = x(t, x) \). Let \( x(t, x^1) \) be a trajectory of the automaton \( \mathcal{H} \) with a hybrid time trajectory \( T_N \) and \( \langle T_N \rangle \). Suppose for any trajectory \( (t, x^2) \) of \( \mathcal{H} \) with the same \( T_N \) and \( \langle T_N \rangle \), the equality \( y(t, x^1) = y(t, x^2) \), a.e. in \( [t_{\text{ini}}, t_{\text{end}}] \) implies \( x(t, x^1) = x(t, x^2) \), a.e. in \( [t_{\text{ini}}, t_{\text{end}}] \), then \( x = x(t, x) \) is \( Z(T_N) \) observable along the trajectory \( x(t, x^1) \).

In this paper the system (1)–(2) is \( Z(T_N) \) observable along any trajectory \( x(t, x) \) and for any possible hybrid time trajectory \( T_N \).

4. Observer design

Consider the following assumptions over system (1)–(2):

**Assumption 1.** Suppose that every vector field \( f_{\mathcal{H}(x)} : \mathcal{X}_{\mathcal{H}(x)} \rightarrow T_{\mathcal{H}(x)} \) can be extended to each subspace \( \mathcal{X}_1, \ldots, \mathcal{X}_q \). Then, there exist smooth vector fields \( g_{\mathcal{X}(x)} \) such that

\[
\begin{align*}
g_j : \mathcal{X}_j &\rightarrow T_j \mathcal{X}_j; \quad g_j = f_1, \quad \forall x \in \mathcal{X}_j, \quad \forall j = 2, 3, \ldots, q, \\
g_k : \mathcal{X}_k &\rightarrow T_k \mathcal{X}_k; \quad g_k = f_2, \quad \forall x \in \mathcal{X}_k, \quad \forall k = 1, 3, \ldots, q, \\
\vdots \\
g_l : \mathcal{X}_l &\rightarrow T_l \mathcal{X}_l; \quad g_l = f_q, \quad \forall x \in \mathcal{X}_l, \quad \forall l = 1, 2, \ldots, q-1.
\end{align*}
\]

Notice that the previous assumption implies that all vector fields \( g_{\mathcal{X}(x)} \) are extensions of every vector field \( f_{\mathcal{X}(x)} \) with the smoothness property satisfied for all the domains \( D_{\mathcal{H}(x)} \).

**Assumption 2.** The vector fields \( f_{\mathcal{H}(x)} \), the matrix \( F \), and the function \( h(x) \) are such that \( \forall x \in \mathcal{X}_{\mathcal{H}(x)}, \forall i = 1, \ldots, q, \) it is satisfied that

\[
d(L_{\mathcal{H}(x)}^k h(x))F = 0, \quad \forall k < n - 1, \quad \forall i = 1, \ldots, q,
\]
Let us define the following mappings:

\[
\Phi_i(x) = \begin{bmatrix}
h(x) \\
L_{f_{i0}(x)}h(x) \\
\vdots \\
L_{f_{i0}(x)}^{n-1}h(x)
\end{bmatrix}, \quad \forall i = 1, \ldots, q.
\] (9)

Their corresponding Jacobian matrices are given by

\[
\frac{\partial \Phi_i(x)}{\partial x} = \begin{bmatrix}
dh(x) \\
dL_{f_{i0}(x)}h(x) \\
\vdots \\
dL_{f_{i0}(x)}^{n-1}h(x)
\end{bmatrix}, \quad \forall i = 1, \ldots, q.
\] (10)

**Assumption 3.** Each matrix (10) is such that

\[
\text{rank}\left(\frac{\partial \Phi_i(x)}{\partial x}\right) = n \quad \text{only} \quad \forall x \in \mathcal{X}_1,
\]

\[
\text{rank}\left(\frac{\partial \Phi_2(x)}{\partial x}\right) = n \quad \text{only} \quad \forall x \in \mathcal{X}_2,
\]

\[
\vdots
\]

\[
\text{rank}\left(\frac{\partial \Phi_q(x)}{\partial x}\right) = n \quad \text{only} \quad \forall x \in \mathcal{X}_q,
\]

Moreover, each mapping \(\Phi_i(x)\) is a diffeomorphism only on its corresponding domain \(\mathcal{X}_i\), \(\forall i = 1, \ldots, q\) (see, for example [20]).

Considering the previous assumptions, it is possible to design the following \(q\) state observers:

\[
\dot{x}_{i\lambda} = f_i(x_{i\lambda}) + \left(\frac{\partial \Phi_i(x_{i\lambda})}{\partial x_{i\lambda}}\right)^{-1}v_{i\lambda}, \quad \forall \lambda = 1, \ldots, q, \quad y_{i\lambda} = h(x_{i\lambda}),
\] (11)

with the estimated state vectors \(x_{i\lambda} \in \mathbb{R}^n\), the estimated outputs \(y_{i\lambda} \in \mathbb{R}\), and the correction terms \(v_{i\lambda} \in \mathbb{R}^n\), that will be designed further on. The solutions of Eq. (11) are understood in the Filippov sense [18] to provide the possibility to use discontinuous signals in the observer and coincide with the usual solutions when the right-hand sides are continuous. It is assumed also that all considered correction terms allow the existence and extension of solutions to the whole semi-axis \(t \geq 0\).

**Remark 1.** Notice that if the \(q\) observability mappings are equal it will be only necessary to design one observer. If \(q - 1\) observability mappings are equal it will be only necessary to design two observers, and so successively.

**Remark 2.** Notice that, due to **Assumptions 2** and **3**, the \(q\) subsystems must be observable only in the corresponding domain \(D_{i(x)}\), outside, according to **Assumption 1**, the observers (11) are well defined.
Let the following assumption be satisfied:

**Assumption 4.** There are known constants $M_\lambda > 0$, $\forall \lambda = 1, \ldots, q$, such that the following inequalities are satisfied:

$$
|L^n_{f_1(x_1)}h(x_1) - L^n_{f_1(x_2)}h(x_2) - d(L^{n-1}_{f_1(x)}h(x))Fw| < M_1, \quad \forall x \in \mathcal{X}_1,
$$

$$
|L^n_{f_1(x_1)}h(x_1) - L^n_{f_1(x_2)}h(x_2) - d(L^{n-1}_{f_1(x)}h(x))Fw| < M_2, \quad \forall x \in \mathcal{X}_2,
$$

$$
\vdots
$$

$$
|L^n_{f_1(x_1)}h(x_1) - L^n_{f_1(x_2)}h(x_2) - d(L^{n-1}_{f_1(x)}h(x))Fw| < M_q, \quad \forall x \in \mathcal{X}_q.
$$

Notice that so the above assumption is satisfied it is necessary, not sufficiently, that the system is bounded-input bounded-state (BIBS) (see, for example [22]) or that it is possible to stabilize it somehow. Nevertheless, notice that only it is sufficient that there exist bounds $M_\lambda$ in the corresponding domains $D(x)$ and not outside.

In the physical meaning (e.g. mechanical systems), Assumption 4 implies knowing an acceleration bound and certain bound for the initial estimation error for each subdynamic. From this point of view, it does not turn out to be so restrictive to know the above-mentioned bounds since the most of the physical signals are bounded and one has certain knowledge from where the trajectories of the system can initiate.

