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Quasi-continuous high-order sliding-mode controllers for reduced-order chaos synchronization

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ABSTRACT

In this paper the problem of the reduced-order synchronization of systems of different order is considered. A control design based on robust exact differentiators and quasi-continuous high-order sliding mode-controllers is proposed ensuring the reduced-order synchronization in spite of master/slave mismatches. Simulation results are provided in order to illustrate the performance of the proposed synchronization scheme.

1. Introduction

1.1. Antecedents

Since the 17th century, the synchronization phenomenon on dynamical systems has been actively researched, starting with the Huygen's work concerning two coupled pendulum clocks that become synchronized in phase [1]. From then on, many others natural and man-made processes and systems have been discovered which exhibit the synchronization phenomenon [2], such as synchronized lightning of fireflies, adjacent organ pipes, biological and physiological systems, synchrony of triode generators and other electronic devices, many rotating mechanical structures, and many classical cases of synchronization of periodic systems [3]. In such context, the meaning of synchronization is understood as an adjustment of rhythms of oscillating objects due to weak interactions [2].

On the other hand, it is well-known that the chaotic systems are nonlinear deterministic systems having a complex and unpredictable behavior. The sensitive dependence on initial conditions and parameter variations is a prominent feature of chaotic behavior, whereby, the synchronization is not trivial in this class of systems. Moreover, chaos synchronization is an interesting subject because many possible applications can be foreseen from a proper understanding of the role of chaotic dynamics in interacting systems [4,5].

In spite of the classical concept of synchrony, varied types of synchronization are known from studies in this field [6–14], which are dependent on signal parameters (frequency, phase, amplitude) to be matched and other technical details. This is very important because different kinds of synchrony can appear or be desired in process for a common objective, as either identical or different systems, for instance, robot coordination and cooperation of manipulators or others electro-mechanical systems [15,16]. Then, some types of synchronization for chaotic systems can be reviewed. On the one hand, the identical synchronization (IS) implies an actual equality (both amplitude and phase) of the corresponding variables of two or more coupled identical systems [6]. The phase synchronization (PS) is displayed if there exists certain relation between the phase of the system variables but the amplitudes remain chaotic and uncorrelated [11]. The generalized synchronization (GS) is a generalization of the above concepts, in this sense it is said that two unidirectionally coupled different systems are synchronized if a (static) functional relation exists between the variables of both systems [7]. That is, for a given master system $\dot{x}=f(x)$ and a slave system $\dot{z}=g(z,u(x))$, where $x\in\Re^n$, $z\in\Re^n$, and $u(x)\equiv u(x_1(t,x_0),\ldots,x_n(t,x_0))$, consider a functional relation or a map $z=\Phi(x)$: $\Re^n\rightarrow\Re^n$ such that (GS) occurs between $x(t)$ and $z(t)$, i.e., $\lim_{t\to\infty}\|z(t,z_0)-\Phi(x(t,x_0))\|=0$, where $u(t)$ is the driving function. In this case we say $z(t,z_0)$ Φ-synchronize with $x(t,x_0)$ [17].

On the other hand, the synchronization schemes are different because of the amount of signals to be synchronized in the systems. Accordingly, chaotic systems with same order achieve complete synchronization if and only if all their trajectories are synchronized of some type [10]. In another way, chaotic systems with same order are partially synchronized if at least one of the trajectories is synchronized and at least one of the trajectories is not synchronized [10,12]. Now, for two unidirectionally coupled systems (called master and

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slave), where the order of the slave system is minor than the master one, the reduced-order synchronization means that all trajectories of the slave system are synchronized, in some way, with projections of the master system [14,18]. Thus, synchronization schemes are associated to the order of the involved systems.

It is worth mentioning that in unidirectional schemes the master system behavior is independent of the slave one. Additionally the slave system behavior evolves under coupling or control actions (forced synchronization) in order to be synchronized with the master one; unlike the bidirectional schemes, where both systems have coupling or control actions in order to be mutually synchronized [19].

At last, the GS in reduced-order is the considered problem in this work, that is, unidirectionally synchronizing a slave system with projections of a master system, taking into account that the dynamics describing the master and slave systems can be different.

1.2. Motivation

Studying synchronization design for the systems of different order is important and interesting and as a matter of fact this phenomenon is displayed in nature, for example: it is observed in the cardiorespiratory system, remarking that cardiac and respiratory systems have a synchronous behavior [20]. Although it is presupposed that both systems are different by nature, also may have different order. Furthermore, the reduced-order synchronization between strictly different systems could play an important role in many fields [21].

However, synchronization is difficult in the most practical situations due to external excitations or disturbances, parametric uncertainties or non-modelled dynamics. Moreover, only partial information (measured states and nominal parameter values) may be available for feedback control laws, or perhaps noise could be added in measurements, and furthermore, the chaotic dynamic is extremely sensitive to their initial conditions. Then, all these obstacles should be taking into account when an algorithm control is designed in order to forcing the slave system states to be synchronized with those of the master. Despite this, a feedback control law is usually expressed in terms of the output function and their derivatives. The designed controller needs the knowledge of all states of the system to be implemented. However, they are not always measurable. In order to avoid this difficulty it is necessary to reconstruct this information from the output and its derivatives. On the other hand, the use of differentiators or some observers cause delay in signals and requirements of filtration [22,23].

The goal of the paper is the synthesis of synchronization for chaotic systems should be focused from a robust viewpoint [24,25].

1.3. Methodology

To ensure the robust reduced-order synchronization two main tools are used:

1. A high-order sliding mode-based exact differentiator is a system ensuring finite-time convergence to the exact value of the derivatives of a function, ensuring a best possible approximation of the derivatives with respect to the upper bound of the measurements noise and sampling step [26].
2. Quasi-continuous high-order sliding-mode controllers provide finite-time stabilization for convergence error [27].

1.4. Main contribution

In this paper, conditions are given providing synchronization for a class of different chaotic systems with different order, basing on the concepts of relative degree and disturbance characteristic index. In addition, the robustness with respect to the presence of certain disturbances and unmodelled dynamics basing on high-order sliding-mode differentiator and a quasi-continuous control is ensured.

The proposed controller is output based, i.e. it can forces the slave to follow the master using outputs of the both systems and ensures their synchronization.

1.5. Structure of the paper

The rest of the paper is organized as follows. The problem description is presented in Section 2. A synchronization scheme constituted by a high-order sliding-mode differentiator combined with a quasi-continuous robust controller is presented in Section 3. Furthermore, simulation results are shown in Section 4 in order to illustrate the GS of the reduced order between two different chaotic systems. Finally, some conclusions are given.

2. Problem description

Now, we analyze the GS problem between two different chaotic systems. Besides, in the considered problem is assumed that the order of the master system is greater than the order of the slave system. Thus, the complete problem will be called GS in reduced-order.

Consider a master system given by

$$\Sigma_{\mathrm{M}}:\begin{cases}\dot{x}_{\mathrm{M}}=f_{\mathrm{M}}(x_{\mathrm{M}})+g_{\mathrm{M}}(x_{\mathrm{M}})u_{\mathrm{M}}\\ y_{\mathrm{M}}=h_{\mathrm{M}}(x_{\mathrm{M}})\end{cases}\tag{1}$$

where $x_{\mathrm{M}}=[x_{1,\mathrm{M}},x_{2,\mathrm{M}},\ldots,x_{n_{\mathrm{M}},\mathrm{M}}]^{\mathrm{T}}\in\Re^{n_{\mathrm{M}}}$ is the state vector, $u_{\mathrm{M}}\in\Re$ is an input, and $y_{\mathrm{M}}\in\Re$ is an output variable of the master system, f_{M} and g_{M} are smooth vector fields, and has relative degree r_{M} (see Appendix A). Next, consider a slave system described by

$$\Sigma_{\mathrm{s}}:\begin{cases}\dot{x}_{\mathrm{s}}=f_{\mathrm{s}}(x_{\mathrm{s}})+g_{\mathrm{s}}(x_{\mathrm{s}})u_{\mathrm{s}}\\ y_{\mathrm{s}}=h_{\mathrm{s}}(x_{\mathrm{s}})\end{cases}\tag{2}$$

where $x_{\mathrm{s}}=[x_{1,\mathrm{s}},x_{2,\mathrm{s}},\ldots,x_{n_{\mathrm{s}},\mathrm{s}}]^{\mathrm{T}}\in\Re^{n_{\mathrm{s}}}$ is the state vector, $u_{\mathrm{s}}\in\Re$ is the control input, and y_{s} an output variable, f_{s} and g_{s} are smooth vector fields, with relative degree r_{s}.

We assume that the order n_{s} of the slave system is less than or equal to the order n_{M} of the master system ($n_{\mathrm{s}}\leqslant n_{\mathrm{M}}$). It is clear that if $n_{\mathrm{s}}=n_{\mathrm{M}}$, then it represents a particular case of reduced-order synchronization.

Now, the synchronization problem considered can be established as follows.

The reduced-order GS objective: Given two chaotic system with different order, find a control to force the states of the slave system (2) to be synchronized with some projections of state vector of the master system (1).

In order to attain the above objective, let us define the synchronization error $\varepsilon: x_{\mathrm{s}}-\Phi(x_{\mathrm{M}})$, where Φ is a functional relation. Then, the GS is defined as follows:

Definition 1 (*Rulkov et al. [7], Kocarev and Partlitz [9], Yang and Duan [17]*). A slave system (2) exhibits GS with the master system (1), if there exists a functional relation Φ, such that

$$\lim_{t\to\infty}\varepsilon=\lim_{t\to\infty}[x_{\mathrm{s}}-\Phi(x_{\mathrm{M}})]=0\tag{3}$$

for all $t\geqslant 0$ and any initial condition $\varepsilon(t_0)=x_{\mathrm{s}}(t_0)-\Phi(x_{\mathrm{M}}(t_0))$.

Definition 1 is meaningful because (3) implies $x_{\mathrm{s}}\to\Phi(x_{\mathrm{M}})$ for all $t\geqslant 0$ and any initial difference $\varepsilon(t_0)=x_{\mathrm{s}}(t_0)-\Phi(x_{\mathrm{M}}(t_0))$, which is understood as the synchronization of system (2) to a projection of state vector of system (1). Then, the dynamical synchronization error is given by

$$\Sigma_\varepsilon : \begin{cases} \dot{\varepsilon} = f_\varepsilon(\varepsilon) + g_\varepsilon(\varepsilon)u_S + q(\varepsilon,\xi) \\ y_\varepsilon = h_\varepsilon(\varepsilon) \end{cases} \tag{4}$$

where $\varepsilon \in \Re^{n_S}$, y_ε is a function of the synchronization error which is available for feedback, f_ε and g_ε are smooth vector fields and $q(\varepsilon,\xi)$ is a smooth vector field containing terms which depends on the synchronization error ε, the states x_S and x_M and control u_M, where $\xi := \xi(x_M, u_M)$. The components of such vector field $q(\varepsilon,\xi)$ are considered as perturbations into the system. In addition, the vector field satisfies $q(\varepsilon, 0) = 0$, $\forall \varepsilon \in \Re^{n_S}$. It is clear that for $\xi = 0$, we obtain the nominal system, i.e. dynamical system without perturbations.

