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History-Dependent Modified Sliding Mode Interception Strategies with Maximal Capture Zone

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Abstract

In order to construct the guidance strategy in a realistic nonlinear noise-corrupted interception endgame against a maneuverable target, a linearized zero-sum differential game is considered. Assuming perfect information in this game, sufficient conditions are established, which guarantee that a continuous interception strategy with memory (history-dependent) has the maximal capture zone. Two examples of such a strategy are analyzed: a modified super-twisting second-order sliding mode control and a modified integral sliding mode control. Simulation results of the original nonlinear interception endgame demonstrate that these strategies considerably reduce the chattering created by the classical game optimal bang-bang strategy without deteriorating the homing performance.

Keywords:
Interception endgame, history-dependent strategy, sliding mode control, maximal capture zone

1. Introduction

The problem of intercepting a maneuverable target admits different mathematical formulations. In [1, 2, 3], the terminal phase of such interception was formulated as a linear perfect information pursuit-evasion differential
game with bounded controls (without penalties on the control usage), where the interceptor is the pursuer and the target is the evader. The cost function in this game is the miss distance.

The planar version of such a game can be reduced to a scalar game, where the state variable is the zero-effort miss distance. The solution of this game [1] includes a singular region of the game space, where the optimal strategies are arbitrary (but bounded) and the game value is constant. Outside of this region, bang-bang type control strategies, i.e. bounded controls with the sign of the zero-effort miss distance, are optimal. It was shown in [1] that if some conditions are satisfied, the constant game value in the singular region is zero, i.e. the closure of this region becomes the capture zone of the game optimal pursuer strategy. By definition, the capture zone of a given strategy is the set of all initial positions from which this strategy guarantees capture robustly with respect to any admissible evader control satisfying the control constraint.

The implementation of an arbitrary (non unique) optimal pursuer strategy in the singular region allows several options for the designer. The most evident option is to use the bang-bang strategy in the entire game space. This strategy has the maximal capture zone, i.e. it contains the capture zones of all other admissible pursuer strategies. Nevertheless, being a first order sliding mode control, it leads to control chattering, which can create implementation difficulties [4, 5]. In particular, it leads to unacceptable wear of the actuators and an unnecessary over expenditure of the control. For this reason a number of studies were devoted to eliminate or at least to reduce considerably this phenomenon [6, 7, 8, 9, 10, 11]. Several alternative (linear, saturated linear and weakly nonlinear) strategies were proposed in order to avoid the chattering [12, 13, 14, 15, 16, 17].

These strategies guarantee capture at a prescribed time \( t_f \) with reduced chattering. In [12], necessary and sufficient conditions were formulated in order to assure that a saturated continuous feedback strategy has the maximal capture zone.

An alternative approach to avoid chattering is utilizing an interceptor strategy with memory, i.e. a strategy depending on a time history of state variables. Higher order sliding mode control algorithms [18, 9, 19, 20, 21, 22, 23] represent such a history-dependent strategy. Such strategies can bring the system onto the time axis before \( t_f \) and keep it there till \( t_f \) without chattering. Interception applications of a second order sliding mode control can be found for example in [24, 25]. In these papers, the interception problem with
nonlinear dynamics is considered. The disturbance is assumed to be a differentiable function of time with known Lipschitz constant of the derivative. The duration of the interception is not prescribed. For this problem, a second order sliding mode guidance law is designed, providing control smoothness and finite time capture. Since control constraints are not specified in the model, this guidance strategy guarantees robust capture from the entire interception space. Another example of a history-dependent strategy is given by an integral sliding mode control [26, 27, 28, 29, 30]. An important feature of this control is tracking the trajectory of the undisturbed system, generated by a nominal control, from the very beginning of the control process. This ensures the insensitivity of the system motion to matched uncertainties.

The structure of the present paper, where the disturbance (the target control) can be also discontinuous, is the following. In the next section, the interception endgame scenario is stated. In a linear framework, which is justified for endgame analysis, the interception duration can be precalculated and prescribed in advance. The first and simplest solution for this problem, which was derived in [1], is briefly recalled in Section 3. It has the maximal capture zone, but yields a chattering control. In the sequel, a general interception strategy with memory (history-dependent) is considered. Sufficient conditions, guaranteeing the maximality of the capture zone for a history-dependent strategy, are established in Section 4. Based on this result, two history-dependent interception strategies are designed: a modification of the super-twisting second order sliding mode control (Section 5) and a modification of the integral sliding mode control (Section 6). In order to evaluate the performance of these strategies in comparison with the bang-bang strategy [1] and a saturated linear strategy used in the singular region [31], Monte Carlo simulations are carried out in a realistic nonlinear noise-corrupted interception scenario (Section 7). These simulations also confirm the validity of the using a linear model for the study. Conclusions are presented in Section 8.

2. Problem Statement

A planar interception endgame scenario between two moving objects - a pursuer and an evader - is considered. The schematic view of this engagement is shown in Fig. 1. The X axis of the coordinate system is aligned with the initial line of sight. The origin is collocated with the initial pursuer position. The points \((x_p, y_p), (x_e, y_e)\) are the current coordinates; \(V_p\) and \(V_e\)
are the velocities and \( a_p \) and \( a_e \) are the lateral accelerations of the pursuer and the evader, respectively; \( \varphi_p, \varphi_e \) are the respective aspect angles between the velocity vectors and the reference line of sight; and \( y = y_e - y_p \) is the relative separation normal to the initial line of sight. The line-of-sight angle \( \lambda \) is the angle between the current and initial lines of sight, \( r \) is the current range between the objects. It is assumed that the dynamics of each object is expressed by a first-order transfer function with the time constants \( \tau_p \) and \( \tau_e \), respectively. The velocities and the bounds of the lateral acceleration commands of both objects are constant.