Now, the high-order sliding-mode differentiator [23] is used as an auxiliary dynamics. The differentiator has the following form:

$$
\dot{\hat{\lambda}}_{\lambda,1} = \theta_{\lambda,2} - \mu_{\lambda,1} M_{\lambda}^{1/n} |\nu_{\lambda,j}|^{(n-1)/n} \text{sign}(\nu_{\lambda,j}),
$$

$$
\dot{\hat{\lambda}}_{\lambda,2} = \theta_{\lambda,3} - \mu_{\lambda,2} M_{\lambda}^{1/(n-1)} |\theta_{\lambda,2} - \hat{\theta}_{\lambda,1}|^{(n-2)/(n-1)} \text{sign}(\theta_{\lambda,2} - \hat{\theta}_{\lambda,1}),
$$

$$
\vdots
$$

$$
\dot{\hat{\lambda}}_{\lambda,n} = -\mu_{\lambda,n} M_{\lambda} \text{sign}(\theta_{\lambda,n} - \hat{\theta}_{\lambda,n-1}),
$$

(15)

where $\nu_{\lambda,j} = \nu_{\lambda,j} - y$, $\forall \lambda = 1, \ldots, q$, are the output errors. The correction terms are taken from Eq. (15) as

$$
\nu_{\lambda} = \begin{bmatrix}
-x_{\lambda,1} M_{\lambda}^{1/n} |\nu_{\lambda,j}|^{(n-1)/n} \text{sign}(\nu_{\lambda,j}) \\
-x_{\lambda,2} M_{\lambda}^{1/(n-1)} |\theta_{\lambda,2} - \hat{\theta}_{\lambda,1}|^{(n-2)/(n-1)} \text{sign}(\theta_{\lambda,2} - \hat{\theta}_{\lambda,1}) \\
\vdots \\
-x_{\lambda,n} M_{\lambda} \text{sign}(\theta_{\lambda,n} - \hat{\theta}_{\lambda,n-1})
\end{bmatrix}.
$$

(16)

The constants $x_{\lambda,j}$ are chosen recursively and sufficiently large. In particular, according to [23], one possible choice is $x_{\lambda,6} = 1.1$, $x_{\lambda,5} = 1.5$, $x_{\lambda,4} = 2$, $x_{\lambda,3} = 3$, $x_{\lambda,2} = 5$, $x_{\lambda,1} = 8$, $\forall \lambda = 1, \ldots, q$, which is enough for the case that $n \leq 6$.

### 4.1. Continuous state estimation between switchings

Let us describe the continuous state estimation during the time interval between switchings. Consider that $\hat{t}(x) \equiv \hat{t}^* = \text{const.}, \forall t \in [0, t_1)$ with the first switching time $t_1 \geq T_\delta$. The system dynamics on the operating mode $\hat{t}^*$ is given by

$$
\dot{x} = f_p(x) + Fw(t, x), \quad \forall t \in [0, t_1).
$$

(17)
In this way, due to Assumption 3, only one of the \( q \) observers (11) can be associated with the corresponding output error \( e_{\bar{x}_i} = \bar{y}_{\lambda_i} - y \) and state error \( e_{\bar{x}_i} = \bar{x}_{\lambda_i} - x \), in the time interval between switchings.

Taking into account the previous explanations, the following theorem can be established.

**Theorem 1.** Consider that the \( \lambda^* \)-th observer of Eq. (11) with the correction terms designed according to Eq. (16) is applied to system (17), and let Assumptions 1–3 be satisfied. Then, provided that constants \( z_i^*, j \) are chosen properly and \( M_{\lambda^*} \) is selected satisfying Assumption 4, the state estimation error converges to zero in finite time, i.e. \( e_{\bar{x}_{\lambda^*}}(t) = 0 \) for all \( t \in [t_{\lambda^*}, t_f) \).

**Proof.** System (17), under Assumptions 2 and 3, can be represented, on new coordinates \( z \), as

\[
\dot{z} = Az + B\varphi_p(z, w), \quad y_z = Cz,
\]

where

\[
A = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{bmatrix}_{n \times n}, \quad B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}_{n \times 1}, \quad C = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}_{1 \times n},
\]

\[
\varphi_p(z, w) = (L^n_{f_{\lambda^*}(x)} h(x) + d(L^{n-1}_{f_{\lambda^*}(x)} h(x)) F_w)x = \Phi_{\lambda^*}(z).
\]

Notice that the functions \( \varphi_p(z, w) \) contain the unknown input that acts on the system. On the other hand, the observers (11), under Assumptions 2 and 3, can be represented as

\[
\dot{\bar{z}}_\lambda = \bar{A} \bar{z}_\lambda + B\varphi_{\lambda}(\bar{z}_\lambda) + v_\lambda, \quad y_{\bar{z}} = C\bar{z}_\lambda,
\]

where \( \varphi_{\lambda}(\bar{z}_\lambda) = (L^n(\bar{z}_\lambda) h(\bar{z}_\lambda))|_{\bar{x}_\lambda} = \Phi^{-1}_{\lambda^*}(z) \).

Define the state observation errors as

\[
e_{\bar{z}}_\lambda = z_{\lambda} - \bar{z}.
\]

Thus, the state observation error dynamics takes the following form:

\[
\dot{e}_{\bar{z}}_\lambda = \bar{A} e_{\bar{z}}_\lambda + \bar{B} \Psi_\lambda(\bar{z}_\lambda, z, w) + v_\lambda,
\]

where \( \Psi_\lambda(\bar{z}_\lambda, z, w) = \varphi_{\lambda}(\bar{z}_\lambda) - \varphi_p(z, w) \).

Now, if it is possible to find appropriate correction terms \( v_\lambda \), which can steer the vector \( \bar{z}_\lambda \) to zero, then equality \( \bar{z}_\lambda = z \) will be satisfied only for the case when \( \lambda = \lambda^* = \bar{\lambda} \). Therefore, only one of the observers can be associated with the corresponding state observation error in the time interval \( t \in [0, t_f) \).

On the other hand, it is not desirable to design the correction terms in the coordinates \( \bar{z}_\lambda \) but in the coordinates \( \bar{x}_\lambda \). Therefore, defining the following output error vector:

\[
e_{\bar{x}} = \begin{bmatrix} e_{\bar{x}, 1} \\ \vdots \\ e_{\bar{x}, n} \end{bmatrix} = \begin{bmatrix} e_{\bar{x}_1} \\ \vdots \\ e_{\bar{x}_{n-1}} \end{bmatrix}.
\]

The state observation error dynamics (21) turns into output observation error dynamics as follows:

\[
\dot{e}_{\bar{x}} = A e_{\bar{x}} + \bar{B} \Psi_\lambda(\bar{x}_\lambda, \Phi_p(x), w) + v_\lambda.
\]
In an extended structure

\[
\begin{align*}
\dot{\epsilon}_{*,1} &= \epsilon_{*,2} + 
\dot{\epsilon}_{*,2} &= \epsilon_{*,3} + 
\vdots &\quad \dot{\epsilon}_{*,n} = \Psi_{\lambda}(\Phi_{\lambda}(\bar{x}_{\lambda}), \Phi_{\lambda}(x), w) + \nu_{\lambda,n}.
\end{align*}
\] (24)

