Adaptive Continuous Twisting Algorithm

Jaime A. Moreno\textsuperscript{a}, Daniel Y. Negrete\textsuperscript{b}\textsuperscript{*}, Victor Torres-González\textsuperscript{b} and Leonid Fridman\textsuperscript{b,c}

\textsuperscript{a} Instituto de ingeniería, Universidad Nacional Autónoma de México,
\textsuperscript{b} Facultad de ingeniería, Universidad Nacional Autónoma de México,
\textsuperscript{c} Institut für Regelungs- und Automatisierungstechnik, TU Graz, 8010 Graz, Kopernikusgasse 24/II.

\textit{(Received 00 Month 20XX; accepted 00 Month 20XX)}

In this paper an Adaptive Continuous Twisting Algorithm (ACTA) is presented. For double integrator, ACTA produces a continuous control signal ensuring finite time convergence of the states to zero. More-\textit{ever, the control signal generated by ACTA, compensates the Lipschitz perturbation in finite time, i.e. its value converges to the opposite value of the perturbation. ACTA also keeps its convergence properties, even in the case that the upper bound of the derivative of the perturbation exists, but it is unknown.}

\textbf{Keywords:} Sliding mode control, adaptive control.

1. Introduction

Sliding-mode control is considered as one of the most efficient methods for control and observation under uncertainty conditions (Utkin, 1992), (Utkin & Guldner, 2009). Such controllers can keep the desired (sliding) variable in zero after finite time, while compensating theoretically exactly \textit{matched bounded} uncertainties. To reach such goal the presence of a discontinuous term in the control law is required. The main disadvantage of the sliding mode controllers is a chattering effect caused by the discontinuous term. The amplitude of the chattering is proportional to the controller gains.

To adjust the sliding mode controller gains, some bounds of the perturbation should be known. However, the bound of the perturbation is usually unknown. This leads to an overestimation of the controller gains and, consequently, to a high amplitude of the chattering. An alternative to reduce the chattering is to adjust on-line the controllers gains.

For a system with relative degree two the Twisting Controller (TC) is historically the first higher order sliding mode controllers (Emelyanov, et. al., 1986) driving the sliding output and its derivative to zero in finite time in the presence of a bounded perturbation, with known bound.

Recently, two approaches to adapt the gain for second order sliding mode controllers were proposed. The first adapts the gains according to the value of the estimated perturbation. Such type of adaptation is proposed, for example, in (Utkin & Poznyak, 2013) for the super-twisting controller. The adaptive gain “follows” the equivalent control, which is obtained by filtering out the control signal by a low-pass filter. The main disadvantage of this method is that the bound of the derivative of the perturbation should be known. Under the same conditions the order of the sliding mode controller could be increased by one and adaptation will be not required. Another disadvantage of the algorithm proposed in (Utkin & Poznyak, 2013) is that for its implementation

\*Corresponding author. Email: danielnegrete@gmail.com
a filter is needed, so that frequency characteristics of perturbation are also required.

The second approach is to increase the sliding mode controller gain until the sliding mode is detected. One can reduce the gain to avoid the effect of chattering until the sliding mode is lost, and then increase the gain again until the sliding mode is attained. In (Shtessel et. al., 2011), (Shtessel & Plestan, 2012) such adaptation of twisting and super-twisting controllers is proposed. In this approach the adaptive gains can increase or decrease. However, only practical sliding mode is ensured, i.e. only convergence to a neighborhood of the origin can be guaranteed. In (Alwi & Edwards, 2013) an adaptive, ideal, super-twisting differentiator is proposed with a non-decreasing gain for fault detection, for the case when the bound of the fault is supposed to exist but it is unknown. In this case, the approach (Alwi & Edwards, 2013) can guarantee convergence to the sliding mode but can not ensure the permanence in the sliding mode.

Recently, a new class of homogeneous continuous SMC for systems with relative degree two based on a generalization of the super-twisting algorithm was announced, (Fridman, et. al., 2015), (Kamal, et. al., 2014). Application of such a controller to the double integrator with a Lipschitz continuous perturbation has the following advantages:

(i) it generates a continuous control signal;
(ii) it compensates (theoretically exactly) Lipschitz uncertainties/perturbations;
(iii) it provides finite-time theoretically exact convergence of the system states to zero. Moreover, the control signal compensates perturbations in finite time, i.e. its value converges to the opposite value of the perturbation;
(iv) it ensures the third order precision of the output with respect to sampling step.

In this paper (see also Torres-González, et. al. (2015)) we propose a Continuous Twisting Algorithm (CTA), based on a generalization of the classical TC, since the part of the controller compensating the derivative of the perturbation has a TC structure. CTA keeps the properties (i)-(iv).