![Interception geometry](image)

**Figure 1: Interception geometry**

The nonlinear equations of motion (to be linearized later) are

\[
\begin{align*}
x_p &= V_p \cos \varphi_p, & \quad x_p(0) &= 0, \\
y_p &= V_p \sin \varphi_p, & \quad y_p(0) &= 0 \\
\dot{\varphi}_p &= \frac{a_p}{V_p}, & \quad \varphi_p(0) &= \varphi_{p0}, \\
\dot{a}_p &= \frac{(a_{p}^{\text{max}}u - a_p)}{\tau_p}, & \quad a_p(0) &= 0, \\
x_e &= -V_e \cos \varphi_e, & \quad x_e(0) &= r_0, \\
y_e &= V_e \sin \varphi_e, & \quad y_e(0) &= 0 \\
\dot{\varphi}_e &= \frac{a_e}{V_e}, & \quad \varphi_e(0) &= \varphi_{e0}, \\
\dot{a}_e &= \frac{(a_{e}^{\text{max}}v - a_e)}{\tau_e}, & \quad a_e(0) &= 0,
\end{align*}
\]

where \( r_0 \) is the initial range between the players, \( a_{p}^{\text{max}} \) and \( a_{e}^{\text{max}} \) are the acceleration bounds of the pursuer and the evader, respectively. The controls
of the pursuer $u$ and the evader $v$ are the respective normalized lateral acceleration commands

$$|u(t)| \leq 1, \quad |v(t)| \leq 1, \quad 0 \leq t \leq t_f. \quad (2)$$

It is assumed that the functions $u(t), v(t)$ are measurable on $[0, t_f]$ and are bounded according to (2).

If the aspect angles $\varphi_p$ and $\varphi_e$, as well as the line of sight angle $\lambda$, either are small or remain close to their initial values during the engagement, which is the case in most interception endgames, then the system (1) can be linearized [1] and written as

$$\dot{x} = Ax + bu + cv, \quad x(0) = x_0, \quad (3)$$

where the state vector is $x = (x_1, x_2, x_3, x_4)^T = (y, \dot{y}, a_e, a_p)^T$ (the superscript $T$ denoting transposition),

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1/\tau_e & 0 \\ 0 & 0 & 0 & -1/\tau_p \end{bmatrix}, \quad (4)$$

$$b = (0, 0, a_p^{\text{max}}/\tau_p)^T, \quad c = (0, 0, a_e^{\text{max}}/\tau_e, 0)^T, \quad (5)$$

$$x_0 = (0, x_{20}, 0, 0)^T, \quad x_{20} = V_e \varphi_{e0} - V_p \varphi_{p0}. \quad (6)$$

The small angles assumption allows calculating the final time of the engagement. For a near head-on engagement with small aspect angles it becomes

$$t_f = r_0/(V_p + V_e). \quad (7)$$

The objective of the pursuer is to nullify the miss distance, expressed by

$$J = |x_1(t_f)|, \quad (8)$$

robustly with respect to all admissible evader control (disturbance) and respecting the constraints (2), by means of a saturated state feedback strategy with memory.

In order to introduce such a strategy, let us consider a continuous function $\vartheta(t) \geq 0, \ t \in [0, t_f]$, which characterizes the memory depth. The actual memory depth is

$$\Delta(t) = \min\{t, \vartheta(t)\}, \quad (9)$$

$$\dot{x} = Ax + bu + cv, \quad x(0) = x_0, \quad (3)$$

where the state vector is $x = (x_1, x_2, x_3, x_4)^T = (y, \dot{y}, a_e, a_p)^T$ (the superscript $T$ denoting transposition),

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$$\Delta(t) = \min\{t, \vartheta(t)\}, \quad (9)$$
i.e. in the case $\vartheta(t) > t$, full memory ($\Delta(t) = t$) is assumed.

Let introduce the saturated strategy

$$u = u^s(t, x_{\vartheta(t)}(\cdot)) = \text{sat}\left(U(t, x_{\vartheta(t)}(\cdot))\right),$$

where

$$x_{\vartheta(t)}(\cdot) \triangleq \{x(\tau) : \rho(t) \leq \tau \leq t\}, \quad 0 \leq t < t_f,$$

$$\rho(t) = t - \Delta(t),$$

$$\text{sat}(w) = \begin{cases} 1, & w > 1, \\ w, & |w| \leq 1, \\ -1, & w < -1. \end{cases}$$

For any given $t \in [0, t_f)$, the function $x_{\vartheta(t)}(\tau)$ is the time history of the motion $x(t)$ with a variable depth $\Delta(t)$. Thus, for any given $t \in [0, t_f)$, $U(t, x_{\vartheta(t)}(\cdot))$ is a functional, defined on the set of absolutely continuous functions $x(\tau)$, $\tau \in [\rho(t), t]$. It is such that there exists an absolutely continuous solution of (3) on the interval $[0, t_f]$ for $u = u^s$ and any admissible disturbance $v(t)$. Note that a pure feedback strategy $U(t, x(t))$ and a feedback strategy with full memory $U(t, x_{\vartheta(t)}(\cdot))$ are particular cases of the functional $U(t, x_{\vartheta(t)}(\cdot))$ for $\vartheta(t) \equiv 0$ and $\vartheta(t) = t$, $t \in [0, t_f]$, respectively.

In the sequel, it is assumed that either (i) $\vartheta(t) < t$ for all $t \in [0, t_f]$, or (ii) there exists a single moment $\bar{t} \in [0, t_f]$ such that $\vartheta(t) > t$ for $t \in [0, \bar{t})$ and $\vartheta(t) < t$ for $t \in (\bar{t}, t_f]$. The assumption (i) means that the memory is not full on the entire control interval. The assumption (ii) means that the memory is full only on the initial interval ($t < \bar{t}$). This feature of a memory depth is important in interception problems.