Notice that the dynamic structures (24) are very similar to the high-order sliding-mode differentiator properties described in Levant [23]. In view of the first row of Eq.(25) it is obtained that

\[
\dot{\epsilon}_{*,1} = \epsilon_{*,2} - \alpha_{*,1} M_{\lambda}^{1/n} |\epsilon_{*,1}|^{(n-1)/n} \text{sign}(\epsilon_{*,1}),
\]

\[
\dot{\epsilon}_{*,2} = \epsilon_{*,3} - \alpha_{*,2} M_{\lambda}^{1/(n-1)} |\epsilon_{*,2} - \dot{\epsilon}_{*,1}|^{(n-2)/(n-1)} \text{sign}(\epsilon_{*,2} - \dot{\epsilon}_{*,1}),
\]

\[
\vdots \quad \dot{\epsilon}_{*,n} = \Psi_{\lambda}(\Phi_{\lambda}(\bar{x}_{\lambda}), \Phi_{\lambda}(x), w) - \alpha_{*,n} M_{\lambda} \text{sign}(\epsilon_{*,n} - \dot{\epsilon}_{*,n-1}).
\] (25)

Let the parameters \(\alpha_{*,j}\) be chosen recursively and sufficiently large according to the high-order sliding-mode differentiator properties described in Levant [23]. In view of the Assumption 4, the following equality is satisfied in finite time when \(\lambda = \lambda^* = \bar{\lambda}^*: \)

\[
[\epsilon_{*,1}, \epsilon_{*,2}, \ldots, \epsilon_{*,n}] \equiv [0, 0, \ldots, 0].
\]

The condition \(\epsilon_{*,1} = 0, \forall t \in [t_{*,1}, t_1]\), implies that \([\epsilon_{*,2}, \ldots, \epsilon_{*,n}] \equiv [0, \ldots, 0], \forall t \in [t_{*,1}, t_1]\) with \(t_1 \geq T_\delta\). To prove that, assume that the condition \(\epsilon_{*,1} = 0\) is satisfied in a nonzero time interval. This condition implies that \(\dot{\epsilon}_{*,1} \equiv 0\) in the same time interval. Thus, from the first row of Eq. (25) it is obtained that \(\epsilon_{*,2} \equiv 0\). Then, since \(\epsilon_{*,2} \equiv 0\) and \(\dot{\epsilon}_{*,1} \equiv 0\) from the second row of Eq. (25) it is obtained that \(\epsilon_{*,3} \equiv 0\). If the same procedure is iterated the following expressions are obtained:

\[\epsilon_{*,j} \equiv 0, \quad \forall j = 1, \ldots, n.\]

Given Assumption 4, the last row of Eq. (25) defines the following differential inclusion:

\[
\dot{\epsilon}_{*,n} \in [-M_{\lambda^*}, M_{\lambda^*}] - \alpha_{*,n} M_{\lambda^*} \text{sign}(\epsilon_{*,n} - \dot{\epsilon}_{*,n-1}),
\]

where \(\Psi_{\lambda^*}(\cdot) \in [-M_{\lambda^*}, M_{\lambda^*}]\). Therefore, according to [23], the dynamics (25), for \(\lambda = \lambda^* = \bar{\lambda}^*\), converges to zero after a finite time, i.e. \(\epsilon_{*,*} \equiv 0 \forall t > t_{*,*}\), and according to Assumption 3, it ensures that the state estimation error \(e_{y,*} = \bar{x}_{\lambda^*} - x\) also converge to zero in finite time in spite of the unknown input. Notice that it is always possible to select the gain \(M_{\lambda^*}\) sufficiently large such that \(t_{*,*} < t_1 \geq T_\delta\). In this way, the theorem is proven. \(\square\)

Therefore, when the active dynamics is the \(\bar{\lambda}^*-\)th, then the observation error \(e_{\lambda^*}\) of the \(\lambda^*-\)th observer tends to zero in finite time.

**Remark 3.** Notice that to design the observers only it is necessary for the calculation of the inverse matrices \((\dot{\partial} \Phi_{\lambda}(\bar{x}_{\lambda})/\dot{\partial} \bar{x}_{\lambda})^{-1}\), not the inverse transformations \(\Phi_{\lambda}^{-1}(\bar{x}_{\lambda})\).

**Theorem 1** establishes that only one of the \(q\) observers, if every observability mappings are different, estimates the continuous state correctly but it does not establish how to know which of them makes it. Nevertheless, according to [3], to detect which of the dynamics

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(25) converge to zero it is sufficient to verify that the following inequality is satisfied
\[ |e_{\hat{y}_\lambda}(t)| \leq \gamma_{\lambda} M_{\lambda} h^n, \quad \forall \lambda = 1, \ldots, q, \forall t \in [0, \gamma_{\lambda} h], \]  
where $\gamma_{\lambda}$ and $\gamma_{\lambda}$ are positive constants and $h$ is the sample time. It is natural to estimate the constants $\gamma_{\lambda}$ and $\gamma_{\lambda}$ through simulation. Thus, in this way it is possible to determine when and which of the $q$ observers has converged to the correct continuous state during the time interval $t \in [0, t_1)$.

**Remark 4.** Notice that the inequality (26) is used for implementation since due to the presence of the sample time it is impossible to reach the equality $|e_{\hat{y}_\lambda}(t)| \equiv 0$ as it is established in the proof of Theorem 1.

In this way, the real estimated state $\hat{x}$ is defined as follows:
\[ \bar{x}_{\lambda}^* = \hat{x}, \quad \forall t \in [t^*_\lambda, t_1). \]  

### 4.2. Discrete state estimation

Once the continuous state is estimated correctly, it is possible to estimate the discrete state with the following discrete state observer:
\[ \hat{\lambda}(\hat{x}) = \begin{cases} 
1, & \forall \hat{x} | C(\hat{x}) \in C_1, \\
2, & \forall \hat{x} | C(\hat{x}) \in C_2, \\
\vdots \\
q, & \forall \hat{x} | C(\hat{x}) \in C_q,
\end{cases} \]  
where the manifolds $C_i$ are known. Then, the discrete state $\hat{i}(x)$ is estimated by means of $\hat{\lambda}(\hat{x})$ since, according to Theorem 1, the following equality is satisfied:
\[ \hat{x}(t) = x(t), \quad \forall t \in [t^*_\lambda, t_1). \]  

Thus it is possible to reconstruct the continuous and discrete state before the first switching happens. Now, the following step is to analyze what it happens in the switching times.

### 4.3. Continuous and discrete state estimation on switching times

Let $t^+_j$ be the time instants after the switching times $t_j$. In order to maintain the state estimation on the switching times the following proposition is done.

**Proposition 2.** The state estimation of system (1)–(2) is maintained in spite of the switchings if the following reset equations are implemented in the bank of observers (11) for all $\lambda \not= \lambda^*$:
\[ \bar{x}_{\lambda}(t^+_j) = \hat{x}(t^-_j), \quad \forall j = 1, 2, \ldots . \]  

**Proof.** Let us consider the dynamics (25) on the time instants before and after the switching time $t_1$, i.e.
\[ \begin{align*}
\dot{e}_{A_x e_{x_{k+1}}} &= e_{A_x e_{x_{k}}} - A_x \dot{e}_{A_x e_{x_{k}}} M_{A_x e_{x_{k}}} \left| e_{A_x e_{x_{k}}} \right|^{(n-1)/n} \text{sign}(e_{A_x e_{x_{k}}}), \\
\dot{e}_{A_x e_{\lambda_{k}}} &= e_{A_x e_{\lambda_{k}}} - A_x \dot{e}_{A_x e_{\lambda_{k}}} M_{A_x e_{\lambda_{k}}} \left| \dot{e}_{A_x e_{\lambda_{k}}} \right|^{(n-1)/n} \text{sign}(e_{A_x e_{\lambda_{k}}} - \dot{e}_{A_x e_{\lambda_{k}}} e_{A_x e_{\lambda_{k}}}),
\end{align*} \]
\[ \dot{e}_{A_{\lambda,k}} = \Psi_{A_{\lambda,k}}(\cdot) - y_{A_{\lambda,k}} M_{A_{\lambda,k}} \text{sign}(e_{A_{\lambda,k}} - \dot{e}_{A_{\lambda,k-1}}), \]  