The main disadvantages of CTA are:

• the CTA requires the calculation of four gains, which complicates the gain adjustment;
• the upper bound of the derivative of the perturbation is needed (usually it is overestimated);
• the CTA gains cannot be adjusted for the case when the upper bound of the derivative of the perturbation is unknown.

To adjust the controller gains for the case when the upper bound of derivative of perturbation is unknown, we propose an adaptive CTA (ACTA) having the properties (i)-(iv). The gain of ACTA is adjusted automatically, ensuring that the states will be remain in zero after finite time despite of the perturbation.

The paper is organized as follows: In section 2, the problem is stated. In section 3, the CTA is presented. In section 4, the ACTA is proposed for the perturbed case, assuming that the upper bound of the derivative of the perturbation exists but it is unknown. The gain of ACTA is adjusted automatically, ensuring that the states will remain in zero and the control signal compensates the value of the perturbation after a finite time. In section 5 a simulation example is given.

**Notation 1:** For a real variable $\zeta \in \mathbb{R}$ to a real power $\rho \in \mathbb{R}$,

$$|\zeta|^\rho = |\zeta|^\rho \text{sign}(\zeta)$$
2. Problem statement

Consider the uncertain double integrator system

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= u + \Delta(t)
\end{align*}
\]  
(1)

where \(x_1, x_2 \in \mathbb{R}\) are the states, \(u \in \mathbb{R}\) is the control input and \(\Delta(t)\) is the perturbation.

For the system (1), the following assumptions are made

**Assumption 1:** The states \(x_1, x_2\) are measurable.

**Assumption 2:** The perturbation \(\Delta(t)\) is a Lipschitz continuous signal, i.e. its derivative exists almost everywhere and it is bounded

\[
|\dot{\Delta}(t)| \leq \delta_p ,
\]  
(2)

with \(\delta_p \geq 0\) unknown.

Firstly, a CTA will be introduced. For the nominal case (\(\Delta(t) \equiv 0\)), it will be shown, that the CTA drives the states of the system to zero in finite time. Secondly, it will be proved that when the bound \(\delta_p\) is known the gains can be adjusted by means of a single parameter such that finite time convergence of the system states to origin is achieved. Finally, assuming that the bound \(\delta_p\) exists but is unknown, an adaptive law for the parameter can be implemented in the CTA ensuring that the system states will be brought to zero after finite time despite of the perturbation.

3. Continuous Twisting Algorithm

In this section we propose the CTA having the following form

\[
\begin{align*}
\dot{u} &= -k_1 [x_1]^{\frac{1}{3}} - k_2 [x_2]^{\frac{1}{3}} + \eta \\
\dot{\eta} &= -k_3 [x_1]^{0} - k_4 [x_2]^{0}
\end{align*}
\]  
(3)

where \(k_1, k_2, k_3, k_4 > 0\) are constant parameters.

The second equation in (3) has the TC structure. Its purpose is to reject the derivative of the Lipschitz perturbation. The integral from TC appears in the controller, producing a continuous control signal.

To simplify the stability analysis, let us define the following virtual state

\[
x_3 = \eta + \Delta(t)
\]  
(4)

System (1) with the new state \(x_3\) and controlled with the CTA (3) has the form

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -k_1 [x_1]^{\frac{1}{3}} - k_2 [x_2]^{\frac{1}{3}} + x_3 \\
\dot{x}_3 &= -k_3 [x_1]^{0} - k_4 [x_2]^{0} + \dot{\Delta}(t)
\end{align*}
\]  
(5)

System (5) has an homogeneous vector field of degree \(\zeta = -1\), with weights of homogeneity \(r = [3 \ 2 \ 1]\) for the variables \(x_1, x_2\) and \(x_3\), respectively (Baccioti & Rosier, 2006). It will be
proved that with the appropriate selection of the controller gains, the state vector \( x = (x_1, x_2, x_3) \) converge to zero in finite time. This implies that \( \eta \) converges to \(-\Delta(t)\), in finite time, proving that the twisting structure (second equation of controller (3)), compensates exactly the derivative of the perturbation \( \Delta(t) \).

3.1 Nominal case (unperturbed case)

In this section the nominal case is considered (unperturbed system, \( \Delta(t) = 0 \)), which yields the following system

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -k_1|x_1|^\frac{3}{2} - k_2|x_2|^\frac{3}{2} + x_3 \\
\dot{x}_3 &= -k_3|x_1|^3 - k_4|x_2|^3
\end{align*}
\]  

(6)

The following Theorem ensures that the controller (3) drives the states of the nominal system (6) to zero in finite time.

**Theorem 1:** For the nominal system (6), if the gains \( k_1, k_2, k_3 \) and \( k_4 \) are selected properly, then, the state vector \( x \) converges to zero in finite time.