Furthermore, we assume that system (4) has relative degree r_S, and taking into account the presence of the perturbation terms associated to ξ, we introduce the following definition, which is important for synchronizing both systems.

Definition 2 (*Disturbance characteristic index, Marino and Tomei [28]*). The disturbance characteristic index v is defined for system (4) as the integer such that

$$L_q L^i_{f_\varepsilon} h_\varepsilon(\varepsilon) = 0, \quad 0 \leqslant i \leqslant v-2, \ \forall e \in \Re^n, \ \forall \xi \in \Omega$$
$$L_q L^{v-1}_{f_\varepsilon} h_\varepsilon(\varepsilon) \neq 0 \quad \text{for some } \xi \in \Omega \text{ for some } \varepsilon \in \Re^n$$

We set $v = \infty$ if

$$L_q L^i_{f_\varepsilon} h_\varepsilon(\varepsilon) = 0, \quad i \geqslant 0, \ \forall \varepsilon \in \Re^n, \ \forall \xi \in \Omega$$

That is, the disturbance characteristic index is the number v of differentiations of the output y_S which are required to show explicitly the term ξ. Thus, the disturbance characteristic index can be interpreted as the dimension of system (4) affected by the function ξ.

Then, the importance of the relationship between v and r_S is clarified by the following result.

Lemma 1 (*Marino and Tomei [28]*). *Assume that relative degree for the nominal system is $r_S \leqslant n_S$. Then there exist $n_S - r_S$ functions $e_i^\perp(\varepsilon)$, $1 \leqslant i \leqslant n_S - r_S$, such that*:

(i) *the functions $h_\varepsilon(\varepsilon), \dots, L^{r_S-1}_{f_\varepsilon} h_\varepsilon(\varepsilon), e_1^\perp(\varepsilon), \dots, e^\perp_{n_S-r_S}(\varepsilon)$ form a local diffeomorphism about the origin*;
(ii) $\langle de_i^\perp, q\rangle = 0$, $1 \leqslant i \leqslant n_S - r_S$.

In local coordinates $\mathbf{e} = [e, e^\perp] = [h_\varepsilon(\varepsilon), \dots, L^{r_S-1}_{f_\varepsilon} h_\varepsilon(\varepsilon), e_1^\perp(\varepsilon), \dots, e^\perp_{n_S-r_S}(\varepsilon)] = \phi(\varepsilon)$ system (4) is expressed as follows:

For the case $v > r_S$:

$$\begin{cases} \dot{e}_i = e_{i+1}, & 1 \leqslant i \leqslant r_S - 1 \\ \dot{e}_{r_S} = L^{r_S}_{f_\varepsilon} h_\varepsilon + u_S L_{g_\varepsilon} L^{r_S-1}_{f_\varepsilon} h_\varepsilon, & \\ \dot{e}_j^\perp = L_{f_\varepsilon} e_j^\perp + L_q e_j^\perp, & 1 \leqslant j \leqslant n_S - r_S \\ y_e = e_1 & \end{cases} \tag{5}$$

For the case $v = r_S$:

$$\begin{cases} \dot{e}_i = e_{i+1}, & 1 \leqslant i \leqslant r_S - 1 \\ \dot{e}_{r_S} = L^{r_S}_{f_\varepsilon} h_\varepsilon + u_S L_{g_\varepsilon} L^{r_S-1}_{f_\varepsilon} h_\varepsilon & \\ \qquad + L_q L^{r_S-1}_{f_\varepsilon} h_\varepsilon, & \\ \dot{e}_j^\perp = L_{f_\varepsilon} e_j^\perp + L_q e_j^\perp, & 1 \leqslant j \leqslant n_S - r_S \\ y_e = e_1 & \end{cases} \tag{6}$$

For the case $v < r_S$:

$$\begin{cases} \dot{e}_i = e_{i+1}, & 1 \leqslant i \leqslant v - 1 \\ \dot{e}_i = e_{i+1} + L_q L^{i-1}_{f_\varepsilon} h_\varepsilon, & v \leqslant i \leqslant r_S - 1 \\ \dot{e}_{r_S} = L^{r_S}_{f_\varepsilon} h_\varepsilon + u_S L_{g_\varepsilon} L^{r_S-1}_{f_\varepsilon} h_\varepsilon & \\ \qquad + L_q L^{r_S-1}_{f_\varepsilon} h_\varepsilon, & \\ \dot{e}_j^\perp = L_{f_\varepsilon} e_j^\perp + L_q e_j^\perp, & 1 \leqslant j \leqslant n_S - r_S \\ y_e = e_1 & \end{cases} \tag{7}$$

There exists a global change of coordinates transforming system (4) *into* (5)–(7), *if the vector fields*

$$\tilde{f}_e = f_\varepsilon - \frac{L^r_{f_\varepsilon} h_\varepsilon}{L_{g_\varepsilon} L^{r-1}_{f_\varepsilon} h_\varepsilon} g_\varepsilon, \quad \tilde{g}_e = \frac{1}{L_{g_\varepsilon} L^{r-1}_{f_\varepsilon} h_\varepsilon} g_\varepsilon$$

are complete and the global relative degree is well defined.

In the sequel, we assume that

Assumption 1. System (4) is assumed to be of minimum phase (see Appendix A).

Remark 1. It is clear that the case $v < r_S$ is not considered because control cannot be designed in order to reject the disturbance and guarantee the synchronization error tends to zero. The case $v > r_S$ is trivial because the disturbance does not affect the control objective, then the synchronization can be achieved. Then, the interesting case is when $v = r_S$, which is considered in this paper.

Then, taking into account the above remark, we introduce the following assumption.

Assumption 2. For synchronization error system (4), the disturbance characteristic index is assumed to be $v = r_S$.

The relative degree r_S of system (4) is the lowest order time derivative of the output y_S for which the control u_S appears explicitly. Consequently, the relative degree can be interpreted as the dimension of system (4) that can be linearized by a change of coordinates and a feedback u_S.

Thus, the synchronization error system (4) is written as

$$\Sigma_e : \begin{cases} \dot{e} = A_e + B_e[\beta_S(\mathbf{e})u_S + \xi(\mathbf{e}, u_M)] \\ \dot{e}^\perp = \eta(e, e^\perp) \\ y_e = C_e \end{cases} \tag{8}$$

with

$$A_e = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \quad B_e = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$
$$C_e(e) = [1 \ 0 \ \cdots \ 0 \ 0]$$

where $\varepsilon := x_S - \Phi(x_M)$, $\mathbf{e} = [e, e^\perp]$, and $\mathbf{e} = \phi(\varepsilon) \in \Re^{n_S}$ is the synchronization error vector, $u_S \in \Re$ is the forcing input, $y_e \in \Re$ is the output of error system (8), the function $\xi(\mathbf{e}, u_M)$ is considered as perturbation term. $\dot{e}^\perp = \eta(e, e^\perp)$ represents the internal dynamic which is assumed to be asymptotically stable, i.e. the system is minimum phase.

It is clear that for obtaining the above representation is necessary to know the functional relation Φ, which usually is unknown. Then, the following result summarized the conditions which allows to find a functional relation Φ and to transform systems (1) and (2) into a normal form.

Lemma 2 (*Marino and Tomei [28]*). *Assume that $r_s \leqslant n_s$ for system (2). Then there exist $n_s - r_s$ functions $z_{i,s}(x_s)$, $1 \leqslant i \leqslant n_s - r_s$, such that:*

(i) *the functions $h_s(x_s), \ldots, L_{f_s}^{r-1} h_s(x_s), \bar{z}_{1,s}^{\perp}(x_s), \ldots, \bar{z}_{n_s-r_s,s}^{\perp}(x_s)$ form a local diffeomorphism $\phi_s(x_s) = [\bar{\phi}_s(x_s), \bar{\phi}_s^{\perp}(x_s)]^T$ about the origin;*

(ii) *$\langle d\bar{z}_{i,s}^{\perp}, g_s \rangle = 0$, $1 \leqslant i \leqslant n_s - r_s$. In local coordinates $[\bar{z}_s, \bar{z}_s^{\perp}] = [h_s(x_s), \ldots, L_{f_s}^{r_s-1} h_s(x_s), \bar{z}_{1,s}^{\perp}(x_s), \ldots, \bar{z}_{n_s-r_s,s}^{\perp}(x_s)]$, i.e. $z_s = [\bar{z}_s, \bar{z}_s^{\perp}]^T = [\bar{\phi}_s(x_s), \bar{\phi}_s^{\perp}(x_s)]^T = \phi_s(x_s)$, system (2) is expressed in normal form*

$$\begin{cases} \dot{\bar{z}}_s = A_s \bar{z}_s + \Gamma_s(\bar{z}_s, \bar{z}_s^{\perp}, u_s) \\ \dot{\bar{z}}_s^{\perp} = \varphi_s(\bar{z}_s, \bar{z}_s^{\perp}) \\ y_s = C_s \bar{z}_s \end{cases} \tag{9}$$

with

$$A_s = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \quad \Gamma_s(z_s, u_s) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \alpha_s(z_s) + \beta_s(z_s) u_s \end{bmatrix}$$

$$C_s = [1 \ 0 \ \cdots \ 0 \ 0]$$

where $\alpha_s(z_s) = L_{f_s}^{r_s} h_s$ and $\beta_s(z_s) = L_{g_s} L_{f_s}^{r_s-1} h_s$. If, in addition, the global relative degree r_s is well defined with $r_s \leqslant n_s$ and

(iii) *the vector fields*

$$\tilde{f}_s = f_s - \frac{L_{f_s}^{r_s} h_s}{L_{g_s} L_{f_s}^{r_s-1} h_s} g_s, \quad \tilde{g}_s = \frac{1}{L_{g_s} L_{f_s}^{r_s-1} h_s} g_s$$

are complete, then exists a global diffeomorphism transforming (2) into the normal form (9).

In this way, consider that:

Lemma 3 (*Marino and Tomei [28]*). *If system (2) has relative degree $r_s \leqslant n_s$, then it is locally partially state feedback linearizable with index r_s.*

Assumption 3. The slave system has relative degree $r_s = n_s$, where n_s is the order of slave system.