(31)

where \( \lambda^{*} \) is the current operating mode, \( \kappa \in \mathcal{Q} \), with \( \kappa \neq \lambda^{*} \), represents the next operating mode, and

\[ e_{A_{\lambda,k}} = e_{A_{\lambda,k}}(t_{1}^{\leftarrow}) - e_{A_{\lambda,k}}(t_{1}^{\rightarrow}), \quad \forall k = 1, \ldots, n, \quad \Psi_{A_{\lambda,k}}(\cdot) = \Psi_{A_{\lambda,k}}(t_{1}^{\leftarrow}) - \Psi_{A_{\lambda,k}}(t_{1}^{\rightarrow}). \]

If the reset equations (30) are applied on each switching time, then, the trajectories of the system (31) always remain in the sliding surface, i.e. \( e_{A_{\lambda,k}} = 0 \). Thus, the state estimation is maintained in spite of the switchings. Notice that due to the nature of the system (i.e. the reset map \( R(\kappa, t, x) = x \)), the transformations \( \Phi_{\kappa}(x) \) do not present jumps in the switching times. Nevertheless, the jumps can appear in the trajectories generated by the observers. However, the reset equations (30) avoid this happening.

Then, once the discrete state has been identified, it is possible to analyze such signal in order to know when some change takes place. Therefore, the reset equations (30) can always be implemented for each observer when the change in the discrete state is detected, i.e. in each switching time.

With the previous explanation, the following identities are obtained:

\[ \hat{x}(t) = x(t), \quad \hat{\lambda}(\hat{x}) = \hat{i}(x), \quad \forall t > t_{\lambda^*}. \]  

(32)

In the following section the unknown input identification problem is studied.

5. Unknown input identification

In steady state, all entries of state error vector \( e_{x} = \hat{x} - x \) are identically zero. Thus, the expression of \( \dot{e}_{x} \) is

\[ \dot{e}_{x} = f_{\hat{\lambda}(\hat{x})}(\hat{x}) \left( \frac{\partial \Phi_{\hat{\lambda}(\hat{x})}(\hat{x})}{\partial \hat{x}} \right)^{-1} v_{\hat{\lambda}(\hat{x})}(x) - Fw(t, x) = 0. \]  

(33)

Then, the discontinuous correction terms of \( v_{\hat{\lambda}(\hat{x})} \) will take the value of the equivalent injection, i.e.

\[ \left( \frac{\partial \Phi_{\hat{\lambda}(\hat{x})}(\hat{x})}{\partial \hat{x}} \right)^{-1} v_{\hat{\lambda}(\hat{x})_{\text{eq}}} = f_{\hat{i}(x)}(\hat{x}) + Fw(t, x). \]  

(34)

At the moment in which the exact reconstruction of the continuous state is reached, the exact estimation of the discrete state also is reached. In this way, since the continuous and discrete state are perfectly known, it is possible to identify the unknown input term as follows:

\[ \hat{w}(t, \hat{x}) = F^{+} \left[ \left( \frac{\partial \Phi_{\hat{\lambda}(\hat{x})}(\hat{x})}{\partial \hat{x}} \right)^{-1} v_{\hat{\lambda}(\hat{x})_{\text{eq}}} \right], \]  

(35)

where \( F^{+} = (F^{T}F)^{-1}F^{T} \). Therefore, if each \( v_{\hat{\lambda}(\hat{x})_{\text{eq}}} \) was available, it might extract the necessary information to determine the unknown input on the switched system.

Theoretically, the equivalent injection is the result of an infinite switching frequency of the discontinuous terms \( v_{\lambda} \). However, the realization of the observer produces finite high switching frequency making necessary the application of a filter.
Thus, let us define the following equivalent injection estimator of $v_{eq}(t)$:

$$
\tau_{eq}(\hat{x}(t)) = \hat{v}_{eq}(\hat{x}(t)) - \hat{v}_{eq}(t),
$$

(36)

where each $\tau_{eq}(\hat{x}(t))$ is designed according to [31], i.e. $h \leq \tau_{eq}(\hat{x}(t)) \leq 1$ with $\tau_{eq}(\hat{x}) = h^{1/2}$, $\forall \hat{x}(\hat{x}) = 1, \ldots , q$, where $h$ is the sample time. Then, the unknown input could be reconstructed using the terms $\hat{v}_{eq}(\hat{x}(t))$ instead of $v_{eq}(\hat{x}(t))$.

Now, to maintain the correct identification of the unknown inputs, like in the observation problem, it is necessary to implement the following reset equations on the equivalent injection estimators (36), i.e.

$$
v_{eq}(t)^{+} = \hat{v}_{eq}(t^{+}), \quad \forall j = 1, 2, \ldots
$$

(37)

The purpose of the reset equations (30) and (37) is that switchings have no effect in the continuous state estimation and in the equivalent injection estimator, respectively. The main idea is to maintain both the state estimation error and the unknown input identification in the “desired surface”, i.e. in zero for the continuous state and in the unknown input for the equivalent injection estimators in spite of the switchings on the system.

Now, the multiple output case is described briefly in the following section.

6. Multiple output case

Consider the system (1)–(2) with $m$ outputs and the same number of unknown inputs, i.e.

$$
\dot{x} = f_{h}(x) + Fw(t), \quad y = h(x),
$$

(38)

where $x \in \mathcal{X} \subseteq \mathbb{R}^{n}$, $y \in \mathcal{Y} \subseteq \mathbb{R}^{m}$ is the output with $h(x) = [h_{1} \ h_{2} \ \cdots \ h_{m}] : \mathcal{X} \rightarrow \mathcal{Y}, \ w \in \mathcal{W} \subseteq \mathbb{R}^{m}$ is the unknown input term and $F = [F_{1} \ F_{2} \ \cdots \ F_{m}] \in \mathbb{R}^{n \times m}$ is a constant matrix. Now, assume that the following assumption is satisfied:

**Assumption 5.** The vector fields $f_{h}(x)$, the matrices $F_{j}$, and the functions $h_{l}(x)$ are such that $\forall x \in \mathcal{X}(x)$, $\forall l = 1, \ldots , m$, it is satisfied that

$$
d(L_{f_{h}(x)}^{k}h_{l}(x))F_{j} = 0, \quad \forall j = 1, \ldots , m, \ \forall k < r_{i, k_{0}} - 1, \ \forall l = 1, \ldots , m,
$$

$$
d(L_{f_{h}(x)}^{r_{i, k_{0}}-1}h_{l}(x))F_{j} \neq 0, \quad \text{for at least one } 1 \leq j \leq m.
$$

(39)

Now, let us define the following mappings:

$$
\Phi_{h}(x) = \begin{bmatrix}
    h_{1}(x) \\
    L_{f_{h}(x)}h_{1}(x) \\
    \vdots \\
    L_{f_{h}(x)}^{r_{i, k_{0}}-1}h_{1}(x) \\
    \vdots \\
    h_{m}(x) \\
    L_{f_{h}(x)}h_{m}(x) \\
    \vdots \\
    L_{f_{h}(x)}^{r_{i, k_{0}}-1}h_{m}(x)
\end{bmatrix}, \quad \forall i = 1, \ldots , q.
$$