**Proof.** We use the following candidate Lyapunov function to prove that the states of system (6) converge to zero in finite time,

\[
V(x) = \alpha_1|x_1|^\frac{5}{2} + \alpha_2x_1x_2 + \alpha_3|x_2|^\frac{5}{2} + \alpha_4x_1[x_3]^2 - \alpha_5x_2x_3^3 + \alpha_6|x_3|^5.
\]

(7)

Its derivative along the trajectories of (6) is

\[
\dot{V}(x) = -W(x),
\]

(8a)

with

\[
W(x) \triangleq \beta_1|x_1|^\frac{5}{2} + \beta_2x_1[x_2]^{\frac{3}{2}} - \beta_3|x_1|^3x_2 + \beta_4|x_1|^{\frac{5}{2}}x_2^{\frac{3}{2}} + \beta_5|x_2|^2 - \beta_6x_1x_3 + \beta_7|x_1||x_3|
\]

\[+ \beta_8x_1[x_2]^0|x_3| - \beta_9|x_2|^\frac{5}{2}x_3 - \beta_{10}x_2[x_3]^2 - \beta_{11}|x_1|^{\frac{3}{2}}x_2|x_3|^2 - \beta_{12}|x_2||x_3|^2
\]

\[- \beta_{13}|x_1|^\frac{5}{2}|x_3|^3 - \beta_{14}|x_2|^\frac{5}{2}|x_3|^3 + \beta_{15}|x_3|^4 + \beta_{16}|x_1|^{\frac{3}{2}}|x_2|^4 + \beta_{17}|x_2|^0|x_3|^4,
\]

(8b)

where

\[
\begin{align*}
\beta_1 &= \alpha_2k_1, & \beta_2 &= \alpha_2k_2, & \beta_3 &= \frac{5}{3}\alpha_1, & \beta_4 &= \frac{5}{2}\alpha_3k_1, & \beta_5 &= \frac{5}{2}\alpha_3k_2 - \alpha_2 \\
\beta_6 &= \alpha_2, & \beta_7 &= 2\alpha_4k_3, & \beta_8 &= 2\alpha_4k_4, & \beta_9 &= \frac{5}{2}\alpha_3, & \beta_{10} &= \alpha_4, & \beta_{11} &= 3\alpha_3k_3, \\
\beta_{12} &= 3\alpha_5k_4, & \beta_{13} &= \alpha_5k_1, & \beta_{14} &= \alpha_5k_2, & \beta_{15} &= \alpha_5, & \beta_{16} &= 5\alpha_6k_3, & \beta_{17} &= 5\alpha_6k_4.
\end{align*}
\]

(8c)

To determine for which values of the parameters \( \alpha_i, i = 1, \cdots, 6 \), and of the gains \( k_1, k_2, k_3 \) the Lyapunov function candidate (7) is positive definite and its derivative (8a) is negative definite, we use the methodology introduced in (Sanchez & Moreno (2014)), based on Polya’s Theorem (Hardy et al. (1951); Sanchez & Moreno (2014)):
Theorem 2: (Pólya’s Theorem). Let \( F : \mathbb{R}^n \to \mathbb{R} \) be a polynomial form and \( P = \{(y_1, \ldots, y_n) | y_i \geq 0, y \neq 0 \} \). If \( F \) is homogeneous and positive definite on \( P \), then there exists a \( p \) large enough such that the coefficients of \( Q(y) \), with
\[
Q(y) = (y_1 + y_2 + \ldots + y_n)^p F(y),
\]
are all positive.

For this we obtain homogeneous forms \( V_j(y) \) and \( W_j(y) \) having the same values of \( V(x) \) and \( W(x) = -\dot{V}(x) \), respectively, in each octant \( j, j = 1, \ldots, 8 \). For example, for the first octant, where \( x_1 > 0, x_2 > 0, x_3 > 0 \), this can be achieved using the change of variables
\[
y_1^3 = |x_1|, \quad y_2^2 = |x_2|, \quad y_3 = |x_3|, \quad y = (y_1, y_2, y_3).
\]
Applying (9) on (7) and (8b) it is possible to write the values of \( V(x) \) and \( W(x) \) on the first octant as the values of the functions
\[
V_1(y) = \alpha_1 y_1^5 + \alpha_2 y_1^3 y_2^2 + \alpha_3 y_1^5 + \alpha_4 y_1^2 y_3^2 - \alpha_5 y_2^2 y_3^3 + \alpha_6 y_3^5,
\]
\[
W_1(y) = \beta_1 y_1^4 + \beta_2 y_1^3 y_2 - \beta_3 y_1^2 y_2^2 + \beta_4 y_1 y_2^3 + \beta_5 y_2^4 - (\beta_6 - \beta_7 - \beta_8) y_1^3 y_3 - \beta_9 y_1 y_3^3 - (\beta_10 + \beta_{11} + \beta_{12}) y_2^2 y_3^2 - \beta_{14} y_2 y_3^3 + \beta_{15} y_1 y_2^2 + \beta_{16} y_1 y_2 + \beta_{17} y_1^4 y_2^4,
\]
on the first octant \( y_1 > 0, y_2 > 0, y_3 > 0 \). Applying Pólya’s theorem to \( V_1(y) \) and \( W_1(y) \) for some value of \( p \) we obtain a set of linear inequalities in the coefficients \( \alpha_i \) and \( \beta_j \) of \( V(x) \) and \( \dot{V}(x) \) that are sufficient for \( V_1(y) \) and \( W_1(y) \) to be positive definite in the set \( P \) of Pólya’s theorem.