Then, it follows that $n_s - n_r = 0$, $z_s = \bar{z}_s$ and

$$\begin{cases} \dot{\bar{z}}_s = A_s \bar{z}_s + \Gamma_s(\bar{z}_s, u_s) \\ y_s = C_s \bar{z}_s \end{cases} \tag{10}$$

Now, from Lemma 2, the master system (1) can be transformed, by the diffeomorphism $z_M = [\bar{z}_M, \bar{z}_M^{\perp}]^T = [\bar{\phi}_M(x_M), \bar{\phi}_M^{\perp}(x_M)]^T = \phi_M(x_M)$, into a normal form:

$$\begin{cases} \dot{\bar{z}}_M = A_M \bar{z}_M + \Gamma_M(\bar{z}_M, \bar{z}_M^{\perp}, u_M) & (11a) \\ \dot{\bar{z}}_M^{\perp} = \varphi_M(\bar{z}_M, \bar{z}_M^{\perp}) & (11b) \\ y_M = C_M \bar{z}_M & (11c) \end{cases}$$

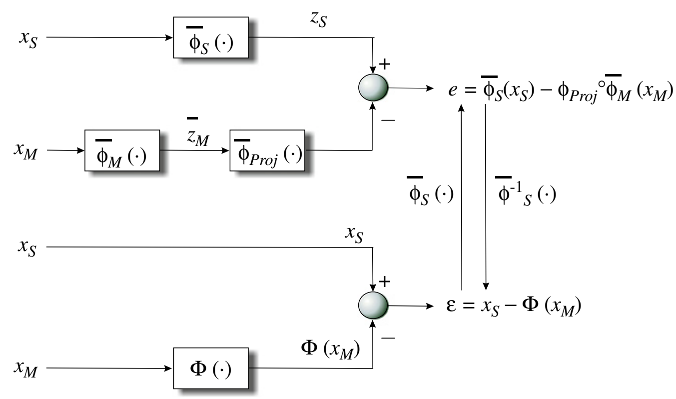

Fig. 1. Block diagram illustrating the relationship between original variables and transformed variables.

with

$$A_M = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix},$$

$$\Gamma_M(\bar{z}_M, \bar{z}_M^{\perp}, u_M) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \alpha_M(\bar{z}_M, \bar{z}_M^{\perp}) + \beta_M(\bar{z}_M, \bar{z}_M^{\perp}) u_M \end{bmatrix}$$

$$C_M = [1 \ 0 \ \cdots \ 0 \ 0]$$

where $z_M = [\bar{z}_M, \bar{z}_M^{\perp}]^T \in \Re^{n_M}$, with $\bar{z}_M \in \Re^{r_M}$, $\bar{z}_M^{\perp} \in \Re^{n_M - r_M}$, $\alpha_M(\bar{z}_M, \bar{z}_M^{\perp})$ and $\beta_M(\bar{z}_M, \bar{z}_M^{\perp})$ are functions of z_M.

The new synchronization error is given by $e = \bar{z}_s - \phi_{proj}(\bar{z}_M)$, where ϕ_{proj} is the projection map from $\Re^{r_M}$ into $\Re^{r_s}$. This synchronization error can be written in original coordinates as follows: $e = \bar{\phi}_s(x_s) - \phi_{proj} \circ \bar{\phi}_M(x_M)$. Then, in order to obtain the functional relation Φ (see Fig. 1), it is easy to see that

$$\begin{aligned} e &= \bar{\phi}_s(x_s) - \phi_{proj} \circ \bar{\phi}_M(x_M) \\ \bar{\phi}_s^{-1}(e) &= x_s - \bar{\phi}_s^{-1} \circ \phi_{proj} \circ \bar{\phi}_M(x_M) \\ \bar{\phi}_s^{-1}(e) &= x_s - \Phi(x_M) \\ \bar{\phi}_s^{-1}(e) &= \varepsilon \end{aligned} \tag{12}$$

Then, it follows that $\Phi(\cdot) = \bar{\phi}_s^{-1} \circ \phi_{proj} \circ \bar{\phi}_M(\cdot)$ and

$$e = \bar{\phi}_s(\varepsilon) \tag{13}$$

Finally, using this transformation and Assumption 1, system (8) is minimum phase, we obtain that

$$\dot{e} = A_e + B_e[\beta_s(\mathbf{e}) u_s + \xi(\mathbf{e}, u_M)] \tag{14}$$

Remark 2. The exact synchronization of (10)–(11a) can be solved by stabilizing the synchronization error system (8) at the origin.

Notice that the synchronization of $e = \bar{z}_s - \phi_{proj}(\bar{z}_M)$ is affected by unknown dynamics or parametric uncertainties present in function $\xi(e, u_M)$ that can be arisen from both models, and due to that the complete states x_M and x_s could not be available for feedback control laws. Then, in the sequel the term $\xi(e, u_M)$ will be considered as external disturbances into the system.

Now, in order to synchronize systems (1) and (2), i.e. the synchronization error $e \to 0$ or equivalently ($\varepsilon \to 0$), it is necessary to design a controller to stabilize system (14) under the presence of external perturbations ξ and/or parametric uncertainties. Then, a quasi-continuous high-order sliding-mode controller is designed. However, this controller requires that the states are available, limiting its implementation. Then, in order to overcome this difficulty, an exact robust differentiator is proposed for solving the state estimation problem.

3. Synchronization approach

Since synchronization between chaotic systems (10) and (11a) can be interpreted as a stabilizing problem for error system (14), it is possible to take advantage of known results for stabilization of dynamical systems around the origin. Furthermore, from the point of view of the control theory, the stabilizing problem is equivalent to analyze the tracking problem of trajectories, i.e. the trajectories of the slave system track some projections of the master system provided that the error between both transformed dynamical systems tends to zero in finite time, which can be seen as a synchronization problem. Hence, in this section, a stabilization approach constituted by an exact robust differentiator and a quasi-continuous high-order sliding-mode controller is considered.

3.1. Quasi-continuous high-order sliding-mode control

Now, we introduce some preliminaries in order to design a quasi-continuous high-order sliding-mode control. Let us consider the dynamical system (8) in the following form:

$$\begin{cases} \dot{e} = a_e(t,e) + b_e(t,e)u_s \\ \sigma = \sigma(t,e) \end{cases} \tag{15}$$

where $e \in \Re^{n_s}$; $a_e, b_e : \Re^{n_s} \times \Re \to \Re^{n_s}$, and $\sigma : \Re^{n_s} \times \Re \to \Re$ are unknown smooth functions, $u_s \in \Re$ is the input variable. It is clear that by comparing with system (8), we have $a_e(t,e) = A_e e$ and $b_e(t,e) = B_e[\beta_s(z_s)u_s + \xi(\mathbf{e}, u_M)]$.

The task is to fulfill the target $\sigma \equiv 0$ in finite time and to keep it exactly by some feedback. Consequently, let system (8) be closed by some possibly dynamical discontinuous feedback and be understood in the Filippov [29] sense. Then, provided that successive total time derivatives $\sigma, \dot{\sigma}, \ldots, \sigma^{(r_s-1)}$ are continuous functions of the closed-system state-space variables and the manifold defined by the equations

$$\sigma = \dot{\sigma} = \cdots = \sigma^{(r_s-1)} = 0 \tag{16}$$

is a non-empty integral set, motion (16) is called r_s-sliding (r_s-th-order sliding) mode [26]. The standard sliding mode used in the most variable structure systems is of the first-order (σ is continuous, and $\dot{\sigma}$ is discontinuous).

Assuming that the relative degree r_s is constant and known (see [30]). Then, the control appears explicitly at the r_s-th total time derivative of σ, which is given by

$$\sigma^{(r_s)} = h(t,e) + g(t,e)u \tag{17}$$

where $h(t,e) = \sigma^{(r_s)}|_{u_s=0}$, $g(t,e) = (\partial/\partial u)\sigma^{(r_s)} \neq 0$. It is supposed that, for some $K_m, K_M, C > 0$, the following inequality holds:

$$0 < K_m \leqslant \frac{\partial}{\partial u}\sigma^{(r_s)} \leqslant K_M, \quad |\sigma^{(r_s)}|_{u_s=0} \leqslant C \tag{18}$$

which is always true at least locally. Trajectories of (15) are assumed infinitely extendible in time for any Lebesgue-measurable bounded control $u_s(t,e)$.

In our case, any continuous control $u_s = U(\sigma, \dot{\sigma}, \ldots, \sigma^{(r_s-1)})$ providing for $\sigma \equiv 0$, would satisfy the equality $U(0,0,\ldots,0) = -h(t,e)/g(t,e)$, whenever (16) holds. Since the problem uncertainty prevents it [31], the control has to be discontinuous at least on the set (16). Hence, the r_s-sliding mode $\sigma = 0$ is to be established.

Now, we present the control design based on sliding mode. As follows from (17) and (18)

$$\sigma^{(r_s)} \in [-C, C] + [K_m, K_M]u. \tag{19}$$

The closed differential inclusion is understood here in the Filippov [29] sense, which means that the right-hand vector set is enlarged in a special way [31], in order to satisfy certain convexity and semi-continuity conditions. This inclusion does not "remember" anything on system (15) except the constants r_s, C, K_m, K_M. Thus, the finite-time stabilization of (19) at the origin solves the stated problem simultaneously for all systems (15) satisfying (18). Let $i = 0, 1, \ldots, r_s - 1$. Denote

$$\begin{cases} \varphi_{0,r_s} = \sigma \\ N_{0,r_s} = |\sigma| \\ \Psi_{0,r_s} = \varphi_{0,r_s}/N_{0,r_s} = \operatorname{sign}\sigma \\ \varphi_{i,r_s} = \sigma^{(i)} + \beta_i N_{i-1,r_s}^{(r_s-i)/(r_s-i+1)}\Psi_{i-1,r_s} \\ N_{i,r_s} = |\sigma^{(i)}| + \beta_i N_{i-1,r_s}^{(r_s-i)/(r_s-i+1)} \\ \Psi_{i,r_s} = \varphi_{i,r_s}/N_{i,r_s} \end{cases}$$

where $\beta_1, \ldots, \beta_{r_s-1}$ are positive numbers.

Remark 3. Recall that according to the Filippov definition values of the control on any set of the zero Lebesgue measure do not affect the solutions.

The following proposition is easily proved by induction.