(40)
Their corresponding Jacobian matrices are given by

\[
\frac{\partial \Phi_{i(x)}(x)}{\partial x} = \begin{bmatrix}
    dh_1(x) \\
    dL_{f_{i\alpha}(x)}h_1(x) \\
    \vdots \\
    dL_{f_{i\alpha}(x)}^{r_{i\alpha}^{-1}}h_1(x) \\
    \vdots \\
    dh_m(x) \\
    dL_{f_{i\alpha}(x)}h_m(x) \\
    \vdots \\
    dL_{f_{i\alpha}(x)}^{r_{i\alpha}^{-1}}h_m(x)
\end{bmatrix}, \quad \forall i = 1, \ldots, q. \tag{41}
\]

**Assumption 6.** Every matrix (41) is such that

\[
\text{rank} \left( \frac{\partial \Phi_1(x)}{\partial x} \right) = r_{1_1} + \cdots + r_{m_1} = n \quad \text{only } \forall x \in X_1,
\]

\[
\text{rank} \left( \frac{\partial \Phi_2(x)}{\partial x} \right) = r_{1_2} + \cdots + r_{m_2} = n \quad \text{only } \forall x \in X_2,
\]

\[
\vdots
\]

\[
\text{rank} \left( \frac{\partial \Phi_q(x)}{\partial x} \right) = r_{1_q} + \cdots + r_{m_q} = n \quad \text{only } \forall x \in X_q.
\]

Moreover, each mapping \( \Phi_{i(x)}(x) \) is a diffeomorphism only on its corresponding domain \( X_{i(x)} \), \( \forall i = 1, \ldots, q \).

Notice that the previous assumption implies the use of all outputs for estimating the state. Nevertheless, it is possible to use only some outputs for the estimation of the state and the rest to facilitate the unknown input identification problem. However, this case is not analyzed in the present work.

Considering the previous assumptions, it is possible to design the observers as in Eq. (11), i.e.

\[
\dot{\bar{x}}_\lambda = f_\lambda(\bar{x}_\lambda) + \left( \frac{\partial \Phi_j(\bar{x}_\lambda)}{\partial \bar{x}_\lambda} \right)^{-1} v_\lambda, \quad \forall \lambda = 1, \ldots, q,
\]

\[
\bar{y}_\lambda = h(\bar{x}_\lambda),
\]

with the estimated state vectors \( \bar{x}_\lambda \in \mathbb{R}^n \), the estimated outputs \( \bar{y}_\lambda \in \mathbb{R}^m \), and the correction terms \( v_\lambda \in \mathbb{R}^n \).

Let Assumption 4 be satisfied for every block formed by each of the outputs \( h_\lambda \), i.e.

**Assumption 7.** There are known constants \( M_{h_\lambda} > 0, \forall \lambda = 1, \ldots, q \), such that the following inequalities are satisfied:

\[
|L_{f_{j(\lambda)}}^{r_{j(\lambda)}}h_\lambda(\bar{x}_1) - L_{f_{j(\lambda)}}^{r_{j(\lambda)}}h_\lambda(x) - d(L_{f_{j(\lambda)}}^{r_{j(\lambda)}-1}h_\lambda(x))F_jw| < M_{1_1}, \quad \forall x \in X_1, \forall j, l = 1, \ldots, m, \tag{43}
\]

\[
|L_{f_{j(\lambda)}}^{r_{j(\lambda)}}h_\lambda(\bar{x}_2) - L_{f_{j(\lambda)}}^{r_{j(\lambda)}}h_\lambda(x) - d(L_{f_{j(\lambda)}}^{r_{j(\lambda)}-1}h_\lambda(x))F_jw| < M_{2_1}, \quad \forall x \in X_2, \forall j, l = 1, \ldots, m, \tag{44}
\]
\begin{equation}
\left| L_{f_{j}(x)}^{r_{j}} h_{i}(x) - L_{f_{j}(x)}^{r_{j}} h_{i}(x) - d(L_{f_{j}(x)}^{r_{j} - 1} h_{i}(x)) F_j \right| < M_{q_i}, \quad \forall x \in X_q, \forall j, l = 1, \ldots, m.
\tag{45}
\end{equation}

Now, the correction terms take the following form:

\begin{equation}
\mathbf{v}_{\lambda} = \begin{bmatrix}
-\alpha_{1,1,1} M_{z_{1_{1}}}^{1/r_{1_{1}}} |e_{y_{1_{1}}}^{l}| (r_{1_{1}} - 1)/r_{1_{1}} \text{ sign}(e_{y_{1_{1}}}) \\
-\alpha_{1,2,1} M_{z_{1_{2}}}^{1/r_{1_{2}}} |\hat{y}_{1,1_{2}} - \hat{y}_{1,1_{1}}| (r_{1_{2}} - 2)/(r_{1_{2}} - 1) \text{ sign}(\hat{y}_{1,1_{2}} - \hat{y}_{1,1_{1}}) \\
\vdots \\
-\alpha_{1,l_{1},1} M_{z_{1_{l_{1}}} - 1} |\hat{y}_{1,l_{1}} - \hat{y}_{1,l_{1} - 1}| \text{ sign}(\hat{y}_{1,l_{1}} - \hat{y}_{1,l_{1} - 1}) \\
-\alpha_{m,2_{1}} M_{z_{2_{1}}}^{1/r_{m_{2}} - 1} |\hat{y}_{m,2_{1}} - \hat{y}_{m,1_{1}}| (r_{m_{2}} - 2)/(r_{m_{2}} - 1) \text{ sign}(\hat{y}_{m,2_{1}} - \hat{y}_{m,1_{1}}) \\
\vdots \\
-\alpha_{m,l_{m},1} M_{z_{m_{1}}}^{1/r_{m_{l_{m}}} - 1} |\hat{y}_{m,l_{m}} - \hat{y}_{m,l_{m} - 1}| \text{ sign}(\hat{y}_{m,l_{m}} - \hat{y}_{m,l_{m} - 1})
\end{bmatrix},
\tag{46}
\end{equation}

where $e_{y_{1_{j}}} = h_{i}(x_{\lambda_{j}}) - h_{i}(x)$, $\forall \lambda = 1, \ldots, q$ and $\forall l = 1, \ldots, m$, are the output errors. The constants $\alpha_{j,k_{k_{j}}}$ are chosen recursively and sufficiently large as in the scalar case.

Notice that the observation and unknown input identification results for the multiple output case are now an extension of the results obtained for the scalar case applied in a block form for every output.

In the following section, simulation results demonstrate the effectiveness of the proposed methods.