In the next section, the first and simplest interception solution, based on a linear pursuit-evasion game formulation with bounded controls [1], is presented.

3. Linear Pursuit-Evasion Game Solution

A convenient way to solve the above stated planar pursuit-evasion problem is by scalarization.
3.1. Scalarization

Let make the following state transformation, called also the terminal projection [32], in the system (3).

\[ z = d \Phi(t_f, t)x, \quad (14) \]

where \( d = (1, 0, 0, 0) \) and \( \Phi(t, t_0) \) is the fundamental matrix of \( \dot{x} = Ax \). The new state variable \( z \) is the zero-effort miss distance [33]. It is expressed explicitly as

\[ z(t) = x_1(t) + (t_f - t)x_2(t) + \tau_e^2 \Psi((t_f - t)/\tau_e)x_3(t) - \tau_p^2 \Psi((t_f - t)/\tau_p)x_4(t), \quad (15) \]

where

\[ \Psi(\xi) = \exp(-\xi) + \xi - 1 > 0, \quad \xi > 0, \quad (16) \]

Based on (15), the system (3) is reduced to the scalar one

\[ \dot{z} = h_1(t)u + h_2(t)v, \quad z(t_0) = z_0, \quad (17) \]

where

\[ h_1(t) \triangleq -h(t; \tau_p, a^\text{max})_p, \quad h_2(t) \triangleq h(t; \tau_e, a^\text{max})_e, \quad (18) \]

\[ h(t; \tau, a^\text{max}) = \tau a^\text{max} \Psi((t_f - t)/\tau). \quad (19) \]

By using (6) and (15), the initial value of \( z(t) \) is

\[ z_0 = t_f x_{20}. \quad (20) \]

Note that

\[ J = |z(t_f)|. \quad (21) \]

The objective of the pursuer in the scalarized interception problem is to achieve \( z(t_f) = 0 \), robustly with respect to any admissible evader control, keeping the constraints (2) satisfied, by means of a strategy

\[ u = u^s(t, z_{\theta(t)}(\cdot)) = \text{sat} \left( u(t, z_{\theta(t)}(\cdot)) \right), \quad (22) \]

where \( z_{\theta(t)}(\cdot) = \{ z(\tau) : \tau \in [\rho(t), t] \} \) is the time history of the state variable of (17) on the interval \([\rho(t), t] \). The solution of this scalarized problem was obtained [1] by formulating it as a linear zero-sum differential game with bounded feedback (no memory depth) controls for the dynamics (17) with the cost function (21) and the control constraints (2).
3.2. The Linear Game Solution

The solution of this linear game is based on the decomposition of the scalarized game space \( D \triangleq \{ (t, z) : t \in [0, t_f], z \in \mathbb{R}^1 \} \) into two regions of different feedback strategies. In the first (regular) region \( D_1 \) the optimal strategies have the "bang-bang" structure:

\[
  u^0(t, z) = v^0(t, z) = \text{sign } z, 
\]

and the value of the game is nonzero, depending on the initial conditions. In the second (singular) region \( D_0 = D \setminus D_1 \) the optimal strategies \( u^0(t, z) \) and \( v^0(t, z) \) are arbitrary subject to (2), and the value of the game is constant.

If the following two inequalities are satisfied

\[
a_{\text{p}}^{\text{max}} > a_{\text{e}}^{\text{max}}, \quad a_{\text{p}}^{\text{max}} / \tau_{\text{p}} > a_{\text{e}}^{\text{max}} / \tau_{\text{e}},
\]

the singular region is

\[
  D_0 = \{ (t, z) : 0 \leq t < t_f, |z| < z^*(t) \},
\]

where

\[
  z^*(t) = \int_t^{t_f} \left[ h(\xi; \tau_{\text{p}}, a_{\text{p}}^{\text{max}}) - h(\xi; \tau_{\text{e}}, a_{\text{e}}^{\text{max}}) \right] d\xi.
\]

The conditions (24) guarantee that

\[
  H(\xi) \triangleq h(\xi; \tau_{\text{p}}, a_{\text{p}}^{\text{max}}) - h(\xi; \tau_{\text{e}}, a_{\text{e}}^{\text{max}}) > 0, \quad \xi > 0,
\]

yielding that the function \( z^*(t) \) is monotonically decreasing. In this case, the singular game value is

\[
  J^0 = 0, \quad (t_0, z_0) \in D_0.
\]

This means that, subject to the conditions (24), the closure of the singular region \( D_{0M} = \text{clo}(D_0) \) becomes the capture zone, i.e. the set of all initial positions, from which the pursuer can guarantee zero miss distance against any admissible evader strategy. This capture zone is maximal, i.e. there is no any other pursuer strategy with a capture zone larger than the capture zone given by the closure of (25). In Fig. 2, the game space decomposition, subject to (24), is presented for the parameters \( \tau_{\text{p}} = \tau_{\text{e}} = 0.2 \text{ s}, a_{\text{p}}^{\text{max}} = 200 \text{ m/s}^2, a_{\text{e}}^{\text{max}} = 100 \text{ m/s}^2, t_f = 4 \text{ s} \).
Remark 1. The definition of $D_{0M}$ implies the following important property. Let $v(t), t \in [0,t_f)$, be any admissible disturbance. If a trajectory of (17) starts from some point $(t_1, z^*(t_1))$ on the upper boundary of $D_{0M}$ with $u(t_1) = 1$ (or from some point $(t_1, -z^*(t_1))$ on the lower boundary of $D_{0M}$ with $u(t_1) = -1$), then it does not leave $D_{0M}$ after $t = t_1$.