Proposition 1. *Let $i = 0, 1, \ldots, r_s - 1$. N_{i,r_s} is positive definite, i.e. $N_{i,r_s} = 0$ if and only if $\sigma = \dot{\sigma} = \cdots = \sigma^{(i)} = 0$. The inequality $|\Psi_{i,r_s}| \leqslant 1$ holds whenever $N_{i,r_s} > 0$. The function $\Psi_{i,r_s}(\sigma, \dot{\sigma}, \ldots, \sigma^{(i-1)})$ is continuous everywhere (i.e., can be redefined by continuity) except the point $\sigma = \dot{\sigma} = \cdots = \sigma^{(i)} = 0$.*

Theorem 1 (*Levant [27]*). *The controller*

$$u_s = -k\Psi_{r_s-1,r_s}(\sigma, \dot{\sigma}, \ldots, \sigma^{(r_s-1)}) \tag{20}$$

is r_s-sliding homogeneous and under proper choice of parameters $\beta_1, \ldots, \beta_{r_s-1}$ and $k > 0$ provides for the finite-time stability of (19) and (20). The finite-time stable r_s-sliding mode $\sigma \equiv 0$ is established in system (15) and (20).

The proof is given in Levant [27]. It follows from Proposition 1 that control (20) is continuous everywhere except the r_s-sliding mode $\sigma = \dot{\sigma} = \cdots = \sigma^{(r_s-1)} = 0$.

Each choice of parameters $\beta_1, \ldots, \beta_{r_s-1}$ determines a controller family applicable to all systems (15) with relative degree r_s. The parameter k is chosen specially for any fixed C, K_m, K_M, most conveniently by computer simulation, avoiding redundantly large estimations of C, K_m, K_M. Obviously, k is to be negative with $(\partial/\partial u)\sigma^{(r_s)} < 0$.

Controller (20) is a continuous function of time everywhere except the r_s-sliding set (16). It may have infinite derivatives when certain surfaces are crossed. However, controller (20) requires the real-exact calculation or direct measurement of output function σ and their derivatives $\dot{\sigma}, \ddot{\sigma}, \ldots, \sigma^{(r_s-1)}$. Furthermore, it is assumed that all states are not available, and system (15) has uncertainties. Then, in what follows, we present a technique allowing to reconstruct the missing information of the state.

3.2. High-order sliding mode exact robust differentiator

It is clear that in order to implement a control, it requires full information on the state that may limit its practical utility. Indeed, even if all state measurements are available they are possibly corrupted by noise. Furthermore, the relative degree of the model with respect to the known outputs dependent on the accuracy of the mathematical model. Then, the use of state observers appears to be useful in the reconstruction of the information of the state.

The main restriction for observer design: the observer should be robust with respect to external perturbations (noise and uncertainties).

In this section, we present a robust exact differentiator, which is designed to ensure the finite-time convergence to the values of the corresponding derivatives, using a high order sliding mode techniques, and allows avoiding the differentiation of the signals which are contaminated by noise.

Furthermore, the main advantages of this high-order sliding-mode differentiator are:

(i) Finite-time convergence is provided.
(ii) Keeping robustness with respect to disturbances and measurement noise.
(iii) Diminishing the dangerous chattering effect.

However, it is noticed that the suggested algorithm is only locally effective.

Now, in order to build a differentiator for computing the real-time derivatives of output function, it is necessary that the following inequality $\sigma^{(r_s)} \leqslant C + \varsigma K_M$ holds, which allows a real-time robust $(r_s - 1)$-th-order differentiation of σ (see for more details [26,31]). Then, the r_s-th order differentiator is given by the following recursive scheme:

$$\begin{cases} \dot{\hat{e}}_1 = -\lambda_1 |\hat{e}_1 - e_1|^{r_s/(r_s+1)} \mathrm{sign}(\hat{e}_1 - e_1) + \hat{e}_2 = v_1 \\ \dot{\hat{e}}_2 = -\lambda_2 |\hat{e}_1 - e_1|^{(r_s-1)/r_s} \mathrm{sign}(\hat{e}_2 - v_1) + \hat{e}_3 = v_2 \\ \vdots \\ \dot{\hat{e}}_{r_s} = -\lambda_{r_s} |\hat{e}_{r_s} - v_{r_s-1}|^{1/2} \mathrm{sign}(\hat{e}_{r_s} - v_{r_s-1}) + \hat{e}_{r_s} = v_{r_s} \\ \dot{\hat{e}}_{r_s+1} = -\lambda_{r_s+1} \mathrm{sign}(\hat{e}_{r_s+1} - v_{r_s}) \end{cases} \tag{21}$$

The convergence of the above r_s-th order differentiator is established in the following theorem.

Theorem 2 (*Levant [26]*). *The parameters λ_i, for $i = 1, 2, \ldots, r_s + 1$; being properly chosen, the following equalities are true in the absence of input noises after a finite time of a transient process*: $\hat{e}_1 = e_1(t)$; $\hat{e}_i = v_i = e_1^{(i)}(t)$, $i = 1, 2, \ldots, r_s + 1$.

Furthermore, the corresponding solutions of the dynamic systems are Lyapunov stable, i.e. finite-time stable.

The above theorem means that the equalities $\hat{e}_i = e_1^{(i)}$ are kept in 2-sliding mode, $i = 1, \ldots, r_s$. The proof of this theorem is given in Levant [26]. Furthermore, the constants λ_i are chosen recursively sufficiently large, and they can be chosen as in [26].

Since the r_s-th order differentiator used in this paper allows to reconstruct the derivatives $\dot{e}_1, \ddot{e}_1, \ldots, e_1^{(r_s)}$ from the output function measurement $\sigma = e_1$ with finite-time convergence. Furthermore, in this case the separation principle theorem is trivial, and hence the r_s-th order differentiator can be designed separately from the controller.

Then, the solution of synchronization problem for two systems with different relative degree is established in the following theorem.

Theorem 3. *Consider the uncertain system* (15) *satisfying the inequality* (18) *and Assumptions* A1–A3. *The uncertain system* (15) *in closed-loop with the control*

$$u_s = -k\Psi_{r_s-1,r_s}(\sigma, \dot{\sigma}, \ldots, \sigma^{(r_s-1)}) \tag{22}$$

using an exact robust differentiator (21) *is such that the synchronization error e, equivalently* (ε), *tends to zero in finite time*.

The proof is a straightforward consequence of applications of the above results.

4. Cases of study

Now, we present three cases of study for illustrating the implementation of the proposed scheme. Let us consider the following cases:

In the 1st case, the trajectories of a 3rd order Rössler system, representing the slave system, track some projections of a generalized Lorenz system, which is a chaotic system of 4th order without external forcing signals and is used as a master system. Similarly, in the 2nd case, we consider a 3rd order Rössler system which will track the projections of the 4th order hyperchaotic Rössler system.

And finally, an additional case is included, where we show the synchronization of two identical systems represented by a 3rd order Rössler system.

4.1. Case 1

A 3rd order Rössler system, representing the slave system, whose trajectories will track some projections of a master system represented by generalized Lorenz system of 4th order.

Then, consider the following the master system, represented by a generalized Lorenz system [32]:

$$\begin{cases} \dot{x}_{1,M} = -a_M x_{1,M} + a_M x_{2,M} + c_M x_{4,M} \\ \dot{x}_{2,M} = h_M x_{1,M} - x_{2,M} - x_{1,M} x_{3,M} \\ \dot{x}_{3,M} = -b_M x_{3,M} + x_{1,M} x_{2,M} \\ \dot{x}_{4,M} = -x_{1,M} - a_M x_{4,M} \end{cases} \tag{23}$$

where a_M, b_M, c_M, and h_M are constant coefficients. The slave system, represented by Rössler system, is given by

$$\begin{cases} \dot{x}_{1,s} = -x_{2,s} - x_{3,s} \\ \dot{x}_{2,s} = x_{1,s} + a_S x_{2,s} \\ \dot{x}_{3,s} = x_{3,s}(x_{1,s} - c_s) + b_s + u_s \end{cases} \tag{24}$$

where a_s, b_s, and c_s are constant coefficients. Systems (23) and (24) are chaotic and have different order and topology. Furthermore, they are not synchronized in any sense (see Figs. 2 and 3).

The objective is that the trajectories of the slave system (24) track some projections of master system (23) taking into account that the master order is greater than the slave one ($n_M = 4$, $n_s = 3$, $r_M = \infty$, $r_s = 3$). Then, in what follows the tracking problem of trajectories will be analyzed.

Let be $y_M := x_{1,M}$ and $y_s := x_{2,s}$ the outputs of master and slave systems, respectively. Consider the map $z_M = \phi_M(x_M)$

$$\begin{bmatrix} \bar{z}_{1,M} \\ \bar{z}_{2,M} \\ \bar{z}_{3,M} \\ \bar{z}_{4,M} \end{bmatrix} = \begin{bmatrix} x_{1,M} \\ -a_M x_{1,M} + a_M x_{2,M} + c_M x_{4,M} \\ (a_M^2 - c_M + a_M h_M) x_{1,M} - a_M(a_M + 1) x_{2,M} \\ \quad -a_M x_{1,M} x_{3,M} - 2a_M c_M x_{4,M} \\ \bar{\phi}_4(x_M) \end{bmatrix}$$

where $\bar{\phi}_4(x_M) = -a_M(a_M^2 + h_M - c_M + 2a_M h_M + 1)x_{1,M} + a_M(a_M^2 + a_M h_M + a_M - c_M + 1)x_{2,M} + (2a_M^2 c_M - a_M^2 + a_M h_M c_M + c_M)x_{4,M} + a_M(2a_M + b_M + 1)x_{1,M}x_{3,M} - a_M^2 x_{2,M} x_{3,M} - a_M c_M x_{3,M} x_{4,M} -$

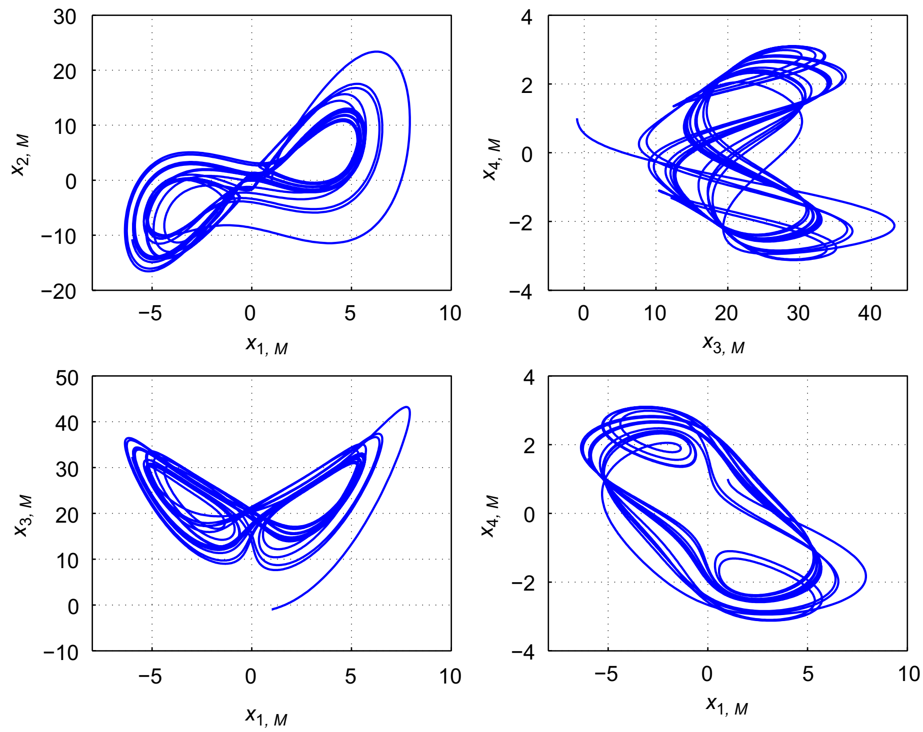

Fig. 2. Case 1. Attractor of generalized Lorenz (master) system.