Using the same procedure for each octant we obtain a set of linear inequalities in the coefficients \( \alpha_i \) and \( \beta_j \) of \( V(x) \) and \( \dot{V}(x) \) that are sufficient for \( V(x) \) to be positive definite and \( \dot{V}(x) \) to be negative definite. Note that, since coefficients \( \beta_j \) are linear in the coefficients \( \alpha_i \) and linear in the gains \( k_i \), the set of inequalities is bilinear in \( \alpha_i \) and \( k_i \). The set of inequalities is too large to be written here, but it can be easily solved iteratively using standard algorithms for solving linear inequalities.

For example, solving the inequalities it can be shown that a set of coefficients \( \alpha_i \) and gains \( k_i \) rendering \( V(x) \) (7) a Lyapunov function is:
\[
\alpha_1 = 0.85, \quad \alpha_2 = 0.37697, \quad \alpha_3 = 0.57107, \quad \alpha_4 = -0.82921, \quad \alpha_5 = 0.86679, \quad \alpha_6 = 3.41417, \quad k_1 = 0.96746, \quad k_2 = 1.40724, \quad k_3 = 0.00844, \quad k_4 = 0.004601.
\]

Note that \( V(x) \) is homogeneous of degree 5 with weights \([3, 2, 1]\), while \( \dot{V}(x) \) (and \( W(x) \)) is homogeneous of degree 4. This implies that the function
\[
\phi(x) = \frac{W(x)}{\dot{V}(x)}
\]
is homogeneous of degree 0, meaning that for every \( \kappa \geq 0 \)
\[
\phi(\kappa^3 x_1, \kappa^2 x_2, \kappa x_3) = \kappa^0 \frac{W(x)}{\dot{V}(x)} = \phi(x).
\]
This implies that all values of φ(x) are attained on the unit sphere in $\mathbb{R}^3$, $S = \{x \in \mathbb{R}^3 | ||x|| = 1\}$. Since $W(x)$ and $V(x)$ are positive definite the function $\phi(x)$ can be bounded on the unit sphere $S$, and therefore globally, as

$$0 < \chi_{\text{min}} \leq \phi(x) = \frac{W(x)}{V^2(x)} = \frac{-\dot{V}(x)}{V^2(x)} \leq \chi_{\text{max}}$$

for some positive constants $\chi_{\text{min}}, \chi_{\text{max}} > 0$. Thus, the derivative of the Lyapunov function can be bounded as

$$\dot{V}(x) \leq -\chi_{\text{min}} V^2(x)$$

From inequality (17), we conclude that the trajectories of system (6) converge to zero in finite time.

We can illustrate graphically that $V(x)$ and $W(x) = -\dot{V}(x)$ are positive definite. For this we use spherical coordinates

$$
\begin{align*}
x_1 &= r \sin(\theta) \cos(\phi) \\
x_2 &= r \sin(\theta) \sin(\phi) \\
x_3 &= r \cos(\theta)
\end{align*}
$$

and find the values of $V(x)$ and $W(x)$ on the unit sphere $S$ (fixing $r = 1$), and represent them in a graph as a function of the two angles $0 < \phi \leq 2\pi$ and $0 \leq \theta \leq \pi$. In Figure 1 we see the projection of $V(\sin(\theta) \cos(\phi), \sin(\theta) \sin(\phi), \cos(\theta))$ on the axis $\phi$. It is easy to see that $V(x) \geq 0.4109$ on the unite sphere $S$, so that $V(x)$ is positive definite. In Figure 2 we see the same graph for function $W(x)$, and it illustrates that $W(x)$ is positive definite, and therefore $\dot{V}(x)$ is negative definite.

**Remark 1:** The same procedure can be used to find other values of the gains $k_1, k_2, k_3$ and $k_4$ and of the parameters of $V(x)$ rendering $V(x)$ and $W(x)$ positive definite. Moreover, similar procedures (Sanchez & Moreno (2014)) can be used to obtain different Lyapunov functions.

### 3.2 Perturbed case

We now consider the perturbed system (1) with controller (3) and the virtual state (4), given by (5). We recall that by Assumption 2 the derivative of the perturbation is bounded $|\dot{\Delta}(t)| \leq \delta_p$. 