7. Simulation example

7.1. Example 1

Consider the following example

\begin{equation}
\Sigma_1 : \begin{cases}
\dot{x} = \begin{bmatrix}
-2x_1 - x_2 - x_3 \\
x_1 \\
-x_3^3 - 2x_3 - x_4 \\
(x_2 - 4) \left( \frac{2x_1 + \sin x_4}{2 + \cos x_3} \right)
\end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} w,
\end{cases}
\end{equation}

\begin{equation}
\Sigma_2 : \begin{cases}
\dot{x} = \begin{bmatrix}
-x_1 - 2x_2 - 0.5x_3 \\
x_1 \\
-2x_3^3 - 3x_3 - 0.5x_4 \\
(x_2 - 4) \left( \frac{2x_2 + \cos x_4}{3 + \sin x_3} \right)
\end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} w,
\end{cases}
\end{equation}
\[
\Sigma_3 : \begin{cases}
\dot{x} = \begin{bmatrix}
-1.5x_1 - 1.5x_2 - 3x_3 \\
x_1 \\
-3x_3^3 - 0.5x_3 - 2.5x_4 \\
(x_3 - 4) \left( \frac{-2x_1^2}{2 - 3 \cos x_1} \right)
\end{bmatrix} + \begin{bmatrix}
0 \\
0 \\
0 \\
1
\end{bmatrix} w,
\end{cases}
\]

\[
\Sigma_4 : \begin{cases}
\dot{x} = \begin{bmatrix}
-3x_1 - 0.5x_2 - 2x_3 \\
x_1 \\
-0.5x_3^3 - 2x_3 - x_4 \\
(x_1 - 4) \sin x_1
\end{bmatrix} + \begin{bmatrix}
0 \\
0 \\
0 \\
1
\end{bmatrix} w.
\end{cases}
\]

The output, the discrete state and the unknown input are the following:

\[ y = x_2, \quad (47) \]

\[ i(x) = \begin{cases}
1, & \forall x | C(x) \in [-1.12, 3.5) \\
2, & \forall x | C(x) \in [-9, -1.12) \\
3, & \forall x | C(x) \in [3.5, 4.68) \\
4, & \forall x | C(x) \in [4.68, 9]
\end{cases}, \quad (48) \]

\[ w = 4x_3 \text{ square}(t) - 0.3 \sin(5t + x_2) + 1, \quad (49) \]

where \( C(x) = x_1 - x_3 \). The system initial conditions are set as \( x(0) = [2, 7, -0.5, 1]^T \). Simulations have been done in the Matlab Simulink environment, with the Euler discretization method and sampling time \( h = 0.0001 \) s.

Now, the assumptions along the paper are going to be analyzed for the example. Assumption 2 is satisfied since

\[ d(h(x)F) = 0, \]

\[ d(L_{h,i}(x)h(x))F = 0, \quad \forall i = 1, 2, 3, 4, \]

\[ d(L_{h,2}(x)h(x))F = 0, \quad \forall i = 1, 2, 3, 4, \]

\[ d(L_{h,3}(x)h(x))F \neq 0, \quad \forall i = 1, 2, 3, 4. \]

The matrix in Assumption 3, for \( i(x) = 1 \), has the following structure:

\[
\frac{\partial \Phi_1(x)}{\partial x} = \begin{bmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
-2 & -1 & -1 & 0 \\
3 & 2 & 4 + x_3^2 & 1
\end{bmatrix}.
\]

It could be computed that \( \text{rank}(\frac{\partial \Phi_1(x)}{\partial x}) = 4, \forall x \in \mathbb{R}^4 \). For the cases \( i(x) = 2, 3, 4 \), it is easy to show that Assumption 3 is satisfied in the same way. Then, since Assumption 2 and the rank condition are satisfied, it is possible to demonstrate that whole Assumption 3 is satisfied too (see [20]).

Now, to estimate the time \( T_\delta \), in Definition 3, it is necessary to compute the derivative of the function \( C(x) \) along the trajectories of the systems \( \Sigma_i \). Assuming that there exist known bounds for the trajectories of the system, it is possible to compute the time \( T_\delta \) estimating...
the variation of the function $C(x)$ w.r.t. the time in a certain arbitrary interval $[0, T]$ with $T > t_1$. Notice that it is necessary to make this for all trajectories of the systems $\Sigma_i$.

Next the procedure using the dynamics of the discrete state $i(x) = 1$ that produces the smallest time $T_\delta$ is shown.

By simulations it is possible to estimate the following bounds for the trajectories of the system in the time interval $[0, 1]$:

\[ |x_1(t)| \leq 2.2, \quad |x_2(t)| \leq 7.2, \quad |x_3(t)| \leq 1.1 \quad \text{and} \quad |x_4(t)| \leq 4. \]

Computing the derivative of the function $C(x)$ using the trajectories of system $\Sigma_1$, the following is obtained:

\[ \dot{C}(x(t)) = \frac{\partial C(x(t))}{\partial x(t)} \ddot{x}(t) = x_3^2(t) + x_3(t) - 2x_1(t) - x_2(t) + x_4(t), \]

\[ \dot{C}(x(t)) = -C(x(t)) + (x_3)^2 - x_1 - x_2 + x_4, \]

\[ \dot{C}(x(t)) \leq -C(x(t)) + 14.61. \]

From the comparison principle (see [22]), it is obtained that

\[ C(t) \leq 14.61 - 12.11 e^{-t}. \]

Now, using $[-1.12, 3.5] \in C_1$, it is possible to obtain that $T_\delta \approx 0.4813$ s. In this way the observers have to converge before $T_\delta \approx 0.4813$ s.

Then, the observers are designed as follows, with $\lambda = 1, 2, 3, 4$:

\[ \dot{\vec{\omega}}_\lambda = f_\lambda(\vec{\omega}_\lambda) + \left( \frac{\partial \phi_\lambda(\vec{\omega}_\lambda)}{\partial \vec{\omega}_\lambda} \right)^{-1} v_\lambda, \]

\[ \vec{y}_\lambda = h(\vec{\omega}_\lambda), \]

where the correction terms are calculated using the following auxiliary dynamics:

\[ \dot{\vec{\omega}}_{\lambda,1} = \vec{\omega}_{\lambda,2} - \vec{\omega}_{\lambda,1} M_{\lambda,1}^{1/4} |e_{\bar{y}_\lambda}|^{3/4} \text{sign}(e_{\bar{y}_\lambda}), \]

\[ \dot{\vec{\omega}}_{\lambda,2} = \vec{\omega}_{\lambda,3} - \vec{\omega}_{\lambda,2} M_{\lambda,2}^{1/3} |\vec{\omega}_{\lambda,2} - \vec{\omega}_{\lambda,1}|^{2/3} \text{sign}(\vec{\omega}_{\lambda,2} - \vec{\omega}_{\lambda,1}), \]

\[ \dot{\vec{\omega}}_{\lambda,3} = \vec{\omega}_{\lambda,4} - \vec{\omega}_{\lambda,3} M_{\lambda,3}^{1/2} |\vec{\omega}_{\lambda,3} - \vec{\omega}_{\lambda,2}|^{1/2} \text{sign}(\vec{\omega}_{\lambda,3} - \vec{\omega}_{\lambda,2}), \]

\[ \dot{\vec{\omega}}_{\lambda,4} = -\vec{\omega}_{\lambda,4} M_{\lambda} \text{sign}(\vec{\omega}_{\lambda,4} - \vec{\omega}_{\lambda,3}). \]

The parameter values of the correction terms are shown in Table 1.\(^3\)

Simulation results are shown in the following figures. Firstly, the output error convergence for every observer is shown in Fig. 1. By means of simulations it is possible to determine that the minimal error band, defined in inequality (26), is approximately $1 \times 10^{-15}$. Now, it is easy to see that the output error $e_{\bar{y}_1}$ satisfies the inequality (26) after $t = 0.420$ s (which is less than the calculated time $T_\delta \approx 0.4813$ s) and before the real first switching time, which is equal to $t_1 = 0.9660$ s, while other output errors not. Therefore, it is possible to determine that the estimation of the Observer 1 is the correct one.

In Fig. 2 the real and estimated continuous state are shown. It is possible to appreciate that the estimated continuous state does not get lost in spite of the switchings on the system and that it shows a finite time convergence. On the other hand, once the estimation of the continuous state is reached, the reconstruction of the discrete state is immediate by means of the switching signal, as shown in Fig. 3.

\(^3\)These parameters provide a minor convergence time to $0.4813$ s.
A comparison between the estimation error convergence with and without reset equation (30) is shown in Fig. 4. It is possible to see the effect of the reset equation which maintains the correct estimation of the continuous state on the switching times.