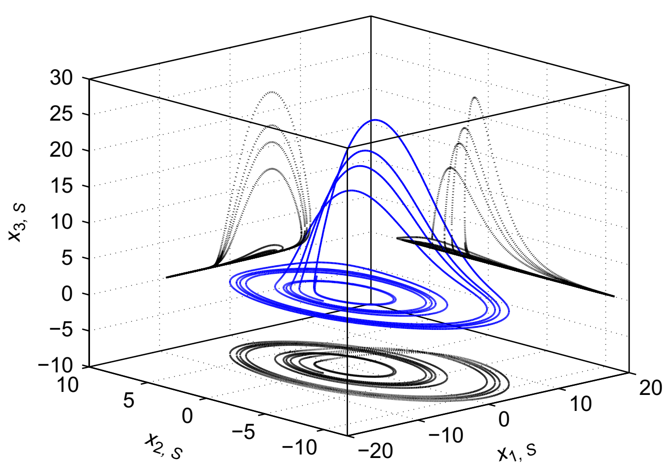

Fig. 3. Case 1. Attractor of Rössler (slave) system.

$a_M x_{1,M}^2 x_{2,M}$, and following the proposed procedure, then system (23) can be transformed into the normal form, as follows:

$$\begin{cases} \dot{\bar{z}}_{1,M} = \bar{z}_{2,M} \\ \dot{\bar{z}}_{2,M} = \bar{z}_{3,M} \\ \dot{\bar{z}}_{3,M} = \bar{z}_{4,M} \\ \dot{\bar{z}}_{4,M} = \alpha_M(\bar{z}_M) \end{cases} \tag{25}$$

Furthermore, defining a smooth map $z_s = \phi_s(x_s)$

$$\begin{bmatrix} \bar{z}_{1,s} \\ \bar{z}_{2,s} \\ \bar{z}_{3,s} \end{bmatrix} = \begin{bmatrix} x_{2,s} \\ x_{1,s} + a_s x_{2,s} \\ (a_s^2 - 1)x_{2,s} - x_{3,s} + a_s x_{1,s} \end{bmatrix}$$

which transform the slave system (24) into the following canonical form:

$$\begin{cases} \dot{\bar{z}}_{1,s} = \bar{z}_{2,s} \\ \dot{\bar{z}}_{2,s} = \bar{z}_{3,s} \\ \dot{\bar{z}}_{3,s} = \alpha_s(\bar{z}_s) + \beta_s(\bar{z}_s)u_s \end{cases} \tag{26}$$

Now, due to the different order it is necessary to look for the map in order to project the trajectories into the suitable space. Thus, trajectories $[x_{1,s}; x_{2,s}; x_{3,s}]^T$ must track the trajectories of master system obtained from the map $\Phi : \Re^4 \to \Re^3$,

$$\begin{aligned} \Phi(x_M) &= \bar{\phi}_s^{-1} \circ \phi_{proj} \circ \bar{\phi}_M(x_M) = [\Phi_1(x_M), \Phi_2(x_M), \Phi_3(x_M)]^T \\ &= \begin{bmatrix} -(a_M + a_s)x_{1,M} + a_M x_{2,M} + c_M x_{4,M} \\ x_{1,M} \\ (c_M - a_M(a_s + a_M + h_M) - 1)x_{1,M} + a_M(1 + a_M + a_s)x_{2,M} \\ \quad + c_M(2a_M + a_s)x_{4,M} + a_M x_{1,M} x_{3,M} \end{bmatrix} \end{aligned} \tag{27}$$

Figs. 3 and 4 show the evolution of $x_{1,s}$, $x_{2,s}, x_{3,s}$ and $\Phi_1(x_M)$, $\Phi_2(x_M), \Phi_3(x_M)$, respectively, without control action. Note that the trajectories are not exactly tracked. Thus, the synchronization error dynamics is given by

$$\begin{cases} \dot{e}_1 = e_2 \\ \dot{e}_2 = e_3 \\ \dot{e}_3 = \xi_1(e, x_M, t) + u_s \end{cases} \tag{28}$$

In this way, the tracking problem of trajectories between (23) and (24) systems can be viewed as the stabilization of system (28) around origin. Then, in order to determine the stabilizing control, the output signal of the synchronization error system is defined as $\sigma := e_1$ with $\dot{\sigma} = \dot{e}_1 = e_2$ and $\ddot{\sigma} = \dot{e}_2 = e_3$. Then, the stabilizing control for the synchronization error system (28) is

$$u_s = -k\frac{[e_3 + 2(|e_2| + |e_1|^{2/3})^{-1/2}(e_2 + |e_1|^{2/3}\,\mathrm{sign}(e_1))]}{[|e_3| + 2(|e_2| + |e_1|^{2/3})^{1/2}]} \tag{29}$$

The choice of the sliding surface as $\sigma := e_1 = 0$, $\dot{\sigma} := e_2 = 0$ and $\ddot{\sigma} := e_3 = 0$ allows to stabilize (28) around origin, which yields $x_{1,s} \to \Phi_1(x_M)$, $x_{2,s} \to \Phi_2(x_M)$ and $x_{3,s} \to \Phi_3(x_M)$ in finite time.

However, the above controller requires direct measurement of e_1, e_2 and e_3, but we have assumed that $e_1 = x_{2,s} - x_{1,M}$ is only available. Then, for implementation purposes and in order to overcome this difficulty, the r_s-th order differentiator (21)

$$\begin{cases} \dot{\hat{e}}_1 = -\lambda_1|\hat{e}_1 - e_1|^{3/4}\,\mathrm{sign}(\hat{e}_1 - e_1) + \hat{e}_2 = v_1 \\ \dot{\hat{e}}_2 = -\lambda_2|\hat{e}_2 - v_1|^{2/3}\,\mathrm{sign}(\hat{e}_2 - v_1) + \hat{e}_3 = v_2 \\ \dot{\hat{e}}_3 = -\lambda_3|\hat{e}_3 - v_2|^{1/2}\,\mathrm{sign}(\hat{e}_3 - v_2) + \hat{e}_3 = v_3 \\ \dot{\hat{e}}_4 = -\lambda_4\,\mathrm{sign}(\hat{e}_4 - v_3) \end{cases} \tag{30}$$

is used to estimate e_1, e_2 and e_3 in the stabilizing control (29). Next, some simulations are shown in order to illustrate the proposed methodology. The following parameters have been used.

- The parameters of generalized Lorenz system (master) were $a_M = 1.0$, $b_M = 0.7$, $c_M = 1.5$ and $h_M = 26$. Initial conditions were chosen at $x_M(0) = (1, 0, -1, 1)^T$.

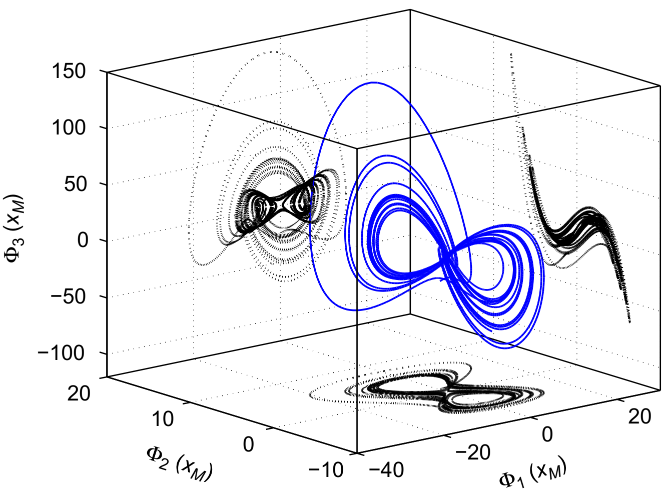

Fig. 4. Case 1. Attractor of master system via functional $\Phi(x_M)$.

- The parameters of Rössler system (slave) were taken as $a_s = 0.2$, $b_s = 0.2$, and $c_s = 5.7$, with initial conditions given by $x_s(0) = (-1, 2, 2)^T$.
- Differentiator's parameters were chosen according to Section 3.2: $\lambda_1 = 3.0M^{1/4}$, $\lambda_2 = 2.0M^{1/3}$, $\lambda_3 = 1.5M^{1/2}$, and $\lambda_4 = 1.1M$; with $M = 500$, and the initial conditions $\hat{e}(0) = [0, 0, 0, 0]^T$.
- The controller parameter was chosen as $k = 100$.

In Fig. 5 is shown that estimation errors $\tilde{e}_1 := e_1 - \hat{e}_1$, $\tilde{e}_2 := e_2 - \hat{e}_2$ and $\tilde{e}_3 := e_3 - \hat{e}_3$ converge to the origin in finite time.

Furthermore, the actual states are replaced in the control law (29) by the corresponding estimated values $\hat{e}_1$, $\hat{e}_2$ and $\hat{e}_3$ obtained from differentiator (30), i.e.

$$u_s = -k\frac{[\hat{e}_3 + 2(|\hat{e}_2| + |\hat{e}_1|^{2/3})^{-1/2}(\hat{e}_2 + |\hat{e}_1|^{2/3}\,\mathrm{sign}(\hat{e}_1))]}{[|\hat{e}_3| + 2(|\hat{e}_2| + |\hat{e}_1|^{2/3})^{1/2}]}. \tag{31}$$

is the control law designed to stabilize the synchronization errors e_1, e_2 and e_3 around the origin.

Thus, the r_s-th differentiator-based control scheme (30) and (31) ensures the trajectory tracking between the Rössler system (*slave*) (24) and the generalized Lorenz system (*master*) (23), by means of the map $\Phi(x_M)$, in spite of differences between the systems. In Fig. 6, the convergence of (a) $x_{1,s}$ to $\Phi_1(x_M)$, (b) $x_{2,s}$ to $\Phi_2(x_M)$ and (c) $x_{3,s}$ to $\Phi_3(x_M)$ are plotted on time-domain, respectively, where the trajectory tracking is achieved in finite time under the control action. Furthermore, the control signal u_s obtained from (31) and applied to the slave system (24) is shown in Fig. 7.