![Figure 1. Lyapunov Function $V(x)$ on unit sphere](image-url)
Proposition 1: Suppose that the gains $k_1, k_2, k_3, k_4$ are such that the origin of system (6) is finite time stable, and that there are coefficients $\alpha_i, i=1,\cdots,6$ rendering $V(x)$ in (7) a Lyapunov Function. Then there exists some value of the bound of the derivative of the perturbation $\delta_p > 0$ such that the origin of the perturbed system (5) is finite time stable.

Proof. The derivative of the Lyapunov function (7) along the trajectories of system (5) is given by

$$\dot{V}_p = -W(x) + U(x)\dot{\Delta}(t) = -\left(1 - \frac{U(x)}{W(x)}\dot{\Delta}(t)\right)W(x),$$

where $W(x)$ is defined in (8b), and $U(x)$ is the continuous function

$$U(x) \triangleq 2\alpha_4x_1|x_3| - 3\alpha_5x_2|x_3|^2 - 5\alpha_6|x_3|^4.$$  

Since $W(x)$ and $U(x)$ are homogeneous of degree 4 and $W(x)$ is positive definite, the fraction $\frac{U(x)}{W(x)}$ is homogeneous of degree 0 and it achieves all its values on the unit sphere $S$, which as compact set. So there is a maximum value on the unit sphere, i.e.

$$0 < \mu \triangleq \max_S \frac{U(x)}{W(x)}.$$  

The inequality $\mu > 0$ in (21) is a consequence of the fact that $U(x)$ takes positive values, e.g. for $x_1 = x_2 = 0$ and $x_3 < 0$ since $\alpha_6 > 0$. If $\delta_p < \frac{1}{\mu}$, then (19) implies that $\dot{V}_p$ is negative definite, and the conclusion follows.

If the perturbation bound $\delta_p$ is given a priori, then the following scaling by a sufficiently large positive parameter $\lambda$ of the gains of the CTA

$$u = -\lambda^{\frac{3}{2}}k_1|x_1|^\frac{1}{2} - \lambda^{\frac{3}{2}}k_2|x_2|^\frac{1}{2} + \eta$$

$$\dot{\eta} = -\lambda(\kappa_3|x_1|^0 + k_4|x_2|^0)$$

will render the origin $x = 0$ for the closed loop system (1) with controller (22) and the virtual state (4)

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\lambda^{\frac{3}{2}}k_1|x_1|^\frac{1}{2} - \lambda^{\frac{3}{2}}k_2|x_2|^\frac{1}{2} + x_3$$

$$\dot{x}_3 = -\lambda(\kappa_3|x_1|^0 + k_4|x_2|^0) + \Delta(t)$$

(23)
finite time stable despite a Lipschitz perturbation satisfying Assumption 2.

**Theorem 3:** If the gains \( k_1, k_2, k_3 \) and \( k_4 \) are such that the origin of system (5) is finite time stable for a perturbation satisfying Assumption 2 with \( \delta_p \), then the origin of system (23) is finite time stable for every perturbation of size \( \lambda \delta_p \).

**Proof.** Consider the following change of variables

\[
    z_1 = \frac{x_1}{\lambda}, \quad z_2 = \frac{x_2}{\lambda}, \quad z_3 = \frac{x_3}{\lambda}
\]

System (23) in the new coordinates \( z \) is given by

\[
    \dot{z}_1 = z_2, \quad \dot{z}_2 = -k_1 [z_1]^{\frac{1}{2}} - k_2 [z_2]^{\frac{1}{2}} + z_3, \quad \dot{z}_3 = -k_3 [z_1]^0 - k_4 [z_2]^0 + \frac{\hat{\Delta}(t)}{\lambda}.
\]

From this expression the affirmation of the Theorem follows easily. \( \square \)

For implementation of CTA using Theorem 3 it is required to know the bound \( \delta_p \) of the perturbation. If this bound does exist but it is unknown, we provide in the next Section an adaptive version of the CTA.

4. **Adaptive Continuous Twisting Algorithm (ACTA)**

We propose a CTA (3) with adaptive gain \( L(t) \) as

\[
    u = -L_1(t)k_1 [x_1]^{\frac{1}{2}} - L_2(t)k_2 [x_2]^{\frac{1}{2}} + \eta, \quad \dot{\eta} = -L(t) \left( k_3 [x_1]^0 + k_4 [x_2]^0 \right),
\]

where the adaptation law of the gain \( L(t) \) is given by

\[
    \dot{L}(t) = \begin{cases} 
    \ell, & \text{if } T_e(t) \neq 0 \text{ or } ||\vec{x}(t)|| \neq 0 \\
    0, & \text{if } T_e(t) = 0 \text{ and } ||\vec{x}(t)|| = 0
    \end{cases}
\]