Although the optimal pursuer strategy of the game solution inside the maximal capture zone is arbitrary (i.e., non-unique), it is proposed in [1] that the interceptor guidance law (which must be unique) employs the bang-bang strategy everywhere. This guidance law, associated with the singular bang-bang strategy is called DGL/1. Using DGL/1 leads to a first order sliding mode [9] along the $t$-axis, yielding control chattering. The chattering is extremely undesirable in practical implementations (see e.g. [4, 5]). Particularly, it causes an unacceptable wear of the actuators and an overexpenditure of the control.

Another known game optimal strategy is a saturated linear strategy of the form [31, 12]

$$u = \text{sat} \left( \frac{z}{z^*(t)} \right).$$

In [12], where this strategy is called the saturated "natural" strategy, it is shown that by using it the control chattering is practically eliminated, but in a noise-corrupted interception scenario the homing performance considerably deteriorates.

The objective of this paper is to apply a feedback strategy of more general
type, depending on \( z_0(t) \cdot \), for the following purposes: (i) to keep the capture zone maximal; (ii) to avoid or at least to substantially reduce the control chattering; (iii) to avoid the deterioration of the homing performance in nonlinear noise-corrupted interception scenarios.

4. Sufficient Condition for Maximal Capture Zone

In this section, the general sufficient condition is established for a saturated history-dependent strategy (22) to have the maximal capture zone.

Let for any \( t \in [0, t_f) \) introduce the class of absolutely continuous functions \( w(\tau), \tau \in [\rho(t), t] \):

\[
W_t = \{ w(\tau) : |w(\tau)| \leq z^*(\tau), \tau \in [\rho(t), t] \},
\]

and two subclasses

\[
W_t^+ = \{ w(\tau) \in W_t : w(t) = z^*(t), \tau \in [\rho(t), t] \},
\]

\[
W_t^- = \{ w(\tau) \in W_t : w(t) = -z^*(t), \tau \in [\rho(t), t] \},
\]

where the function \( z^*(t) \) is given by (26).

**Theorem 1.** Assume that for any \( t \in [0, t_f) \), and for any \( w^+(\cdot) \in W_t^+ \) and \( w^-(\cdot) \in W_t^- \), the inequalities

\[
\begin{align*}
u(t, w^+ ( \cdot )) & \geq 1, \\
u(t, w^- ( \cdot )) & \leq -1,
\end{align*}
\]

are valid. Then the capture zone of the saturated strategy (22) is maximal.

**Proof.** In the proof, the idea of Lemma 2 in [12] is extended to the case of history-dependent pursuer strategy.

In [12], it was shown that for a given function \( L(t, z) \), the saturated strategy \( u = \text{sat} \left( L(t, z(t)) \right) \) has the maximal capture zone if and only if

\[
L(t, z^*(t)) \geq 1, \quad L(t, -z^*(t)) \leq -1, \quad t \in [0, t_f).
\]

For any (but fixed) admissible disturbance, consider a trajectory \( z(t) \) of the closed-loop system (17), (22), emanating from \( (t_0, z_0) \in D_{0M} \). Let \( t_1 \geq t_0 \) be the first moment when this trajectory reaches the boundary of \( D_{0M} \).
Consider the case where it reaches the upper boundary, i.e \( z(t_1) = z^*(t_1) \).
This means that \( z_{\theta(t_1)}(\cdot) \in W_{t_1}^{*+} \) and, by virtue of (33),
\[
 u(t_1) \triangleq u(t_1, z_{\theta(t_1)}(\cdot)) \geq 1. 
\] (36)
Then, due to equations (13), (22), \( u^*(t_1) = 1 \). Therefore, by Remark 1, this trajectory will not leave \( D_{0M} \) after \( t = t_1 \). Thus, the trajectory \( z(t) \), emanating from \( (t_0, z_0) \in D_{0M} \) does not leave the maximal capture zone through its upper boundary. Similarly, based on the condition (34), it is shown that such a trajectory does not leave the maximal capture zone through its lower boundary. This completes the proof of the theorem.

The following corollary is a direct consequence of Theorem 1.

**Corollary 1.** Assume that for any \( t \in [0, t_f) \), the functional \( u(t, w(\cdot)) \) is defined for all \( w(\cdot) \in W_t \) and is odd with respect to \( w(\cdot) \):
\[
 u(t, -w(\cdot)) = -u(t, w(\cdot)). 
\] (37)
If the condition (33) holds for any \( w^+(\cdot) \in W_t^{*+} \) then the capture zone of the saturated strategy (22) is maximal.

In the following two sections two saturated history-dependent feedback strategies that satisfy the conditions of Theorem 1 are presented.

### 5. Modified Super-Twisting Strategy

The classical super-twisting (STW) control for the scalar system (17) is of the form \( [9, 20, 34, 35, 23] \)
\[
 u = u^{\text{STW}}(t, z_t(\cdot)) = k_1 |z(t)|^{1/2} \text{sign} z(t) + k_2 \int_0^t \text{sign} z(\tau) d\tau, 
\] (38)
where \( z_t(\cdot) = \{ z(\tau) : \tau \in [0, t] \} \) is the full time history of state variable; \( k_1 > 0, k_2 > 0 \) are constant gain coefficients. In a recent work \( [36] \), the gain coefficients are assumed to be time-varying:
\[
 u = k_1(t, z)|z(t)|^{1/2} \text{sign} z(t) + \int_0^t k_2(\tau, z(\tau)) \text{sign} z(\tau) d\tau, 
\] (39)
where the functions \( k_i = k_i(t, z) \), \( i = 1, 2 \), are continuous and bounded from below. Under the assumptions of boundedness and sufficient smoothness of the disturbance function \( h_2(t)v(t, z) \), the gain coefficients can be chosen in order to provide robust finite-time stability of the closed-loop system (17), (38), i.e. the sliding "manifold" \( z = 0 \) is reached by the system in a finite time, robustly with respect to the disturbance function. Some estimates of this reaching time are known in the literature [18, 35, 23]. These estimates, as well as the magnitude of the control time realization, depend on the gain coefficients. Note that the functions \( h_1(t) \) and \( h_2(t) \) in the system (17) satisfy the conditions:

\[
h_i(t) \neq 0, \quad t \in [0, t_f), \quad i = 1, 2,
\]

and there exists a finite limit

\[
\lim_{t \to t_f^-} \frac{h_2(t)}{h_1(t)} = -\frac{\tau_p a_p^{\max}}{\tau_e a_p^{\max}}.
\]

Therefore, the uncertainty term \( h_2(t)v \) is matched.