4.2. *Case 2*

Now, consider the following master system, now represented by a hyperchaotic Rössler system:

$$\begin{cases} \dot{x}_{1,M} = -x_{2,M} - x_{3,M} \\ \dot{x}_{2,M} = x_{1,M} + a_M x_{2,M} + x_{4,M} \\ \dot{x}_{3,M} = b_M + x_{1,M}x_{3,M} \\ \dot{x}_{4,M} = -h_M x_{3,M} + d_M x_{4,M} \end{cases} \tag{32}$$

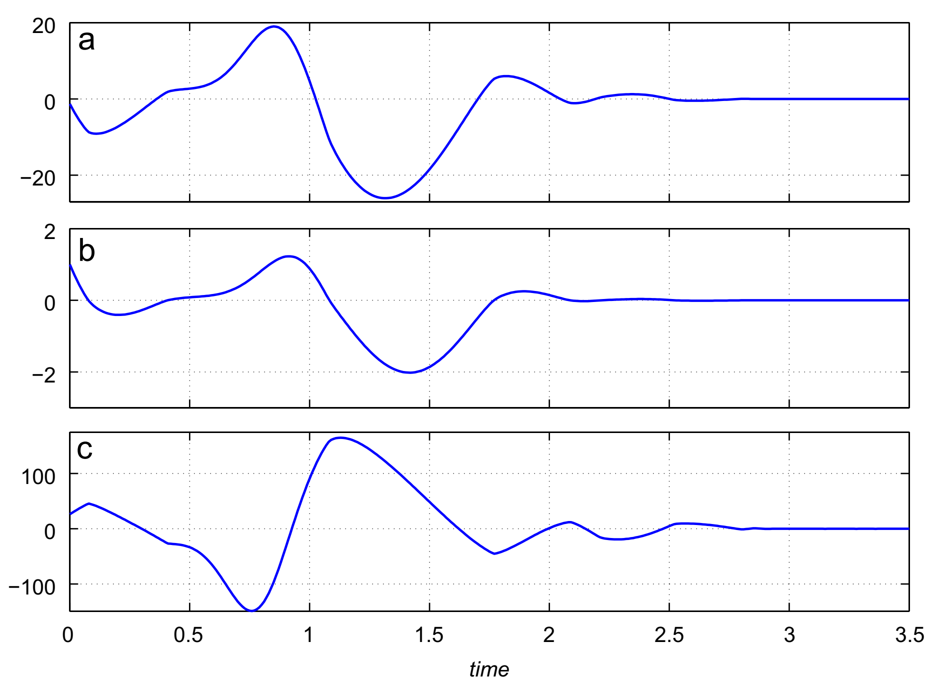

Fig. 5. Case 1. Estimation errors: (a) $\tilde{e}_1 = e_1 - \hat{e}_1$, (b) $\tilde{e}_2 = e_2 - \hat{e}_2$, (c) $\tilde{e}_3 = e_3 - \hat{e}_3$.

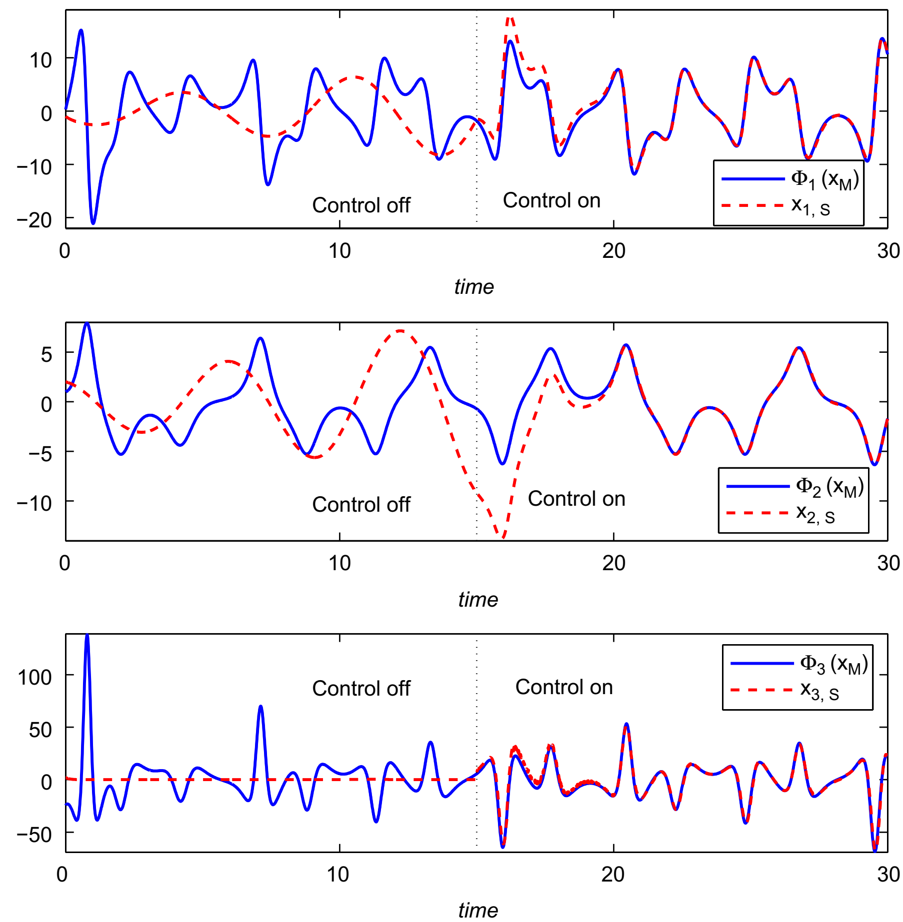

Fig. 6. Case 1. Tracking of x_s to $\Phi(x_M)$.

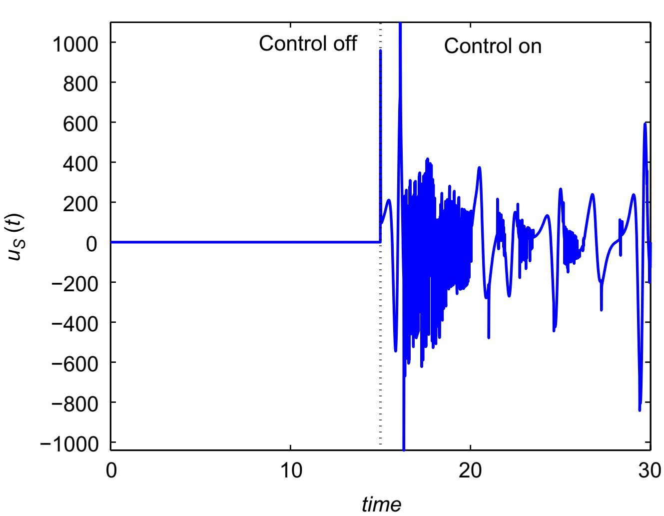

Fig. 7. Case 1. Control signal u_s.

where a_M, b_M, d_M and h_M are constant coefficients and whose dynamics are plotted in Fig. 8. The slave system is represented by a Rössler system given in (24), whose dynamics are plotted in Fig. 3. Furthermore, set $y_M := x_{2,M}$ and $y_s := x_{2,s}$ the outputs of master and slave systems, respectively. Following the same procedure presented in case 1, using the smooth map $z_M = \phi_M(x_M)$

$$\begin{bmatrix} \bar{z}_{1,M} \\ \bar{z}_{2,M} \\ \bar{z}_{3,M} \\ \bar{z}_{4,M} \end{bmatrix} = \begin{bmatrix} x_{2,M} \\ x_{1,M} + a_M x_{2,M} + x_{4,M} \\ (a_M^2 - 1)x_{2,M} - (h_M + 1)x_{3,M} \\ \quad + a_M x_{1,M} + (a_M + d_M)x_{4,M} \\ \bar{\phi}_4(x_M) \end{bmatrix}$$

where $\bar{\phi}_4(x_M) = (a_M^2 - 1)(x_{1,M} + a_M x_{2,M} + c_M x_4) - (h_M + 1)(b_M + x_{1,M}x_{3,M}) + a_M(-x_{2,M} - x_{3,M}) + (a_M + d_M)(d_M x_{4,M} - h_M x_{3,M})$, the master system (32) is transformed into the following system:

$$\begin{cases} \dot{\bar{z}}_{1,M} = \bar{z}_{2,M} \\ \dot{\bar{z}}_{2,M} = \bar{z}_{3,M} \\ \dot{\bar{z}}_{3,M} = \bar{z}_{4,M} \\ \dot{\bar{z}}_{4,M} = \alpha_M(\bar{z}_M) \end{cases} \tag{33}$$

Furthermore, using the same transformation given in case 1 for slave system (24), slave system is transformed into the canonical form (26).

Thus, taking the same projection map that case 1, the trajectories $[x_{1,s}; x_{2,s}; x_{3,s}]^T$ track the trajectories of master system obtained from

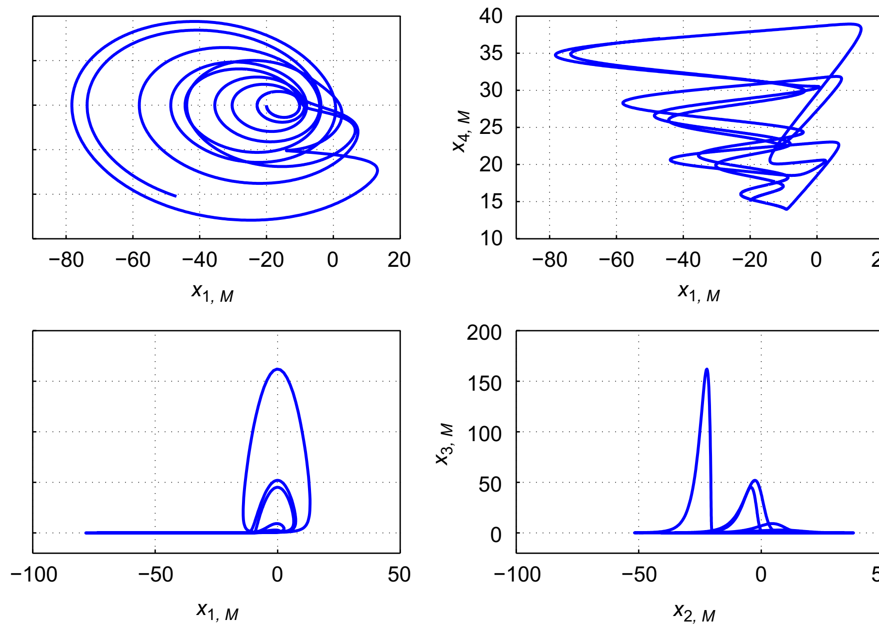

Fig. 8. Case 2. Attractor of hyperchaotic Rössler (master) system.