The unknown input identification is depicted in Fig. 5. The equivalent injection estimators were designed as in Eq. (36) with $\tau_j = 0.01$ $\forall \lambda = 1,2,3,4$. It is important to remark that to realize the identification of the unknown input it is necessary for the information of the four equivalent injections on the corresponding discrete state. A comparison between the identification with and without reset equation for equivalent injection estimators (37) is shown in Fig. 6. It is easy to see the effect of the reset equation which maintains the correct unknown input identification on the switchings times.

### Table 1

Correction term parameters.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>$\lambda = 1$</th>
<th>$\lambda = 2$</th>
<th>$\lambda = 3$</th>
<th>$\lambda = 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_{\lambda,1}$</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>$a_{\lambda,2}$</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>$a_{\lambda,3}$</td>
<td>1.5</td>
<td>1.5</td>
<td>1.5</td>
<td>1.5</td>
</tr>
<tr>
<td>$a_{\lambda,4}$</td>
<td>1.1</td>
<td>1.1</td>
<td>1.1</td>
<td>1.1</td>
</tr>
<tr>
<td>$M_\lambda$</td>
<td>200</td>
<td>50</td>
<td>100</td>
<td>100</td>
</tr>
</tbody>
</table>

7.2. Example 2

Consider a simplified particular 2-DOF model of a vertical oilwell drillstring (see [27] for more details). The drillstring torsional behavior is described by a torsional pendulum.
Fig. 2. Continuous trajectories of the nonlinear switched system.

Fig. 3. Switching signal reconstruction and discrete state estimation.

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driven by an electrical motor and the bit-rock contact is described by a dry friction term. The drill pipes are modeled by a linear spring with torsional stiffness $k_t$ and a torsional damping $c_t$, which connect the inertias $J_r$ and $J_b$ (Fig. 7 illustrates the $n$-DOF model of the drillstring).

The state vector $x \in \mathbb{R}^3$ composed of the angular displacements ($\phi_r$ and $\phi_b$) and angular velocities ($\dot{\phi}_r$ and $\dot{\phi}_b$) of the top-rotary system and the bit, i.e. $x = [\dot{\phi}_r, \phi_r - \phi_b, \dot{\phi}_b]^T$. A viscous damping torque $c_t x_1$ is considered at the top-drive system. The constant input torque applied by a motor at the surface is $T_m = u$. The torque on the bit is $T_b(x_3) = c_p x_3 + T_{fb}(x_3)$ where $c_p x_3$ approximating the influence of the mud drilling on the bit behavior.

![Fig. 4. Comparison between estimation errors convergence.](image)

![Fig. 5. Unknown input identification.](image)
and $T_{fb}(x_3)$ is the friction modeling the bit-rock contact, and $T_{fb}(x_3) = f_b(x_3) \text{sign}(x_3)$ with

$$f_b(x_3) = W_{ob}R_b \left[ \mu_{cb} + (\mu_{sb} - \mu_{cb}) \exp \left( -\frac{\gamma_b}{v_f} |x_3| \right) \right],$$

where $W_{ob} > 0$ is the constant weight on the bit, $R_b > 0$ is the bit radius. The static and Coulomb friction coefficients associated with $J_b$ are represented by $\mu_{sb}, \mu_{cb} \in (1, 0)$, and the positive constants $0 < \gamma_b < 1$ and $v_f > 0$. In addition, the Coulomb and static friction torque are $T_{cb} = W_{ob}R_b\mu_{cb}$ and $T_{sb} = W_{ob}R_b\mu_{sb}$, respectively.

Then, the drillstring switched dynamics with unknown input is given by

$$\Sigma_1 : \dot{x} = \begin{bmatrix} \frac{1}{J_r}[-(c_t + c_r)x_1 - k_r x_2 + c_t x_3 + u] \\ c_t x_1 - x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ k_c \frac{c_t}{J_r} \\ k_t \frac{k_t}{J_r} \end{bmatrix} w,$$

$$\Sigma_2 : \dot{x} = \begin{bmatrix} \frac{1}{J_r}[-(c_t + c_r)x_1 - k_r x_2 + c_t x_3 + u] \\ c_t x_1 - x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ k_c \frac{c_t}{J_r} \\ k_t \frac{k_t}{J_r} \end{bmatrix} w,$$

$$\Sigma_3 : \dot{x} = \begin{bmatrix} \frac{1}{J_r}[-(c_t + c_r)x_1 - k_r x_2 + u] \\ x_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ k_c \frac{k_t}{J_r} \\ k_t \frac{k_t}{J_r} \end{bmatrix} w.$$

Fig. 6. Unknown input identification without reset equation.
where $T^+_b$ and $T^-_b$ are $T_b(x_3)$ for $x_3>0$ and $x_3<0$, respectively. The output, the discrete state and the unknown input are the following:

$$y = [x_1, x_3]^T, \quad (50)$$

$$i(x) = \begin{cases} 
1, & \forall x | C(x) \in (T_s, \infty), \\
2, & \forall x | C(x) \in (-\infty, -T_s), \\
3, & \forall x | C(x) \in [-T_s, T_s],
\end{cases} \quad (51)$$

$$w = 2 \cos(1.5t) + 6 \text{ square}(0.5t) + 1, \quad (52)$$

Fig. 7. Mechanical model of the torsional behavior of a drillstring.
where $C(x) = c_x x_1 + k_t x_2$ and the unknown input $w$ could represent some non-desired oscillation (parameter $k$ represents a constant coupling of the unknown input). The parameter values of the drillstring dynamics are shown in Table 2.

The system initial conditions are set as $x(0) = [-10, -50, -10]^T$. The simulations have been done with the Euler discretization method and sampling time $h = 0.00001$ s.

Then, to design the observers 1 and 2 only will use the output $h_1 = x_1$ to reconstruct the whole state vector (scalar output case), while for the observer 3 will use the whole output $y$ to reconstruct the state (multiple output case). Now, the assumptions are going to be analyzed.

**Assumption 2**, for scalar output case, is satisfied since

$$d(h_1(x))F = 0,$$

$$d(L_{f_{i_{0}}(x)} h_1(x))F = 0, \quad \forall i = 1, 2,$$

$$d(L_{f_{i_{0}}(x)}^2 h_1(x))F \neq 0, \quad \forall i = 1, 2.$$  

**Assumption 5**, for multiple output case, is satisfied since

$$d(h_1(x))F = 0,$$

$$d(L_{f_{i_{0}}(x)} h_1(x))F = 0, \quad \forall i = 3,$$

$$d(h_2(x))F = 0.$$

The matrices in Eq. (10), for $i(x) = 1, 2$, have the following structure:

$$
\frac{\partial \Phi_{i(x)}(x)}{\partial x} = \begin{bmatrix}
1 & 0 & 0 \\
-0.26608 & -0.32896 & 6.5795 \times 10^{-2} \\
-0.23870 & 0.18484 & 0.35134 \exp(-0.9|x_3|) + 0.28502
\end{bmatrix}.
$$

Notice that in this case the observability mappings are equal. Therefore, it is possible to design only one observer for $i(x) = 1, 2$. 