The solution of the linear interception problem has to respect the following requirements:

(i) prescribed control constraint (2);

(ii) prescribed final time \( t_f \);

(iii) possible discontinuity of the disturbance (evader control);

(iv) memory restriction.

The above described classical super-twisting control does not satisfy these requirements. Therefore, it should be modified for the purposes of interception guidance.

The modified super-twisting interception strategy suggested in this paper, is a particular case of (22):

\[
u = u_{m}^{STW}(t, z_{\theta(t)}(\cdot)) = \text{sat}\left(\bar{u}(t, z_{\theta(t)}(\cdot))\right),
\]

where

\[
\bar{u}(t, z_{\theta(t)}(\cdot)) = K_1(t)|z(t)|^{1/2} \text{sign} z(t) + K_2(t) \int_{\rho(t)}^{t} \text{sign} z(\tau) d\tau,
\]

\[
u = u_{m}^{STW}(t, z_{\theta(t)}(\cdot)) = \text{sat}\left(\bar{u}(t, z_{\theta(t)}(\cdot))\right),
\]

and there exists a finite limit

\[
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\]

where

\[
\bar{u}(t, z_{\theta(t)}(\cdot)) = K_1(t)|z(t)|^{1/2} \text{sign} z(t) + K_2(t) \int_{\rho(t)}^{t} \text{sign} z(\tau) d\tau,
\]
\[ K_1(t) \triangleq \frac{k_1}{(t_f - t)^{p/2}}, \quad K_2(t) \triangleq \frac{k_2}{(t_f - t)^{(p-3)/2}}, \]  

where \( k_1 > 0, k_2 > 0, p \geq 3 \) are some constants.

Note that the gains \( K_1(t) \) (for \( p > 0 \)) and \( K_2(t) \) (for \( p > 3 \)) tend to infinity for \( t \to t_f - 0 \).

In [18, 35, 23, 36] a Lyapunov function was used to establish the finite-time stability and to estimate the reaching time. In the present paper, the prescribed time convergence, provided by the modified super-twisting control (42), is obtained by employing the capture zone technique.

Let introduce the function

\[ F(t, k_1, k_2, p) \triangleq \frac{k_1}{(t_f - t)^{p/2}} \sqrt{z^*(t)} - \frac{k_2}{(t_f - t)^{(p-3)/2}} \Delta(t), \]  

where \( z^*(t) \) is given by (26), \( \Delta(t) \) is given by (9).

**Theorem 2.** If the constants \( k_1, k_2 \) and \( p \) satisfy the inequality

\[ \inf_{t \in [0,t_f]} F(t, k_1, k_2, p) \geq 1, \]  

then the capture zone of the strategy (42) is maximal, i.e it coincides with \( D_{0M} \).

**Proof.** In order to prove the theorem, it is sufficient to show that the conditions of Corollary 1 are valid for the functional (43). The fulfilment of the condition (37) directly follows from (43).

Proceed to the condition (33). Let \( w(\cdot) \in W_t^{*+} \) be arbitrary. Then, due to (43) and (9), for any \( t \in [0,t_f] \),

\[ \bar{u}(t, w(\cdot)) \geq K_1(t)|z^*(t)|^{1/2} - K_2(t)\Delta(t) = F(t, k_1, k_2, p). \]  

This inequality, along with the condition (46), directly yields (33). \( \square \)

**Remark 2.** By using (16), (18), (24), (26) and the L’Hôpital rule

\[ \lim_{t \to t_f-0} \frac{z^*(t)}{(t_f - t)^3} = \frac{1}{6} \left( \frac{a_p^\text{max}}{\tau_p} - \frac{a_e^\text{max}}{\tau_e} \right) \neq 0. \]  

The equation (48), along with (45), yields that for \( p < 3 \) and for any \( k_1 > 0, k_2 > 0 \),

\[ \lim_{t \to t_f-0} F(t, k_1, k_2, p) = 0, \]  

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i.e. the condition (46) is not satisfied.

**Example 1.** Consider a noise free interception with the parameters $\tau_p = \tau_e = 0.2\,\text{s}$, $a_{p}^{\text{max}} = 200\,\text{m/s}^2$, $a_{e}^{\text{max}} = 100\,\text{m/s}^2$, $t_f = 4\,\text{s}$, and $\vartheta(t) = t$ (full memory case). In Fig. 3, the domain of the coefficients $k_1, k_2$, satisfying the condition (46) for different values of $p$, is shown for $k_1 \in [0, 5]$ and $k_2 \in [0, 10]$. It is seen that this domain is maximal for $p = 3$.

![Figure 3: Domain of $k_1, k_2$, guaranteeing maximal capture zone](image)

In Fig. 4a, the trajectories of (17) are depicted for the bang-bang strategy and for the modified super-twisting strategy (42) with $k_1 = 0.3$, $k_2 = 0.1$, $p = 3$, $\vartheta(t) \equiv 2\,\text{s}$. The evader control is $v = \text{sign } z$, optimal in the differential

![Figure 4: Trajectories generated by the modified STW and the bang-bang strategies](image)
game with bounded controls. The initial point is \((t_0, z_0) = (0, 600)\). In Fig. 4b, the zoom of the final segments of these trajectories is shown. It is seen that the trajectory, generated by the bang-bang strategy, reaches the \(t\)-axis at \(t = 2.24\) s, after which it slides along this axis, yielding control chattering. The trajectory, generated by the modified super-twisting strategy, reaches the \(t\)-axis exactly at \(t = t_f\).