where \( \vec{x} = (x_1, x_2) \) and \( \ell > 0 \) is a positive constant. Function \( T_e(t) \) represents a timer with behavior given by

\[
    T_e(t) = \begin{cases} 
    t_i + \tau - t & \text{if } t_i \leq t \leq t_i + \tau \\
    0 & \text{if } t > t_i + \tau
    \end{cases}
\]

where \( \tau > 0 \) is a constant dwell time. The times \( t_i \) are defined as the instants when \( ||x(t)|| \) changes from zero to a non zero value. For \( i = 0 \), \( t_0 = 0 \), and for \( i > 0 \) \( t_i \) are defined as the instants such
that
\[ \exists \epsilon^- > 0, \ \forall \mu \in (0, \epsilon^-) \ |x(t_i - \mu)| = 0 \]
\[ \exists \epsilon^+ > 0, \ \forall \mu \in (0, \epsilon^+) \ |x(t_i + \mu)| \neq 0. \]

The idea of this adaptation law is to let the adaptive gain grow until \( |\bar{x}(t) = 0| \), but for at least a fixed time \( \tau \). Every time \( t_i \), when \( \bar{x} \) deviates from zero due to an increase of the size of the perturbation, the gain will grow again for at least a time \( \tau \) until it becomes zero again. This process is repeated until \( |x(t)| \) remains in zero for all future times. It is possible to show that this happens in finite time. Of course, in practice the ideal condition \( x = 0 \) will be replaced by \( x \) belonging to a small neighborhood of zero.

System (1) with control input (24) and the virtual state (4) can be written as
\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -L z_1 x_2 - L \dot{z}_1 k_2 + x_3 \\
\dot{x}_3 &= -L (k_3 x_1^0 + k_4 x_2^0) + \dot{z}_1 \\
\end{align*}
\]
where the perturbation term \( \Delta(t) \) satisfies Assumption 2, i.e. \( |\dot{\Delta}(t)| \leq \delta_p \). The following theorem states the main result of the paper:

**Theorem 4:** Assume that the gains \( k_1, k_2, k_3, \) and \( k_4 \) in system (26) are selected as in the proof of Theorem 1, that the gain \( L(t) \) is started at an initial positive value \( L(0) > 0 \) and is adapted according to the law (25) and that the derivative of the perturbation \( \dot{\Delta}(t) \) is bounded by an unknown constant \( \delta_p \), i.e. \( |\dot{\Delta}(t)| \leq \delta_p \). Under these conditions the states of system (26) will finally converge to the origin in finite time and they will remain there for all future times. Moreover, the adaptive gain \( L(t) \) will be bounded and it will finally reach in finite time the value ensuring that the states of the system (26) remain in zero.

**Proof.** For this proof we use the ideas introduced in (Negrete & Moreno (2014)). To prove that the states of system (26) converge to zero in finite time and remain there afterwards we consider the following change of variables:
\[
\begin{align*}
z_1 &= \frac{x_1}{L(t)^{2q+1}}, \quad z_2 = \frac{x_2}{L(t)^{2q+1}}, \quad z_3 = \frac{x_3}{L(t)^{q+1}},
\end{align*}
\]
where \( L(t) > 0 \forall t \geq 0 \) and \( 0 < q \in \mathbb{R} \) is to be selected. System (26) in the new coordinates is given by
\[
\begin{align*}
\dot{z}_1 &= -(3q+1) \frac{L(t) \dot{L}(t)}{L(t)} z_1 + \frac{z_2}{L(t)} \\
\dot{z}_2 &= -(2q+1) \frac{L(t) \dot{L}(t)}{L(t)} z_2 - \frac{k_1 L(t)}{L^q(t)} z_3^3 + \frac{k_2 L(t)}{L^q(t)} z_2^3 + \frac{z_3}{L(t)} \\
\dot{z}_3 &= -(q+1) \frac{L(t) \dot{L}(t)}{L(t)} z_3 - \frac{k_3 z_1^0}{L^q(t)} - \frac{k_4 z_2^0}{L^q(t)} + \frac{\Delta(t)}{L^{q+1}(t)}.
\end{align*}
\]