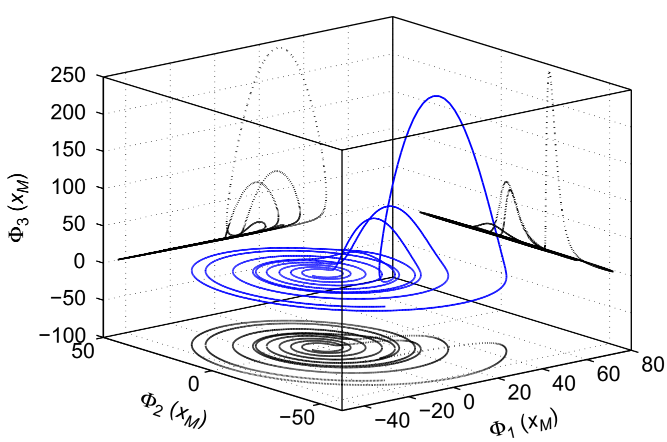

Fig. 9. Case 2. Attractor of master system via functional $\Phi(x_M)$.

the map $\Phi : \Re^4 \rightarrow \Re^3$,

$$\Phi(x_M) = \bar{\phi}_s^{-1} \circ \phi_{proj} \circ \bar{\phi}_M(x_M) = [\Phi_1(x_M), \Phi_2(x_M), \Phi_3(x_M)]^T$$

$$\Phi(x_M) = \begin{bmatrix} 1 & (a_M - a_s) & 0 & 1 \\ 0 & 1 & 0 & 0 \\ a_s - a_M & a_s a_M - a_M^2 & 1 + h_M & a_s - a_M - d_M \end{bmatrix} \times \begin{bmatrix} x_{1,M} \\ x_{2,M} \\ x_{3,M} \\ x_{4,M} \end{bmatrix} \quad (34)$$

Figs. 3 and 9 show the evolution without control action of the states $x_{1,s}$, $x_{2,s}$, $x_{3,s}$ and the projections $\Phi_1(x_M)$, $\Phi_2(x_M)$, $\Phi_3(x_M)$, respectively.

Thus, the synchronization error dynamics is of the form

$$\begin{cases} \dot{e}_1 = e_2 \\ \dot{e}_2 = e_3 \\ \dot{e}_3 = \xi_2(e, x_M, t) + u_s \end{cases} \quad (35)$$

with $e_1 = y_s - y_M = x_{2,s} - x_{2,M}$. To design the controller stabilizing the synchronization error system, the output signal is defined as $\sigma := e_1$ with $\dot{\sigma} = \dot{e}_1 = e_2$ and $\ddot{\sigma} = \dot{e}_2 = e_3$. Then, the controller stabilizing system (35) is of form (29). The choice of the sliding surface as $\sigma := e_1 = 0$, $\dot{\sigma} := e_2 = 0$ and $\ddot{\sigma} := e_3 = 0$ allows to stabilize (35) around origin, which induces the convergence of $x_{1,s} \rightarrow \Phi_1(x_M)$, $x_{2,s} \rightarrow \Phi_2(x_M)$ and $x_{3,s} \rightarrow \Phi_3(x_M)$ in finite time. In order to implement the above controller, the differentiator (30) is used to estimate the errors e_1, e_2, e_3 and e_4. The parameters used for simulation were selected as follows:

- The parameters of hyperchaotic Rössler system (master) were $a_M = 0.25$, $b_M = 3.0$, $d_M = 0.5$ and $h_M = 0.05$. Initial conditions were chosen at $x_M(0) = (-20, 0, 0, 15)^T$.
- The parameters of Rössler system (slave) were taken as $a_s = 0.2$, $b_s = 0.2$, and $c_s = 5.7$, with initial conditions given by $x_s(0) = (-1, 1, 2)^T$.
- Differentiator's parameters were chosen according to Section 3.2: $\lambda_1 = 3.0M^{1/4}$, $\lambda_2 = 2.0M^{1/3}$, $\lambda_3 = 1.5M^{1/2}$, and $\lambda_4 = 1.1M$; with $M = 100$, and the initial conditions $\hat{e}(0) = [0, 0, 0, 0]^T$.
- The controller parameter was chosen as $k = 120$.

In Fig. 10 is shown that estimation errors $\tilde{e}_1 := e_1 - \hat{e}_1$, $\tilde{e}_2 := e_2 - \hat{e}_2$ and $\tilde{e}_3 := e_3 - \hat{e}_3$, which converge to the origin in finite time. Furthermore, replacing the actual states by the corresponding estimated values $\hat{e}_1$, $\hat{e}_2$ and $\hat{e}_3$ given by the differentiator (30) in the control law (31), then the errors e_1, e_2 and e_3 are stabilized around the origin. In Fig. 11, the convergence of (a) $x_{1,s}$ towards the projection $\Phi_1(x_M)$, (b) $x_{2,s}$ towards the projection $\Phi_2(x_M)$ and (c) $x_{3,s}$ towards the projection $\Phi_3(x_M)$ are illustrated on time-domain, respectively. Then, trajectory tracking is achieved under the control action u_s which is plotted in Fig. 12.

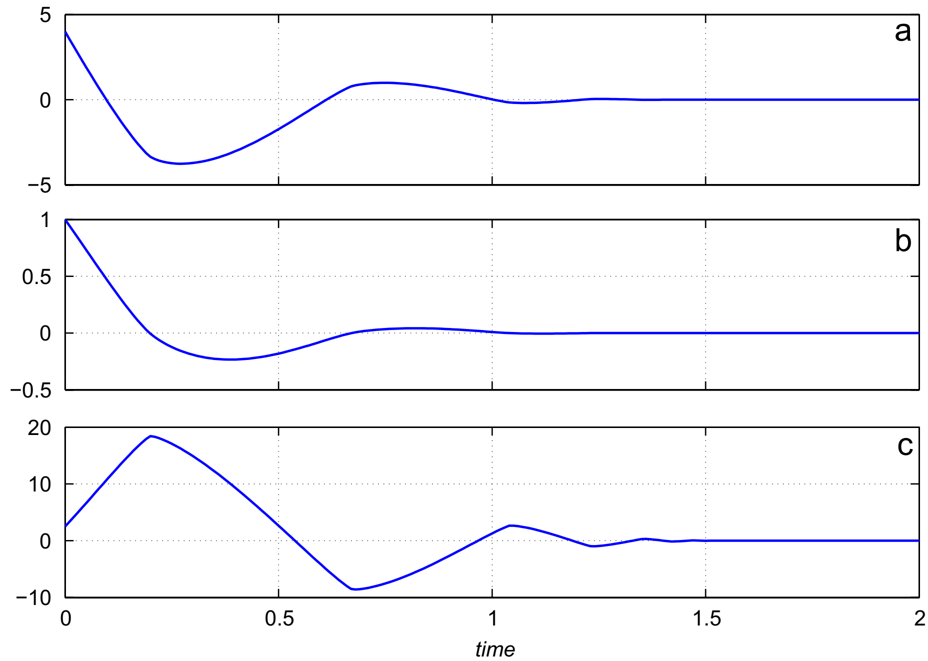

Fig. 10. Case 2. Estimation errors: (a) $\tilde{e}_1 = e_1 - \hat{e}_1$, (b) $\tilde{e}_2 = e_2 - \hat{e}_2$, (c) $\tilde{e}_3 = e_3 - \hat{e}_3$.

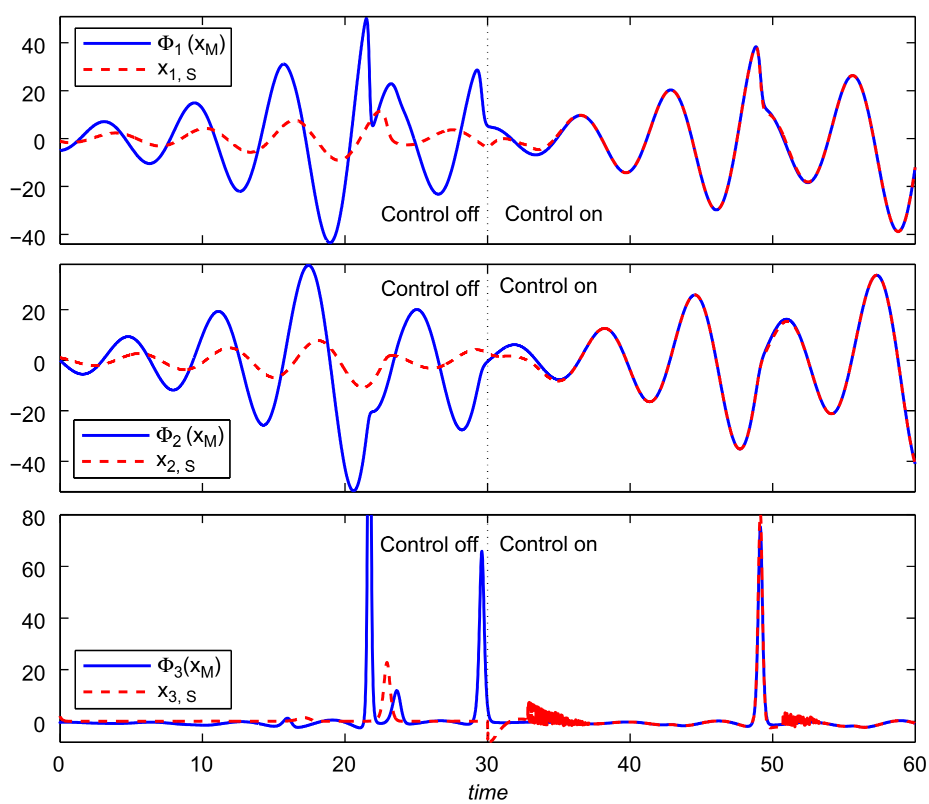

Fig. 11. Case 2. Tracking of x_s to $\Phi(x_{\rm M})$.

4.3. Case 3

Now, we consider the trajectory tracking (synchronization) between two identical Rössler system, which are represented by (24). Following the same procedure and using the following parameters:

- The parameters of Rössler system (master) were $a_{\rm M} = 0.3$, $b_{\rm M} = 0.3$ and $c_{\rm M} = 5.1$. Initial conditions were chosen at $x_{\rm M}(0) = (1, -1, 1)^{\rm T}$.