In Fig. 5, the respective time realizations of the pursuer control are presented. It is seen that in a noise free scenario by using the strategy (42) the chattering effect is completely eliminated.

6. Integral Sliding Mode Strategy

In this section, another history-dependent interception strategy is studied. It is the modification of the integral sliding mode control [28]. In order to construct the classical integral sliding mode (ISM) control, first a nominal control \(u^{\text{nom}}(t, z(t))\) has to be designed. For the system (17), this control provides \(z(t_f) = 0\) in the non-disturbed system \((v \equiv 0)\). The second step is constructing an integral sliding "surface"

\[
\sigma = \sigma(t, z_{t}(\cdot)) \triangleq z(t) - z(0) - \int_{0}^{t} h_1(\xi)u^{\text{nom}}(\xi, z(\xi)) = 0. \tag{50}
\]

Then the integral sliding mode control is given as

\[
u = u^{\text{ISM}}(t, z_{t}(\cdot)) = u^{\text{nom}}(t, z(t)) + \chi u^{\text{vs}}(t, z_{t}(\cdot)), \tag{51}\]
where the variable structure control $u^{vs}(t, z_t(\cdot))$ has the form

$$u^{vs}(t, z_t(\cdot)) = \begin{cases} 
\text{sign} \sigma(t, z_t(\cdot)), & \sigma(t, z_t(\cdot)) \neq 0, \\
0, & \sigma(t, z_t(\cdot)) = 0,
\end{cases} \quad (52)$$

and $\chi$ is a sufficiently large positive number.

In accordance with the requirements for the interception strategy formulated in Section 5, the classical integral sliding mode control (51) is modified as follows:

$$u = u^{ISM}_{m}(t, z_{\vartheta(t)}(\cdot)) = \text{sat} \left( \bar{u}(t, z_{\vartheta(t)}(\cdot)) \right), \quad (53)$$

where

$$\bar{u}(t, z_{\vartheta(t)}(\cdot)) = u^{nom}(t, z(t)) + \chi u^{vs}(t, z_{\vartheta(t)}(\cdot)), \quad (54)$$

and the modified integral sliding ”surface” has the form

$$\sigma = \sigma_{m}(t, z_{\vartheta(t)}(\cdot)) \triangleq z(t) - z(\rho(t)) - \int_{\rho(t)}^{t} h_1(\xi)u^{nom}(\xi, z(\xi)) = 0. \quad (55)$$

Moreover, the modified variable structure control has the form

$$u^{vs}_{m}(t, z_{\vartheta(t)}(\cdot)) = \begin{cases} 
\text{sign} \sigma_{m}(t, z_{\vartheta(t)}(\cdot)), & \sigma_{m}(t, z_{\vartheta(t)}(\cdot)) \neq 0, \\
0, & \sigma_{m}(t, z_{\vartheta(t)}(\cdot)) = 0.
\end{cases} \quad (56)$$

In the sequel, the nominal control is chosen as the linear strategy

$$u^{nom}(t, z(t)) = \frac{kz(t)}{(t_f - t)^p}, \quad (57)$$

where $k > 0$, $p \geq 3$ are some constants. Due to [17], for sufficiently large $k$, this strategy provides $z(t_f) = 0$ in the non-disturbed system (17).

Let introduce the function

$$G(t, k, \chi, p) \triangleq \frac{kz^*(t)}{(t_f - t)^p} - \chi, \quad (58)$$

where $z^*(t)$ is given by (26).
Theorem 3. If the constants $k$, $\chi$ and $p$ satisfy the inequality

$$\inf_{t \in [0,t_f)} G(t, k, \chi, p) \geq 1,$$  \hspace{1cm} (59)

then the capture zone of the strategy (53) with the nominal control (57) is maximal, i.e. it coincides with $D_{0M}$.

Proof. Due to Theorem 1, it is sufficient to prove that the strategy (54) satisfies the conditions (33) and (34) for any $w^+(\cdot) \in W_{t^*}^+$ and for any $w^-(\cdot) \in W_{t^*}^-$, respectively.

Let $w(\cdot) \in W_{t^*}^+$ be arbitrary. Then, due to (54) and (57), for any $t \in [0,t_f)$,

$$\bar{u}(t, w(\cdot)) \geq k z^*(t)/(t_f - t)^p - \chi = G(t, k, \chi, p).$$  \hspace{1cm} (60)

This inequality, along with the condition (59), directly yields (33).

Now, let $w(\cdot) \in W_{t^*}^-$ be arbitrary. Then, for any $t \in [0,t_f)$,

$$\bar{u}(t, w(\cdot)) \leq -k z^*(t)/(t_f - t)^p + \chi = -G(t, k, \chi, p).$$  \hspace{1cm} (61)

Hence, the condition (34) is satisfied if

$$\sup_{t \in [0,t_f)} \left[ -G(t, k, \chi, p) \right] \leq -1,$$  \hspace{1cm} (62)

which coincides with (59). This completes the proof of the theorem. \qed

Example 2. Consider the same interception problem as in Example 1.

![Figure 6: Domain of $k$, $\chi$, guaranteeing maximal capture zone](image-url)
In Fig. 6, the domain of the coefficients $k, \chi$, satisfying the condition (59) for different values of $p$, is shown for $k \in [0, 5]$ and for $\chi \in [0, 60]$. It is seen that this domain is maximal for $p = 3$. Such a feature is similar to the case of the modified super-twisting strategy.