If the state vector \( z = (z_1, z_2, z_3) \) of system (28) converges to zero in finite time, the states of system (26) will also converge to zero in finite time. The Lyapunov function (7), in the new variables (27), has the form
\[
V(z) = \alpha_1 z_1^3 + \alpha_2 z_2^2 + \alpha_3 z_2^2 + \alpha_4 z_1^3 - \alpha_5 z_2 z_3^3 + \alpha_6 z_3^5.
\]
The derivative of the function (29), along the trajectories of system (28) is
\[
\dot{V} = -\frac{1}{L(t)} \left( 1 - \frac{U(z) \Delta(t)}{W(z) L(t)} \right) W(z) - 5q \frac{\dot{L}(t)}{L(t)} H(z)
\] (30)
where \(W(z)\) is defined in (8b), \(U(z)\) is given by (20) and
\[
H(z) = \left( 1 + \frac{1}{3q} \right) \alpha_1 |z_1|^\frac{5}{2} + \left( 1 + \frac{2}{5q} \right) \alpha_2 z_1 z_2 + \left( 1 + \frac{1}{2q} \right) \alpha_3 |z_2|^\frac{5}{2} + \left( 1 + \frac{3}{5q} \right) \alpha_4 z_1 |z_3|^2 - \left( 1 + \frac{3}{5q} \right) \alpha_5 z_2 z_3^2 + \left( 1 + \frac{1}{q} \right) \alpha_6 |z_3|^5.
\] (31)

Selecting the parameters \(\alpha_i, i = 1, \ldots, 6\), as in Theorem 1, e.g. taking the values (12), functions \(V(z)\) (29) and \(W(z)\) (8b) will be positive definite.

Note that for large values of \(q\) the coefficients of \(H(z)\) are near to the coefficients of \(V(z)\) in (29), and therefore, by continuity of \(V(z)\) with respect to its coefficients, for sufficiently large values of \(q\), the function \(H(z)\) is positive definite if \(V(z)\) is positive definite. In this case, using similar arguments to those in the proof of Proposition 1, it is possible to show that \(\eta_{min} V(z) \leq H(z) \leq \eta_{max} V(z)\), for some \(\eta_{max} \geq \eta_{min} > 0\). Moreover, from inequality (16) it follows that \(\chi_{min} V^\frac{\ast}{5}(z) \leq W(z) \leq \chi_{max} V^\frac{\ast}{5}(z)\).

If the state is not at the origin \(x = 0\), then according to the adaptive law (25) the gain \(L(t)\) will grow at a rate \(\ell\). The second term on the right hand side (rhs) of (30) is therefore negative when \(L(t)\) is growing and the first term on the rhs is negative for every perturbation (recall the analysis in the proof of Proposition 1) when \(L(t) > L^* \triangleq \delta_0 \mu\), where \(\mu\) is given in (21). In the worst case, the first rhs term in (30) will be positive because e.g. \(\Delta(t)\) grows faster than \(L(t)\). However, this cannot happen for all the time, since \(\Delta(t)\) is bounded, and therefore after a finite time \(L(t) > L^*\) and the first rhs term in (30) will become negative. In this case we can write
\[
\dot{V} \leq -\frac{\chi_{min}}{L^2(t)} \left( 1 - \mu \frac{\dot{\Delta}(t)}{L(t)} \right) V^\frac{\ast}{5}(z) - 5q \eta_{min} \frac{\dot{L}(t)}{L(t)} V(z).
\] (32)

We note that the Bernoulli differential equation
\[
\dot{v} = -h(t)v^\frac{\ast}{5} - 5q \eta_{min} \frac{\dot{L}(t)}{L(t)} v,
\] (33)
has as solution
\[
v^\frac{\ast}{5}(t) = \exp \left( -q \eta_{min} \int_{t_0}^{t} \frac{\dot{L}(\tau)}{L(\tau)} d\tau \right) v^\frac{\ast}{5}(t_0) - \frac{1}{5} \int_{t_0}^{t} \exp \left( -q \eta_{min} \int_{s}^{t} \frac{\dot{L}(\tau)}{L(\tau)} d\tau \right) h(s) ds
\]
\[
= \exp \left( -q \eta_{min} \ln \left( \frac{L(t)}{L(t_0)} \right) \right) v^\frac{\ast}{5}(t_0) - \frac{1}{5} \int_{t_0}^{t} \exp \left( -q \eta_{min} \ln \left( \frac{L(t)}{L(s)} \right) \right) h(s) ds.
\]
From the last expression it is clear that if \(L(t)\) is growing and \(h(t) > 0\), then \(v(t)\) will become zero in finite time.

Using the comparison Lemma (Khalil (2002)) on inequality (32), and considering the solution of the differential equation (33) with \(h(t) = \frac{\dot{\Delta}(t)}{L(t)} \left( 1 - \mu \frac{\dot{\Delta}(t)}{L(t)} \right)\), which is assumed to be positive, it
follows that $V(z)$ will become zero in finite time, and therefore $x(t) = 0$ will be reached. After this time instant $L(t)$ will stop growing and will remain constant.