- The parameters of Rössler system (slave) were taken as $a_s = 0.3$, $b_s = 0.3$, and $c_s = 5.1$, with initial conditions given by $x_s(0) = (-1, 1, -2)^T$.
- Differentiator's parameters were chosen according to Section 3.2: $\lambda_1 = 3.0M^{1/4}$, $\lambda_2 = 2.0M^{1/3}$, $\lambda_3 = 1.5M^{1/2}$, and $\lambda_4 = 1.1M$; with $M = 20$, and the initial conditions $\hat{e}(0) = [0, 0, 0, 0]^T$.
- The controller parameter was chosen as $k = 10$.

Fig. 13 shows the estimation errors $\tilde{e}_1 := e_1 - \hat{e}_1$, $\tilde{e}_2 := e_2 - \hat{e}_2$ and $\tilde{e}_3 := e_3 - \hat{e}_3$, which converge to the origin in finite time.

In Fig. 14, the convergence of (a) $x_{1,s}$ to $x_{1,M}$, (b) $x_{2,s}$ to $x_{2,M}$ and (c) $x_{3,s}$ to $x_{3,M}$ are illustrated on time-domain, respectively; where Φ is the identity map.

Furthermore, the control signal u_s is shown in Fig. 15.

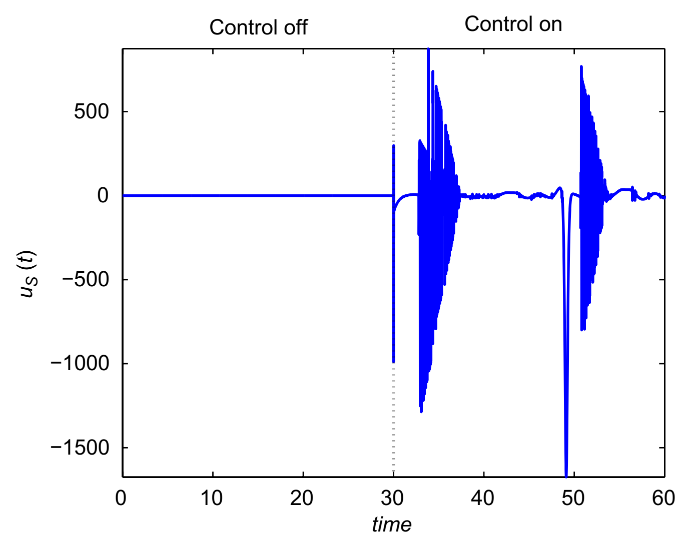

Fig. 12. Case 2. Control signal u_s.

5. Conclusions

In this paper, the reduced-order synchronization problem for a class of chaotic systems has been studied as a tracking problem of trajectories. Using higher-order sliding-mode techniques, a scheme for synchronizing chaotic systems of different order has been proposed. It is concluded that generalized synchronization in reduced-order, which can be viewed as a tracking problem of trajectories, can be ensured if the disturbance characteristic index ν is greater or equal than relative degree r_s for the synchronization error system (4). Furthermore, a stabilizing control algorithm, based on quasi-continuous high-order sliding mode, has been designed. In order to implement that controller, the time derivatives of the output have been computed by means of a high-order sliding-mode robust exact differentiator. The simulations are given illustrating the performance of controller.

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Appendix A. Concepts and definitions

Now, we introduced the following. Consider the following nonlinear system:

$$\Sigma : \begin{cases} \dot{x} = f(x) + g(x)u \\ y = h(x) \end{cases} \tag{36}$$

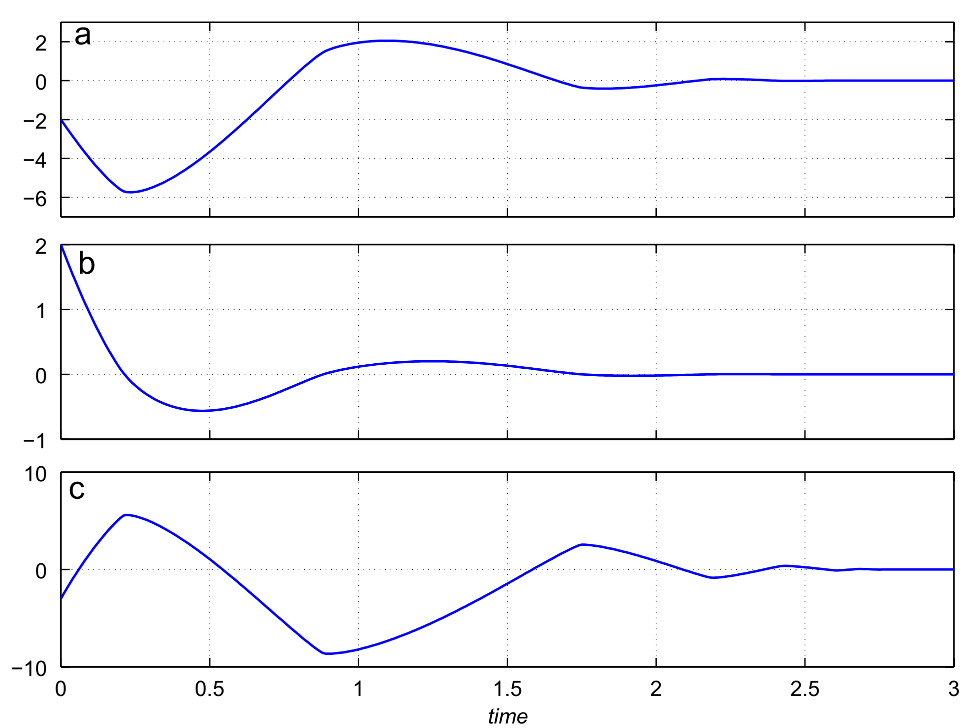

Fig. 13. Case 3. Estimation errors: (a) $\tilde{e}_1 = e_1 - \hat{e}_1$, (b) $\tilde{e}_2 = e_2 - \hat{e}_2$, (c) $\tilde{e}_3 = e_3 - \hat{e}_3$.

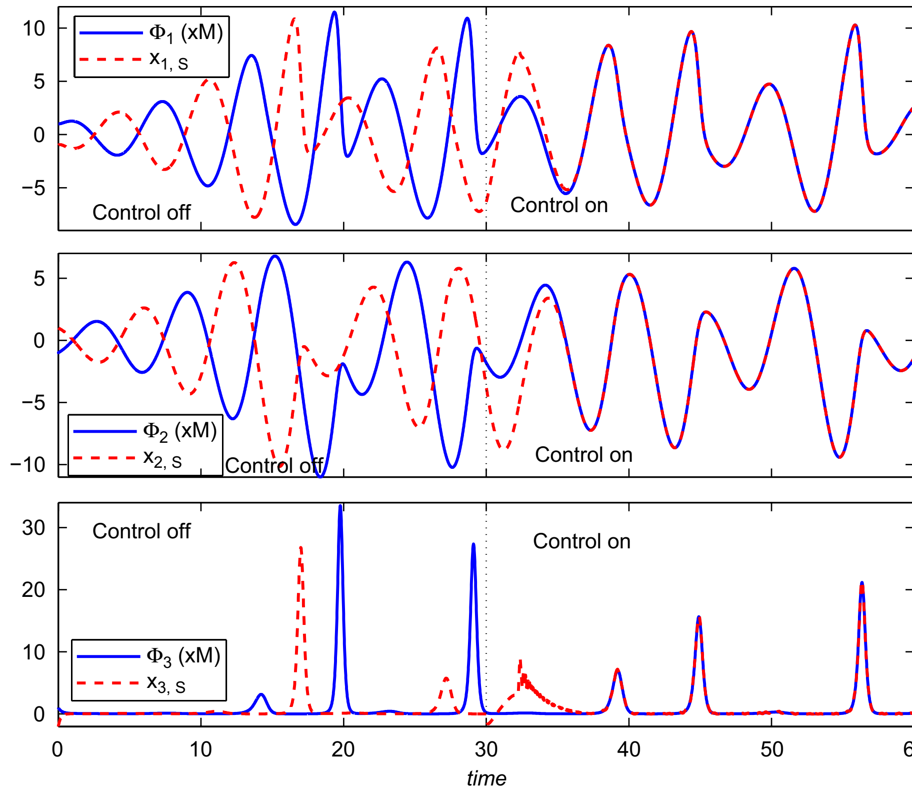

Fig. 14. Case 3. Tracking of x_s to $\Phi(x_M)$.

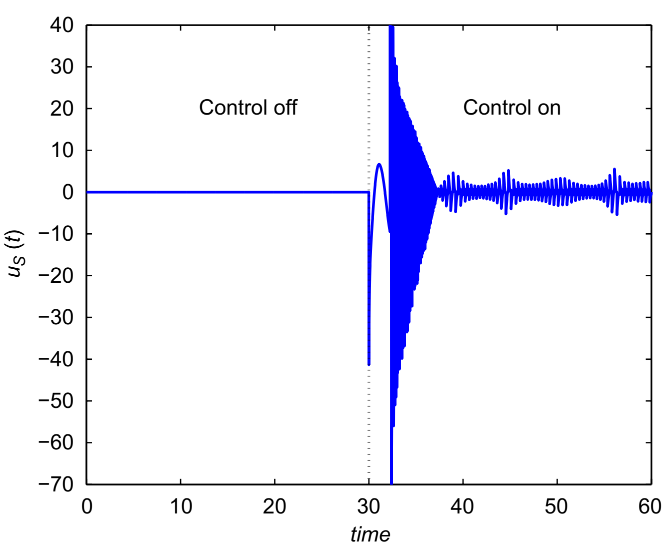

Fig. 15. Case 3. Control signal u_s.

where $x=[x_1,x_2,\ldots,x_n]^T \in \Re^n$ is the state vector, $u \in \Re$ is an input, and $y \in \Re$ is an output variable of the system, f and g are smooth vector fields.

A *vector field* is said to be *complete* if the solutions to the differential equation $\dot{x}=f(x)$ may be defined for all $t \in \Re$ [28].

Definition 3 (*Global relative degree, Marino and Tomei [28]*). The global relative degree r of (36) is defined as the integer such that

$$L_g L_f^i h(x)=0, \quad \forall x \in \Re^n, \ 0 \leqslant i \leqslant r-2$$
$$L_g L_f^{r-1} h_s(x) \neq 0, \quad \forall x \in \Re^n.$$

If

$$L_g L_f^i h_s(x)=0, \quad \forall x \in \Re^n, \ \forall i \geqslant 0$$

we say that $r=\infty$.

Now, we introduce the following definition

Definition 4 (*Minimum phase, Marino and Tomei [28]*). Systems (5)–(7), with $r_s \leqslant n_s$ are said to be minimum phase if the origin $\bar{e}^{\perp}=0$ is an asymptotically stable equilibrium point for the zero dynamics. A system which is not minimum phase is said to be non-minimum phase.

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