![Figure 7: Trajectories generated by the modified ISM and the bang-bang strategies](image1)

In Fig. 7a, the trajectories of (17) are depicted for the bang-bang strategy and for the modified integral sliding mode strategy (53) with $k = 0.3, \chi = 0.1, p = 3, \theta(t) \equiv 2s$. Like in the previous section, the evader control is $v = \text{sign} \, z$ and the initial point is $(t_0, z_0) = (0, 600)$. In Fig. 7b, the zoom of the final segments of these trajectories is shown. It is seen that, in contrast with the
trajectory generated by the bang-bang strategy, the trajectory generated by the modified integral sliding mode strategy reaches the $t$-axis exactly at $t = t_f$.

In Fig. 8, the respective time realizations of the bang-bang strategy and of the modified integral sliding mode strategy are presented. It is seen that these time realizations equal to each other on the interval $[0, 2.11]$. This explains that the respective trajectories coincide with each other on this interval (see Fig. 7a). It is also seen that using the strategy (53) completely eliminates the chattering effect.

7. Simulation Results

The validity of the modified super-twisting strategy (42) and the modified integral sliding mode strategy (53) was tested by Monte Carlo simulations in a realistic interception scenario, including nonlinear equations of motion (1), noise-corrupted measurements of the line-of-sight angle and an estimator in the loop. Four pursuer strategies were compared: the bang-bang strategy $u = \text{sign } z$, the saturated "natural" strategy (29), the modified STW strategy (42) and the modified ISM strategy (53). It was assumed that the parameter triplets $(k_1, k_2, p)$ of (42) and $(k, \chi, p)$ of (53) satisfy Theorems 2 and 3, respectively, thus guaranteeing that the capture zones of these strategies are maximal.

Remark 3. Since the strategies (42) and (53) are saturated and have the maximal capture zone, these strategies, as well as the bang-bang strategy and the saturated "natural" strategy, are optimal in the perfect information pursuit-evasion differential game for the system (17) with the cost function (21) and the control constraints (2).

7.1. Logic-Based Estimator

In the simulations, it was assumed that the only information, which is available to the pursuer, is the sampled noisy measurements of the line-of-sight angle $\lambda$: $\tilde{\lambda}_i = \lambda_i + \eta$, where $\eta$ is a zero-mean Gaussian measurement noise, the index $i$ refers to the sampling time moments. The Gaussian model of the angular measurement noise has been widely accepted in the literature [37, 38, 39]. Actual values of the relative separation, the relative velocity and the evader acceleration cannot be directly measured. Therefore, their estimates, produced by an estimator in the loop, have to be used in the pursuer guidance strategies. In practice, the estimator input is the reconstructed
value of the line-of-sight angle in inertial coordinates, obtained from the seeker [40].

In the Monte Carlo simulations, a logic based estimator [41] was utilized. It consisted of a bank of Kalman filters, using an exponentially correlated acceleration model (ECA, [42]) and working in parallel. The ECA filter has two tuning parameters: the decorrelation time $\tau_f$ of the ECA model and the proportionality factor $C_f$ of the process noise covariance of the model $\sigma_f^2 = (a_{\text{max}}^\text{e}/C_f)^2$.

The bank of the filters was organized as follows. The first filter was of narrow bandwidth. The others were of wide bandwidth and they are tuned to the timing of different specific switch events, covering the interval from some critical time $t_{cr}$ to $t_f$.

It was assumed that the evader uses a bang-bang maneuver

$$v = \begin{cases} 
1, & t \in [0, t_{sw}], \\
-1, & t \in (t_{sw}, t_f],
\end{cases}$$

and the switch event at $t_{sw}$ can be detected sufficiently fast (in this simulation, a constant detection delay $\Delta t_{sw}$ was assumed to be known). If the switch event was detected before a critical time ($t_{sw} + \Delta t_{sw} < t_{cr}$), then the guidance strategy used the output of the first (narrow-bandwidth) filter in the bank. Otherwise, this output was replaced by the output of a wide-bandwidth filter, tuned to the switch event, nearest to the detection at $t_{sw} + \Delta t_{sw}$.

The estimator produced the estimates of the relative separation $\hat{y}$, of the relative velocity $\dot{\hat{y}}$ and of the evader acceleration $\dot{a}_e$. Due to (15), the corresponding estimate of the zero-effort miss distance can be expressed explicitly as

$$\hat{z}(t) = \hat{y}(t) + (t_f - t)\dot{\hat{y}}(t) +$$

$$\tau_e^2 \Psi((t_f - t)/\tau_e)\dot{a}_e(t) - \tau_p^2 \Psi((t_f - t)/\tau_p)a_p(t).$$

(64)

### 7.2. Control Dead Zone

In [41], the bang-bang pursuer strategy was modified by using a time-varying dead zone version of the signum function after the critical time $t_{cr}$:

$$\text{sign}_{dz}(\hat{z}) = \begin{cases} 
1, & \hat{z} > z_{dz}, \\
0, & |\hat{z}| \leq z_{dz}, \\
-1, & \hat{z} < -z_{dz},
\end{cases}$$

(65)
where

\[ z_{dz}(t) = A_{dz} \exp(-k_{dz}t), \]  

(66)

\( A_{dz}, k_{dz} \) are positive design parameters. It was shown [41] that using the function \( \text{sign}_{dz} \) instead of the traditional signum function reduced the error, created during the period of detection delay.

In this paper, a time-varying dead zone was implemented also for the other strategies by nullifying their history-independent parts after the critical time in the case where \( \dot{z} \) is inside the dead zone.

7.3. Simulation Parameters

The engagement parameters of this simulation were the same as in the example of the previous section: \( \tau_p = \tau_e = 0.2 \) s, \( a_{p}^{\text{max}} = 200 \text{ m/s}^2 \), \( a_{e}^{\text{max}} = 100 \text{ m/s}^2 \). The players velocities were \( V_p = 2300 \text{ m/s} \), \( V_e = 2700 \text{ m/s} \), yielding, by (7), \( t_f = 4 \) s. The initial aspect angles were \( \varphi_e^0 = 0 \), \( \varphi_p^0 = 3.67 \text{ deg} \), yielding, by (20), \( |z_0| = 600 \text{ m} \) (as in the example of the previous section).