It may happen that $x$ becomes zero in finite time but that $L(t) < L^*$. In this case, the perturbation can grow further and the state $x(t)$ will leave the origin. In this case the gain $L(t)$ will grow at a rate $\ell$ for at least a duration of $\tau$, i.e. $L(t)$ will increase its value by at least $\tau \ell$, but it will grow until $x(t)$ becomes again zero. If still $L(t) < L^*$ this process can be repeated again. However, since $L^*$ is finite and at every such event $L(t)$ is increased by at least $\tau \ell$, these events can happen only a finite number of times, before $L(t)$ reaches the value $L^*$, i.e. the number of such events $N$ is at the most

$$N \leq \frac{L^*}{\tau \ell} + 1.$$ 

It follows that $x(t) = 0$ after a finite time, and it will remain there for all future times.

5. Simulation Example

In this Section some simulations will be presented to illustrate the results. Consider the system

$$\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= u + \Delta(t) \\
u &= -L^\frac{\dot{z}_1}{2}(t)k_1[x_1]^\frac{1}{2} - L^\frac{\dot{z}_2}{2}(t)k_2[x_2]^\frac{1}{2} + \eta \\
\dot{\eta} &= -L(t) (k_3[x_1]^0 + k_4[x_2]^0)
\end{align*}$$

where $L(t)$ is the adaptive gain with adaptation law (25). The parameters have been chosen as $\ell = 15$, $\tau = 1$ s and the gains $k_1, k_2, k_3$ and $k_4$ are selected as in (13). The perturbation term $\Delta(t)$ is given by

$$\Delta(t) = \begin{cases} 
5 & \text{if } t \leq 20 \text{ s} \\
3t - 55 & \text{if } 20 < t \leq 30 \text{ s} \\
35 & \text{if } 30 < t \leq 40 \text{ s} \\
6t - 205 & \text{if } 40 < t \leq 50 \text{ s} \\
95 & \text{if } 50 < t \leq 60 \text{ s} \\
9t - 445 & \text{if } t > 60 \text{ s}.
\end{cases}$$

Figure 3 shows the behavior of the perturbation $\Delta(t)$, and the growth of the gain $L(t)$. The derivative of the perturbation $\dot{\Delta}(t)$ reaches its upper bound $\delta_p = 9$ at $t = 60$ s, while the gain $L$ stops growing at $t = 220$ s.

Figure 4 shows, that the state initially converges to zero at time $12.45$ s, and therefore the adaptive gain $L(t)$ stops growing (Figure 3), since it has reached a value large enough to compensate the perturbation. In the interval $20 < t \leq 30$ the perturbation grows again, and shortly after the state deviates from zero. As a consequence the adaptive gain $L(t)$ starts growing again until reaching a new constant value, large enough to compensate the new magnitude of the perturbation. This process is repeated at the interval $50 < t \leq 60$ s. Note that for the interval $20 < t \leq 30$ the derivative of the perturbation is 3, for the interval $40 < t \leq 50$ is 6 and after time 60 it is 9. In every case the gain $L(t)$ adapts its value to the derivative of the perturbation.

Figure 5 shows the behavior of the control signal $u(t)$ and of the perturbation $\Delta(t)$. Note that $u(t)$ is continuous, and as soon as $x(t)$ (see Figure 4) reaches the origin, $u(t)$ compensates exactly.
the perturbation $\Delta(t)$, i.e. $u(t) = -\Delta(t)$. This is also clear from Figure 6, which presents the value of the perturbation $\Delta(t)$ and the negative of the control signal $-u(t)$. In this Figure it is also observed, that during the time intervals $I_1 = [30, 32]$ and $I_2 = [51, 57]$, in which the sliding mode is lost (see Figure 4), the control signal $u(t)$ does not compensate exactly the perturbation $\Delta(t)$.

![Adaptive gain L and perturbation Δ(t)](image)

**Figure 3.** Adaptive gain $L(t)$ and perturbation $\Delta(t)$

![Euclidean norm of state vector x](image)

**Figure 4.** Euclidean norm of state vector $x(t)$

6. Conclusions

For the double integrator with a Lipschitz continuous perturbation we propose an Adaptive Continuous Twisting Algorithm (ACTA), which is a controller providing a continuous control signal and assuring the convergence of the state to the origin in finite time in spite of the perturbation. The only assumption is that the derivative of the perturbation is bounded, but this bound is unknown.
Similar to the classical super-twisting the ACTA contains a discontinuous integral term, resembling the Twisting Controller, and which is able to compensate perfectly the perturbation. Since the control signal is continuous the chattering effect is reduced. ACTA ensures that the states of the uncertain double integrator will remain in zero after a finite time.

7. ACKNOWLEDGMENT

The authors are grateful for the financial support of CONACyT (Consejo Nacional de Ciencia y Tecnología) grants: 261737, 241171 CVU 419644 and Project 241171; PAPIIT-UNAM (Programa de Apoyo a Proyectos de Investigación e Innovación Tecnológica) IN 113614, IN 113613; DGAPA PASPA Program, and Fondo de Colaboración del II-FI UNAM IISGBAS-122-2014.
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