**Table 2**

Drillstring parameters.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>$J_r$</td>
<td>2122</td>
<td>kg m$^2$</td>
</tr>
<tr>
<td>$J_b$</td>
<td>471.9698</td>
<td>kg m$^2$</td>
</tr>
<tr>
<td>$R_b$</td>
<td>0.155575</td>
<td>m</td>
</tr>
<tr>
<td>$k_t$</td>
<td>698.063</td>
<td>N m/rad</td>
</tr>
<tr>
<td>$c_t$</td>
<td>139.6126</td>
<td>N m s/rad</td>
</tr>
<tr>
<td>$c_b$</td>
<td>425</td>
<td>N m s/rad</td>
</tr>
<tr>
<td>$W_{ob}$</td>
<td>60</td>
<td>kN</td>
</tr>
<tr>
<td>$T_{sb}$</td>
<td>7.4676</td>
<td>kN m</td>
</tr>
<tr>
<td>$\mu_{ts}$</td>
<td>0.5</td>
<td>–</td>
</tr>
<tr>
<td>$\mu_{hs}$</td>
<td>0.8</td>
<td>–</td>
</tr>
<tr>
<td>$\gamma_b$</td>
<td>0.9</td>
<td>–</td>
</tr>
<tr>
<td>$v_f$</td>
<td>1</td>
<td>–</td>
</tr>
</tbody>
</table>

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The matrix in Eq. (41), for \( i(x) = 3 \), has the following structure:

\[
\frac{\partial \Phi_3(x)}{\partial x} = \begin{bmatrix} 1 & 0 & 0 \\ -0.26608 & -0.32896 & 0 \\ 0 & 0 & 1 \end{bmatrix}.
\]

It could be computed that \( \text{rank}(\partial \Phi_i(x)/\partial x) = 3 \), \( \forall x \in \mathbb{R}^3 \) for \( i(x) = 1, 2, 3 \). Then, since Assumptions 2 and 5 are satisfied, it is possible to demonstrate that Assumptions 3 and 6 are satisfied too (see [20]). As for Definition 3, it is considered that the first switching time \( t_1 = 1.146 \) s \( \geq T_0 \).

Then, the observers are designed as follows, with \( \lambda = 1, 3 \):

\[
\begin{align*}
\dot{x}_\lambda &= f_\lambda(x_\lambda) + \left( \frac{\partial \Phi_\lambda(x_\lambda)}{\partial x_\lambda} \right)^{-1} y_\lambda, \\
\dot{y}_\lambda &= h(x_\lambda),
\end{align*}
\]

where the correction terms, for \( \lambda = 1 \), are calculated using the following auxiliary dynamics:

\[
\begin{align*}
\dot{\lambda}_{1,1} &= \lambda_{1,2} - \lambda_{1,1} M_{\lambda}^{1/3} |e_{y_{1,1}}|^{2/3} \text{sign}(e_{y_{1,1}}), \\
\dot{\lambda}_{1,2} &= \lambda_{1,3} - \lambda_{1,2} M_{\lambda}^{1/2} |\lambda_{1,2} - \lambda_{1,1}|^{1/2} \text{sign}(\lambda_{1,2} - \lambda_{1,1}), \\
\dot{\lambda}_{1,3} &= -\lambda_{1,3} M_{\lambda} \text{sign}(\lambda_{1,3} - \lambda_{1,2}).
\end{align*}
\]

For \( \lambda = 3 \) the correction terms are calculated using the following auxiliary dynamics:

\[
\begin{align*}
\dot{\lambda}_{1,1} &= \lambda_{1,2} - \lambda_{1,1} M_{\lambda}^{1/2} |e_{y_{1,1}}|^{1/2} \text{sign}(e_{y_{1,1}}), \\
\dot{\lambda}_{1,2} &= -\lambda_{1,2} M_{\lambda} \text{sign}(\lambda_{1,2} - \lambda_{1,1}), \\
\dot{\lambda}_{1,3} &= -\lambda_{1,3} M_{\lambda} \text{sign}(e_{y_{1,2}}),
\end{align*}
\]

where \( e_{y_{1,1}} = h_1(x_\lambda) - h_1(x) \) and \( e_{y_{1,2}} = h_2(x_\lambda) - h_2(x) \). The parameter values of the corrections terms are shown in Table 3.5

**Table 3**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>( \lambda = 1 )</th>
<th>( \lambda = 3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_{1,1} )</td>
<td>2</td>
<td>–</td>
</tr>
<tr>
<td>( a_{1,2} )</td>
<td>1.5</td>
<td>–</td>
</tr>
<tr>
<td>( a_{1,3} )</td>
<td>1.1</td>
<td>–</td>
</tr>
<tr>
<td>( M_{\lambda} )</td>
<td>100</td>
<td>–</td>
</tr>
<tr>
<td>( a_{1,11} )</td>
<td>–</td>
<td>1.5</td>
</tr>
<tr>
<td>( a_{1,21} )</td>
<td>–</td>
<td>1.1</td>
</tr>
<tr>
<td>( a_{2,11} )</td>
<td>–</td>
<td>1.1</td>
</tr>
<tr>
<td>( M_{\lambda} )</td>
<td>10</td>
<td>–</td>
</tr>
<tr>
<td>( M_{\lambda} )</td>
<td>10</td>
<td>–</td>
</tr>
</tbody>
</table>

The parameter values of the corrections terms are shown in Table 3.5

Simulation results are shown in the following figures. The output error convergence for every observer is shown in Fig. 8. By means of simulations it is possible to determine that the minimal error band, defined in inequality (26), is approximately \( 1 \times 10^{-10} \). It is possible

---

4Notice that the observer \( \lambda = 1 \) is working for \( i(x) = 1, 2 \) since its observability mappings are equal.

5These parameters provide a minor convergence time to 1.146 s.
Fig. 8. Output error convergence for each observer, drillstring.

Fig. 9. Continuous trajectories of the nonlinear switched system, drillstring.

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Fig. 10. Switching signal reconstruction and discrete state estimation, drillstring.

Fig. 11. Comparison between estimation errors convergence, drillstring.
to see that the output error $e_{y_{11}}$ satisfies the inequality (26) after $t = 1.028$ s and before the real first switching time, which is equal to $t_1 = 1.146$ s, while other output errors are not. Then, it is possible to determine that the estimation of the Observer 1 is the correct one and then estimate the correct discrete state.

The real and estimated continuous state are shown in Fig. 9. It is possible to see that the estimated continuous state does not get lost in spite of the switchings on the system and that it shows a finite time convergence. On the other hand, once the estimation of the continuous state is reached, the reconstruction of the discrete state is immediate by means of the switching signal, as shown in Fig. 10. Notice that there exists an interval of time $t \in (13, 14)$ s where the system presents high frequency switchings. Nevertheless, the estimation of the continuous state never gets lost.

![Fig. 12. Unknown input identification, drillstring.](image1.png)

![Fig. 13. Unknown input identification without reset equations, drillstring.](image2.png)
A comparison between the estimation error convergence with and without reset equation (30) is shown in Fig. 11. It is clear that the effect of the reset equation which maintain the correct estimation of the continuous state on the switching times.

Finally, in Fig. 12 the unknown input identification is shown. The equivalent injection estimators were designed as in Eq. (36) with \( t_\lambda = 0.003162 \) \( \forall \lambda = 1,2,3 \). A comparison between the identification with and without reset equation for equivalent injection estimators (37) is shown in Fig. 13. It is possible to appreciate the effect of reset estimator equation which maintains the correct unknown input identification on the switching times.

8. Conclusions

A solution to the state estimation and the unknown input identification problems for a class of nonlinear autonomous switched systems has been proposed. A robust observer has been established that it is able to estimate the continuous and discrete state in the presence of unknown inputs. On the other hand, a method for identification of unknown inputs using the equivalent injection is proposed. Simulation results confirmed the effectiveness of the proposed approach.

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