The standard deviation of the measurement noise was \( \sigma_\lambda = 0.1 \text{ mrad} \) and the sampling frequency was 200 Hz. The parameters of the narrow-bandwidth ECA filter of the logic-based estimator bank were \( \tau_f = 1.5 \) s, \( C_f = 2 \). Additionally, the bank included three wide-bandwidth ECA filters with \( \tau_f = 0.2 \) s, \( C_f = 3 \). They were tuned to the switch events at \( t_{sw}^1 = 2.4 \) s, \( t_{sw}^2 = 3 \) s and \( t_{sw}^3 = 3.5 \) s, respectively. The critical time was \( t_{cr} = 2.4 \) s, the detection delay was \( \Delta t_{sw} = 0.1 \) s. The parameters of the control dead zone were \( A_{dz} = 50 \text{ m}, k_{dz} = 1 \text{ s}^{-1} \).

7.4. Comparison of Strategies

Based on the estimate (64), four strategies, namely the bang-bang, the saturated ”natural”, the modified super-twisting and the modified integral sliding mode strategies were implemented as

\[ \hat{u}^{\text{BB}} = \begin{cases} 
\text{sign} \, \dot{z}, & t < t_{cr} \text{ or } t \geq t_{cr} \text{ & } |\dot{z}| > z_{dz}(t), \\
0, & t \geq t_{cr} \text{ & } |\dot{z}| \leq z_{dz}(t), \end{cases} \]  

(67)

\[ \hat{u}^{\text{NAT}} = \begin{cases} 
\text{sat}(\dot{z}/z^*(t)), & t < t_{cr} \text{ or } t \geq t_{cr} \text{ & } |\dot{z}| > z_{dz}(t), \\
0, & t \geq t_{cr} \text{ & } |\dot{z}| \leq z_{dz}(t), \end{cases} \]  

(68)
\[ \hat{u}_{m}^{STW} = \begin{cases} \text{sat} \left( \hat{u}(t, \hat{z}_{\theta(t)}(\cdot)) \right), & t < t_{cr} \text{ or } t \geq t_{cr} \& |\hat{z}| > z_{dz}(t), \\ \text{sat} \left( K_2(t) \int_{\rho(t)}^{t} \text{sign} \hat{z}(\tau)d\tau \right), & t \geq t_{cr} \& |\hat{z}| \leq z_{dz}(t), \end{cases} \]

and

\[ \hat{u}_{m}^{ISM} = \begin{cases} \text{sat} \left( \hat{u}(t, \hat{z}_{\theta(t)}(\cdot)) \right), & t < t_{cr} \text{ or } t \geq t_{cr} \& |\hat{z}| > z_{dz}(t), \\ \text{sat} \left( \chi u^{\text{ss}}(t, \hat{z}_{\theta(t)}) \right), & t \geq t_{cr} \& |\hat{z}| \leq z_{dz}(t). \end{cases} \]

(69)

(70)

In the simulation phase, \( N = 500 \) Monte Carlo runs were carried out, in which the timing of the evader switch \( t_{sw} \) was uniformly distributed on the engagement interval \([0, t_f]\), for all four strategies. The performance of each guidance strategy was characterized by the 95% percentile of the miss distance cumulative distribution \( r_{95} \), the average miss distance \( r_{av} \) and the average number of sign switches \( N_u \) in the control time realizations. The values of \( r_{av} \) and \( r_{95} \) characterize the homing accuracy, while the value of \( N_u \) is a measure of the pursuer control chattering.

The exponent \( p \) and the memory depth \( \theta(t) \) in the modified super-twisting and the integral sliding mode strategies were chosen as \( p = 3 \) and \( \theta(t) \equiv 2 \) s. Based on extensive simulations, the other parameters of (42) and (53) were chosen as \( k_1 = 0.4, k_2 = 0.1, k = 0.3, \chi = 0.3 \). These values allowed to reduce \( N_u \) considerably (i.e., eliminate the control chattering) without deteriorating the homing accuracy.

The numerical results of this simulation are presented in Table 1 and the respective cumulative miss distance distributions are shown in Fig. 9.

From these results it can be seen that chattering, characterizing the bang-bang strategy, is practically eliminated by the other three strategies. The homing performance of both modified strategies is slightly better in comparison with the bang-bang strategy, while the saturated ”natural” has much inferior homing accuracy, as it was found also in [12]. In the example scenario against a randomly maneuvering target, the modified integral sliding mode strategy performs slightly better than the modified super-twisting strategy.
8. Conclusions

In this paper, a planar interception endgame scenario with constant velocities is considered. For this problem, two saturated history-dependent strategies, namely, the modified super-twisting second-order sliding mode strategy and the modified integral sliding mode strategy are constructed and analyzed. Sufficient conditions for such strategies to have maximal capture zone are established. In contrast with the classical sliding mode controls (super-twisting and integral), the modified strategies, being saturated, respect the prescribed control constraint. Moreover, they guarantee robust capture at the prescribed time. Examples demonstrate that the capture zone maximality conditions are feasible.
Monte Carlo simulations in a realistic nonlinear environment corrupted with Gaussian noise show that both modified strategies eliminate control chattering. Moreover, their homing accuracy is slightly improved in comparison with the bang-bang strategy, while the saturated "natural" strategy, which also eliminates chattering, has much inferior homing accuracy in noise corrupted scenarios.

The paper assumed that in the interception endgame scenarios of interest the state equations can be linearized and the results of the Monte Carlo simulations confirmed the validity of this assumption.

References


